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Boundary behavior of p-precise functions on a half space of R^n

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1. Introduction

Let u be a function which is locally p-precise in $D=\{x=(x_1,\ldots, x_n);\ x_n>0\}$, $n\geqq 2$, and satisfies

$$(1) \qquad \int_D |\mathrm{grad}\, u(x)|^p x_n^\alpha dx < \infty, \quad 1<p<\infty, \quad -1<\alpha<p-1$$

(see Ohtsuka [12] for (locally) p-precise functions). Many authors have tried to find a set $F\subset D$ such that $u(x)$ has a finite limit as x tends to the boundary ∂D along F (see Aikawa [1], Carleson [2], Mizuta [5], [7], [8], [9], Wallin [13]). They were mainly concerned with the nontangential case, that is, the case where $F=\ell_\xi\equiv\{\xi+(0, t);\ t>0\}$ or $F=\Gamma(\xi, a)\equiv\{x=(x', x_n)\in R^{n-1}\times R^1;\ |x'-\xi'|<ax_n\}$; if $u(x)$ has a finite limit as $x_n\downarrow 0$ along ℓ_ξ, then u is said to have a perpendicular limit at ξ, and if $u(x)$ has a finite limit as $x\to\xi$ along $\Gamma(\xi, a)$ for any $a>0$, then u is said to have a nontangential limit at ξ. The existence of tangential limits of u at ξ was discussed by Aikawa [1] and Mizuta [9]. The proof of the existence of these limits can be carried out by local arguments; in fact it requires to find conditions near ξ which assure the existence of limits.

In this paper we investigate a global behavior of u near the boundary ∂D. More precisely, we aim to find a function $A(x)$ such that $A(x)u(x)$ tends to zero as x tends to ∂D along a set $F\subset D$. In order to evaluate the size of F, we use the capacity:

$$C_p(E;\ G) = \inf \|f\|_p^p,$$

where the infimum is taken over all nonnegative measurable functions f on R^n such that $f=0$ outside G and $\int_G |x-y|^{1-n}f(y)dy\geqq 1$ for every $x\in E$; $\|\cdot\|_p$ denotes the L^p-norm in R^n. As in Aikawa [1], we introduce a notion of thinness of a set in D, near the boundary ∂D; we say that a set E is *C_p-thin near ∂D* if there exists a positive integer j_0 such that

$$\text{in case } p<n, \quad \textstyle\sum_{j=j_0}^\infty 2^{j(n-p)}C_p(E_j;\ D)<\infty,$$

$$\text{in case } p=n, \quad \textstyle\sum_{j=j_0}^\infty C_p(E_j\cap G_1;\ G_2)<\infty$$

for any bounded open sets G_1, G_2 such that $\bar{G}_1$ (the closure of G_1) is included in G_2, and

$$\text{in case } p > n, \quad \cup_{j=j_0}^{\infty} E_j \text{ is empty,}$$

where $E_j = \{x = (x', x_n) \in E;\ 2^{-j} \leqq x_n < 2^{-j+1}\}$.

First we shall establish the following result.

THEOREM 1. *Let* $-1 < \alpha < p-1$. *If* u *is a function which is locally* p*-precise in* D *and satisfies* (1), *then there exists a set* $E \subset D$ *such that* E *is* C_p*-thin near* ∂D *and*

$$\lim_{x_n \to 0, x \in D-E} x_n^{(n-p+\alpha)/p} u(x) = 0, \qquad \text{in case } n - p + \alpha > 0,$$

$$\lim_{x_n \to 0, x \in D-E} [\log (x_n^{-1}(|x|+1))]^{1/p-1} u(x) = 0, \qquad \text{in case } n - p + \alpha = 0,$$

$$\limsup_{x_n \to 0, x \in D-E} (|x|+1)^{(n-p+\alpha)/p} |u(x)| < \infty, \qquad \text{in case } n - p + \alpha < 0.$$

Next we study the boundary behavior of functions u satisfying the additional condition that $\lim_{x_n \downarrow 0} u(x', x_n) = 0$ for almost every $x' \in R^{n-1}$; for such a function u we can prove later the existence of a sequence $\{\varphi_j\}$ of functions in $C_0^{\infty}(D)$ such that $\int_D |\mathrm{grad}\,(u - \varphi_j)|^p x_n^{\alpha} dx \to 0$ as $j \to \infty$ (see Proposition 3). It will be expected naturally that such functions behave better than those in Theorem 1, near the boundary ∂D. In fact, we can prove the following result.

THEOREM 2. *Let* α *and* p *be as in Theorem* 1. *Let* u *be a function which is locally* p*-precise in* D *and satisfies* (1). *If* $\lim_{t \downarrow 0} u(x', t) = 0$ *for almost every* $x' \in R^{n-1}$, *then there exists a set* E *which is* C_p*-thin near* ∂D *and satisfies*

$$\lim_{x_n \to 0, x \in D-E} x_n^{(n-p+\alpha)/p} u(x) = 0.$$

As applications of Theorems 1 and 2, we shall discuss the existence of radial and perpendicular limits of u multiplied by a suitable weight function. If in addition u is assumed to be harmonic in D, then it will be shown that u multiplied by a weight has a limit as the variable tends to the boundary of D. Naturally, if we apply the same methods, then we can prove the existence of nontangential and parabolic limits in the usual sense; for related results, see Cruzeiro [3], Mizuta [10], Nagel, Rudin and Shapiro [11] and Wallin [13].

2. Lemmas

In order to prove Theorem 1, we prepare several lemmas. First we establish an integral representation of functions satisfying (1), which is a main tool in our discussions. For this purpose, we consider the functions $k_j(x, y) = (x_j - y_j)|x-y|^{-n} - (-y_j)|y|^{-n}$ if $|y|>1$ and $k_j(x, y) = (x_j - y_j)|x-y|^{-n}$ if $|y| \leqq 1$, for $j = 1, \ldots, n$. Then it is easy to see that

(2) $$|k_j(x, y)| \leqq M|x|\,|y|^{-n} \quad \text{whenever} \quad |y| \geqq 2|x| > 2$$

with a positive constant M.

LEMMA 1 (cf. [5; Lemma 6]). *Let* $-1 < \alpha < p-1$ *and* f *be a nonnegative function in* $L^p(R^n)$, *and define*

$$u(x) = \int k_j(x, y) f(y)\,|y_n|^{-\alpha/p} dy.$$

Then u *is locally* p*-precise in* D *and locally* q*-precise in* R^n *for* q *such that* $1 < q < \min\{p, p/(\alpha+1)\}$. *Further,* u *satisfies*

$$\int |\mathrm{grad}\, u(x)|^p |x_n|^\alpha dx \leqq M \|f\|_p^p$$

with a positive constant M *independent of* f.

PROOF. With the aid of (2), it follows from Hölder's inequality that $\int (1 + |y|)^{-n} |f(y)| |y_n|^{-\alpha/p} dy < \infty$ for $f \in L^p(R^n)$. For $R > 1$, letting $B(0, R)$ denote the open ball with center at the origin and radius R, we write

$$\begin{aligned} u(x) &= \int_{B(0,R)} k_j(x, y) f(y)\,|y_n|^{-\alpha/p} dy + \int_{R^n - B(0,R)} k_j(x, y) f(y)\,|y_n|^{-\alpha/p} dy \\ &= u'(x) + u''(x). \end{aligned}$$

Then u' is locally p-precise in $D \cap B(0, R)$ in view of Lemma 3.3 in [4] and u'' is continuously differentiable in $B(0, R)$. Hence it is seen that u is locally p-precise in D. If $1 < q < \min\{p, p/(\alpha+1)\}$, then we have by Hölder's inequality

$$\int_G (f(y)|y_n|^{-\alpha/p})^q dy \leqq \left(\int_G f(y)^p dy\right)^{q/p} \left(\int_G |y_n|^{-(\alpha q/p)/(1-q/p)} dy\right)^{1-q/p} < \infty$$

for any bounded open set $G \subset R^n$. Consequently we see as above that u is locally q-precise in R^n.

