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The weak supersolution-subsolution method for second order quasilinear elliptic equations

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1. Introduction

This paper is concerned with the Dirichlet problem for second order quasilinear elliptic equations of the type

$$-\operatorname{div} A(x, \nabla u) + B(x, u, \nabla u) = 0 \qquad \text{in } \Omega\,, \tag{1.1}$$

$$u = g \qquad \text{on } \partial\Omega\,, \tag{1.2}$$

where Ω is either a bounded domain or an exterior domain in $\boldsymbol{R}^N$, A is a given N-vector function of the variables x and $\nabla u = (\partial u/\partial x_1, \ldots, \partial u/\partial x_N)$, B is a given scalar function of the variables x, u and ∇u, and g is a function given on the boundary $\partial\Omega$ of Ω. We allow the domain Ω to be the entire space $\boldsymbol{R}^N$, in which case the boundary condition (1.2) is void. Equation (1.1) is allowed to be degenerate so that the nonlinear pseudo-Laplacian equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + B(x, u, \nabla u) = 0 \qquad \text{in } \Omega\,, \qquad p > 1\,, \tag{1.3}$$

is included as a special case of it. Our objective here is to develop the method of supersolutions and subsolutions for constructing weak solutions of the problem (1.1)–(1.2) and for analyzing the structure of the set of weak solutions thus constructed.

A systematic study of nonlinear elliptic boundary problems by means of the supersolution-subsolution method was initiated by Nagumo [21], who considered the semilinear equation

$$-\textstyle\sum_{i,j=1}^N a_{ij}(x)\dfrac{\partial^2 u}{\partial x_i\,\partial x_j} + B(x, u, \nabla u) = 0 \tag{1.4}$$

in a bounded domain Ω and established an existence theorem asserting that the problem (1.4)–(1.2) has a classical solution if suitable classical quasi-supersolutions and quasi-subsolutions are known to exist. (By a quasi-supersolution (quasi-subsolution) we mean a function which is expressed locally as the minimum (maximum) of a finite number of supersolutions (subsolutions) of the problem.) Nagumo's existence theory has been generalized and extended in various directions. Among other things Akô [1] (see also Hirai and Akô [14]) proved the existence of classical minimal and maximal solutions to the Dirichlet problem for general uniformly elliptic quasilinear equations of the form

$$(1.5)\qquad -\sum_{i,j=1}^{N} a_{ij}(x, u, \nabla u)\frac{\partial^2 u}{\partial x_i\, \partial x_j} + B(x, u, \nabla u) = 0 \qquad \text{in } \Omega$$

and showed moreover that, in case all the a_{ij} are independent of u and B is nondecreasing in u, a Peano type theorem holds for the problem (1.5)–(1.2), that is, the interval between the minimal and maximal solutions is filled with the set of solutions contained between these two extremal solutions. Akô and Kusano [2] applied the supersolution-subsolution method to find classical entire solutions of equation (1.5), i.e. those solutions of (1.5) which are guaranteed to exist throughout $\boldsymbol{R}^N$.

It was only recently that the supersolution-subsolution approach was attempted to the solvability of nonlinear elliptic problems in the framework of weak or generalized solutions; see e.g. Boccardo, Murat and Puel [3], Các [5], Hess [12, 13], and Deuel and Hess [6]. The papers [3, 6, 12] deal with the Dirichlet problem for equations of the form

$$(1.6)\qquad -\operatorname{div} A(x, u, \nabla u) + B(x, u, \nabla u) = 0$$

in a bounded domain Ω and give sufficient conditions for the existence of a weak solution between a weak supersolution and a weak subsolution. In case A is independent of u and B is independent of ∇u, Diaz [7] has established, by means of the monotone method, the existence of weak maximal and minimal solutions between weak super- and subsolutions (see also [13]). The Dirichlet problem for (1.6) in unbounded domains is studied in the papers [5, 13], in each of which it is shown that the existence of a weak solution in $W^{1,p}(\Omega)$ of the problem is implied by the existence of suitable weak super- and subsolutions in $W^{1,p}(\Omega)$.

A survey of the previous results sketched above raises the following questions.

(1) Is it possible to develop an analogue of the Nagumo-Akô existence theory for weak solutions of the problem (1.6)–(1.2)? More precisely, is it possible to establish an existence theorem for (1.6)–(1.2) in terms of weak quasi-supersolutions and quasi-subsolutions?

(2) Is it possible to prove for the problem (1.6)–(1.2) weak versions of Akô's theorem on the existence of maximal and minimal solutions and a Peano-Akô type theorem on the structure of the set of solutions?

The purpose of this paper is to make an attempt to answer the above questions. Partial answers to these questions will be given for the problem (1.1)–(1.2). In the case of bounded domains Ω, we introduce three kinds of weak quasi-supersolutions and quasi-subsolutions, called super- and subsolutions of class W, L or C, depending on the structure of equation (1.1), and show that the existence of a quasi-subsolution φ_1 and a quasi-supersolution φ_2 of any kind satisfying $\varphi_1 \leq \varphi_2$ a.e. in Ω and $\varphi_1 \leq g \leq \varphi_2$ a.e. on $\partial\Omega$ implies the existence of a weak solution u of (1.1)–(1.2) such that $\varphi_1 \leq u \leq \varphi_2$ a.e. in Ω; furthermore we show that when φ_1, φ_2 are of class W or L, the maximal and minimal weak solutions of (1.1)–(1.2) are guaranteed to exist between φ_1 and φ_2, and that the interval between these extremal solutions is filled with the set of solutions of (1.1)–(1.2). In the case of unbounded domains Ω, we intend to solve the problem (1.1)–(1.2) in the framework of $W^{1,p}_{loc}(\Omega)$; it is shown that all the results for bounded domains can be carried over to the case where Ω is either an exterior domain in $\mathbf{R}^N$ or coincides with the entire space $\mathbf{R}^N$. Examples illustrating our main results will be presented; in particular, sufficient conditions will be given under which the equation (1.3) possesses bounded positive weak solutions defined in the entire space $\mathbf{R}^N$.

Finally we refer to Tolksdorf [23], DiBenedetto [8] and Reshetnjak [22] for the regularity of bounded weak solutions of equation (1.6) or (1.1).

2. Preliminaries

Throughout this paper all functions are real-valued. We define $x \cdot y$ and $|x|$ by $x \cdot y = \sum_{i=1}^{N} x_i y_i$ for $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N) \in \mathbf{R}^N$, and $|x| = (x \cdot x)^{1/2}$. Let $\mathbf{N}$ be the set of positive natural numbers. We put $\mathbf{R}_+ = (0, \infty)$ and $\bar{\mathbf{R}}_+ = [0, \infty)$. We let $t^+ = \max(t, 0)$ for $t \in \mathbf{R}$. Let p and q be fixed constants satisfying $1 < p < \infty$ and $q = p/(p-1)$. Let Ω be a bounded domain or an exterior domain in $\mathbf{R}^N$ ($N \geq 1$); the possibility $\Omega = \mathbf{R}^N$ is not excluded. Let $\partial\Omega$ be the boundary of Ω. We assume that $\partial\Omega$ belongs to the class C^1 if $\partial\Omega$ is not empty. Let $W^{1,p}(\Omega)$ be the Sobolev space and $W_0^{1,p}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$. In the trace sense we write $u = (\leq, \geq)\ v$ a.e. on $\partial\Omega$ for functions u and v in $W^{1,p}(\Omega)$. The norms in $L^p(\Omega)$ and $W^{1,p}(\Omega)$ are defined by

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p \, dx \right)^{1/p}, \qquad \|u\|_{W^{1,p}(\Omega)} = \sum_{|\beta| \leq 1} \|D^\beta u\|_{L^p(\Omega)} .$$

We shall use $\|u\|_p = \|u\|_{p;\Omega} = \|u\|_{L^p(\Omega)}$ and $\|u\| = \|u\|_{W^{1,p}(\Omega)}$ when there is no ambiguity. Let $W^{1,p}_{loc}(\Omega)$ be the set of all functions belonging to $W^{1,p}(\Omega_0)$ for all bounded subdomains Ω_0 of Ω with $\bar{\Omega}_0 \subset \Omega$.

We assume in the boundary condition (1.2) that $g \in W^{1,p}(\Omega)$ is a given function. In the equation (1.1) we assume that the functions $A\colon \Omega \times \mathbf{R}^N \to \mathbf{R}^N$, $A(x, \xi) = (A_1(x, \xi), \ldots, A_N(x, \xi))$, and $B\colon \Omega \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ satisfy the Carathéodory condition, that is, each $A_i(x, \xi)$ is measurable in $x \in \Omega$ for every fixed $\xi \in \mathbf{R}^N$ and continuous in $\xi \in \mathbf{R}^N$ for almost every fixed $x \in \Omega$, and $B(x, t, \xi)$ is measurable in $x \in \Omega$ for every fixed $(t, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^N$ and continuous in $(t, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^N$ for almost every fixed $x \in \Omega$. Furthermore we assume that the function $\boldsymbol{A}$ satisfies the following conditions:

$$(\mathrm{H}_1) \qquad |A_i(x, \xi)| \le |f_0(x)| + |c_0(x)||\xi|^{p-1}, \qquad i = 1, \dots, N,$$

for a.e. $x \in \Omega$, $^\forall \xi \in \boldsymbol{R}^N$, where f_0 is a measurable function in Ω and $c_0 \in L^\infty_{loc}(\boldsymbol{R}^N)$;

$$(\mathrm{H}_2) \qquad (\boldsymbol{A}(x, \xi) - \boldsymbol{A}(x, \xi'))\cdot(\xi - \xi') > 0$$

for a.e. $x \in \Omega$, $^\forall\ \xi, \xi' \in \boldsymbol{R}^N$ with $\xi \ne \xi'$;

$$(\mathrm{H}_3) \qquad \boldsymbol{A}(x, \xi)\cdot\xi \ge \alpha(x)|\xi|^p - |f_1(x)||\xi|^{p-1} - |f_2(x)|,$$

for a.e. $x \in \Omega$, $^\forall\ \xi \in \boldsymbol{R}^N$, where $\alpha\colon \boldsymbol{R}^N \to \boldsymbol{R}_+$ is a continuous function and f_1 and f_2 are measurable functions in Ω.

