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On the steady state of the heat conduction on a Riemannian symmetric space

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§1. Introduction

Let (X, g) be a Riemannian manifold and Δ the Laplacian associated to the Riemannian metric g. For any bounded continuous function f on X, the heat equation which has the initial value f is given by the following:

$$\begin{cases} \partial u/\partial t = -\Delta u & \text{on} \quad X \times (0, \infty) \\ u(x, 0) = f(x) & \text{for} \quad x \in X, \end{cases}$$

where the solution $u(x, t)$ is a function in $C^0(X \times [0, \infty))$ and is assumed to be twice continuously differentiable in x and once continuously differentiable in t, for $(x, t) \in X \times (0, \infty)$. These equations describe the conduction of heat through the homogeneous medium X. When X is a compact manifold or X is a bounded domain with smooth boundary ∂X in a larger Riemannian manifold (in this case we impose in addition, the boundary condition that

$$u(b, t) = \psi(b) \qquad \text{for} \quad (b, t) \in \partial X \times (0, \infty),$$

where ψ is a bounded continuous function on ∂X and the solution u is in $C^0(\overline{X} \times [0, \infty))$), it is known that $u(x, t)$ converges uniformly to a function which does not depend on t and is harmonic on X as the time t becomes large (cf. [1] Ch. VI, VII). The limit function is called the steady state. The purpose of the present article is to describe the steady state when X is a Riemannian symmetric space of the noncompact type under the condition that the initial value has the limit along the Martin boundary in the Oshima compactification $\tilde{X}$ of X. When X is a noncompact manifold, even in the case for $X = \boldsymbol{R}$ there exists an example of the initial value which does not converge to a steady state (but it has many ω-limits each of which is a constant function [8]).

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§2. The steady state on a symmetric space

Let X be a Riemannian symmetric space of the noncompact type. Then X

is isometric to a coset space G/K where G is a noncompact connected semisimple Lie group with finite center and K is a maximal compact subgroup. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K respectively and let B denote the Killing form of $\mathfrak{g}$. B is nondegenerate since $\mathfrak{g}$ is semisimple. Let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. Then $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ is the Cartan decomposition and let θ be the Cartan involution. $\mathfrak{p}$ is identified with the tangent space $T_o(X)$ at $o=\{K\}\in X$. The restriction $B\,|\,\mathfrak{p}\times\mathfrak{p}$ is positive definite, so defines an invariant Riemannian metric g on X. Let Δ be the corresponding Laplacian. Fix a maximal abelian subspace $\mathfrak{a}\subset\mathfrak{p}$ and let M be the centralizer of $\mathfrak{a}$ in K. If α is a linear function on $\mathfrak{a}$ and $\alpha\neq 0$, let $\mathfrak{g}_\alpha=\{X\in\mathfrak{g}\,|\,[H, X]=\alpha(H)X$ for all $H\in\mathfrak{a}\}$. α is called a restricted root if $\mathfrak{g}_\alpha\neq 0$. Let $\mathfrak{a}'$ be the open subset of $\mathfrak{a}$ where all restricted roots are $\neq 0$. Fix a Weyl chamber $\mathfrak{a}^+$ in $\mathfrak{a}$, i.e., a connected component of $\mathfrak{a}'$. A restricted root α is called positive (denoted by $\alpha>0$) if its values on $\mathfrak{a}^+$ are positive and let $\Pi=\{\alpha_1,\dots,\alpha_l\}$ be the corresponding set of simple roots. Let the linear function ρ on $\mathfrak{a}$ be defined by $2\rho=\sum_{\alpha>0}m_\alpha\alpha$ where $m_\alpha=\dim\mathfrak{g}_\alpha$ and denote by $\mathfrak{n}$ the subalgebra $\sum_{\alpha>0}\mathfrak{g}_\alpha$ and put $\bar{\mathfrak{n}}=\theta\mathfrak{n}$. Let A, N and $\bar{N}$ be the analytic subgroups of G corresponding to $\mathfrak{a}$, $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ respectively. Then $G=KAN$ is an Iwasawa decomposition. For $g\in G$ we write $g=\kappa(g)\exp H(g)n(g)$ with $\kappa(g)\in K$, $H(g)\in\mathfrak{a}$ and $n(g)\in N$. Put $A^+=\exp\mathfrak{a}^+$. Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{a})$, i.e., $W=N_K(\mathfrak{a})/M$ where N_K denotes the normalizer in K. In order to describe the behavior at infinity of functions on X we embed X into the Oshima compactification $\tilde{X}$. For the detailed definition of this compactification, see [10]. In this compactification we have the map $A^+o\to[0, 1]\subset\tilde{X}$ defined by $(\exp H)o\to(e^{-\alpha_1(H)},\dots, e^{-\alpha_l(H)})$ and the G-orbit B of the point o_∞ corresponding to $(0,\dots, 0)$ by the above embedding is called the Martin boundary of X. The stabilizer of o_∞ is $P=MAN$ and $B=G/P=K/M$. The normalized Haar measure dk on K induces a K-invariant measure on B. For h in $L^\infty(B)$ of the bounded measurable functions on B its Poisson integral on X is given by

$$\mathscr{P}h(g)=\int_K h(gk)dk.$$

The Poisson transformation $\mathscr{P}$ is a bijection of $L^\infty(B)$ onto the space of all the bounded solutions of Laplace equation $\Delta u=0$ on X.

