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Study of three-dimensional algebras with straightening laws which are Gorenstein domains III

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Introduction

Since the previous papers [6] and [7] were published, some related topics [2], [3], [4] and [5] have been studied. Among them, in [5], we obtain certain combinatorial information about a partially ordered set from a ring-theoretical property of an affine semigroup ring which is an algebra with straightening laws on the partially ordered set.

Let k be a field, A a polynomial ring in a finite number of indeterminates over k and H a finite partially ordered set (*poset* for short) with an injection ρ: $H \hookrightarrow A$ such that $\rho(\alpha)$ is a monomial of A for any $\alpha \in H$. Then the couple (H, ρ) is called a *toroidal* poset if the subring

$$R_\rho := k[\{\rho(\alpha)\}_{\alpha \in H}]$$

is a homogeneous (cf. [6, (1.4)]) algebra with straightening laws (abbreviated ASL) on the poset H, with respect to the embedding ρ, over k. A toroidal poset (H, ρ) is called *Gorenstein* if R_ρ is Gorenstein. Also, we say that two toroidal posets (H, ρ) and (H', ρ') are *equivalent* if there exists a poset isomorphism ψ: $H \cong H'$ such that R_ρ and $R_{\rho' \circ \psi}$ are equivalent as ASL's in the sense of [6, §4]. To describe a toroidal poset (H, ρ) we will write the monomial $\rho(\alpha)$ near the vertex α in the Hasse diagram of the poset H.

Now, the purpose of this paper is to classify all the Gorenstein toroidal posets (H, ρ) with $\dim R_\rho = 3$. Our result is

THEOREM. *The Gorenstein toroidal posets* (H, ρ) *with* $\dim R_\rho = 3$ *are, up to equivalence as toroidal posets, as follows*:

Fig. 1

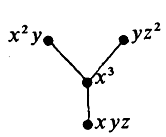


Fig. 2

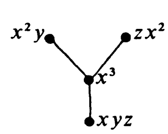


Fig. 3

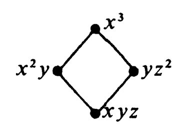


Fig. 4

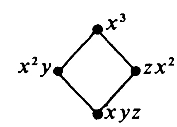


Fig. 5

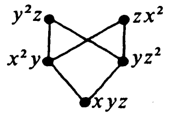


Fig. 6

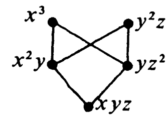


Fig. 7

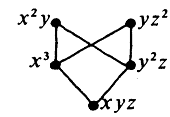


Fig. 8

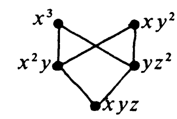


Fig. 9

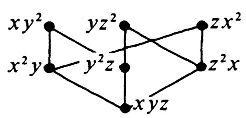


Fig. 10

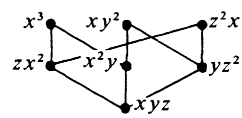


Fig. 11

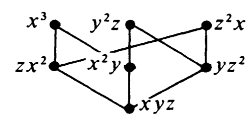


Fig. 12

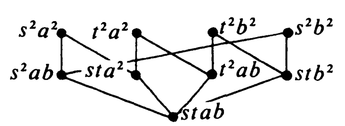


Fig. 13

Throughout this paper, we fix a field k. We shall refer to [6] for the basic definition and terminologies on commutative algebra and combinatorics and, unless otherwise stated, keep the notation in [6].