For simplicity, it is assumed in (H_1) and (H_3) that c_0 and α are defined on $\boldsymbol{R}^N$. Typical conditions to be imposed on the functions f_0, f_1, f_2 are as follows:

$$(\mathrm{H}_4) \qquad f_0 \in L^q(\Omega), \qquad f_1 \in L^p(\Omega), \qquad f_2 \in L^1(\Omega);$$

$$(\mathrm{H}_5) \qquad f_0 \in L^q_{loc}(\boldsymbol{R}^N), \qquad f_1 \in L^p_{loc}(\boldsymbol{R}^N), \qquad f_2 \in L^1_{loc}(\boldsymbol{R}^N);$$

$$(\mathrm{H}_6) \qquad f_0, f_1, f_2 \in L^\infty_{loc}(\boldsymbol{R}^N).$$

Here we state four lemmas which will be used in the later sections.

LEMMA 2.1. *Let Ω be a bounded open set in $\boldsymbol{R}^N$. Suppose that* (H_1), (H_2), (H_3) *and* (H_4) *hold. Let $\{u_n\}_{n \in \boldsymbol{N}}$ be a sequence in $W^{1,p}(\Omega)$ and $u \in W^{1,p}(\Omega)$ such that*

$$u_n \to u \qquad \text{weakly in } W^{1,p}(\Omega),$$
$$u_n \to u \qquad \text{strongly in } L^p(\Omega).$$

If

$$\int_\Omega (\boldsymbol{A}(x, \nabla u_n) - \boldsymbol{A}(x, \nabla u))\cdot(\nabla u_n - \nabla u)\, dx \to 0 \qquad \text{as } n \to \infty,$$

then u_n converges strongly to u in $W^{1,p}(\Omega)$.

LEMMA 2.2. *Let Ω be a measurable set in $\boldsymbol{R}^N$ and let $m \in \boldsymbol{N}$ and $1 \le p_i < \infty$ $(i = 0, \dots, m)$ be constants. Assume that a function $f\colon \Omega \times \boldsymbol{R}^m \to \boldsymbol{R}$ satisfies the Carathéodory condition and $f(x, u_1(x), \dots, u_m(x)) \in L^{p_0}(\Omega)$ if $u_i \in L^{p_i}(\Omega)$ $(i = 1, \dots, m)$. Then $F\colon L^{p_1}(\Omega) \times \cdots \times L^{p_m}(\Omega) \to L^{p_0}(\Omega)$, $F(u_1, \dots, u_m)(x) = f(x, u_1(x), \dots, u_m(x))$, is continuous in the strong topology.*

LEMMA 2.3. *Let Ω be a bounded open set in $\boldsymbol{R}^N$ $(N \geq 2)$. Suppose that A_i and B satisfy the conditions*

$$|A(x, \xi)|(1 + |\xi|) + |B(x, t, \xi)| \leq \mu(|t|)(1 + |\xi|)^p,$$

$$A(x, \xi) \cdot \xi \geq \nu |\xi|^p - \mu(0),$$

for a.e. $x \in \Omega$, $^\forall$ $(t, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^N$, where $\mu: \bar{\boldsymbol{R}}_+ \to \boldsymbol{R}_+$ is a nondecreasing function and ν is a positive constant. If u is a bounded solution of (1.1) with $\|u\|_{\infty;\Omega} \leq M$ then $u \in C^\gamma(\Omega)$ and

$$\|u\|_{C^\gamma(\bar{\Omega}_0)} \leq C$$

for any subdomain $\Omega_0 \subset\subset \Omega$, where $0 < \gamma < 1$, $\gamma = \gamma(N, p, M, \nu, \mu(M))$ and $C = C(\gamma, \operatorname{dist}(\Omega_0, \partial\Omega))$ are positive constants.

Lemma 2.1 is proved in [4, p. 13, Lemma 3]. The proof of Lemma 2.2 is given in [17, p. 22, Theorem 2.1]. Lemma 2.3 is due to Ladyzhenskaya and Ural'tseva [19, p. 251, Theorem 1.1].

We shall employ the theory of monotone operators. Let V be a real reflexive Banach space and V^* be its dual space. A map $F: V \to V^*$ is called *pseudo-monotone* if F satisfies the following conditions:

(i) F is a bounded map;

(ii) if u_i, $u \in V$, $u_i \to u$ weakly in V and $\limsup_{i\to\infty} \langle F(u_i), u_i - u\rangle \leq 0$ then $\liminf_{i\to\infty} \langle F(u_i), u_i - v\rangle \geq \langle F(u), u - v\rangle$ for all $v \in V$.

Let $v \in W^{1,p}(\Omega)$ be given. Let $\tilde{A}_i(x, \xi) = A_i(x, \xi + \nabla v(x))$, $i = 1, \ldots, N$, for a.e. $x \in \Omega$, $^\forall$ $\xi \in \boldsymbol{R}^N$. Put $\tilde{A}(x, \xi) = (\tilde{A}_1(x, \xi), \ldots, \tilde{A}_N(x, \xi))$. Then $\tilde{A}_i$ $(i = 1, \ldots, N)$ satisfy the Carathéodory condition (see [24, p. 152, Theorem 18.3]). It follows from (H_1), (H_2) and (H_3) that for a.e. $x \in \Omega$, $^\forall$ $\xi \in \boldsymbol{R}^N$

$$\begin{aligned}|\tilde{A}_i(x, \xi)| &\leq |f_0(x)| + |c_0(x)||\xi + \nabla v(x)|^{p-1} \\ &\leq |\tilde{f}_0(x)| + 2^p|c_0(x)||\xi|^{p-1}, \qquad i = 1, \ldots, N,\end{aligned}$$

where $\tilde{f}_0(x) = |f_0(x)| + 2^p|c_0(x)||\nabla u(x)|^{p-1} \in L^q(\Omega)$, and that

$$\begin{aligned}\tilde{A}(x, \xi) \cdot \xi &= A(x, \xi + \nabla v(x)) \cdot (\xi + \nabla v(x)) - \tilde{A}(x, \xi) \cdot \nabla v(x) \\ &\geq 2^{-p}\alpha(x)|\xi|^p - |\tilde{f}_1(x)||\xi|^{p-1} - |\tilde{f}_2(x)|,\end{aligned}$$

where $\tilde{f}_1(x) = 2^p(|f_1(x)| + N|c_0(x)||\nabla v(x)|) \in L^p(\Omega)$ and $\tilde{f}_2(x) = \alpha(x)|\nabla v(x)|^p + 2^p|f_1(x)||\nabla v(x)|^{p-1} + |f_2(x)| + N|\tilde{f}_0(x)||\nabla v(x)| \in L^1(\Omega)$. Consequently, we can assume that $\tilde{A}$ satisfies (H_1), (H_2) and (H_3).

Let $B: \Omega \times \boldsymbol{R} \times \boldsymbol{R}^N \to \boldsymbol{R}$ satisfy the Carathéodory condition and the following condition:

$$B(x, u_0, u_1, \dots, u_N) \in L^{p_0}(\Omega) \quad \text{for all} \quad u_i \in L^p(\Omega), \quad i = 0, 1, \dots, N,$$

where $1 \le p_0 < \infty$ is a constant. For $\varphi_i, \psi_i \in W^{1,p}(\Omega)$ with $\varphi_i \le 0 \le \psi_i$ a.e. in Ω, $i = 1, 2$, we define $T_i(x, t)$ by

$$T_i(x, t) = \begin{cases} \varphi_i(x) & \text{if } t < \varphi_i(x) \\ t & \text{if } \varphi_i(x) \le t \le \psi_i(x) \\ \psi_i(x) & \text{if } \psi_i(x) < t \end{cases}$$

for a.e. $x \in \Omega$, $^\forall t \in \boldsymbol{R}$. We see that for $v \in W^{1,p}(\Omega)$

$$\text{(2.1)} \qquad \nabla T_i(v) = \begin{cases} \nabla \varphi_i & \text{in } \{v < \varphi_i\} \\ \nabla v & \text{in } \{\varphi_i \le v \le \psi_i\} \\ \nabla \psi_i & \text{in } \{\psi_i < v\} \end{cases}$$

where $(T_i(v))(x) = T_i(x, v(x))$. We also define the maps F_1, F_2, G: $W^{1,p}(\Omega) \to L^{p_0}(\Omega)$ by

$$F_i(u)(x) = B(x, T_i(u)(x), \nabla T_i(u)(x)), \qquad i = 1, 2,$$
$$G(u)(x) = |F_1(u)(x) - F_2(u)(x)| \operatorname{sgn} u(x),$$

where $(T_i(u))(x) = T_i(x, u(x))$. The following lemma holds.