Let f be a bounded continuous function on X. We consider the heat equation on X which has the initial value f:

$$\text{(1)}\qquad \begin{cases} \partial u/\partial t+\Delta u=0 & \text{on}\quad X\times(0, \infty)\\ u(x, 0)=f(x) & x\in X.\end{cases}$$

For the existence and uniqueness of the solution, see [1], Ch. VIII, Theorems 3 and 4 or [2]. Our theorem is as follows:

THEOREM. *Suppose that the initial value f has the limit along the Martin boundary B, i.e., there exists a bounded function f_∞ on B such that $f(x)$ converges to $f_\infty(b)$ as x tends in $\tilde{X}$ to a boundary point b. Then the solution u of the heat equation* (1) *has the steady state which is the harmonic function given by the Poisson integral of f_∞*

$$\lim_{t\to\infty} u(x, t) = \mathscr{P} f_\infty(x),$$

uniformly for x in every compact subset of X.

PROOF. The heat equation on a symmetric space $X=G/K$ has the following Gauss kernel by the Plancherel theorem (cf. [4]).

$$g_t(x) = \int_{\mathfrak{a}^*} e^{-t(|\nu|^2+|\rho|^2)}\phi_\nu(x)\,|c(\nu)|^{-2}d\nu/w, \quad x\in G,$$

where $\mathfrak{a}^*$ is the dual space of $\mathfrak{a}$, $\phi_\nu(x)=\int_K e^{(i\nu-\rho)H(xk)}dk$ is the elementary spherical function corresponding to ν in $\mathfrak{a}^*$, $c(\nu)$ is the Harish-Chandra c-function and $w=\sharp W$ the order of the Weyl group. By this kernel function the solution u of (1) is given by

$$u(x, t) = \int_{G/K} g_t(x^{-1}y)f(y)d\dot{y} = \int_{G/K} g_t(y)f(xy)d\dot{y}, \quad x\in G.$$

The integral formula for the Cartan decomposition $G=KAK$ yields that for $x=kak'$, $dx=D(a)dkdadk'$, $D=\prod_{\alpha>0}|\sinh\alpha|^{m_\alpha}$ and

$$u(x, t) = \int_{A^+} g_t(a)\Big(\int_K f(xka)dk\Big)D(a)da.$$

We know that (cf. [4], Prop. 3.1) $g_t\geqq 0$ on X and

$$\int_G g_t(x)dx = \int_{A^+} g_t(a)D(a)da = 1 \quad \text{for each} \quad t>0.$$

We shall prove the following Lemma in the next section.

LEMMA 1. *For any compact set C in the closure $\bar{X}$ in $\tilde{X}$ such that $C\cap B=\emptyset$, we have*

$$\int_{C\cap B} g_t(x)dx \longrightarrow 0 \quad \textit{when} \quad t\longrightarrow\infty.$$

Taking this Lemma for granted we proceed as follows. Given $\varepsilon>0$, by the assumption (2) we can take an open neighborhood U of o_∞ in the closure $\mathrm{Cl}(A^+o)\cong[0, 1]$ such that

$$|f(xka)-f_\infty(xko_\infty)| \leqq \varepsilon \quad \text{for any} \quad a\in U, \quad k\in K.$$

Then KU is an open set in $\bar{X}$ containing B, therefore $C=\bar{X}-KU$ is compact and $C\cap B=\emptyset$. By Lemma 1, we have

$$\int_{C\cap X} g_t(x)dx = \int_{A^+-U} g_t(a)D(a)da \longrightarrow 0 \quad \text{when} \quad t \longrightarrow \infty.$$

Hence we have

$$u(x, t) = \int_K f(xko_\infty)dk + \int_{A^+} g_t(a)D(a)da \int_K (f(xka)-f(xko_\infty))dk,$$

$$\int_{A^+} g_t(a)D(a)da \int_K (f(xka)-f(xko_\infty))dk = \int_{A^+-U} + \int_U .$$

As for the first term put $M=\sup_X |f|$, then $\sup_B |f_\infty| \leqq M$ and

$$\int_{A^+-U} g_t(a)D(a)da \int_K |f(xka)-f_\infty(xko)_\infty|dk$$

$$\leqq 2M \int_{A^+-U} g_t(a)D(a)da \longrightarrow 0 \quad \text{when} \quad t \longrightarrow \infty.$$