§1. A fundamental lemma of toroidal posets

Let H be a connected poset with $\operatorname{rank}(H)=2$. We denote by f_0 the cardinality $\sharp(H)$ of H as a set and write f_1 for the number of chains of length two contained in H. Then, since H is connected, the inequality $f_1-f_0+1\geq 0$ holds. Also, $f_1-f_0+1=0$ if and only if H contains no cycle (cf. [6, Fig. 14]). In general, a rank two poset is called a *tree* (cf. [4]) if it is connected without cycles. Recall that an element P of H is called an *upper* (resp. a *lower*) *branch* if there exists a unique element A (resp. X) such that $P>A$ (resp. $P<X$). Also, consult [6, p. 32] for the definition of *branch sequences.*

Throughout the remainder of this section, let (H, ρ) be a toroidal poset with $\dim R_\rho=\operatorname{rank}(H)=3$. Since R_ρ is an integral domain, H has a unique minimal element T and $H-\{T\}$ is connected by [6]. Also, thanks to [3], $H-\{T\}$ has neither lower branches nor branch sequences if $H-\{T\}$ is not a tree. Moreover, somewhat surprisingly, we can prove the following

LEMMA. *$H-\{T\}$ has no upper branch if $H-\{T\}$ is not a tree.*

PROOF. Let P be an upper branch of $H-\{T\}$ and A a unique element of $H-\{T\}$ with $A<P$. Note that, for any minimal element B $(\neq A)$ of $H-\{T\}$, there exists $X \in H-\{T\}$ such that $A<X$ and $B<X$ by [6, Prop. A].

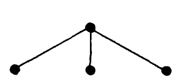
First, assume that for any minimal element B $(\neq A)$ of $H-\{T\}$ there exists exactly one element X of $H-\{T\}$ such that $A<X$ and $B<X$. Then $H-\{T\}$ has at least three minimal elements since $H-\{T\}$ is not a tree. Let B $(\neq A)$ and C $(\neq A)$ be two minimal elements of $H-\{T\}$. Write ξ and η for the elements of $H-\{T\}$ with $A<\xi$, $B<\xi$ and $A<\eta$, $C<\eta$. Then, in R_ρ, the set $[PB]$ (cf. [6, (1.3)]) is contained in $\{T^2, TA, T\xi\}$ by [6, Lemma 2]. Hence $AB=T\xi$ by [6, Lemma 1]. Thus $[PB]\subset\{T^2, TA\}$. Similarly, we obtain $AC=T\eta$, $[PC]\subset\{T^2, TA\}$. Hence we may assume $AB=T\xi$, $AC=T\eta$, $PB=T^2$ and $PC=TA$. Then we have $AB=TC$ since $(PC)B=(PB)C$, a contradiction.

Secondly, assume that for some minimal element B $(\neq A)$ of $H-\{T\}$ there exist at least two maximal elements X and Y of $H-\{T\}$ which are greater than both A and B. Then, in R_ρ, we may assume $AB=TX$ and $[PB]\subset\{T^2, TA, TY\}$. To begin with, if $PB=TY$ then $PX=AY$, however, this is impossible by [6, Lemma 4]. On the other hand, if $PB=TA$ then $PX=A^2$. Let $XY=T\alpha$, $\alpha\in H$ (resp. $XY=\beta^2$, $\beta\in H-\{T\}$ with $\beta<X$ and $\beta<Y$). Then $PXY=TP\alpha$ (resp. $PXY=P\beta^2$). However, $TP\alpha$ (resp. $P\beta^2$) cannot be equal to the standard monomial A^2Y. Finally, if $PB=T^2$ then $PX=TA$, hence $PXY=TAY$. Let $XY=T\alpha$, $\alpha\in H$ (resp. $XY=\beta^2$, $\beta\in H-\{T\}$ with $\beta<X$ and $\beta<Y$). Then $PXY=TP\alpha$ (resp. $PXY=P\beta^2$). Hence, $AY=P\alpha$ (resp. $\beta\neq A$ and $AY=\beta\gamma$ if $P\beta=T\gamma$, $\gamma\in H$), which is also impossible. Q.E.D.

§2. Classification of troidal trees

Let (H, ρ) be a toroidal poset with $\dim R_\rho=\operatorname{rank}(H)=3$. Then (H, ρ) is called a toroidal tree if $H-\{T\}$ is a tree.