Let $c_n = (2-n)^{-1}$ if $n \geqq 3$ and $c_n = 2^{-1}$ if $n = 2$. Define $k_\varepsilon(x) = c_n(|x|^2 + \varepsilon^2)^{(2-n)/2}$ in case $n \geqq 3$ and $k_\varepsilon(x) = c_2 \log(|x-y|^2 + \varepsilon^2)$ in case $n = 2$, and set $k_{\varepsilon,j}(x, y) = ((\partial/\partial x_j)k_\varepsilon)(x-y)$ if $|y| \leqq 1$ and $k_{\varepsilon,j}(x, y) = ((\partial/\partial x_j)k_\varepsilon)(x-y) - ((\partial/\partial x_j)k_\varepsilon)(-y)$ if $|y| > 1$. We further define

$$u_\varepsilon(x) = \int k_{\varepsilon,j}(x, y) f(y)\,|y_n|^{-\alpha/p} dy.$$

In view of Lemma 3.3 in [4], we see that $(\partial/\partial x_i)u_\varepsilon(x)$ tends to $(\partial/\partial x_i)u(x)$ in $L^p_{loc}(R^n - \partial D)$ as $\varepsilon \to 0$. Thus we have only to prove

$$(3) \qquad \int |\mathrm{grad}\, u_\varepsilon(x)|^p |x_n|^\alpha dx \leqq M_2 \|f\|_p^p$$

with a positive constant M_2 independent of ε and f. For this, we first note that $(\partial/\partial x_i)u_\varepsilon(x) = \int (\partial/\partial x_i)(\partial/\partial x_j)\, k_\varepsilon(x-y) f(y)|y_n|^{-\alpha/p} dy$. Setting $v_\varepsilon(x) = \int (\partial/\partial x_j)$ $k_\varepsilon(x-y)f(y)dy$, we have

$$(4) \qquad \int |\mathrm{grad}\, v_\varepsilon(x)|^p dx \leqq M_3 \|f\|_p^p$$

by the proof of Lemma 3.2 in [4], and further

$$||x_n|^{\alpha/p}(\partial/\partial x_i)u_\varepsilon(x) - (\partial/\partial x_i)v_\varepsilon(x)| \leqq M_4 \int \frac{|1-(|x_n|/|y_n|)^{\alpha/p}|}{|x-y|^n} f(y)dy,$$

where M_3 and M_4 are positive constants independent of ε and f. By the proof of Lemma 6 in [5], the L^p-norm in R^n of the right hand side is dominated by $M_5\|f\|_p$ as long as $\int_0^\infty |1-y_n^{-\alpha/p}||1-y_n|^{-1} y_n^{-1/p} dy_n < \infty$, or $-1<\alpha<p-1$, with a positive constant M_5. Thus, with the aid of (4), we can establish (3), and the proof of Lemma 1 is completed.

LEMMA 2 (cf. Ohtsuka [12; Lemma 9.16]). *If h is a function which is harmonic in R^n and satisfies* (1) *with D replaced by R^n and with α such that $-1<\alpha<p-1$, then h is constant.*

PROOF. By the mean value property of harmonic functions and Hölder's inequality, we have

$$\begin{aligned} |(\partial/\partial x_i)h(x)| &= |M_1 r^{-n} \int_{B(x,r)} (\partial/\partial y_i)h(y)dy| \\ &\leqq M_1 r^{-n} \left(\int_{B(x,r)} |y_n|^{-\alpha p'/p} dy \right)^{1/p'} \left(\int_{B(x,r)} |\mathrm{grad}\, h(y)|^p |y_n|^\alpha dy \right)^{1/p} \\ &\leqq M_2 \left(\frac{r+|x_n|}{r} \right)^n (r+|x_n|)^{-(n+\alpha)/p} \left(\int_{B(x,r)} |\mathrm{grad}\, h(y)|^p |y_n|^\alpha dy \right)^{1/p}, \end{aligned}$$

where M_1, M_2 are positive constants independent of x, r and $1/p+1/p'=1$. Letting $r\to\infty$, we establish

$$(\partial/\partial x_i)h(x) = 0,$$

from which it follows that h is constant.

By Lemmas 1 and 2, we establish an integral representation of functions satisfying (1).

LEMMA 3. *Let $-1<\alpha<p-1$. For functions u, v which are locally p-precise in D and satisfy* (1), *set $w(x', x_n) = u(x', x_n)$ when $x_n>0$ and $w(x', x_n) = v(x', -x_n)$ when $x_n<0$. If $\lim_{t\downarrow 0} u(x', t) = \lim_{t\downarrow 0} v(x', t)$ for almost every x', then w*

is extended to a function w^* which is locally q-precise in R^n for any q such that $1<q<\min\{p, p/(\alpha+1)\}$. Further there exist a number A and a set E such that $C_p(E \cap G; G)=0$ for any bounded open set $G \subset R^n$ and

$$w(x) = c \sum_{j=1}^n \int k_j(x, y)(\partial/\partial y_j)w^*(y)dy + A$$

for every $x \in D-E$, where c is a constant depending only on the dimension n.

REMARK. If $p>n$, then any locally p-precise function on D is continuous there, and the above integrals converge absolutely at any $x \in D$ and are continuous on D. Moreover, if $p>n$ and $C_p(E \cap G; G)=0$ for any bounded open set $G \subset R^n$, then E is empty.

PROOF OF LEMMA 3. If $1<q<\min\{p, p/(\alpha+1)\}$, then, as in the proof of Lemma 1, Hölder's inequality yields $\int_G |\mathrm{grad}\ u|^q dx<\infty$ for any bounded open set $G \subset D$. In view of Ohtsuka [12; Theorem 5.6], w is extended to a function w^* which is locally q-precise in R^n; here we remark that w^* is an ACL function on R^n if we define $w^*(x', 0)=\liminf_{t \downarrow 0} u(x', t)$, and hence grad w^* is well-defined almost everywhere and measurable on R^n.

Set $W(x)=\sum_{j=1}^n \int k_j(x, y)(\partial/\partial y_j)w^*(y)dy$. Then, in view of Lemma 1, W is locally q-precise in R^n and satisfies (1) with D replaced by the whole space R^n. We shall prove that $\Delta(w^*-cW)=0$ for some constant c. For this purpose, let $\varphi \in C_0^\infty(R^n)$ and note by Fubini's theorem that

$$\begin{aligned}\int W(x)\Delta\varphi(x)dx &= \sum_{j=1}^n \int\left(\int k_j(x, y)\Delta\varphi(x)dx\right)(\partial/\partial y_j)w^*(y)dy \\ &= -c' \sum_{j=1}^n \int (\partial/\partial y_j)\varphi(y)(\partial/\partial y_j)w^*(y)dy \\ &= c' \int w^*(y)\Delta\varphi(y)dy\end{aligned}$$

with a positive constant c' depending only on n. By Lemma 2, by letting $c=c'^{-1}$, we see that w^*-cW is equal to a constant A a.e. on R^n. Since w and W are locally p-precise in D, $E=\{x \in D; w(x) \neq cW(x)+A\}$ satisfies the required conditions.

COROLLARY. *Let $-1<\alpha<p-1$, $n-p+\alpha>0$ and u be a function which is locally p-precise in D and satisfies (1). Then the function $u(x', |x_n|)$ on $R^n-\partial D$ is extended to a function $\bar{u}$ which is locally q-precise in R^n for q such that $1<q<\min\{p, p/(\alpha+1)\}$. Moreover, there exist a number A and a set E such that $C_p(E \cap G; G)=0$ for any bounded open set $G \subset R^n$ and*

$$u(x) = c \sum_{j=1}^n \int (x_j-y_j)|x-y|^{-n}(\partial/\partial y_j)\bar{u}(y)dy + A$$

for every $x \in D-E$, *where c is the same constant as above.*

This is an easy consequence of Lemma 3, since, in case $n-p+\alpha>0$, $\int(1+|y|)^{1-n}|f(y)|dy<\infty$ for any measurable function f on R^n such that $\int |f(y)|^p \cdot |y_n|^\alpha dy<\infty$.

We here give a technical lemma for later use.

LEMMA 4. *Let* $\beta<n, \gamma>-1$ *and* $r_1>2r_2>0$. *If* $x=(x', x_n)\in D$ *and* $x_n \leqq 2r_2$, *then*

$$\int_{B(0,r_1)-B(x,r_2)} |x-y|^{\beta-n}|y_n|^\gamma dy \leqq M \begin{cases} (r_1^{\beta+\gamma}+r_2^{\beta+\gamma}) & \text{in case } \beta+\gamma \neq 0, \\ \log(r_1/r_2) & \text{in case } \beta+\gamma = 0, \end{cases}$$

where M is a positive constant independent of x, r_1 *and* r_2.