And for the second term,

$$\int_U g_t(a)D(a)da \int_K |f(xka)-f_\infty(xko_\infty)|dk \leqq \varepsilon \int_{A^+} g_t(a)D(a)da \int_K dk = \varepsilon.$$

Since we can take $\varepsilon>0$ arbitrarily small we obtain that

$$\int_{A^+} g_t(a)D(a)da \int_K |f(xka)-f_\infty(xko_\infty)|dk \longrightarrow 0 \quad \text{when} \quad t \longrightarrow \infty.$$

§ 3. Proof of Lemma 1

By the compactness of C it suffices to prove for any $x_0 \in \bar{X}-B$ there exists a neighborhood V of x_0 in $\bar{X}$ such that $V\cap B=\phi$ and $\int_{V\cap X} g_t(x)dx \to 0$ when $t\to\infty$. First for $x_0 \in X$, take any compact neighborhood V of x_0 in X. Then the inequality

$$g_t(x) \leqq \int_{\mathfrak{a}^*} e^{-t(|\nu|^2+|\rho|^2)} |\phi_\nu(x)|\, |c(\nu)|^{-2} d\nu/w$$

$$\leqq \phi_0(x) \int_{\mathfrak{a}^*} e^{-t(|\nu|^2+|\rho|^2)} |c(\nu)|^{-2} d\nu/w,$$

and the Lebesgue convergence theorem yield that when $t\to\infty$,

$$\int_V g_t(x)dx \leqq \int_V \phi_0(x)dx \int_{\mathfrak{a}^*} e^{-t(|\nu|^2+|\rho|^2)} |c(\nu)|^{-2} d\nu/w \longrightarrow 0,$$

since $|c(\nu)|^{-2}$ is at most polynomial growth. Next as for $x_0 \in \overline{X}-(X \cup B)$, there exist $g \in G$ and $H \in \mathrm{Cl}(\mathfrak{a}^+)$, $\neq 0$ such that $x_0 = \lim_{s\to\infty} g \exp sHo$ in $\tilde{X}$. We shall fix such g and H for the rest of this section. Since $x_0 \in B$, there exists a simple root $\alpha \in \Pi$ such that $\alpha(H)=0$. Note in this case that rank $G/K \geqq 2$. Here we recall some of the basic facts on parabolic subalgebras and establish the notation. For more details refer to [11], 1.2. Now let Θ_H be the set of simple roots vanishing at H. Θ_H (or H) defines a parabolic subalgebra $\mathfrak{p}_H = \mathfrak{g}_H + \mathfrak{n}^H$ where $\mathfrak{g}_H$ is the centralizer of H in $\mathfrak{g}$ and $\mathfrak{n}^H = \sum_{\alpha(H)>0} \mathfrak{g}_\alpha$. We have the direct decomposition $\mathfrak{g}_H = \mathfrak{m}_H + \mathfrak{a}^H = \mathfrak{m} + \mathfrak{a} + \sum_{\alpha(H)=0} \mathfrak{g}_\alpha$ where $\mathfrak{m}_H = \bar{\mathfrak{n}}_H + \mathfrak{m} + \mathfrak{a}_H + \mathfrak{n}_H$, $\mathfrak{n}_H = \sum_{\alpha(H)=0,\alpha>0} \mathfrak{g}_\alpha$, $\mathfrak{a}^H = \{X \in \mathfrak{a} \mid \alpha(X)=0 \text{ for all } \alpha \in \Theta_H\}$, $\mathfrak{a}_H = \{X \in \mathfrak{a} \mid B(X, \mathfrak{a}^H)=0\}$, $\bar{\mathfrak{n}}_H = \theta\mathfrak{n}_H$ and $\mathfrak{m}$ = the centralizer of $\mathfrak{a}$ in $\mathfrak{h}$. Note that $H \in \mathfrak{a}^H \neq \{0\}$, $\mathfrak{a}_H \neq \{0\}$ and $\mathfrak{a} = \mathfrak{a}^H + \mathfrak{a}_H$ (direct sum). Put $\bar{\mathfrak{p}}_H = \theta\mathfrak{p}_H$ and $\bar{\mathfrak{n}}_H = \theta\mathfrak{n}_H$. Let P_H, M_H, N_H, N^H, A_H, A^H, $\bar{P}_H$ $\bar{N}^H$ be the analytic subgroups of G with the corresponding Lie algebras. Then $P_H = M_H A^H N^H$ is the Langlands decomposition and P_H is the stabilizer of x_0 in G and also $\bar{P}_H = M_H A^H \bar{N}^H$. Every $\alpha \in \Theta_H$ has restriction zero on $\mathfrak{a}^H$ and restrictions of Θ_H to $\mathfrak{a}_H$ precisely form the roots of the pair $(\mathfrak{m}_H, \mathfrak{a}_H)$. Put $K_H = M_H \cap K$. Then we have the analytic diffeomorphism ψ: $\bar{N}^H \times A^H \times M_H/K_H \to G/K$ defined by $\psi(\bar{n}, a, yK_H) = \bar{n}ayK$ and for a suitable normalization of the measures, the invariant measure on G/K is written by $dx = a^{2\rho^H} d\bar{n}\, da\, d\dot{y}$ where $x = \bar{n}ayK$ and $2\rho^H = \sum_{\alpha(H)>0} \alpha$ (see [9], §9 or [11], Theorem 1.2.4.11). The following Lemma holds.