LEMMA 2.4. *The maps F_1, F_2, G: $W^{1,p}(\Omega) \to L^{p_0}(\Omega)$ are continuous in the strong topology.*

PROOF. Since $T_i(u) = \varphi_i + (u - \varphi_i)^+ - (u - \psi_i)^+$ for $u \in L^p(\Omega)$, $i = 1, 2$, it follows from Lemmas 3.1 and 3.2 in [19, pp. 50–51] that T_i: $L^p(\Omega) \to L^p(\Omega)$ and T_i: $W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ are continuous in the strong topology. Lemma 2.2 implies that F_i: $W^{1,p}(\Omega) \to L^{p_0}(\Omega)$ $(i = 1, 2)$ are continuous in the strong topology. We shall show the continuity of G. Let u_n, $u \in W^{1,p}(\Omega)$ and $u_n \to u$ strongly in $W^{1,p}(\Omega)$. Put

$$\Omega_n^{(1)} = \{x \in \Omega : u_n(x) \ge 0 \text{ and } u(x) \ge 0\},$$
$$\Omega_n^{(2)} = \{x \in \Omega : u_n(x) \ge 0 \text{ and } u(x) < 0\},$$
$$\Omega_n^{(3)} = \{x \in \Omega : u_n(x) \le 0 \text{ and } u(x) > 0\},$$
$$\Omega_n^{(4)} = \{x \in \Omega : u_n(x) \le 0 \text{ and } u(x) \le 0\}.$$

It follows from Lemma 3.1 in [19, p. 50] that $\mathscr{L}^N(\Omega_n^{(2)} \cup \Omega_n^{(3)}) \to 0$ as $n \to \infty$, where $\mathscr{L}^N$ is the Lebesgue measure in $\boldsymbol{R}^N$. Since $\Omega = \bigcup_{i=1}^4 \Omega_n^{(i)}$ for all $n \in \boldsymbol{N}$ we have

$$\|G(u_n) - G(u)\|_{p_0;\Omega}^{p_0} \le \|G(u_n) - G(u)\|_{p_0;\Omega_n^{(1)} \cup \Omega_n^{(4)}}^{p_0} + 2^{p_0} \|F_1(u_n) - F_2(u_n)\|_{p_0;\Omega_n^{(2)} \cup \Omega_n^{(3)}}^{p_0}$$
$$+ 2^{p_0} \|F_1(u) - F_2(u)\|_{p_0;\Omega_n^{(2)} \cup \Omega_n^{(3)}}^{p_0},$$

$$\|F_1(u_n) - F_2(u_n)\|_{p_0;\Omega_n^{(2)}\cup\Omega_n^{(3)}} \le \|F_1(u_n) - F_1(u) + F_2(u) - F_2(u_n)\|_{p_0;\Omega_n^{(2)}\cup\Omega_n^{(3)}} + \|F_1(u) - F_2(u)\|_{p_0;\Omega_n^{(2)}\cup\Omega_n^{(3)}}.$$

Put

$$E_n^{(1)} = \{x \in \Omega : u_n(x) > 0 \text{ and } u(x) > 0\},$$
$$E_n^{(2)} = \{x \in \Omega : u_n(x) > 0 \text{ and } u(x) = 0\},$$
$$E_n^{(3)} = \{x \in \Omega : u_n(x) = 0 \text{ and } u(x) > 0\}.$$

From (2.1) we have $F_1(u) = F_2(u)$ in $E_n^{(2)}$ and $F_1(u_n) = F_2(u_n)$ in $E_n^{(3)}$. Hence

$$\|G(u_n) - G(u)\|_{p_0;\Omega_n^{(1)}} \le \|F_1(u_n) - F_1(u) + F_2(u) - F_2(u_n)\|_{p_0;E_n^{(1)}\cup E_n^{(2)}\cup E_n^{(3)}} \le \|F_1(u_n) - F_1(u) + F_2(u) - F_2(u_n)\|_{p_0;\Omega}.$$

Similarly, we obtain

$$\|G(u_n) - G(u)\|_{p_0;\Omega_n^{(4)}} \le \|F_1(u_n) - F_1(u) + F_2(u) - F_2(u_n)\|_{p_0;\Omega}.$$

Consequently, $\|G(u_n) - G(u)\|_{p_0;\Omega} \to 0$ as $n \to \infty$, which implies that $G: W^{1,p}(\Omega) \to L^{p_0}(\Omega)$ is continuous in the strong topology. This completes the proof of Lemma 2.4.

3. Equations in bounded domains

Throughout this section we assume that Ω is bounded and that the conditions (H_1)–(H_4) are satisfied for (1.1). Let $\alpha_0 = \inf\{\alpha(x) : x \in \Omega\}$ and $d = \|c_0\|_{\infty;\Omega}$, where $\alpha(x)$ and $c_0(x)$ are functions appearing in (H_1) and (H_3). Note that $\alpha_0 > 0$ since α is a positive continuous function on $\boldsymbol{R}^N$.

DEFINITION 1. A function u is said to be a solution (subsolution, supersolution) of equation (1.1) in Ω if $u \in W^{1,p}(\Omega)$, $B(x, u, \nabla u) \in L^1_{loc}(\Omega)$ and

$$\int_\Omega \{A(x, \nabla u)\cdot\nabla\varphi + B(x, u, \nabla u)\varphi\}\, dx = 0 \quad (\le 0, \ge 0), \tag{3.1}$$

for all $\varphi \in C_0^\infty(\Omega)$ with $\varphi \ge 0$ in Ω.

DEFINITION 2. A function u is said to be a W-subsolution (L-subsolution, C-subsolution) of equation (1.1) in Ω if $u = \max\{u_i : i = 1, \dots, m\}$ a.e. in Ω for some $m \in \boldsymbol{N}$, where each u_i is a subsolution of (1.1) in Ω and $u_i \in W^{1,p}(\Omega)$ ($u_i \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $u_i \in C^{0,1}(\bar{\Omega})$). Here $C^{0,1}(\bar{\Omega})$ is the space of Lipschitz continuous functions in Ω.

A function u is said to be a W-supersolution (L-supersolution, C-supersolution) of equation (1.1) in Ω if $u = \min\{u_i : i = 1, \dots, m\}$ a.e. in Ω for some

$m \in \boldsymbol{N}$, where each u_i is a supersolution of (1.1) in Ω and $u_i \in W^{1,p}(\Omega)$ ($u_i \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$, $u_i \in C^{0,1}(\bar{\Omega})$).

The notion of W, L, C-subsolutions (-supersolutions) is not a complete weak version of Nagumo's quasi-subsolutions (-supersolutions). However, these are sufficient for the existence of weak minimal and maximal solutions and for the formulation of Peano-Akô type theorems. We shall use W, L, C-subsolutions (-supersolutions) depending on the conditions of $B(x, u, \nabla u)$, which influence technically the restriction on test functions φ in (3.1).

It follows from the definition that if u_1 and u_2 are W-subsolutions (L-subsolutions, C-subsolutions) of (1.1), then $\max(u_1, u_2)$ is a W-subsolution (L-subsolution, C-subsolution) of (1.1), and that if u_1 and u_2 are W-supersolutions (L-supersolutions, C-supersolutions) of (1.1), then $\min(u_1, u_2)$ is a W-supersolution (L-supersolution, C-supersolution) of (1.1). It is not known in general whether W, L, C-subsolutions (-supersolutions) of (1.1) are subsolutions (supersolutions) of (1.1). However, in the following situation, we can prove that an L-subsolution (-supersolution) of (1.1) is indeed a subsolution (supersolution) of (1.1).

Proposition 1. *Let equation* (1.1) *be of the form*

$$-\operatorname{div} A(x, \nabla u) + B(x, u) = 0 \qquad \text{in } \Omega . \tag{3.2}$$

Assume that $B(x, u)$ is nondecreasing with respect to $u \in \boldsymbol{R}$ for almost every fixed $x \in \Omega$ and satisfies the following condition:

$$|B(x, t)| \le |f_3(x)| + h(|t|) \qquad \text{for a.e. } x \in \Omega , \quad \forall\, t \in \boldsymbol{R} , \tag{3.3}$$

where $f_3 \in L^q(\Omega)$ and $h: \bar{\boldsymbol{R}}_+ \to \bar{\boldsymbol{R}}_+$ is a nondecreasing function. If u is an L-subsolution (L*-supersolution*) *of* (3.2), *then u is a subsolution* (*supersolution*) *of* (3.2).

We give the proof of Proposition 1 in the last part of this section.

3.1. W-subsolutions and W-supersolutions

Theorem 3.1. *Let φ_1 and φ_2 be respectively a W-subsolution and a W-supersolution of* (1.1) *in Ω such that $\varphi_1 \le \varphi_2$ a.e. in Ω and $\varphi_1 \le g \le \varphi_2$ a.e. on $\partial\Omega$. Suppose that there exist a positive constant c_1 and a function $f_3 \in L^q(\Omega)$ such that*

$$|B(x, t, \xi)| \le |f_3(x)| + h(|t|) + c_1 |\xi|^{p-1} , \tag{3.4}$$

for a.e. $x \in \Omega$, $\forall\, (t, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^N$, where $h: \bar{\boldsymbol{R}}_+ \to \bar{\boldsymbol{R}}_+$ is a nondecreasing function such that $h(|\varphi|) \in L^q(\Omega)$ for $\varphi \in L^p(\Omega)$. Then the problem (1.1)–(1.2) *has a solution u such that $\varphi_1 \le u \le \varphi_2$ a.e. in Ω.*

PROOF. The functions φ_1 and φ_2 are of the form

(3.5) $$\varphi_1 = \max\{\underline{\psi_i}: i = 1, \dots, m\},\ \varphi_2 = \min\{\overline{\psi}_i: i = 1, \dots, n\}$$

a.e. in Ω, where $\underline{\psi_i}$ and $\overline{\psi}_i$ are respectively subsolutions and supersolutions of (1.1) in Ω. By adding the same functions to $\{\underline{\psi_i}\}$ or $\{\overline{\psi}_i\}$, we can assume that $m = n$. By taking $\varphi_1 + (g - \varphi_1)^+ - (g - \varphi_2)^+$ instead of g, without loss of generality, we can assume that $\varphi_1 \le g \le \varphi_2$ a.e. in Ω. Let $\tilde{A}(x, \xi) = A(x, \xi + \nabla g(x))$ and $\tilde{B}(x, t, \xi) = B(x, t + g(x), \xi + \nabla g(x))$ for a.e. $x \in \Omega$, $^\forall$ $(t, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^N$. Then we can assume that $\tilde{A}$ satisfies (H_1), (H_2), (H_3) and (H_4). Let $u_i = \underline{\psi_i} - g$ and $v_i = \overline{\psi}_i - g$ $(i = 1, \dots, m)$ and let $u_0 = \max\{u_i: i = 1, \dots, m\}$ and $v_0 = \min\{v_i: i = 1, \dots, m\}$. We note that $u_0 = \varphi_1 - g \le 0 \le v_0 = \varphi_2 - g$ a.e. in Ω and that u_i and v_i $(i = 1, \dots, m)$ are respectively subsolutions and supersolutions of the equation