PROOF. Let $x=(x', x_n)$ satisfy $0<x_n\leqq 2r_2$. First we note that

$$\begin{aligned} &\int_{B(0,r_1)-B(x,r_2)} |x-y|^{\beta-n}|y_n|^\gamma dy \\ &\leqq \int_{B(0,r_1)-B(x,r_1)} |x-y|^{\beta-n}|y_n|^\gamma dy + \int_{B(x,r_1)-B(x,r_2)} |x-y|^{\beta-n}|y_n|^\gamma dy \\ &\leqq r_1^{\beta-n}\int_{B(0,r_1)} |y_n|^\gamma dy + \int_{\{y\in B(x,r_1)-B(x,r_2);\, y_n\geqq x_n/2\}} |x-y|^{\beta-n}|y_n|^\gamma dy \\ &\quad + \int_{\{y\in B(x,r_1)-B(x,r_2);\, y_n< x_n/2\}} |x-y|^{\beta-n}|y_n|^\gamma dy = I_1+I_2+I_3. \end{aligned}$$

Since $\gamma>-1$, $I_1=M_1 r_1^{\beta+\gamma}$ with a positive constant M_1. Letting $z=(x', 0)$, since $|x-y|>|z-y|$ if $y_n<x_n/2$, we see that $\{y\in B(x, r_1);\, y_n<x_n/2\}\subset B(z, r_1)$, so that we obtain

$$\begin{aligned} I_3 &\leqq \int_{B(z,r_1)-B(z,r_2)} |z-y|^{\beta-n}|y_n|^\gamma dy + \int_{B(z,r_2)-B(x,r_2)} |x-y|^{\beta-n}|y_n|^\gamma dy \\ &\leqq \int_{B(0,r_1)-B(0,r_2)} |y|^{\beta-n}|y_n|^\gamma dy + M_1 r_2^{\beta+\gamma}. \end{aligned}$$

If $\gamma<0$, then

$$\begin{aligned} I_2 &\leqq \int_{B(x,r_1)-B(x,r_2)} |x-y|^{\beta-n}|x_n-y_n|^\gamma dy \\ &= \int_{B(0,r_1)-B(0,r_2)} |y|^{\beta-n}|y_n|^\gamma dy. \end{aligned}$$

If $\gamma\geqq 0$, then $|y_n/|x-y||\leqq 1+x_n/|x-y|\leqq 3$ if $|x-y|>x_n/2$, so that

$$I_2 \leqq 3^\gamma \int_{B(x,r_1)-B(x,r_2)} |x-y|^{\beta+\gamma-n}dy = 3^\gamma \int_{B(0,r_1)-B(0,r_2)} |y|^{\beta+\gamma-n}dy.$$

Thus the lemma is proved.

LEMMA 5. *Let p and α be as in Theorem 1. Let f be a nonnegative function in $L^p(R^n)$ and set $u(x)=\int_{R^n-B(0,2|x|)} k_j(x, y)f(y)|y_n|^{-\alpha/p}dy$. Then there exists a positive constant $M>0$ independent of f such that*

$$|u(x)| \leqq M|x|^{-(n-p+\alpha)/p}\|f\|_p$$

for any $x \in D-B(0, 1/2)$.

PROOF. Since there exists $M_1>0$ such that $|k_j(x, y)| \leqq M_1|x||y|^{-n}$ whenever $|y|>1$ and $|y| \geqq 2|x|$, we have by Hölder's inequality

$$\begin{aligned}
&\left|\int_{R^n-B(0,2|x|)} k_j(x, y)f(y)|y_n|^{-\alpha/p}dy\right| \\
&\leqq M_1|x|\left(\int_{R^n-B(0,2|x|)} |y|^{-np'}|y_n|^{-\alpha p'/p}dy\right)^{1/p'} \|f\|_p \\
&= M_2|x|^{-(n-p+\alpha)/p}\|f\|_p
\end{aligned}$$

for any $x \in R^n-B(0, 1/2)$.

3. Proof of Theorem 1

Let u be a function which is locally p-precise in D and satisfies condition (1). Then, in view of the corollary to Lemma 3, there exist a number A and a set $F \subset D$ such that $C_p(F \cap G; G)=0$ for any bounded open set $G \subset R^n$ and

$$u(x) = c \sum_{j=1}^n \int k_j(x, y)(\partial/\partial y_j)\bar{u}(y)dy + A$$

holds for any $x \in D-F$, where $\bar{u}$ is defined as in the corollary to Lemma 3. It is easy to see that F is C_p-thin near ∂D. Therefore, letting f be a nonnegative function in $L^p(R^n)$, we have only to prove Theorem 1 for the function

$$U(x) = \int k_j(x, y)f(y)|y_n|^{-\alpha/p}dy.$$

Let

$$U_1(x) = \int_{R^n-B(0,2|x|)} k_j(x, y)f(y)|y_n|^{-\alpha/p}dy,$$

$$U_2(x) = \int_{B(0,2|x|)-B(x,x_n/2)} k_j(x, y)f(y)|y_n|^{-\alpha/p}dy$$

and

$$U_3(x) = \int_{B(x,x_n/2)} k_j(x, y)f(y)|y_n|^{-\alpha/p}dy.$$

Then we see that $U_1(x)$ and $U_2(x)$ are finite for $x \in D$ but $U_3(x)$ is finite for $x \in D$

except those in a set F' satisfying $C_p(F' \cap G;\, G)=0$ for any bounded open set $G \subset R^n$.

First we treat the function U_1.

LEMMA 6. *If $n-p+\alpha>0$, then* $\lim_{x_n \downarrow 0} x_n^{(n-p+\alpha)/p} U_1(x)=0$.

PROOF. If $x \in D - B(0,\, 1/2)$, then Lemma 5 implies

$$x_n^{(n-p+\alpha)/p}|U_1(x)| \leqq M_1(2x_n)^{(n-p+\alpha)/p}\|f\|_p$$

for some positive constant M_1 independent of x. Hence we have

$$\lim_{x_n \downarrow 0,\, x \in D-B(0,1/2)} x_n^{(n-p+\alpha)/p} U_1(x) = 0.$$

We next assume that $x \in B(0,\, 1/2)$. If $0<2x_n<\varepsilon$, then it follows from Hölder's inequality that

$$\begin{aligned} |U_1(x)| \leqq\ & M_2\Big(|x|\int_{R^n-B(0,1)} |y|^{-n}f(y)|y_n|^{-\alpha/p}dy \\ & + \int_{\{y \in B(0,1)-B(0,2|x|);\, |y_n| \geqq \varepsilon\}} |x-y|^{1-n}f(y)|y_n|^{-\alpha/p}dy \\ & + \int_{\{y \in B(0,1)-B(0,2|x|);\, |y_n| < \varepsilon\}} |x-y|^{1-n}f(y)|y_n|^{-\alpha/p}\, dy\Big) \\ \leqq\ & M_3\Big\{\|f\|_p + \varepsilon^{1-n}\int_{B(0,1)} f(y)|y_n|^{-\alpha/p}dy \\ & + |x|^{-(n-p+\alpha)/p}\Big(\int_{\{y;\, |y_n|<\varepsilon\}} f(y)^p dy\Big)^{1/p}\Big\} \end{aligned}$$

with positive constants M_2 and M_3 independent of x and ε. Consequently, we obtain

$$\limsup_{x_n \downarrow 0,\, x \in B(0,1/2)} x_n^{(n-p+\alpha)/p}|U_1(x)| \leqq M_3\Big(\int_{\{y;\, |y_n|<\varepsilon\}} f(y)^p dy\Big)^{1/p},$$

which implies by arbitrariness of ε that the left hand side in equal to zero. Thus the required statement is established.

In the same manner as Lemma 6 we can derive the following two results.

LEMMA 7. *If $n-p+\alpha<0$, then $(|x|+1)^{(n-p+\alpha)/p}U_1(x)$ is bounded on D.*

LEMMA 8. *If $n-p+\alpha=0$, then* $\lim_{x_n \downarrow 0} [\log(1/x_n)]^{-1/p'}U_1(x)=0$.

Next we treat the function U_2 in the case $n-p+\alpha=0$, that would be the most difficult case.

LEMMA 9. *If $n-p+\alpha=0$, then* $\lim_{x_n \downarrow 0} [\log((|x|+1)/x_n)]^{-1/p'}U_2(x)=0$.

PROOF. For $x \in D - B(0, 1/2)$, we have

$$|U_2(x)| \leqq M_1\Big(\int_{B(0,2|x|)-B(x,x_n/2)} |x-y|^{1-n} f(y)\,|y_n|^{-\alpha/p} dy + \int_{B(0,2|x|)-B(0,1)} |y|^{1-n} f(y)|y_n|^{-\alpha/p} dy\Big)$$

with a positive constant M_1 independent of x. If $0 < 4x_n < 2\delta_2 < 2 < \delta_1$, then we have by Lemma 4

$$\int_{B(0,\delta_1)-B(x,\delta_2)} |x-y|^{p'(1-n)}|y_n|^{-\alpha p'/p} dy \leqq M_2 \log(\delta_1/\delta_2)$$

with a positive constant M_2 independent of δ_1, δ_2 and x. Hence it follows that

$$\int_{B(0,2|x|)-B(0,\delta_1)-B(x,x_n/2)} |x-y|^{1-n} f(y)\,|y_n|^{-\alpha/p} dy \leqq M_3[\log((|x|+1)/x_n)]^{1/p'} \Big(\int_{R^n - B(0,\delta_1)} f(y)^p dy\Big)^{1/p},$$

$$\int_{B(0,\delta_1)-B(x,\delta_2)} |x-y|^{1-n} f(y)\,|y_n|^{-\alpha/p} dy \leqq M_3[\log(\delta_1/\delta_2)]^{1/p'} \|f\|_p$$