It is easy to see that if $H-\{T\}$ is either



or

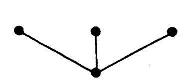


Fig. 14 Fig. 15

then (H, ρ) is never toroidal for any embedding ρ. Hence, thanks to [4] and [6, Prop. B], it is a routine work to prove the following

PROPOSITION. *The troidal trees are, up to equivalence as toroidal posets, as follows*:

Fig. 16

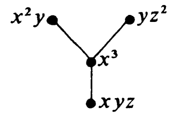


Fig. 17

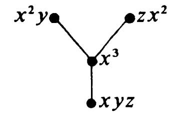


Fig. 18

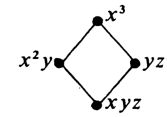


Fig. 19

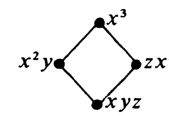


Fig. 20

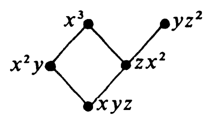


Fig. 21

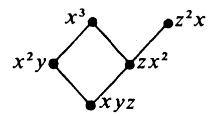


Fig. 22

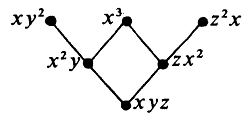


Fig. 23

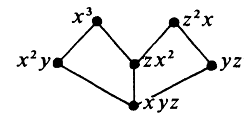


Fig. 24

§ 3. Classification of toroidal cycles

We now turn to the problem of finding the embeddings ρ on the cycles

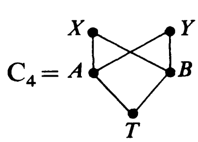


Fig. 25

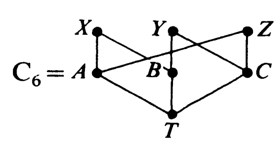


Fig. 26

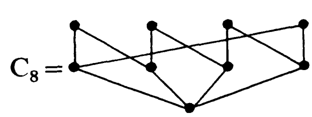


Fig. 27

To begin with, if (C_4, ρ) is toroidal then, in R_ρ, either $AB = T^2$ or $XY = T^2$ holds. In fact, assume that neither the standard monomial expression for AB nor that for XY in R_ρ coincides with T^2, say $AB = TX$ and $XY = TA$ (resp. $XY = A^2$). Then $(AB)Y = T^2A$ (resp. $(AB)Y = TA^2$), which is absurd since R_ρ is an integral domain. Hence we can easily classify the toroidal posets (C_4, ρ) and obtain the toroidal posets of Fig. 6–9.

On the other hand, if (C_6, ρ) is toroidal then, in R_ρ, we may assume that (i) $AB = TX$, $BC = TY$, $CA = TZ$ or (ii) $AB = TX$, $BC = T^2$, $CA = TZ$. In case (i), we have $CX = AY = BZ$, thus $CX = AY = BZ = T^2$. Hence $XY = TB$, $YZ = TC$, $ZX = TA$. This is the toroidal poset of Fig. 10. Now, in case (ii), $CX = BZ = TA$ and $ZX = A^2$. The possibility of the standard monomial expression for AY is either TB (resp. TC) or T^2. If $AY = TB$ (resp. TC) then $XY = B^2$ (resp. T^2) and $YZ = T^2$ (resp. C^2). Hence we obtain the toroidal poset of Fig. 11. If $AY = T^2$ then $XY = TB$ and $YZ = TC$, which is the toroidal poset of Fig. 12.

Finally, concerning the cycle C_8, we refer to [6, Example b)].

§4. The Veronese subring $k[x, y, z]^{(3)}$

As soon as we obtain the toroidal posets of Fig. 1–12 and Fig. 21–24, we cannot escape the temptation to classify all the toroidal posets (H, ρ) with $\dim R_\rho = \mathrm{rank}(H) = 3$ such that $\rho(H)$ is contained in the set $\mathscr{M}_3^{(3)}$ of monomials of degree three in three-indeterminates x, y and z.