(3.6) $$-\operatorname{div} \tilde{A}(x, \nabla u) + \tilde{B}(x, u, \nabla u) = 0 \qquad \text{in } \Omega$$

Therefore u_0 and v_0 are respectively a W-subsolution and a W-supersolution of (3.6) in Ω. For $i \in \{0, \dots, m\}$, a.e. $x \in \Omega$, $^\forall$ $t \in \boldsymbol{R}$ we define

$$T_i(x, t) = \begin{cases} u_i(x) & \text{if } t < u_i(x) \\ t & \text{if } u_i(x) \le t \le v_i(x)\,, \\ v_i(x) & \text{if } v_i(x) < t \end{cases}$$

and

$$h(x, t) = |t - T_0(x, t)|^{p-1} \operatorname{sgn}(t - T_0(x, t))\,.$$

The functions $T_i(x, t)$ and $h(x, t)$ satisfy the Carathéodory condition. Consider the function $\tilde{B}(x, T_i(u), \nabla T_i(u))$ where $T_i(u)(x) = T_i(x, u(x))$ (cf. (2.1)). Put $w = \max\{|u_i| + |v_i|: i = 1, \dots, m\}$. From (2.1) and (3.4) we have for $u \in W^{1,p}(\Omega)$

$$\begin{aligned} |\tilde{B}(x, T_i(u), \nabla T_i(u))| &\le |f_3(x)| + h(w + |g|) + c_1 |\nabla T_i(u) + \nabla g|^{p-1} \\ &\le |f_4(x)| + 2^p c_1 |\nabla u|^{p-1}\,, \end{aligned}$$

where

$$\begin{aligned} f_4(x) = |f_3(x)| &+ h(w(x) + |g(x)|) \\ &+ 2^p c_1 \textstyle\sum_{l=0}^m (|\nabla g(x)| + |\nabla u_l(x)| + |\nabla v_l(x)|)^{p-1}\,. \end{aligned}$$

We also have for a.e. $x \in \Omega$, $^\forall$ $t \in \boldsymbol{R}$

$$|h(x, t)| \le (|u_0(x)| + |v_0(x)| + |t|)^{p-1} \le |f_5(x)| + 2^p |t|^{p-1}$$

where $f_5(x) = 2^p(|u_0(x)| + |v_0(x)|)^{p-1}$. Consequently, the following estimates hold:

(3.7) $$|\tilde{B}(x, T_i(u), \nabla T_i(u))| \le |f_4(x)| + 2^p c_1 |\nabla u|^{p-1}, \qquad i = 0, 1, \dots, m,$$

for $\forall\, u \in W^{1,p}(\Omega)$, a.e. $x \in \Omega$, where $f_4 \in L^q(\Omega)$,

(3.8) $$|h(x, t)| \le |f_5(x)| + 2^p |t|^{p-1},$$

for a.e. $x \in \Omega$, $\forall\, t \in \boldsymbol{R}$, where $f_5 \in L^q(\Omega)$. For $i \in \{0, 1, \dots, m\}$ we define $\tilde{B}_i$, B_i: $W^{1,p}(\Omega) \to L^q(\Omega)$ by

(3.9) $$\begin{cases} \tilde{B}_i(u)(x) = \tilde{B}(x, T_i(u)(x), \nabla T_i(u)(x)) \\ B_i(u)(x) = |\tilde{B}_i(u)(x) - \tilde{B}_0(u)(x)| \operatorname{sgn} u(x). \end{cases}$$

We consider the following problem

(3.10) $$\begin{cases} -\operatorname{div} \tilde{A}(x, \nabla v) + \tilde{B}_0(v) + \sum_{i=1}^m B_i(v) + \beta h(x, v) = 0 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

where $\beta = 4^{p(p+1)}(2m + 1)^p c_1^p \alpha_0^{1-p} + 1$. Theorem 3.1 is proved if the following two lemmas are proved.

LEMMA 3.1. *If* $v \in W_0^{1,p}(\Omega)$ *is a solution of* (3.10) *then* $u = v + g$ *is a solution of* (1.1)–(1.2) *such that* $\varphi_1 \le u \le \varphi_2$ *a.e. in* Ω.

LEMMA 3.2. *There exists a solution* $v \in W_0^{1,p}(\Omega)$ *of* (3.10).

PROOF OF LEMMA 3.1. Since $(v - v_i)^+ \in W_0^{1,p}(\Omega)$ $(i = 1, \dots, m)$ and $h(x, v) = |v - v_0|^{p-1}$ in $\{v > v_i\}$, we have from (2.1)

$$\int_{v>v_i} [\tilde{A}(x, \nabla v)\cdot \nabla(v - v_i) + (v - v_i)\{\tilde{B}(x, v_0, \nabla v_0) + \sum_{l=1}^m |\tilde{B}(x, T_l(v), \nabla T_l(v)) - \tilde{B}(x, v_0, \nabla v_0)| + \beta |v - v_0|^{p-1}\}]\, dx = 0.$$

On the other hand, since v_i is a supersolution of (3.6), we obtain

$$\int_{v>v_i} \{\tilde{A}(x, \nabla v_i)\cdot \nabla(v - v_i) + \tilde{B}(x, v_i, \nabla v_i)(v - v_i)\}\, dx \ge 0$$

and hence, by (H_2),

$$\begin{aligned} 0 &\le \int_{v>v_i} (\tilde{A}(x, \nabla v) - \tilde{A}(x, \nabla v_i))\cdot \nabla(v - v_i)\, dx \\ &\le \int_{v>v_i} (v - v_i)\{\tilde{B}(x, v_i, \nabla v_i) - \tilde{B}(x, v_0, \nabla v_0) - |\tilde{B}(x, v_i, \nabla v_i) - \tilde{B}(x, v_0, \nabla v_0)| \\ &\quad - \beta |v - v_0|^{p-1}\}\, dx \le 0. \end{aligned}$$

Therefore we have

$$0 = \int_{v > v_i} (v - v_i)|v - v_0|^{p-1}\, dx \geq \int_\Omega |(v - v_i)^+|^p\, dx\,,$$

which shows that $v \leq v_i$ a.e. in Ω and hence $v \leq v_0$ a.e. in Ω. Similarly, we obtain $u_0 \leq v$ a.e. in Ω. Consequently, v is a solution of (3.6), so that $u = v + g$ is a solution of (1.1)–(1.2). This proves the assertion of Lemma 3.1.

PROOF OF LEMMA 3.2. Let $V = W_0^{1,p}(\Omega)$ and V^* be its dual space. For u, $v \in V$ we define

$$\langle a_1(u), v \rangle = \int_\Omega \tilde{A}(x, \nabla u) \cdot \nabla v\, dx\,,$$

$$\langle a_2(u), v \rangle = \int_\Omega \{\tilde{B}_0(u) + \textstyle\sum_{l=1}^m B_l(u) + \beta h(x, u)\} v\, dx\,.$$

It follows from (H_1), (H_4), (3.7) and (3.8) that a_1, a_2: $V \to V^*$ are bounded maps. We define F: $V \to V^*$ by $F(u) = a_1(u) + a_2(u)$. We shall show that F is pseudo-monotone. Let u_i, $u \in V$, $u_i \to u$ weakly in V and $\limsup_{i \to \infty} \langle F(u_i), u_i - u \rangle \leq 0$. Then $\{u_i\}_{i \in N}$ is bounded in V and $u_i \to u$ strongly in $L^p(\Omega)$. By (3.7), (3.8) and Hölder's inequality we have

$$|\langle a_2(u_i), u_i - u \rangle| \leq \|u_i - u\|_p \{\|\tilde{B}_0(u_i)\|_q + \textstyle\sum_{l=1}^m \|B_l(u_i)\|$$
$$+ \beta(\|f_5\|_q + 2^p \|u_i\|_p^{p-1})\} \to 0 \quad \text{as } i \to \infty\,.$$

From (H_2) we obtain

$$\langle F(u_i), u_i - u \rangle \geq \langle a_1(u), u_i - u \rangle + \langle a_2(u_i), u_i - u \rangle\,,$$

so that

$$\langle F(u_i), u_i - u \rangle \to 0 \qquad \text{as } i \to \infty\,,$$

which implies

$$\langle a_1(u_i), u_i - u \rangle \to 0 \qquad \text{as } i \to \infty\,.$$

Consequently we have

$$\langle a_1(u_i) - a_1(u), u_i - u \rangle \to 0 \qquad \text{as } i \to \infty\,.$$

By Lemma 2.1 we have $u_i \to u$ strongly in V, so that, by Lemmas 2.2 and 2.4, for all $v \in V$

$$\langle a_1(u_i) - a_1(u), u - v \rangle \to 0\,, \qquad \langle a_2(u_i) - a_2(u), u - v \rangle \to 0\,,$$

and thus

$$\langle F(u_i), u_i - v \rangle = \langle F(u_i), u_i - u \rangle + \langle F(u_i) - F(u), u - v \rangle + \langle F(u), u - v \rangle$$
$$\to \langle F(u), u - v \rangle \qquad \text{as } i \to \infty\,,$$

which implies that F is pseudo-monotone. From (H_3) and (3.7) we have for $u \in V$

$$\begin{aligned}\langle F(u), u\rangle \geq \alpha_0 \|\nabla u\|_p^p &- \|f_1\|_p \|\nabla u\|_p^{p-1} - \|f_2\|_1 \\ &- (2m+1)\|u\|_p(\|f_4\|_q + 2^p c_1 \|\nabla u\|_p^{p-1}) + 2^{-p}\beta\|u\|_p^p \\ &- 2^p\beta(\|u_0\|_p^p + \|v_0\|_p^p) - \beta(\|u_0\|_p^{p-1} + \|v_0\|_p^{p-1})\|u\|_p ,\end{aligned}$$

and so, by the definition of β,

$$\begin{aligned}\langle F(u), u\rangle &\geq 2^{-p}\{\alpha_0 \|\nabla u\|_p^p - (2m+1)4^p c_1 \|u\|_p \|\nabla u\|_p^{p-1} + \beta\|u\|_p^p\} + o(\|u\|^p) \\ &\geq 2^{-p-1}(\alpha_0 \|\nabla u\|_p^p + \|u\|_p^p) + o(\|u\|^p) \text{ as } \|u\| \to \infty .\end{aligned}$$

Hence

$$\frac{1}{\|u\|}\langle F(u), u\rangle \to \infty \qquad \text{as } \|u\| \to \infty . \tag{3.11}$$

From Theorem 2.7 in [20, p. 180] there exists a solution $v \in V$ of

$$\langle F(v), \varphi\rangle = 0 \qquad \text{for all } \varphi \in \mathrm{V} ,$$

implying that v is a solution of (3.10). This completes the proof of Lemma 3.2.