and

$$\int_{B(x,\delta_2)-B(x,x_n/2)} |x-y|^{1-n} f(y)\,|y_n|^{-\alpha/p} dy \leqq M_3[\log(\delta_2/x_n)]^{1/p'} \Big(\int_{\{y;|y_n|\leqq\delta_2+x_n\}} f(y)^p dy\Big)^{1/p}$$

with a positive constant M_3. In the same manner we have

$$\int_{B(0,2|x|)-B(0,1)} |y|^{1-n} f(y)\,|y_n|^{-\alpha/p} dy \leqq M_4\Big\{[\log(4|x|)]^{1/p'}\Big(\int_{R^n-B(0,\delta_1)} f(x)^p dy\Big)^{1/p} + (\log\delta_1)^{1/p'}\|f\|_p\Big\},$$

where $\delta_1 > 2$ and M_4 is a positive constant independent of δ_1 and x. From these facts we obtain

$$\limsup_{x_n \to 0,\, x \in D - B(0,1/2)} [\log((|x|+1)/x_n)]^{-1/p'} U_2(x) \leqq (M_3 + M_4)\Big(\int_{R^n-B(0,\delta_1)} f(y)^p dy\Big)^{1/p} + M_3\Big(\int_{\{y;|y_n|\leqq\delta_2\}} f(y)^p dy\Big)^{1/p},$$

which implies that the left hand side is equal to zero. If $x \in D \cap B(0, 1/2)$, then

$$|U_2(x)| \leqq M_5 \int_{B(0,2|x|)-B(x,x_n/2)} |x-y|^{1-n} f(y)\,|y_n|^{-\alpha/p} dy$$

with a positive constant M_5. Hence, by the same considerations as above, we deduce

$$\lim_{x_n \downarrow 0, x \in D \cap B(0,1/2)} [\log(1/x_n)]^{-1/p'} U_2(x) = 0,$$

and Lemma 9 is established.

In the same manner we can prove the following results.

LEMMA 10. *If* $n-p+\alpha>0$, *then* $\lim_{x_n \downarrow 0} x_n^{(n-p+\alpha)/p} U_2(x)=0$.

LEMMA 11. *If* $n-p+\alpha<0$, *then* $|x|^{(n-p+\alpha)/p} U_2(x)$ *is bounded on* D.

REMARK. If $n-p+\alpha<0$ and $\xi \in \partial D$, then we can show that $\int |k_j(\xi, y)| f(y) \cdot |y_n|^{-\alpha/p} dy < \infty$ and $\lim_{x \to \xi, x \in \Gamma(\xi, a)} (U_1(x)+U_2(x)) = \int k_j(\xi, y) f(y) |y_n|^{-\alpha/p} dy = U(\xi)$ for any $a>0$.

LEMMA 12. *If* $p \leqq n$, *then there exists a set* $E \subset D$ *which is* C_p-*thin near* ∂D *such that*

$$\lim_{x_n \to 0, x \in D-E} x_n^{(n-p+\alpha)/p} U_3(x) = 0.$$

PROOF. First we note that $\sum_{j=1}^{\infty} \int_{D_j} f(y)^p dy < \infty$, where $D_j = \{y=(y', y_n); 2^{-j-1} < y_n < 2^{-j+2}\}$. Hence we find a sequence $\{a_j\}$ of positive numbers such that $\lim_{j \to \infty} a_j = \infty$ and $\sum_{j=1}^{\infty} a_j \int_{D_j} f(y)^p dy < \infty$. Consider the sets

$$E_j = \{x=(x', x_n) \in D;\ 2^{-j} \leqq x_n < 2^{-j+1},\ |U_3(x)| > 2^{j(n-p+\alpha)/p} a_j^{-1/p}\}$$

and $E = \cup_{j=1}^{\infty} E_j$. Since $B(x, x_n/2) \subset D_j$ if $2^{-j} \leqq x_n < 2^{-j+1}$, we can find a positive constant M_1 independent of j, x such that

$$|U_3(x)| \leqq M_1 2^{j\alpha/p} \int_{B(x, x_n/2)} |x-y|^{1-n} f(y) dy \tag{5}$$

whenever $2^{-j} \leqq x_n < 2^{-j+1}$. Let G_1 and G_2 be open sets for which there exists a number c such that $0<c<1/2$ and $B(x, cx_n) \subset G_2$ for any $x \in G_1$. Then easy calculation gives

$$\begin{aligned}\int_{B(x,x_n/2)-B(x,cx_n)} |x-y|^{1-n} f(y) dy &\leqq M_2 2^{j(n-p)/p} \left(\int_{D_j} f(y)^p dy \right)^{1/p} \\ &= M_2 [2^{j(n-p)/p} a_j^{-1/p}] \left(a_j \int_{D_j} f(y)^p dy \right)^{1/p}\end{aligned}$$

for x such that $2^{-j} \leqq x_n < 2^{-j+1}$, where M_2 is a positive constant independent of x and j. Consequently, if j is large enough, say $j \geqq j_0$, then we see from (5) that

$$\int_{B(x,cx_n)} |x-y|^{1-n} f(y) dy \geqq (2M_1)^{-1} 2^{j(n-p)/p} a_j^{-1/p},$$

whenever $x \in E_j$. Hence we have by the definition of C_p

$$C_p(E_j \cap G_1; D_j \cap G_2) \leqq (2M_1)^p 2^{-j(n-p)} a_j \int_{D_j} f(y)^p dy$$

for $j \geqq j_0$, from which it follows that

$$\sum_{j=j_0}^{\infty} 2^{j(n-p)} C_p(E_j \cap G_1; D_j \cap G_2) < \infty. \tag{6}$$

If $p<n$, then (6) with $G_1=G_2=D$ means the C_p-thinness of E near ∂D. If $p=n$, then (6) implies the C_p-thinness of E near ∂D. Clearly,

$$\limsup_{x_n \to 0, x \in D-E} x_n^{(n-p+\alpha)/p} |U_3(x)| \leqq 2^{|n-p+\alpha|/p} \limsup_{j \to \infty} a_j^{-1/p} = 0.$$

Hence E satisfies all the conditions in Lemma 12, and the proof of Lemma 12 is completed.

LEMMA 13. *If $p>n$, then* $\lim_{x_n \downarrow 0} x_n^{(n-p+\alpha)/p} U_3(x)=0$.

PROOF. By Hölder's inequality we have

$$\begin{aligned} |U_3(x)| &\leqq M_1 x_n^{-\alpha/p} \int_{B(x,x_n/2)} |x-y|^{1-n} f(y) dy \\ &\leqq M_2 x_n^{-\alpha/p} x_n^{(p-n)/p} \left(\int_{B(x,x_n/2)} f(y)^p dy \right)^{1/p} \end{aligned}$$

with positive constants M_1 and M_2. Hence the required equality follows readily.

PROOF OF THEOREM 1. By Lemmas $6 \sim 13$, the proof of Theorem 1 is completed.

For simplicity, we define $A(x)=x_n^{(n-p+\alpha)/p}$ if $n-p+\alpha>0$, $A(x)=[\log((|x|+1)/x_n)]^{-1/p'}$ if $n-p+\alpha=0$ and $A(x)=(|x|+1)^{(n-p+\alpha)/p}$ if $n-p+\alpha<0$. Further we set $a_j=2^{j(n-p)}$ if $n-p+\alpha>0$, $a_j=j^{p-1}2^{j(n-p)}$ if $n-p+\alpha=0$ and $a_j=2^{-\alpha j}$ if $n-p+\alpha<0$ for each positive integer j. In view of the proof of Theorem 1 we can establish the following result.

PROPOSITION 1. *Let $-1<\alpha<p-1$, $p \leqq n$ and u be a function which is locally p-precise in D and satisfies $\int_D |\mathrm{grad}\, u|^p x_n^{\alpha} dx<\infty$. Then there exists a set $E \subset D$ satisfying*

$$\sum_{j=1}^{\infty} a_j C_p(E_j \cap G_1; D_j \cap G_2) < \infty \tag{7}$$

for any open sets G_1 and G_2 for which there exists a number $c>0$ such that $B(x, cx_n) \subset G_2$ whenever $x \in G_1$, and

$$\lim_{x_n \downarrow 0, x \in D-E} A(x)u(x) = 0 \qquad \textit{in case} \quad n-p+\alpha \geqq 0,$$

$$\limsup_{x_n \downarrow 0, x \in D-E} A(x)u(x) < \infty \quad \textit{in case} \quad n - p + \alpha < 0.$$

REMARK. If $n-p+\alpha>0$, then (7) is equivalent to the C_p-thinness of E near ∂D.

We shall show below that Proposition 1 is best possible as to the size of the exceptional sets.