(4.1) Let m and n be positive integers. Write Q_m^n for the rank two poset $\{\alpha_1, \alpha_2, \ldots, \alpha_m, \beta_1, \beta_2, \ldots, \beta_n\}$ with $\alpha_i < \beta_j$ for any i and j. For example,

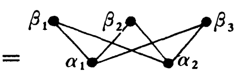
$Q_2^3 =$ 

Fig. 28

Also, we denote by H_m^n the rank three poset $Q_m^n \cup \{T\}$, where T is a unique minimal element of H_m^n. Then

LEMMA. *If (H_m^n, ρ) is toroidal, then $m \leq 2$ and $n \leq 2$.*

PROOF. First, assume $m \geq 3$ and, in R_ρ, let $\alpha_1\alpha_2 = T\gamma_1$, $\alpha_2\alpha_3 = T\gamma_2$. Here $\gamma_1, \gamma_2 \in \{T, \beta_1, \beta_2, \ldots, \beta_n\}$. Then we have $\gamma_1\alpha_3 = \gamma_2\alpha_1$, which contradicts the axiom (ASL-1).

Now, in R_ρ, $\beta_i\beta_j \neq \alpha_p\beta_q$ for any $1 \leq p \leq m$ and $1 \leq q \leq n$ by [6, Lemma 4]. On the other hand, $\beta_i\beta_j \neq T\beta_q$ for any $1 \leq q \leq n$. In fact, let $\beta_i\beta_j = T\beta_q$. If $\beta_i\beta_q = \alpha_p^2$ then $T\beta_q^2 = \alpha_p^2\beta_j$, a contradiction. Also, if $\beta_i\beta_q = T\gamma$, $\gamma \in H_m^n$, then $\beta_q^2 = \gamma\beta_j$, which is impossible. Hence, any β_q, $1 \leq q \leq n$, does not appear in the standard monomial expression for $\beta_i\beta_j$, thus we easily see that (H_m^n, ρ) is never toroidal if $n \geq 3$. Q. E. D.

However, it should be remarked that, for any positive integers m and n, we can construct a homogeneous ASL domain on the poset H_m^n if k is infinite.

(4.2) We now prove the following effective lemma which plays an essential role in our classification.

LEMMA. *Let (H, ρ) be a toroidal poset with* $\dim R_\rho = \mathrm{rank}(H) = 3$. *Assume that $H - \{T\}$ has at least three minimal elements and at least three maximal elements. Then, there exists no minimal element A of $H - \{T\}$ such that A is comparable with any maximal elements of $H - \{T\}$.*

PROOF. On the contrary, assume that there exists a minimal element A of $H - \{T\}$ such that A is comparable with any maximal element of $H - \{T\}$. Let

$B\ (\neq A)$ and $C\ (\neq A)$ be two minimal elements of $H-\{T\}$. Then, in R_ρ, $BC=T^2$. Thus, $H-\{T\}$ has no minimal element except A, B and C. Now, $AB=TX$ and $CA=TY$ for some elements X and Y of $H-\{T\}$ with $X\neq Y$, $B<X$ and $C<Y$. Thus $CX=BY=TA$ and $XY=A^2$. In particular, $B \nsim Y$ and $C \nsim X$ (the symbol "$\nsim$" stands incomparability). Let $Z\ (\neq X, Y)$ be another maximal element of $H-\{T\}$. Since the set $[ZX]$ is contained in $\{T\gamma;\ \gamma\in H\}\cup\{B^2\}$, the standard monomial expression for $(ZX)Y$ cannot coincide with the standard monomial $((XY)Z=)A^2Z$, a contradiction. Q. E. D.

(4.3) Thanks to (4.1) and (4.2), if (H, ρ) is a toroidal poset with $\dim R_\rho = \mathrm{rank}(H)=3$, $\rho(H)\subset\mathscr{M}_3^{(3)}$ and $\sharp(H)\leq 7$, then (H, ρ) is equivalent to one of the toroidal posets of Fig. 1–12 and Fig. 21–24. On the other hand, (H, ρ) is never toroidal if $\rho(H)=\mathscr{M}_3^{(3)}$ (cf. [6, Example c)]).