An essential device in the above proof is to consider the equation (3.10). By using Theorem 3.1 we prove the existence of minimal and maximal solutions of (1.1)–(1.2) between W-subsolutions and W-supersolutions and establish a Peano-Akô type theorem. We employ the techniques of Hirai and Akô [14] and Akô [1].

LEMMA 3.3. *Let the hypotheses of Theorem* 3.1 *hold. If u is a solution of* (1.1)–(1.2) *such that $\varphi_1 \leq u \leq \varphi_2$ a.e. in Ω, then we have the estimate*

$$\|u\|_{W^{1,p}(\Omega)} \leq C ,$$

where C is a constant independent of u.

PROOF. We can assume that $\varphi_1 \leq g \leq \varphi_2$ a.e. in Ω. Since $u - g \in W_0^{1,p}(\Omega)$ we have

$$\int_\Omega \{A(x, \nabla u)\cdot\nabla(u-g) + B(x, u, \nabla u)(u-g)\}\, dx = 0 .$$

From (H_1), (H_3) and (H_4) we can estimate

$$\alpha_0 \int_\Omega |\nabla u|^p \, dx \leq \int_\Omega \{|f_1||\nabla u|^{p-1} + |f_2| + N|\nabla g|(|f_0| + d|\nabla u|^{p-1}) + |u - g|(|f_3| + h(|\varphi_1| + |\varphi_2|) + c_1|\nabla u|^{p-1})\} \, dx$$
$$\leq \int_\Omega \{|f_2| + N|\nabla g||f_0| + (|\varphi_1| + |\varphi_2|)(|f_3| + h(|\varphi_1| + |\varphi_2|)\} \, dx + \int_\Omega \{|f_1| + Nd|\nabla g| + c_1(|\varphi_1| + |\varphi_2|)\}|\nabla u|^{p-1} \, dx .$$

We have for $\varepsilon > 0$

$$(|f_1| + Nd|\nabla g| + c_1(|\varphi_1| + |\varphi_2|))|\nabla u|^{p-1} \leq \varepsilon^{-p}(|f_1| + Nd|\nabla g| + c_1(|\varphi_1| + |\varphi_2|))^p + \varepsilon^q|\nabla u|^p .$$

The conclusion of Lemma 3.3 follows by choosing ε such that $\varepsilon^q = \alpha_0/2$.

THEOREM 3.2. *Let the hypotheses of Theorem* 3.1 *hold. Suppose that* f_0, $f_3 \in L^q(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$, $f_1 \in L^p(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ *and* $f_2 \in L^1(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ *in* (H_4) *and* (3.4). *Moreover, suppose that* $\varphi_1, \varphi_2 \in L^\infty_{\mathrm{loc}}(\Omega)$. *Then the problem* (1.1)–(1.2) *has a minimal solution* $\underline{u}$ *and a maximal solution* $\bar{u}$ *such that* $\varphi_1 \leq \underline{u} \leq \bar{u} \leq \varphi_2$ *a.e. in* Ω *in the sense that if* $u \in W^{1,p}(\Omega)$ *is any solution of* (1.1)–(1.2) *with* $\varphi_1 \leq u \leq \varphi_2$ *a.e. in* Ω, *then* $\underline{u} \leq u \leq \bar{u}$ *a.e. in* Ω.

PROOF. Put

(3.12) $\mathscr{S} = \{u: u \text{ is a solution of (1.1)–(1.2) with } \varphi_1 \leq u \leq \varphi_2 \text{ a.e. in } \Omega\}$.

By Theorem 3.1 we see that $\mathscr{S} \neq \emptyset$. It follows from Lemma 2.3 that for any subdomain $\Omega_0 \subset\subset \Omega$ there exists $\beta \in (0, 1)$ such that the restriction of $\mathscr{S}$ on Ω_0 is bounded in $C^\beta(\bar{\Omega}_0)$. We define the functions $\underline{u}$ and $\bar{u}$ by

(3.13) $\underline{u}(x) = \inf\{u(x): u \in \mathscr{S}\}$, $\bar{u}(x) = \sup\{u(x): u \in \mathscr{S}\}$,

for $x \in \Omega$. Then we see that $\underline{u}, \bar{u} \in C(\Omega)$. We shall show that $\underline{u}, \bar{u} \in \mathscr{S}$. Let $\{x^i\}_{i \in N}$ be the set of all rational points of Ω and for $i \in N$ let $\{v_n^{(i)}\}_{n \in N}$ be a sequence in $\mathscr{S}$ such that $\lim_{n\to\infty} v_n^{(i)}(x^i) = \bar{u}(x^i)$. Put $u_1 = v_1^{(1)}$ and $\lambda_2 = \max(u_1, v_2^{(1)}, v_2^{(2)})$. Then λ_2 is a W-subsolution of (1.1) in Ω with $\varphi_1 \leq \lambda_2 \leq \varphi_2$ a.e. in Ω and $\lambda_2 = g$ a.e. on $\partial\Omega$. From Theorem 3.1 we see that there exists $u_2 \in \mathscr{S}$ such that $\lambda_2 \leq u_2 \leq \varphi_2$ a.e. in Ω. Inductively we can choose a nondecreasing sequence $\{u_n\}_{n \in N} \subset \mathscr{S}$ such that for $n \geq 2$, $\lambda_n \leq u_n \leq \varphi_1$ a.e. in Ω, where $\lambda_n = \max(u_{n-1}, v_n^{(1)}, \ldots, v_n^{(n)})$. Let $u(x) = \lim_{n\to\infty} u_n(x)$ for $x \in \Omega$. By virtue of Ascoli-Arzela's Theorem, we see that u_n converges to u uniformly on any compact subsets of Ω and hence $u \in C(\Omega)$. Since $v_n^{(i)}(x^i) \leq \lambda_n(x^i) \leq u_n(x^i) \leq \bar{u}(x^i)$ for $n \geq i$, we have $u(x^i) = \bar{u}(x^i)$ for all $i \in N$. Therefore $u = \bar{u}$ in Ω.

By Lemma 3.3 we see that u_n is bounded in $W^{1,p}(\Omega)$ and hence we can extract a subsequence, still denoted by u_n, such that

$$u_n \to u \qquad \text{weakly in } W^{1,p}(\Omega),$$
$$u_n \to u \qquad \text{strongly in } L^p(\Omega).$$

Since $u = g$ a.e. on $\partial\Omega$, we obtain $u_n - u \in W_0^{1,p}(\Omega)$ and hence

$$\int_\Omega (A(x, \nabla u_n) - A(x, \nabla u)) \cdot \nabla(u_n - u)\, dx = -\int_\Omega A(x, \nabla u) \cdot \nabla(u_n - u)\, dx - \int_\Omega B(x, u_n, \nabla u_n)(u_n - u)\, dx.$$

The first term on the right hand side of the above equality tends to zero as $n \to \infty$ since u_n converges to u weakly in $W^{1,p}(\Omega)$. From (3.4) we obtain

$$\int_\Omega |B(x, u_n, \nabla u_n)(u_n - u)|\, dx \le \|u_n - u\|_p \{\|f_3\|_q + \|h(|\varphi_1| + |\varphi_2|)\|_q + c_1 \|\nabla u_n\|_p^{p-1}\} \to 0 \qquad \text{as } n \to \infty.$$

Therefore we have

$$\int_\Omega (A(x, \nabla u_n) - A(x, \nabla u)) \cdot \nabla(u_n - u)\, dx \to 0 \qquad \text{as } n \to \infty.$$

Hence, by Lemmas 2.1 and 2.2, we obtain

$$A(x, \nabla u_n) \to A(x, \nabla u), \qquad B(x, u_n, \nabla u_n) \to B(x, u, \nabla u) \qquad \text{strongly in } L^q(\Omega),$$

which proves that $u = \bar{u} \in \mathscr{S}$. Similarly we have $\underline{u} \in \mathscr{S}$. This completes the proof of Theorem 3.2.

Under the assumptions of Theorem 3.2, we denote by $\mathscr{S}$ the set defined by (3.12). By virtue of Lemma 2.3, we see that $\mathscr{S} \subset C(\Omega)$. We can derive the following Peano-Akô type theorem for the problem (1.1)–(1.2).