PROPOSITION 2. *Let $-1<\alpha<p-1$, $p \leqq n$ and E be a bounded subset of D satisfying $\sum_{j=1}^{\infty} a_j C_p(E_j;\ G \cap D_j)<\infty$, where G is a bounded open set including the closure of E. Then there exists a nonnegative function $f \in L^p(R^n)$ such that $u(x)=\int |x-y|^{1-n}f(y)|y_n|^{-\alpha/p}dy \not\equiv \infty$ and $\lim_{x_n \downarrow 0, x \in E} A(x)u(x)=\infty$.*

PROOF. By the definition of C_p, for each j we can find a nonnegative measurable function f_j such that $f_j=0$ outside $G \cap D_j$, $\|f_j\|_p^p<C_p(E_j;\ G \cap D_j)+\varepsilon_j$ and $\int_{G \cap D_j} |x-y|^{1-n}f_j(y)dy \geqq 1$ for every $x \in E_j$, where $\{\varepsilon_j\}$ is a sequence of positive numbers such that $\sum_{j=1}^{\infty} a_j\varepsilon_j<\infty$. Further we can find a sequence $\{b_j\}$ of positive numbers such that $\lim_{j\to\infty} b_j=\infty$ and $\sum_{j=1}^{\infty} b_j a_j\{C_p(E_j;\ G \cap D_j)+\varepsilon_j\}<\infty$. We now consider the function $f=\sum_{j=1}^{\infty} b_j^{1/p}a_j^{1/p}f_j$. Then

$$\int f(y)^p dy \leqq 3 \sum_{j=1}^{\infty} b_j a_j \int f_j(y)^p dy \leqq 3 \sum_{j=1}^{\infty} b_j a_j \{C_p(E_j;\ G \cap D_j)+\varepsilon_j\} < \infty.$$

Moreover, if $x \in E_j$, then we have

$$u(x) \geqq b_j^{1/p}a_j^{1/p}\int |x-y|^{1-n}f_j(y)\,|y_n|^{-\alpha/p}dy \geqq M b_j^{1/p}A(x)^{-1},$$

where M is a positive constant. Since f vanishes outside G, $u(x) \not\equiv \infty$. Hence f has the required properties in the proposition.

REMARK. In view of the proof of Lemma 1, the above function u satisfies $\int |\mathrm{grad}\, u|^p|x_n|^\alpha dx<\infty$.

4. Proof of Theorem 2

We begin with the following result.

LEMMA 14. *Let $-1<\alpha<p-1$ and let u be a locally p-precise function on D satisfying* (1). *If $\lim_{t \downarrow 0} u(x', t)=0$ for almost every $x' \in R^{n-1}$, then there exists a set $E \subset D$ such that $C_p(E \cap G;\ G)=0$ for any bounded open set $G \subset D$ and*

$$u(x) = c \sum_{j=1}^{n} \int_D (x_j-y_j)(|x-y|^{-n}-|\bar{x}-y|^{-n})(\partial u/\partial y_j)(y)dy$$
$$+ 2cx_n \int_D |\bar{x}-y|^{-n}(\partial u/\partial y_n)(y)dy$$

for any $x \in D-E$, *where* $\bar{x}=(x', -x_n)$ *for* $x=(x', x_n)$ *and* c *is the absolute constant given in Lemma* 3.

PROOF. Setting $u^*(x', x_n)=u(x', x_n)$ if $x_n>0$ and $u^*(x)=0$ otherwise, we note that u^* is locally q-precise in R^n for q, $1<q<\min\{p, p/(\alpha+1)\}$. Hence we can apply Lemma 3 and obtain

$$u^*(x) = c\sum_{j=1}^{n}\int k_j(x, y)(\partial u^*/\partial y_j)dy + A$$

for $x \in R^n - E$, where A is a constant depending on u and $C_p(E \cap G; G)=0$ for any bounded open set $G \subset R^n$. If $x \in D-E$ and $\bar{x} \notin E$, then

$$u(x) = u^*(x) - u^*(\bar{x}) = c\sum_{j=1}^{n}\int (k_j(x, y)-k_j(\bar{x}, y))(\partial u^*/\partial y_j)dy,$$

which implies that u satisfies the required equality.

By this lemma we can establish the following result.

PROPOSITION 3. *If* u *is as in Lemma* 14, *then there exists a sequence* $\{\varphi_j\} \subset C_0^\infty(D)$ *such that* $\int_D |\mathrm{grad}\,(\varphi_j-u)|^p x_n^\alpha dx$ *tends to zero as* $j\to\infty$.

PROOF. For $N>0$, set

$$u_N(x) = c\sum_{j=1}^{n}\int_{D\cap B(0,N)}(k_j(x, y)-k_j(\bar{x}, y))(\partial u/\partial y_j)dy$$

with the constant c given above. In view of Lemma 1, we find a positive number M_1 (independent of N) such that

$$\int_D |\mathrm{grad}\,(u_N-u)|^p x_n^\alpha dx \leqq M_1\int_{D-B(0,N)} |\mathrm{grad}\,u|^p x_n^\alpha dx,$$

from which the left hand side tends to zero as $N\to\infty$. For $\varepsilon>0$, define

$$u_{N,\varepsilon}(x) = c\sum_{j=1}^{n}\int_{\{y=(y',y_n);\,y_n>\varepsilon\}\cap B(0,N)}\{k_j(x, y)-k_j(\bar{x}, y)\}(\partial u/\partial y_j)dy.$$

Then $u_{N,\varepsilon}$ is continuous on ∂D and vanishes there. Moreover, $u_{N,\varepsilon}(x)$ tends to zero as $|x|\to\infty$, and, again by Lemma 1,

$$\int_D |\mathrm{grad}\,(u_{N,\varepsilon}-u_N)|^p x_n^\alpha dx \leqq M_1\int_{\{y\in B(0,N);\,0<y_n<\varepsilon\}} |\mathrm{grad}\,u|^p y_n^\alpha dy.$$

Finally, we set $u_{N,\varepsilon,\delta}(x)=\max\{u_{N,\varepsilon}(x)-\delta, 0\}+\min\{u_{N,\varepsilon}(x)+\delta, 0\}$ for $\delta>0$. Then $u_{N,\varepsilon,\delta}$ vanishes outside some compact set in D and

$$\int_D |\mathrm{grad}\,(u_{N,\varepsilon,\delta}-u_{N,\varepsilon})|^p x_n^\alpha dx \longrightarrow 0 \quad \text{as} \quad \delta \downarrow 0.$$

Thus we can find a sequence $\{v_j\}$ such that each v_j is a p-precise function on D with compact support in D and

$$\int_D |\mathrm{grad}\,(v_j-u)|^p x_n^\alpha dx \longrightarrow 0 \quad \text{as} \quad j \longrightarrow \infty.$$

By a routine method of regularization of functions v_j, we obtain a sequence $\{\varphi_j\}$ with the required properties.

PROOF OF THEOREM 2. Let u be as in Theorem 2. In view of Lemma 14, the equality

$$u(x) = c\sum_{j=1}^n \int_D (k_j(x, y)-k_j(\bar{x}, y))(\partial u/\partial y_j)dy$$

holds for $x \in D-E$, where $C_p(E \cap G; G)=0$ for any bounded open set G. We note here that E is C_p-thin near ∂D.

We see from elementary calculation that $|k_j(x, y)-k_j(\bar{x}, y)| \leqq M_1 x_n(y_n|x-y|^{1-n}|\bar{x}-y|^{-2}+|\bar{x}-y|^{-n})$ for any x and y in D, with a positive constant M_1. Hence we can find a positive constant M_2 such that

$$\begin{aligned}|u(x)| \leqq M_2\Big(x_n \int_{D-B(x,x_n/2)} |x-y|^{-n}|\mathrm{grad}\,u|dy \\ + \int_{B(x,x_n/2)} |x-y|^{1-n}|\mathrm{grad}\,u|dy\Big) = M_2(U_1(x)+U_2(x))\end{aligned}$$

for $x \in D-E$. For $\delta > x_n/2$ we have by Hölder's inequality and Lemma 4

$$\begin{aligned}U_1(x) \leqq M_3 x_n^{1-(n+\alpha)/p}\Big(\int_{D\cap B(x,\delta)-B(x,x_n/2)} |\mathrm{grad}\ u|^p y_n^\alpha\, dy\Big)^{1/p} \\ + M_3\delta^{-(n+\alpha)/p}x_n\Big(\int_{D-B(x,\delta)} |\mathrm{grad}\ u|^p y_n^\alpha dy\Big)^{1/p}\end{aligned}$$

with a positive constant M_3. Therefore it follows that

$$\limsup_{x_n \downarrow 0} x_n^{(n-p+\alpha)/p} U_1(x) \leqq M_3\Big(\int_{\{y\in D;\, y_n<\delta\}} |\mathrm{grad}\ u|^p x_n^\alpha dx\Big)^{1/p},$$

which implies that the left hand side is equal to zero. As in the proofs of Lemmas 12 and 13, we can find a set $E' \subset D$ which is C_p-thin near ∂D and satisfies

$$\lim_{x_n \to 0, x\in D-E'} x_n^{(n-p+\alpha)/p} U_2(x) = 0.$$

Now the proof of Theorem 2 is completed.