(4.4) Before studying the problem of finding the toroidal posets (H, ρ) with $\dim R_\rho=\mathrm{rank}(H)=3$, $\rho(H)\subset\mathscr{M}_3^{(3)}$ and $8\leq\sharp(H)\leq 9$, we had better show the following

Lemma. *Let (H, ρ) be a toroidal poset with* $\dim R_\rho=\mathrm{rank}(H)=3$. *Assume that there exists a minimal element A of $H-\{T\}$ such that $\sharp\{\alpha\in H-\{T\};\alpha>A\}\geq 3$. Then, for any two minimal elements $B\ (\neq A)$, $C\ (\neq A)$ of $H-\{T\}$, in R_ρ, the standard monomial expression for BC does not coincide with T^2.*

Proof. Suppose that, in R_ρ, $BC=T^2$. Then $AB=T\alpha$, $CA=T\beta$ for some elements $\alpha, \beta\in H-\{T\}$ with $\alpha>A$ and $\beta>A$. Hence $C\alpha=B\beta=TA$, thus $\alpha\beta=A^2$. Now, let $\gamma\ (\neq\alpha, \beta)$ be another element of $H-\{T\}$ with $\gamma>A$. Then the standard monomial expression for $(\alpha\gamma)\beta$ coincides with $A^2\gamma$, however, this is impossible because the standard monomial expression for $\alpha\gamma$ is of the form either $T\delta$ $(\delta\in H)$ or D^2 $(D\in H-\{T\}$ with $D<\alpha$, $D<\gamma)$. Q. E. D.

(4.5) Let (H, ρ) be a toroidal poset with $\dim R_\rho=\mathrm{rank}(H)=3$ and $\sharp(H)=8$. Then, thanks to (4.1) and (4.2), the poset H is among the followings:

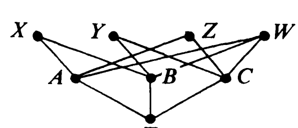


Fig. 29

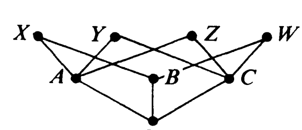


Fig. 30

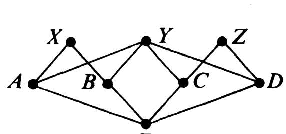


Fig. 31

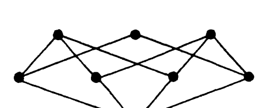


Fig. 32

LEMMA. *Any poset among* Fig. 29–32 *is never toroidal for any embedding* ρ.

PROOF. Let H be the poset of Fig. 29 and assume that (H, ρ) is toroidal. Then, by (4.4), in R_ρ, $AB=T\alpha$, $BC=T\beta$, $CA=T\gamma$, where $\alpha, \beta, \gamma\in\{X, Y, Z, W\}$. Hence $C\alpha=A\beta=B\gamma$, thus $\alpha\nsim C$, $\beta\nsim A$, $\gamma\nsim B$. So, $\alpha=X$, $\beta=Y$, $\gamma=Z$. If $CX=AY=BZ=TW$ then $XY=BW$, which contradicts [6, Lemma 4]. Thus $CX=AY=BZ=T^2$, hence $XY=TB$, $YZ=TC$, $ZX=TA$. Now, let $XW=A^2$ (resp. B^2). Then $A^2Y=TBW$, i.e., $T^2A=TBW$ (resp. $B^2Y=TBW$), a contradiction. On the other hand, if $XW=T\delta$, $\delta\in H$, then $\delta Y=BW$, however, there exists no $\delta\in H$ which satisfies $\delta Y=BW$ in R_ρ.

Let H be the poset of Fig. 30 and assume that (H, ρ) is toroidal. Then, in R_ρ, $AB=TX$ and $BC=TW$. Let $BY=T\alpha$, $BZ=T\beta$ $(\alpha, \beta\in H)$. Then, $\alpha\neq B$, X, Y, W and $\beta\neq B$, X, Z, W. Also, since $\alpha Z=\beta Y$, we have $\alpha\neq A$, C, Z and $\beta\neq A$, C, Y, thus $\alpha=\beta=T$, a contradiction.