THEOREM 3.3. *Let the hypotheses of Theorem* 3.2 *hold. Suppose that* φ_1, $\varphi_2 \in L^\infty(\Omega)$. *Moreover, suppose that* $B(x, t, \xi)$ *is nondecreasing with respect to* $t \in [\varphi_1(x), \varphi_2(x)]$ *for almost every fixed* $x \in \Omega$ *and every fixed* $\xi \in \boldsymbol{R}^N$. *Then we have for every* $x_0 \in \Omega$

$$\{u(x_0) : u \in \mathscr{S}\} = [\underline{u}(x_0), \bar{u}(x_0)],$$

where $\underline{u}$ *and* $\bar{u}$ *are, respectively, the minimal solution and the maximal solution of* (1.1)–(1.2) *between* φ_1 *and* φ_2.

PROOF. It follows from Theorem 3.2 that

$$\mathscr{S} = \{u \in \mathscr{S} : \underline{u} \le u \le \bar{u} \text{ a.e. in } \Omega\}.$$

Hence we have

$$\{u(x_0) : u \in \mathcal{S}\} \subset [\underline{u}(x_0), \bar{u}(x_0)] .$$

To prove Theorem 3.3 it suffices to derive a contradiction from the assumption that there exists $u_0 \in \boldsymbol{R}$ such that $\underline{u}(x_0) < u_0 < \bar{u}(x_0)$ and $u_0 \notin \{u(x_0) : u \in \mathcal{S}\}$. We denote by $T(x, t)$ the truncated function

$$T(x, t) = \begin{cases} \varphi_1(x) & \text{if } t < \varphi_1(x) \\ t & \text{if } \varphi_1(x) \leq t \leq \varphi_2(x) \\ \varphi_2(x) & \text{if } \varphi_2(x) < t \end{cases}$$

for a.e. $x \in \Omega$, $\forall\, t \in \boldsymbol{R}$. We set $\tilde{B}(x, t, \xi) = B(x, T(x, t), \xi)$. From (3.4) we have

$$|\tilde{B}(x, t, \xi)| \leq |f_3(x)| + h(|\varphi_1(x)| + |\varphi_2(x)|) + c_1 |\xi|^{p-1}$$

for a.e. $x \in \Omega$, $\forall\, (t, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^N$. We consider the equation

$$-\operatorname{div} A(x, \nabla u) + \tilde{B}(x, u, \nabla u) = 0 \qquad \text{in } \Omega . \tag{3.14}$$

Let $u_1 = \underline{u}$, $v_1 = \bar{u}$ and $d_1 = \|v_1 - u_1\|_\infty$. Since $\tilde{B}(x, t, \xi)$ is nondecreasing with respect to $t \in \boldsymbol{R}$, we have for all non-negative functions $\varphi \in C_0^\infty(\Omega)$

$$\begin{aligned} &\int_\Omega \{A(x, \nabla(u_1 + d_1/2)) \cdot \nabla\varphi + \tilde{B}(x, u_1 + d_1/2, \nabla(u_1 + d_1/2))\varphi\}\, dx \\ &\geq \int_\Omega \{A(x, \nabla u_1) \cdot \nabla\varphi + \tilde{B}(x, u_1, \nabla u_1)\varphi\}\, dx = 0 . \end{aligned}$$

Hence $u_1 + d_1/2$ is a supersolution of (3.14). It is easy to see that any $u \in \mathcal{S}$ is a solution of (3.14) in Ω and hence $\bar{\lambda}_1 = \min(u_1 + d_1/2, v_1)$ is a W-supersolution of (3.14) in Ω. Similarly, $v_1 - d_1/2$ is a subsolution of (3.14) in Ω and $\underline{\lambda}_1 = \max(u_1, v_1 - d_1/2)$ is a W-subsolution of (3.14) in Ω. Since $\underline{\lambda}_1 \leq \bar{\lambda}_1$ in Ω and $\underline{\lambda}_1 = g = \bar{\lambda}_1$ a.e. on $\partial\Omega$, it follows from Theorem 3.1 that there exists a solution u of the problem (3.14)–(1.2) such that $\underline{\lambda}_1 \leq u \leq \bar{\lambda}_1$ in Ω. Hence, we have

$$u_1 \leq u \leq u_1 + d_1/2 , \qquad v_1 - d_1/2 \leq u \leq v_1 \qquad \text{in } \Omega .$$

Therefore, $u \in \mathcal{S}$. By our assumption we see that $u(x_0) \neq u_0$. Let $u_2 = u_1$, $v_2 = u$ if $u(x_0) > u_0$ and let $u_2 = u$, $v_2 = v_1$ if $u(x_0) < u_0$. Then we have $u_1 \leq u_2 \leq v_2 \leq v_1$ in Ω, $u_2(x_0) < u_0 < v_2(x_0)$, $\|v_2 - u_2\|_\infty \leq d_1/2$. Put $d_2 = \|v_2 - u_2\|_\infty$. Proceeding as above, there exist $u_3, v_3 \in \mathcal{S}$ such that $u_2 \leq u_3 \leq v_3 \leq v_2$ in Ω, $u_3(x_0) < u_0 < v_3(x_0)$, $\|v_3 - u_3\|_\infty \leq d_2/2 \leq 2^{-2} d_1$. By an inductive process we can construct sequences $\{u_n\}_{n \in N}$ and $\{v_n\}_{n \in N}$ of $\mathcal{S}$ such that for $n \in \boldsymbol{N}$, $u_n \leq u_{n+1} \leq v_{n+1} \leq v_n$ in Ω, $u_n(x_0) < u_0 < v_n(x_0)$, $\|v_n - u_n\|_\infty \leq 2^{1-n} d_1$. Let $u^*(x) = \lim_{n\to\infty} u_n(x)$ for $x \in \Omega$. Then, $u^*(x_0) = u_0$. From Lemma 2.3 it follows that u_n

converges to u^* uniformly on any compact subsets of Ω. By an argument similar to that of Theorem 3.2, we obtain $u^* \in \mathscr{S}$, which contradicts $u^*(x_0) = u_0$. This completes the proof of Theorem 3.3.

3.2. L-subsolutions and L-supersolutions

THEOREM 3.4. *Let φ_1 and φ_2 be respectively an L-subsolution and an L-supersolution of* (1.1) *in Ω such that $\varphi_1 \leq \varphi_2$ a.e. in Ω and $\varphi_1 \leq g \leq \varphi_2$ a.e. on $\partial\Omega$. Suppose that there exist a constant $\varepsilon \in (0, 1]$ and a function $f_3 \in L^1(\Omega)$ such that*

$$|B(x, t, \xi)| \leq h(|t|)(|f_3(x)| + |\xi|^{p-\varepsilon}) \tag{3.15}$$

for a.e. $x \in \Omega$, $^\forall$ $(t, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^N$, where $h: \bar{\boldsymbol{R}}_+ \to \bar{\boldsymbol{R}}_+$ is a nondecreasing function. Then the problem (1.1)–(1.2) *has a solution u such that $\varphi_1 \leq u \leq \varphi_2$ a.e. in Ω.*

PROOF. This result follows from an argument similar to that of [12]. We give a proof for the sake of completeness. Without loss of generality we can assume that $\varphi_1 \leq g \leq \varphi_2$ a.e. in Ω and that φ_1 and φ_2 are of the form (3.5) with $n = m$. Let $\tilde{A}_i(x, \xi)$ $(i = 1, \ldots, N)$, $T_l(x, t)$, u_l, v_l $(l = 0, 1, \ldots, m)$, $\tilde{B}(x, t, \xi)$ and $h(x, t)$ be as in the proof of Theorem 3.1. To prove Theorem 3.4 it suffices to solve the problem

$$\begin{cases} -\mathrm{div}\, \tilde{A}(x, \nabla u) + \tilde{B}(x, u, \nabla u) = 0 \text{ in } \Omega\,, \qquad u_0 \leq u \leq v_0 \text{ in } \Omega\,, \\ \qquad\qquad u = 0 \text{ on } \partial\Omega\,. \end{cases} \tag{3.16}$$

Since u_l and $v_l \in L^\infty(\Omega)$ $(l = 1, \ldots, m)$ are respectively subsolutions and supersolutions of (3.16) in Ω, u_0 and v_0 are respectively an L-subsolution and an L-supersolution of (3.16) in Ω. Put

$$M = 1 + \|\varphi_1\|_\infty + \|\varphi_2\|_\infty + \textstyle\sum_{l=1}^m (\|u_l\|_\infty + \|v_l\|_\infty)\,.$$

By a calculation similar to that of (3.7) and (3.8), we have

$$|\tilde{B}(x, T_l(u), \nabla T_l(u))| \leq |f_4(x)| + 2^p h(M)|\nabla u|^{p-\varepsilon}\,, \tag{3.17}$$

$l = 0, 1, \ldots, m$, for $^\forall$ $u \in W^{1,p}(\Omega)$, a.e. $x \in \Omega$, where $f_4 \in L^1(\Omega)$,

$$|h(x, t)| \leq 2^p(M^{p-1} + |t|^{p-1}) \text{ for a.e. } x \in \Omega,\ ^\forall\, t \in \boldsymbol{R}\,. \tag{3.18}$$

We define u_* by $u^* = \min\{u_l: l = 1, \ldots, m\}$ and $v^* = \max\{v_l: l = 1, \ldots, m\}$. Put

$$K = \{\varphi \in V: u_* - 1 \leq \varphi \leq v^* + 1 \text{ a.e. in } \Omega\}$$

where $V = W_0^{1,p}(\Omega)$. Then K is a closed convex subset of V. Let $\tilde{B}_l$, $B_l: V \to L^1(\Omega)$ $(l = 0, 1, \ldots, m)$ be the maps defined by (3.9). For $u, v \in V$ we define a_1,

a_2 and a_3 by

$$\langle a_1(u), v\rangle = \int_\Omega \tilde{A}(x, 8u)\cdot \nabla v\, dx\,, \qquad \langle a_2(u), v\rangle = \int_\Omega h(x, u)v\, dx\,,$$

$$\langle a_3(u), v\rangle = \int_\Omega \{\tilde{B}_0(u) + \textstyle\sum_{l=1}^m B_l(u)\}v\, dx\,.$$

We note that a_1, a_2: $V \to V^*$. We also define F: $V \to V^*$ by $F(u) = a_1(u) + a_2(u)$. We consider the variational inequality

$$(3.19) \qquad \langle F(u), \varphi - u\rangle + \langle a_3(u), \varphi - u\rangle \geq 0 \qquad \text{for } ^\forall\, \varphi \in K\,.$$

Theorem 3.4 is proved if the following two lemmas are proved.