Set $G_1(x, y)=|x-y|^{1-n}-|\bar{x}-y|^{1-n}$. Then by elementary calculation we find $M>0$ such that

$$M^{-1}x_n y_n|x-y|^{1-n}|\bar{x}-y|^{-2} < G_1(x, y) < Mx_n y_n|x-y|^{1-n}|\bar{x}-y|^{-2}$$

whenever x any y are in D. Hence we can find a positive number M' such that $|x-y|^{1-n} \leqq M'G_1(x, y)$ whenever $y \in B(x, x_n/2)$. Thus we obtain the following result.

THEOREM 2′. *If u is as in Theorem 2, then there exists a set $E \subset D$ such that*

$$\lim_{x_n \downarrow 0, x \in D-E} x_n^{(n-p+\alpha)/p} u(x) = 0$$

and

$$\sum_{j=1}^{\infty} 2^{j(n-p)} C_{G_1}(E_j; D_j) < \infty, \tag{8}$$

where $C_{G_1}(F; G) = \inf \|g\|_p^p$, the infimum being taken over all nonnegative measurable functions g on R^n such that $g=0$ outside an open set G and $\int_G G_1(x, y)g(y)dy \geqq 1$ for any x in a set F.

REMARK. If E satisfies (8), then E is C_p-thin near ∂D; in case $p<n$, (8) is equivalent to the C_p-thinness near ∂D.

We shall show below that Theorem 2′ is best possible as to the size of the exceptional sets.

PROPOSITION 4. *Let $-1<\alpha<p-1$ and $p \leqq n$. If $E \subset D$ satisfies* (8), *then there exists a function u such that $\int |\text{grad}\, u|^p x_n^\alpha dx < \infty$, $\lim_{t \downarrow 0} u(x', t) = 0$ for almost every $x' \in R^{n-1}$ and $\lim_{x_n \downarrow 0, x \in E} x_n^{(n-p+\alpha)/p} u(x) = \infty$.*

PROOF. By the definition of C_{G_1}, we can find a nonnegative measurable function f_j such that $f_j = 0$ outside D_j, $\int_{D_j} G_1(x, y) f_j(y) dy \geqq 1$ and $\|f_j\|_p^p < C_{G_1}(E_j; D_j) + \varepsilon_j$, where $\{\varepsilon_j\}$ is a sequence of positive numbers such that $\sum_{j=1}^{\infty} 2^{j(n-p)} \varepsilon_j < \infty$. Letting $\{b_j\}$ be a sequence of positive numbers such that $\lim_{j \to \infty} b_j = \infty$ and $\sum_{j=1}^{\infty} b_j 2^{j(n-p)} \{C_{G_1}(E_j; D_j) + \varepsilon_j\} < \infty$, we consider the function $u(x) = \int_D G_1(x, y) f(y) dy$, where $f = \sum_{j=1}^{\infty} b_j^{1/p} 2^{j(n-p+\alpha)/p} f_j$. Then f vanishes outside D and

$$\int_D f(y)^p y_n^\alpha dy \leqq M_1 \sum_{j=1}^{\infty} b_j 2^{j(n-p+\alpha)} \int_D f_j(y)^p y_n^\alpha dy$$
$$\leqq M_1 \sum_{j=1}^{\infty} b_j 2^{j(n-p)} \{C_{G_1}(E_j; D_j) + \varepsilon_j\} < \infty$$

with a positive constant M_1. Thus, in the same way as in the proof of Lemma 1, we can prove that $\int_D |\text{grad}\, u|^p x_n^\alpha dx < \infty$. On the other hand, we have for $x \in E_j$

$$x_n^{(n-p+\alpha)/p} u(x) \geqq M_2 b_j^{1/p} \int_D G_1(x, y) f_j(y) dy \geqq M_2 b_j^{1/p},$$

where M_2 is a positive constant. This implies that $\lim_{x_n \downarrow 0, x \in E} x_n^{(n-p+\alpha)/p} u(x) = \infty$.

What remains is to show that $\lim_{t\downarrow 0} u(x', t)=0$ for almost every $x' \in R^{n-1}$. For this, it suffices to note that if $N>0$, then $\int_{B(0,N)} |x-y|^{1-n} f(y)dy$ is q-precise in R^n for q with $1<q<\min\{p, p/(\alpha+1)\}$, and hence it is absolutely continuous on the line $\ell_{x'}=\{(x', t); t\in R^1\}$ for almost every $x' \in R^{n-1}$.

5. Boundary behavior near the origin

We say that a set E is C_p-*thin at* the origin 0 if

$$\sum_{j=1}^{\infty} 2^{j(n-p)} C_p(E\cap B(0, 2^{-j+1})-B(0, 2^{-j}); B(0, 2^{-j+2})) < \infty.$$

For $a>0$, we set $\Gamma(a)=\{x=(x', x_n); |x'|<ax_n\}$.

LEMMA 15. *For any $a>0$, $\Gamma(a)$ is not C_p-thin at* 0.

PROOF. For each nonnegative integer j, set

$$\Gamma_j(a) = \Gamma(a) \cap B(0, 2^{-j+1}) - B(0, 2^{-j}).$$

Then $C_p(\Gamma_j(a); B(0, 2^{-j+2}))=2^{-j(n-p)}C_p(\Gamma_0(a); B(0, 4))$ and $C_p(\Gamma_0(a); B(0, 4))>0$, so that $\Gamma(a)$ is not C_p-thin at 0.

LEMMA 16. *Let $E\subset\Gamma(a)$, $a>0$. If $p\leqq n$ and $\sum_{j=1}^{\infty} a_j C_p(E\cap\Gamma_j(a); B(0, 2))<\infty$, then E is C_p-thin at* 0, *where $a_j=2^{j(n-p)}$ if $p<n$ and $a_j=j^{n-1}$ if $p=n$.*

PROOF. We shall give a proof only in the case $p=n$. For simplicity, set $E_j=E\cap\Gamma_j(a)$. Assume that $\sum_{j=1}^{\infty} j^{n-1}C_n(E_j; B(0, 2))<\infty$. Let f_j be a nonnegative measurable function on R^n such that $\int_{B(0,2)} |x-y|^{1-n} f_j(y)dy\geqq 1$ for any $x\in E_j$, $f_j=0$ outside $B(0, 2)$ and $\|f_j\|_n^n<C_n(E_j; B(0, 2))+j^{-n}$. Then, by Lemma 4, we have for $x\in E_j$

$$\begin{aligned}\int_{B(0,2)-B(x,x_n/2)} |x-y|^{1-n} f_j(y)dy &\leqq M_1(\log(4/x_n))^{1-1/n}\|f_j\|_n \\ &\leqq M_2(j^{n-1}C_n(E_j; B(0, 2))+j^{-1})^{1/n}\end{aligned}$$

with positive constants M_1 and M_2. Since $\sum_{j=1}^{\infty} j^{n-1}C_n(E_j; B(0, 2))<\infty$ by our assumption, if j is large enough, then

$$\int_{B(x,x_n/2)} |x-y|^{1-n} f_j(y)dy > 2^{-1}$$

for any $x\in E_j$. If $x\in E_j$, then $B(x, x_n/2)\subset B(0, 2^{-j+2})$, so that

$$C_n(E_j; B(0, 2^{-j+2})) \leqq 2^n\|f_j\|_n^n < 2^n[C_n(E_j; B(0, 2))+j^{-n}]$$

for large j, which implies easily that E is C_n-thin at 0.

The above proof shows that if $p<n$ and $E\subset B(0, 1)\cap D$, then the C_p-thinness of E near ∂D is equivalent to $\sum_{j=1}^{\infty} 2^{j(n-p)}C_p(E_j;\ B(0, 2)\cap D_j)<\infty$. For $a>0$, if we take k_0 such that $2^{k_0}>(a^2+1)^{1/2}$, then $E\cap\Gamma_j(a)\subset\bigcup_{k=0}^{k_0} E_{j+k}$, so that $a_jC_p(E\cap\Gamma_j(a);\ B(0, 2^{-j+2}))\leqq\sum_{k=0}^{k_0} a_{j+k}C_p(E_{j+k}\cap\Gamma(a);\ B(0, 2))$. Hence we obtain

COROLLARY. *If $p<n$ and $E\cap\Gamma(a)$, $a>0$, is C_p-thin near ∂D, then $E\cap\Gamma(a)$ is C_p-thin at* 0.