LEMMA 2. *Let $\bar{n} \in \bar{N}^H$, a, $a' \in A^H$ and y, $y' \in M_H$. We have the formula*:

$$a^{2\rho^H} \int_{\bar{N}^H} g_t(y^{-1}a^{-1}\bar{n}a'y')d\bar{n}$$
$$= (4\pi t)^{-l^H/2} e^{-|\log a' - \log a - 2tH_0|^2/4t}\, g_t'(y^{-1}y'),$$

where $l^H = \dim A^H$, g_t' is the Gauss kernel for the symmetric space M_H/K_H and $H_0 \in \mathfrak{a}^H$ is such that $B(H_0, X) = \rho^H(X)$ for $X \in \mathfrak{a}^H$.

For a proof, see [9] Theorem 16.4.1.

Under these preparation we proceed as follows: Recall $x_0 = \lim_{s\to\infty} g \exp sHo$. Let $g^{-1} \in \bar{n}ayK$ where $\bar{n} \in \bar{N}^H$, $a \in A^H$ and $y \in M_H$. Take any compact neighborhood U of $\{K_H\}$ in M_H/K_H. Then the set V= the closure of $\psi(\bar{N}^H \times A^H \times U)$ in $\overline{X}$ contains the geodesic $(\exp sH)o$ and gV forms a neighborhood of x_0 in $\overline{X}$. We have by Lemma 2,

$$\int_{gV\cap X} g_t(x)dx = \int_{V\cap X} g_t(gx)dx$$
$$= \int_{\bar{N}^H} \int_{A^H} \int_U g_t(y^{-1}a^{-1}\bar{n}a'y')a^{2\rho^H}\, d\bar{n}\, da'\, d\dot{y}'$$

$$=(4\pi t)^{-l^H/2}\int_{\mathfrak{a}^H} e^{-|X-\log a-2tH_0|^2/4t}dX\int_U g_t'(y^{-1}y')d\dot{y}'$$

$$=(4\pi t)^{-l^H/2}\int_{\mathfrak{a}^H} e^{-|X|^2/4t}dX\int_{y^{-1}U} g_t'(y')d\dot{y}'$$

$$=\int_{y^{-1}U} g_t'(y')\,d\dot{y}' \longrightarrow 0 \text{ when } t \longrightarrow \infty$$

since $y^{-1}U$ is compact in M_H/K_H. This completes the proof of Lemma 1.

References

[1] I. Chavel, Eigenvalues in Riemannian Geometry, Academic Press, Orlando, Florida (1984).

[2] J. Dodziuk, Maximum principle for parabolic inequalities and the heat flow on open manifolds, Indiana Univ. Math. J. **32** (1983), 703–716.

[3] H. Donnelly and P. Li, Heat equation and compactifications of complete Riemannian manifolds, Duke Math. J., **51**, No. 3, (1984) 667–673.

[4] R. Gangolli, Asymptotic behavior of spectra of compact quotient of certain symmetric spaces, Acta Math. J., **121** (1968), 151–192.

[5] Harish-Chandra, Spherical functions on a semisimple Lie group, I. Amer. J. Math. **80** (1958), 241–310, II. Ibid, pp. 553–613.

[6] S. Helgason, Groups and Geometric Analysis. Integral Geometry, Invariant Differential Operators and Spherical Functions. Academic Press, New York (1984).

[7] S. Helgason and A. Koranyi, A Fatou-type theorem for harmonic functions on symmetric spaces, Bull. Amer. Math. Soc., **74** (1968) 258–263.

[8] H. Matano, personal communication.

[9] F. I. Karapelevič, The geometry of geodesics and the eigenfunctions of the Laplace-Beltrami operator on symmetric spaces, Trans. Moscow Math. Soc., **14** (1965), 48–185.

[10] T. Oshima, A realization of Riemannian symmetric spaces, J. Math. Soc. Japan., **30** (1978), 117–132.

[11] G. Warner, Harmonic Analysis on Semi-Simple Lie Groups, I, II, Springer-Verlag (1972).

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