LEMMA 3.4. *If $u \in K$ is a solution of* (3.19) *then u is a solution of* (3.16).

LEMMA 3.5. *There exists a solution $u \in K$ of* (3.19).

PROOF OF LEMMA 3.4. We note that $u - \min(u, v_i) = (u - v_i)^+$ and $\max(u, u_i) - u = (u_i - u)^+$ for $1 \leq i \leq m$. Since $\min(u, v_i)$, $\max(u, u_i) \in K$ we have from (3.19)

$$\langle F(u), (u - v_i)^+\rangle + \langle a_3(u), (u - v_i)^+\rangle \leq 0\,,$$

$$\langle F(u), (u_i - u)^+\rangle + \langle a_3(u), (u_i - u)^+\rangle \geq 0\,.$$

By an argument similar to that of Lemma 3.1, we have $u_0 \leq u \leq v_0$ a.e. in Ω. Therefore,

$$\langle a_1(u), \varphi - \mathrm{u}\rangle + \int_\Omega \tilde{B}(x, u, \nabla u)(\varphi - u)\, \mathrm{dx} \geq 0 \qquad \text{for } ^\forall\, \varphi \in K.$$

For arbitrary non-negative function $\psi \in C_0^\infty(\Omega)$, we can choose a positive constant δ such that $u \pm \delta\psi \in K$. From the above inequality with $\varphi = u \pm \delta\psi$ we have

$$\int_\Omega \{\tilde{A}(x, \nabla u)\cdot \tilde{B}(x, u, \nabla \mathrm{u})\psi\}\, dx = 0.$$

PROOF OF LEMMA 3.5.

Step 1. For arbitrary $z \in L^1(\Omega)$ there exists a unique $u \in K$ such that

$$\langle F(u), \varphi - u\rangle + \int_\Omega z(\varphi - u)\, dx \geq 0 \qquad \text{for } ^\forall\, \varphi \in K\,.$$

In fact, let $\{z_n\}_{n \in N} \subset L^q(\Omega)$ be a sequence such that

$$z_n \to v \qquad \text{strongly in } L^1(\Omega)\,,$$

$$z_n \to \mathrm{z} \text{ a.e. in } \Omega\,.$$

It follows from the proof of Lemma 3.2 that $F: V \to V^*$ is pseudo-monotone and that (3.11) holds. From Theorem 8.2 in [20, p. 247] there exists a $u_n \in K$ such that

$$\langle F(u_n), \varphi - u_n \rangle + \int_\Omega z_n(\varphi - u_n)\, dx \geq 0 \qquad \text{for } ^\forall \varphi \in K . \tag{3.20}$$

By (3.18) we have

$$\begin{aligned} \langle a_1(u_n), u_n \rangle &\leq -\langle a_2(u_n), u_n \rangle - \int_\Omega z_n u_n\, dx \\ &\leq \int_\Omega (2^{p+1}M^p + M|z_n|)\, dx , \end{aligned}$$

and hence, from (H_3), $\{u_n\}_{n \in N}$ is bounded in V. We can extract a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that for some $u \in K$

$$\begin{aligned} u_n &\to u \qquad \text{weakly in } V, \\ u_n &\to u \qquad \text{a.e. in } \Omega . \end{aligned}$$

Put $\varphi = u$ in (3.20). Then we obtain

$$\langle a_1(u_n), u_n - u \rangle \leq -\langle a_2(u_n), u_n - u \rangle - \int_\Omega z_n(u_n - u)\, dx \to 0 \qquad \text{as } n \to \infty ,$$

and hence

$$\langle a_1(u_n) - a_1(u), u_n - u \rangle \to 0 \qquad \text{as } n \to \infty .$$

We have from Lemma 2.1 $u_n \to u$ strongly in V. Therefore, letting $n \to \infty$ in (3.20), we see that

$$\langle F(u), \varphi - u \rangle + \int_\Omega z(\varphi - \mathrm{u})\, dx \geq 0 \qquad \text{for } ^\forall \varphi \in K .$$

Let u_1 and $u_2 \in K$ satisfy the above inequality then we obtain

$$0 \leq \langle a_1(u_1) - a_1(u_2), u_1 - u_2 \rangle \leq -\langle a_2(u_1) - a_2(u_2), u_1 - u_2 \rangle \leq 0$$

since $h(x, t)$ is nondecreasing with respect to $t \in \boldsymbol{R}$. By (H_2) and Poincaré's inequality, we have $u_1 = u_2$.

Step 2. It follows from Step 1 that for arbitrary $u \in K$ there exists a unique $v \in K$ such that

$$\langle F(v), \varphi - v \rangle + \langle a_3(u), \varphi - v \rangle \geq 0 \qquad \text{for } ^\forall \varphi \in K .$$

We define $S: K \to K$ by letting $v = S(u)$ be the unique solution of the above problem for $u \in K$. The following assertion holds:

$$S\colon K_R \to K_R \qquad \text{for some } R > 0\,,$$

where $K_R = K \cap \{\varphi \in K\colon \|\varphi\| \le R\}$.

In fact, set $v = S(u)$ for $u \in K$. From (3.17) and (3.18) we have

$$\begin{aligned}\langle a_1(v), v\rangle &\le -\langle a_2(v), v\rangle - \langle a_3(u), v\rangle \\ &\le \int_\Omega \{2^{p+1}M^p + 2(m+1)M(|f_4| + 2^p f(M)|\nabla u|^{p-\varepsilon})\}\, dx\end{aligned}$$

and hence from (H_3)

$$\|S(u)\|^p \le C(1 + \|u\|^{p-\varepsilon})\,,$$

where C is a constant independent of u. Taking $R > 0$ such that $C(1 + R^{p-\varepsilon}) \le R^p$, we see that $S\colon K_R \to K_R$.

Step 3. The map $S\colon K_R \to K_R$ is compact the continuous in the strong topology.

In fact, let $\{u_n\}_{n \in N} \subset K_R$. Since K_R is a bounded closed convex subset of V, we can extract a subsequence, still denoted by u_n, such that for some $u \in K_R$

$$\begin{aligned} u_n &\to u \qquad \text{weakly in } V\,, \\ u_n &\to u \qquad \text{a.e. in } \Omega\,. \end{aligned}$$

Since $\{S(u_n)\} \subset K_R$, we can assume that for some $w \in K_R$

$$\begin{aligned} S(u_n) &\to w \qquad \text{weakly in } V\,, \\ S(u_n) &\to w \qquad \text{a.e. in } \Omega\,. \end{aligned}$$

We see that for $n \in N$

$$\langle a_1(S(u_n)), S(u_n) - w\rangle \le -\langle a_2(S(u_n)), S(u_n) - w\rangle - \langle a_3(u_n), S(u_n) - w\rangle\,.$$

It is easy to see that $\langle a_2(S(u_n)), S(u_n) - w\rangle \to 0$ as $n \to \infty$. By (3.17) we have

$$|\langle a_3(u_n), S(u_n) - w\rangle| \le 2(m+1)\int_\Omega (|f_4| + 2^p h(M)|\nabla u_n|^{p-\varepsilon})|S(u_n) - w|\, dx\,.$$

We observe that

$$\int_\Omega |f_4||S(u_n) - w|\, dx \to \infty \qquad \text{as } n \to \infty\,,$$

and

$$\int_\Omega |\nabla u_n|^{p-\varepsilon}|S(u_n) - w|\, dx \le \|\nabla u_n\|_p^{p-\varepsilon}\|S(u_n) - w\|_{p/\varepsilon} \to 0 \qquad \text{as } n \to \infty\,.$$

Thus

$$\langle a_3(u_n), S(u_n) - w\rangle \to 0 \qquad \text{as } n \to \infty .$$

Consequently, we have

$$\langle a_1(S(u_n)) - a_1(w), S(u_n) - w\rangle \to 0 .$$

By Lemma 2.1 we obtain that $S(u_n) \to w$ strongly in V, which shows the compactness of $S: K_R \to K_R$. Let u_n, $u \in K_R$, $u_n \to u$ strongly in V. Since S: $K_R \to K_R$ is compact we can extract a subsequence $\{u_n'\}_{n \in N} \subset \{u_n\}_{n \in N}$ such that for some $w \in K_R$

$$S(u_n') \to w \qquad \text{strongly in } V ,$$

$$S(u_n') \to w \qquad \text{a.e. in } \Omega .$$

On the other hand, we have for all $\varphi \in K$

$$\langle F(S(u_n')), \varphi - S(u_n')\rangle + \langle a_3(u_n'), \varphi - S(u_n')\rangle \geq 0 .$$

Letting $n \to \infty$, we see that from Lemma 2.4 with $p_0 = 1$

$$\begin{aligned}\langle F(S(u_n')), \varphi - S(u_n')\rangle &= \langle F(S(u_n')), \varphi - w\rangle + \langle F(S(u_n')), w - S(u_n')\rangle \\ &\to \langle F(w), \varphi - w\rangle ,\end{aligned}$$

and

$$\begin{aligned}\langle a_3(u_n'), \varphi - S(u_n')\rangle &= \langle a_3(u_n'), w - S(u_n')\rangle + \langle a_3(u_n'), \varphi - w\rangle \\ &\to \langle a_3(u), \varphi - w\rangle .\end{aligned}$$

Hence we have

$$\langle F(w), \varphi - w\rangle + \langle a_3(u), \varphi - w\rangle \geq 0 .$$

It follows from the definition of S that $w = S(u)$ and hence $S(u_n') \to S(u)$ strongly in V, which proves that $S: K_R \to K_R$ is continuous in the strong topology.