PROPOSITION 5. *If u is as in Theorem* 1, *then there exists a set $E\subset D$ such that $E\cap\Gamma(a)$ is C_p-thin at* 0 *for any $a>0$ and*

$$\lim_{x\to 0,x\in D-E} x_n^{(n-p+\alpha)/p}u(x) = 0, \qquad \text{in case} \quad n-p+\alpha>0,$$

$$\lim_{x\to 0,x\in D-E} [\log(1/x_n)]^{-1/p'}u(x) = 0, \quad \text{in case} \quad n-p+\alpha=0,$$

$$\lim_{x\to 0,x\in D-E} u(x) \text{ exists and is finite}, \quad \text{in case} \quad n-p+\alpha<0.$$

PROOF. The case where $p\leqq n$ and $n-p+\alpha\geqq 0$ is proved by Proposition 1 together with Lemma 16. The case $p>n$ and $n-p+\alpha\geqq 0$ is a consequence of Theorem 1. In case $n-p+\alpha<0$, with the notation in the proof of Theorem 1, we see that

$$\lim_{x\to 0,x\in D}\int_{R^n-B(x,|x|/2)} k_j(x, y)(\partial/\partial y_j)\bar{u}(y)dy$$
$$=\int k_j(0, y)(\partial/\partial y_j)\bar{u}(y)dy$$

for $j=1,\dots, n$, where the integrals converge absolutely. Moreover, as in the proof of Lemma 12, we see that $\int_{B(x,|x|/2)} k_j(x, y)(\partial/\partial y_j)\bar{u}(y)dy$ tends to zero as $x\to 0$ outside an exceptional set E such that $E\cap\Gamma(a)$ is C_p-thin at 0 for any $a>0$.

In the same manner we can establish the following result.

PROPOSITION 6. *If u is as in Theorem* 2, *then there exists a set $E\subset D$ such that $E\cap\Gamma(a)$ is C_p-thin at* 0 *for any $a>0$ and*

$$\lim_{x\to 0,x\in D-E} x_n^{(n-p+\alpha)/p}u(x) = 0.$$

The next two propositions show the best possibility of Propositions 5 and 6 as to the order of convergence.

PROPOSITION 7. *Let $-1<\alpha<p-1$ and $n-p+\alpha\geqq 0$. If h is a nonincreasing positive function on $(0, \infty)$ such that $\lim_{t\downarrow 0} h(t)=\infty$, then there exists a function $u\in C^\infty(D)$ satisfying* (1) *such that $\lim_{t\downarrow 0} u(x', t)=0$ for $x'\in R^{n-1}-\{0\}$ and $\lim_{x\to 0,x\in A} h(x_n)x_n^{(n-p+\alpha)/p}u(x)=\infty$ for some A which is not C_p-thin at* 0.

PROOF. Take a sequence $\{i_j\}$ of positive integers such that $i_j+2<i_{j+1}$ and $\sum_{j=1}^{\infty} a_j^{-p}<\infty$, where $a_j=h(2^{-i_j+1})$. Further take a sequence $\{b_j\}$ of positive numbers such that $\lim_{j\to\infty} a_j b_j=\infty$ and $\sum_{j=1}^{\infty} b_j^p<\infty$. Let φ be a function in $C_0^\infty(R^n)$ such that $\varphi=1$ on $B(0, 1/4)$ and $\varphi=0$ outside $B(0, 1/2)$. Setting $e^{(j)}=(0, 2^{-j})\in D$, we define

$$u(x) = \sum_{j=1}^{\infty} b_j 2^{i_j(n-p+\alpha)/p}\varphi(2^{i_j}(x-e^{(i_j)})).$$

Then it is easy to see that $\lim_{t\downarrow 0} u(x', t)=0$ for $x'\neq 0$ and

$$\int |\mathrm{grad}\, u|^p|x_n|^\alpha dx \leqq \sum_{j=1}^{\infty} b_j^p 2^{i_j(n-p+\alpha)}\int |\mathrm{grad}\, \varphi(2^{i_j}(x-e^{(i_j)}))|^p|x_n|^\alpha dx$$

$$\leqq \text{const.} \sum_{j=1}^{\infty} b_j^p < \infty.$$

If we set $A=\cup_{j=1}^{\infty} B(e^{(i_j)}, 2^{-i_j-2})$, then $A\subset\Gamma(1/2)$. Since $C_p(B(e^{(i_j)}, 2^{-i_j-2}); B(0, 2^{-i_j+2}))=2^{-i_j(n-p)}C_p(B(e^{(0)}, 1/4); B(0, 4))$, A is not C_p-thin at 0. Further, if $x\in B(e^{(i_j)}, 2^{-i_j-2})$, then $h(x_n)x_n^{(n-p+\alpha)/p}u(x)\geqq 2^{-(n-p+\alpha)/p}a_j b_j$, so that $\lim_{x\to 0, x\in A} h(x_n)x_n^{(n-p+\alpha)p}u(x)=\infty$.

PROPOSITION 8. *Let $\alpha=p-n>-1$. If h is as above, then there exists a nonnegative measurable function f such that $f=0$ outside $D\cap B(0, 1)$, $\int_D f(y)^p y_n^\alpha dy<\infty$ and*

$$\lim_{x\to 0, x\in A} h(x_n)(\log(1/x_n))^{-1/p'}u(x) = \infty$$

for some A which is not C_p-thin at 0, where $u(x)=\int |x-y|^{1-n}f(y)dy$.

REMARK. Since f has compact support, u satisfies (1).

PROOF OF PROPOSITION 8. As in the proof of Proposition 7 take a sequence $\{b_j\}$ of positive numbers such that $\lim_{j\to\infty} b_j h(2^{-2i_j+1})=\infty$ and $\sum_{j=1}^{\infty} b_j^p<\infty$; here we assume that $2i_j<i_{j+1}$. Define $f_j(y)=b_j|y|^{-1}(\log|y|^{-1})^{-1/p}$ if $y\in\Gamma(1)\cap B(0, 2^{-i_j})-B(0, 2^{-2i_j})$ and $f_j=0$ otherwise. Set $f=\sum_{j=1}^{\infty} f_j(y)$. Then

$$\int_D f(y)^p y_n^\alpha dy = \sum_{j=1}^{\infty}\int_D f_j(y)^p y_n^\alpha dy \leqq M_1\sum_{j=1}^{\infty} b_j^p < \infty,$$

where M_1 is a positive constant. Consider the sets $A_j=\{x\in\Gamma(1);\ 2^{-2i_j}<|x|<2^{-2i_j+1}\}$ and $A=\cup_{j=1}^{\infty} A_j$. Then, as in the proof of Lemma 15, we see that A is not C_p-thin at 0. Further, if $x\in A_j$, then

$$u(x) = \int |x-y|^{1-n}f(y)dy \geqq 3^{1-n}\int |y|^{1-n}f_j(y)dy \geqq M_2 b_j(\log(1/x_n))^{1/p'}$$

with a positive constant M_2, so that

$$\lim_{x\to 0, x\in A} h(x_n)(\log(1/x_n))^{-1/p'}u(x) = \infty.$$

6. Radial and perpendicular limits

In this section, as applications of Theorems 1 and 2, we study the existence of radial and perpendicular limits of functions satisfying (1).

THEOREM 3. *Let u be a function which is locally p-precise in D and satisfies* (1) *with α such that $-1<\alpha<p-1$. Then there exists a set $E'\subset\partial D$ such that $C_p(E'\cap G;\, G)=0$ for any bounded open set $G\subset R^n$ and*

$$\lim_{t\downarrow 0} t^{(n-p+\alpha)/p}u(x',t)=0, \qquad \text{in case } n-p+\alpha>0,$$

$$\lim_{t\downarrow 0} (\log(1/t))^{-1/p'}u(x',t)=0, \qquad \text{in case } n-p+\alpha=0,$$

$$u(x',t) \text{ has a finite limit as } t\downarrow 0, \qquad \text{in case } n-p+\alpha<0,$$

for any x' such that $(x',0)\notin E'$.

REMARK. In case $\alpha\geqq 0$, we can find a set $E''\subset\partial D$ such that $B_{1-\alpha/p,p}(E'')=0$ and $u(x',t)$ has a finite limit as $t\downarrow 0$ for any x' with $(x',0)\in\partial D-E''$, where $B_{\beta,p}$ denotes the Bessel capacity of index (β, p) (see [8; Theorem 3] for details). In case $\alpha<0$, from this fact we can find $E''\subset\partial D$ such that $B_{1,p}(E'')=0$ and $u(x',t)$ has a finite limit as $t\downarrow 0$ for any x' with $(x',0)\in\partial D-E''$. We note here that $B_{1,p}(F)=0$ if and only if $C_p(F\cap G;\, G)=0$ for any bounded open set $G\subset R^n$. Hence we see that in case $\alpha\leqq 0$, Theorem 3 follows readily from [8; Theorem 3].

For a proof of Theorem 3, we need the following fact.