Let H be the poset of Fig. 31 and suppose that (H, ρ) is toroidal. Then, in R_ρ, we may assume $AB=TX$, $AC=T^2$, $AD=TY$, $BC=TY$, $BD=T^2$ and $CD=TZ$. Hence $AY=CX$, a contradiction. A similar technique is also valid for the poset of Fig. 32. Q.E.D.

(4.6) We now try to find the toroidal posets (H, ρ) with $\dim R_\rho=\operatorname{rank}(H)=3$, $\rho(H)\subset\mathcal{M}_3^{(3)}$ and $\#(H)=9$.

To begin with, let $\mathcal{N}$ be an arbitrary subset of $\mathcal{M}_3^{(3)}$ with $\#(\mathcal{N})=9$ and $R=\oplus_{n\geq 0}R_n$, $R_0=k$ and $\mathcal{N}\subset R_1$, the subring of $k[x, y, z]^{(3)}$ generated by all monomials contained in $\mathcal{N}$. Then, $25\leq\dim_k R_2\leq 28$ and $\dim_k R_2\neq 26$.

On the other hand, let (H, ρ) be a toroidal poset with $\dim R_\rho=\operatorname{rank}(H)=3$ and $\#(H)=9$. Write f_1 for the number of chains of length two contained in $H-\{T\}$. Then $\dim_k(R_\rho)_2=f_1+17$. Here, $R_\rho=\oplus_{n\geq 0}(R_\rho)_n$ with $(R_\rho)_0=k$ and $\rho(H)\subset(R_\rho)_1$.

Hence, if (H, ρ) is a toroidal poset with $\dim R_\rho=\operatorname{rank}(H)=3$, $\rho(H)\subset\mathcal{M}_3^{(3)}$ and $\#(H)=9$, then $8\leq f_1\leq 11$ and $f_1\neq 9$.

(4.7) let (H, ρ) be a toroidal poset with $\dim R_\rho=\operatorname{rank}(H)=3$, $\rho(H)\subset\mathcal{M}_3^{(3)}$ and $\#(H)=9$. Write c_1 (resp. c_2) for the number of minimal (resp. maximal) elements of $H-\{T\}$. Also, as in (4.6), we denote by f_1 the number of chians of length two contained in $H-\{T\}$.

LEMMA. *If* $(c_1, c_2)=(4, 4)$ *then* $f_1=8$.

PROOF. Obviously, $f_1\geq 8$. Suppose $f_1>8$. Let A, B, C, D (resp. X, Y, Z, W) be minimal (resp. maximal) elements of $H-\{T\}$. We may assume $\#\{\alpha\in H-\{T\};\ D<\alpha\}\geq 3$. Let $AB=TX$, $BC=TY$ and $CA=TZ$ by (4.4). Since the sets $[AD]$, $[BD]$, $[CD]$ are contained in $\{T^2, TX, TY, TZ, TW\}$, we may assume

$AD \neq T^2$, TW, thus $AD = TY$, hence $BY = DX$, in particular, $B \nsim Y$, however, this contradicts $BC = TY$, i.e., $B < Y$. Q.E.D.

Hence, if $(c_1, c_2) = (4, 4)$ then the poset H looks like

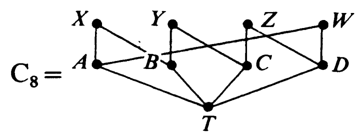


Fig. 33

Now, let $(c_1, c_2, f_1) = (3, 5, 10)$. Then, thanks to (4.2), the poset H is either

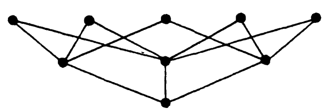


Fig. 34

or

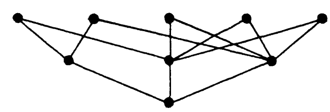


Fig. 35

Also, if $(c_1, c_2, f_1) = (3, 5, 11)$ then the poset H looks like

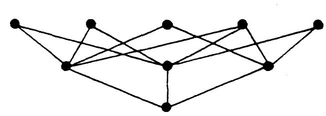


Fig. 36

On the other hand, if $(c_1, c_2, f_1) = (5, 3, 10)$ then the poset H is one of the followings:

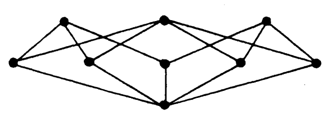


Fig. 37

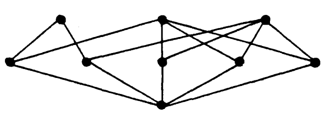


Fig. 38

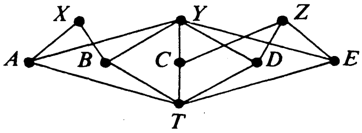


Fig. 39

Finally, if $(c_1, c_2, f_1)=(5, 3, 11)$ then the poset H is one of the followings:

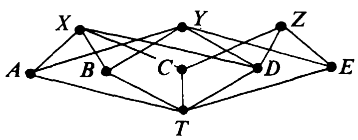


Fig. 40

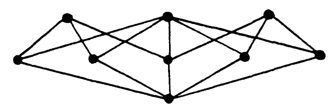


Fig. 41

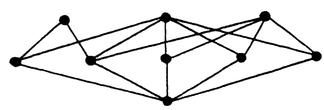


Fig. 42

LEMMA. *Any poset among* Fig. 34–42 *is never toroidal for any embedding* ρ.

PROOF. Our standard technique enables us to see that any poset of Fig. 34–36 is never toroidal. The routine details are omitted.

Let H be the poset of Fig. 39 and assume that (H, ρ) is toroidal. Then, in R_ρ, the three dimensional vector space spanned by AC, AD and AE over k is contained in the two dimensional vector space spanned by T^2 and TY over k, which is absurd. The similar technique is valid for the posets of Fig. 37–38.

On the other hand, let H be the poset of Fig. 40. If (H, ρ) is toroidal then, in R_ρ, thanks to (4.4), the three dimensional vector space spanned by AB, AC and AE over k is contained in the two dimensional vector space spanned by TX and TY over k, a contradiction. The same argument is also applied to the posets of Fig. 41–42. Q. E. D.

Our final work is to examine whether the poset C_8 of Fig. 33 can be embedded into $\mathscr{M}_3^{(3)}$ as toroidal posets. Assume that (C_8, ρ) is toroidal with $\rho(C_8)\subset\mathscr{M}_3^{(3)}$. Since $f_1=8$, by (4.6), we may assume $\rho(C_8)=\mathscr{M}_3^{(3)}-\{x^3\}$. Thanks to [6, Example b)], in R_ρ, $AB=TX$, $BC=TY$, $CD=TZ$, $DA=TW$ and $(*)\ CA=BD=T^2$. Then, by $(*)$, $\rho(T)=xyz$. Hence y^3, $z^3 \notin \{\rho(A), \rho(B), \rho(C), \rho(D)\}$. Let $\rho(X)=y^3$ and $\rho(A)=xy^2$, $\rho(B)=y^2z$. Thus $\rho(C)=xz^2$, $\rho(D)=x^2z$ by $(*)$. However, $\rho(C)\cdot\rho(D)=x^3z^3$ cannot be divided by $\rho(T)=xyz$. So, (C_8, ρ) is never toroidal if $\rho(C_8)\subset\mathscr{M}_3^{(3)}$.

(4.8) Summarizing our discussion we obtain the following

SUMMARY. *Assume that* (H, ρ) *is a toroidal poset with* $\dim R_\rho=\operatorname{rank}(H)=3$ *and* $\rho(\mathrm{H})\subset\mathscr{M}_3^{(3)}$. *Then* (H, ρ) *is equivalent to one of the toroidal posets of*

Fig. 1–12 *and* Fig. 21–24. *In particular, if* $H-\{T\}$ *is not a tree then* R_ρ *is Gorenstein.*

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