LEMMA 17. *Let $E\subset D$ satisfy*

$$\sum_{j=1}^{\infty} C_p(E_j\cap B(0,r);\, B(0,2r))<\infty \qquad \text{for any} \quad r>0. \tag{9}$$

Then there exists a set $E'\subset\partial D$ having the following properties:

(i) *$C_p(E'\cap G;\, G)=0$ for any bounded open set $G\subset R^n$.*

(ii) *For each $\xi\in\partial D-E'$ there exists $\delta>0$ such that $\xi+(0,t)\notin E$ whenever $0<t<\delta$.*

PROOF. In view of Lemma 1 and its proof in [6], we first note that $C_p(E_j\cap B(0,r);\, B(0,2r))\geqq C_p(E_j^*\cap B(0,r);\, B(0,2r))$, where E_j^* denotes the projection of E_j to the hyperplane ∂D. Set $E'=\bigcap_{k=1}^{\infty}(\bigcup_{j=1}^{\infty} E_j^*)$. Then $C_p(E'\cap B(0,r);\, B(0,2r))\leqq\sum_{j=k}^{\infty} C_p(E_j^*\cap B(0,r);\, B(0,2r))$ for any k. Hence it follows that $C_p(E'\cap B(0,r);\, B(0,2r))=0$. On the other hand, if $\xi\in\partial D\cap B(0,r)-E'$, then there exists k such that $\xi\notin\bigcup_{j=k}^{\infty} E_j^*$. This implies that $\xi+(0,t)\notin E$ whenever $0<t<2^{-k+1}$. Thus the lemma is proved.

If E is C_p-thin near ∂D, then it satisfies (9). Hence, by the aid of Theorem 1, we obtain Theorem 3; in case $n-p+\alpha<0$, we need to notice the Remark after Lemma 11.

Theorem 2 together with Lemma 17 gives the following result.

THEOREM 4. *If u is as in Theorem 2, then there exists $E'\subset\partial D$ such that $C_p(E'\cap G; G)=0$ for any bounded open set $G\subset D$ and*

$$\lim_{t\downarrow 0} t^{(n-p+\alpha)/p}u(x', t) = 0 \quad \textit{whenever} \quad (x', 0)\in\partial D - E'.$$

Next we give radial limit theorems for functions satisfying (1).

THEOREM 5. *Let u be as in Theorem 1. Then, for each $\xi\in\partial D$, there exist a set $E_\xi\subset\partial B(\xi, 1)\cap D$ and a number c_ξ such that $C_p(E_\xi; B(\xi, 2))=0$ and*

$$\lim_{r\downarrow 0} A(r)u(\xi+r(\eta-\xi)) = c_\xi \quad \textit{if} \quad \eta\in D\cap\partial B(\xi, 1) - E_\xi,$$

where $A(r)=r^{(n-p+\alpha)/p}$ if $n-p+\alpha>0$, $A(r)=(\log(1/r))^{-1/p'}$ if $n-p+\alpha=0$ and $A(r)=1$ if $n-p+\alpha<0$.

Theorem 5 is a consequence of Proposition 5; instead of Lemma 17, we have only to note the following

LEMMA 18. *Let $E\subset D$. If E is C_p-thin at 0, then there exists a set $E^\sim\subset D\cap\partial B(0, 1)$ satisfying the following conditions:*

(i) *$C_p(E^\sim; B(0, 2))=0$.*

(ii) *For each $\eta\in D\cap\partial B(0, 1)-E^\sim$, there exists $\delta>0$ such that $r\eta\notin E$ whenever $0<r<\delta$.*

By Proposition 6 and Lemma 18 we can establish the following theorem.

THEOREM 6. *If u is as in Theorem 2, then, for each $\xi\in\partial D$ there exists a set $E_\xi\subset\partial B(\xi, 1)\cap D$ such that $C_p(E_\xi; B(\xi, 2))=0$ and*

$$\lim_{r\downarrow 0} r^{(n-p+\alpha)/p}u(\xi+r(\eta-\xi)) = 0 \qquad \textit{for every} \quad \eta\in D\cap\partial B(\xi, 1) - E_\xi.$$

7. Boundary behavior of harmonic functions

If u is harmonic in D, then, by Green's formula,

$$\sum_{j=1}^n \int_{B(x,x_n/2)} (x_j-y_j)|x-y|^{-n}(\partial u/\partial y_j)dy = 0$$

for $x\in D$. Consequently, the proof of Theorem 1 gives the following result.

THEOREM 7. *Let u be a function which is harmonic in D and satisfies* (1) *with α such that $-1<\alpha<p-1$. Then*

$\lim_{x_n \downarrow 0} x_n^{(n-p+\alpha)/p} u(x) = 0,$ *in case* $n - p + \alpha > 0,$

$\lim_{x_n \downarrow 0} [\log (x_n^{-1}(|x|+1))]^{-1/p'} u(x) = 0,$ *in case* $n - p + \alpha = 0,$

$\limsup_{x_n \downarrow 0} (|x|+1)^{(n-p+\alpha)/p} u(x) < \infty,$ *in case* $n - p + \alpha < 0.$

We can also prove the existence of tangential boundary limits of harmonic functions in D.

THEOREM 8. *Let u be a function which is harmonic in D and satisfies* (1) *with α such that $n-p+\alpha \geqq 0$ and $\alpha > -1$. Letting h be a positive nondecreasing function on the interval* $(0, \infty)$ *such that $h(2r) < Mh(r)$ for $r > 0$ with a positive constant M, we set*

$$E_1 = \left\{\xi \in \partial D;\ \int_{B(\xi,1)\cap D} |\xi - y|^{1-n} |\mathrm{grad}\, u(y)| dy < \infty\right\},$$

$$E_2 = \left\{\xi \in \partial D;\ \lim_{r\to 0} h(r)^{-1} \int_{B(\xi,r)} |\mathrm{grad}\, u(y)|^p |y_n|^\alpha dy = 0\right\}.$$

If $\xi \in \partial D - E_1 \cup E_2$, then $u(x)$ has a finite limit as $x \to \xi$, $x \in T_h(\xi, a) \equiv \{x \in D;$ $h(|x-\xi|) \leqq a\tilde{A}(x-\xi)\}$, for any $a > 0$, where $\tilde{A}(x) = x_n^{n-p+\alpha}$ if $n-p+\alpha > 0$ and $\tilde{A}(x) = [\log (2|x|/x_n)]^{1-p}$ if $n-p+\alpha = 0$.

REMARK 1. In view of [8; Lemma 4], $B_{1-\alpha/p,p}(E_1) = 0$. On the other hand we can prove that $H_h(E_2) = 0$ in the same way as Lemma 2 in [9], where H_h denotes the Hausdorff measure with the measure function h. If $h(r) = r^{\gamma(n-p+\alpha)}$ in case $n-p+\alpha > 0$ and $h(r) = [\log(2+r^{-1})]^{1-p}$ in case $n-p+\alpha = 0$, then $T_\gamma(\xi, a)$ is included in some $T_h(\xi, b)$, where $T_\gamma(\xi, a) = \{x = (x', x_n);\ |(x', 0) - \xi|^\gamma < a x_n\}$. Hence Theorem 8 implies the existence of limits of u along the sets $T_\gamma(\xi, a)$ (cf. Cruzeiro [3], Mizuta [10], Nagel, Rudin and Shapiro [11]).

REMARK 2. If u is a function on D which is harmonic in D and satisfies (1) with α such that $-1 < \alpha < p - n$, then u has a finite limit at any boundary point.

In fact, the sets E_1 and E_2 with $h \equiv 1$ in the theorem are shown to be empty, and, moreover, the proof below will show that u has a finite limit at any $\xi \in \partial D - E_1 \cup E_2$; see also [10; Theorem (iii)].

PROOF OF THEOREM 8. To prove Theorem 8, we use the integral representation of u given in Lemma 3 and write u as

$$u(x) = c\sum_{j=1}^n \int k_j(x, y)(\partial \bar{u}/\partial y_j) dy + C$$

$$= c\sum_{j=1}^n \int_{R^n - B(\xi, 2|x-\xi|)} k_j(x, y)(\partial \bar{u}/\partial y_j) dy$$

$$+ c \sum_{j=1}^{n} \int_{B(\xi, 2|x-\xi|)} k_j(x, y)(\partial \bar{u}/\partial y_j) dy + C$$
$$= u_1(x) + u_2(x) + C.$$

We remark here that since $\partial \bar{u}/\partial y_j$ are continuous on D, the integrals are continuous on D and the equalities hold everywhere on D. If $\xi \in \partial D - E_1$, then $\int |k_j(\xi, y)| \cdot |\text{grad}\, u(y)| dy < \infty$ for each j and u_1 has a finite limit as $x \to \xi$, $x \in D$. Since, as in the proof of Lemma 9, $|u_2(x)| \leqq M' \left(\tilde{A}(x-\xi)^{-1} \int_{B(\xi, 2|x-\xi|)} |\text{grad}\ u(y)|^p |y_n|^\alpha dy \right)^{1/p}$ with a positive constant M', $u_2(x)$ tends to zero as $x \to \xi$, $x \in T_h(\xi, a)$, if $\xi \in \partial D - E_2$. Thus the theorem is obtained.

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