Applying the Schauder fix point theorem we can find a $u \in K_R$ such that $u = S(u)$. This completes the proof of Lemma 3.5.

LEMMA 3.6. *Let the hypotheses of Theorem* 3.4 *hold. If u is a solution of* (1.1)–(1.2) *such that* $\varphi_1 \leq u \leq \varphi_2$ *a.e. in* Ω, *then we have the estimate*

$$\|u\|_{W^{1,p}(\Omega)} \leq C ,$$

where C is a constant independent of u.

Lemma 3.6 follows from an argument similar to that of Lemma 3.3. By applying an argument similar to the proof of Theorems 3.2 and 3.3, we can conclude from Lemma 3.6 the following theorems

THEOREM 3.5. *Let the hypotheses of Theorem* 3.4 *hold. Suppose that* $f_0 \in L^q(\Omega) \cap L^\infty_{loc}(\Omega)$, $f_1 \in L^p(\Omega) \cap L^\infty_{loc}(\Omega)$ *and* f_2, $f_3 \in L^1(\Omega) \cap L^\infty_{loc}(\Omega)$ *in* (H_4) *and* (3.15). *Then the problem* (1.1)–(1.2) *has a minimal solution* $\underline{u}$ *and a maximal solution* $\bar{u}$ *such that* $\varphi_1 \leq \underline{u} \leq \bar{u} \leq \varphi_2$ *a.e. in* Ω *in the sense that if* $u \in W^{1,p}(\Omega)$ *is any solution of* (1.1)–(1.2) *with* $\varphi_1 \leq u \leq \varphi_2$ *a.e. in* Ω, *then* $\underline{u} \leq u \leq \bar{u}$ *a.e. in* Ω.

THEOREM 3.6. *Let the hypotheses of Theorem* 3.5 *hold. Suppose that* $B(x, t, \xi)$ *is nondecreasing with respect to* $t \in [\varphi_1(x), \varphi_2(x)]$ *for almost every fixed* $x \in \Omega$ *and every fixed* $\xi \in \boldsymbol{R}^N$. *Then we have for every* $x_0 \in \Omega$

$$\{u(x_0) \colon u \in \mathscr{S}\} = [\underline{u}(x_0), \bar{u}(x_0)]\,,$$

where $\mathscr{S}$ *is the set defined by* (3.12) *and* $\underline{u}$, $\bar{u}$ *are respectively the minimal solution and the maximal solution of* (1.1)–(1.2) *between* φ_1 *and* φ_2.

3.3. *C*-subsolutions and *C*-supersolutions

THEOREM 3.7. *Let* φ_1 *and* φ_2 *be respectively a C-subsolution and a C-supersolution of* (1.1) *in* Ω *such that* $\varphi_1 \leq \varphi_2$ *a.e. in* Ω *and* $\varphi_1 \leq g \leq \varphi_2$ *a.e. on* $\partial\Omega$. *Suppose that* $f_0 \in L^{q+\varepsilon}(\Omega)$, $f_1 \in L^{p+\varepsilon}(\Omega)$ *and* $f_2 \in L^{1+\varepsilon}(\Omega)$ *for some positive constant* ε *in* (H_1) *and* (H_3). *Moreover, suppose that there exists a function* $f_3 \in L^q(\Omega)$ *such that*

$$|B(x, t, \xi)| \leq |f_3(x)| + h(|t|)(1 + |\xi|^p) \tag{3.21}$$

for a.e. $x \in \Omega$, $\forall\, (t, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^N$, *where* $h \colon \bar{\boldsymbol{R}}_+ \to \bar{\boldsymbol{R}}_+$ *is a nondecreasing function. Then the problem* (1.1)–(1.2) *has a solution u such that* $\varphi_1 \leq u \leq \varphi_2$ *a.e. in* Ω.

PROOF. Without loss of generality we can assume that $\varphi_1 \leq g \leq \varphi_2$ a.e. in Ω and that φ_1 and φ_2 are of the form (3.5) where $\underline{\psi}_i$, $\bar{\psi}_i \in C^{0,1}(\bar{\Omega})$ and $\underline{n} = m$. Put

$$M = \textstyle\sum_{i=1}^m \left(\|\underline{\psi}_i\|_\infty + \|\bar{\psi}_i\|_\infty + \|\nabla\underline{\psi}_i\|_\infty + \|\nabla\bar{\psi}_i\|_\infty\right).$$

Let for $n \in \boldsymbol{N}$

$$B_n(x, t, \xi) = \begin{cases} B(x, T(t), \xi) & \text{if } |\xi| \leq n + M \\ B(x, T(t), (n + M)\xi/|\xi|) & \text{if } |\xi| > n + M \end{cases}$$

where

$$T(t) = \begin{cases} -M & \text{if } t < -M \\ t & \text{if } -M \leq t \leq M \\ M & \text{if } M < t\,. \end{cases}$$

We see that $B_n \colon \Omega \times \boldsymbol{R} \times \boldsymbol{R}^N \to \boldsymbol{R}$ satisfies the Carathéodory condition and that φ_1 and φ_2 are respectively a *W*-subsolution and a *W*-supersolution of the equation

$$(3.22)_n \qquad -\operatorname{div} A(x, \nabla u) + B_n(x, u, \nabla u) = 0 \qquad \text{in } \Omega .$$

From (3.21) we have

$$|B_n(x, t, \xi)| \le |f_3(x)| + h(M)(1 + |n + M|^p)$$

for a.e. $x \in \Omega$, $^\forall (t, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^N$, and hence by Theorem 3.1 there exists a solution u_n of the problem $(3.22)_n$–(1.2) such that $\varphi_1 \le u_n \le \varphi_2$ a.e. in Ω. From the proof of Theorem 2.1 in [3, pp. 225–233] we can extract a subsequence of $\{u_n\}_{n \in N}$ which converges to a solution of (1.1)–(1.2) strongly in $W^{1,p}_{loc}(\Omega)$.

3.4. Examples and remark

EXAMPLE 3.1. We consider the problem (1.3)–(1.2). Let $B(x, t, \xi)$ be nondecreasing with respect to $t \in \boldsymbol{R}$ for almost every fixed $x \in \Omega$ and every fixed $\xi \in \boldsymbol{R}^N$. Suppose that B satisfies the condition

$$|B(x, t, \xi)| \le h(|t|)(1 + |\xi|^{p-1})$$

for a.e. $x \in \Omega$, $^\forall (t, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^N$, where $h\colon \bar{\boldsymbol{R}}_+ \to \bar{\boldsymbol{R}}_+$ is a nondecreasing function. Moreover, suppose that $g \in W^{1,p}(\Omega) \cap L^\infty(\partial\Omega)$. By following Remark 2 in [1], we can construct an L-subsolution φ_1 and an L-supersolution φ_2 of (1.3) such that $\varphi_1 \le \varphi_2$ a.e. in Ω and $\varphi_1 \le g \le \varphi_2$ a.e. on $\partial\Omega$. In fact, because of the boundedness of Ω, there exists a positive constant M such that

$$\Omega \subset \{x \in \boldsymbol{R}^N : -M < x_1 < M\} .$$

We choose positive constants γ and C such that

$$\gamma = (h(0) + 1)/(p - 1) , \qquad C \ge \gamma^{-1} e^{2M\gamma} h(0)^{1/(p-1)} + \|g\|_{\infty;\partial\Omega} .$$

We define the functions φ_1 and φ_2 by

$$\varphi_1(x) = -\varphi_2(x) , \qquad \varphi_2(x) = C(2 - e^{-\gamma(x_1 + M)}) .$$

Since $C \le \varphi_2(x) \le 2C$ in Ω, we see that $\varphi_1 \le \varphi_2$ in Ω and $\varphi_1 \le g \le \varphi_2$ a.e. on $\partial\Omega$. We have for all non-negative functions $\varphi \in C_0^\infty(\Omega)$

$$\begin{aligned} &\int_\Omega \{|\nabla\varphi_2|^{p-2}\nabla\varphi_2 \cdot \nabla\varphi + B(x, \varphi_2, \nabla\varphi_2)\varphi\}\, dx \\ &\quad \ge \int_\Omega \{|\nabla\varphi_2|^{p-2}\nabla\varphi_2 \cdot \nabla\varphi + B(x, 0, \nabla\varphi_2)\varphi\}\, dx \\ &\quad \ge \int_\Omega \{|\nabla\varphi_2|^{p-2}\nabla\varphi_2 \cdot \nabla\varphi - h(0)(1 + |\nabla\varphi_2|^{p-1})\varphi\}\, dx \\ &\quad = \int_\Omega \{|C\gamma|^{p-1} e^{-\gamma(p-1)(x_1 + M)} - h(0)\}\varphi\, dx \ge 0 , \end{aligned}$$

which implies that φ_2 is a supersolution of (1.3). Similarly we see that φ_1 is a subsolution of (1.3). Since the pseudo-Laplacian operator satisfies the conditions (H_1)–(H_3) (see, for example, [7, p. 264, Lemma 4.10]), by virtue of Theorems 3.2 and 3.3 (or Theorems 3.5 and 3.6), there exist a minimal solution $\underline{u}$ and a maximal solution $\bar{u}$ of (1.3)–(1.2) between φ_1 and φ_2 and the interval between $\underline{u}$ and $\bar{u}$ is filled with the set of solutions of (1.3)–(1.2).

EXAMPLE 3.2. In Theorem 3.6 we assumed that $B(x, t, \xi)$ is nondecreasing with respect to $t \in [\varphi_1(x), \varphi_2(x)]$. If $B(x, t, \xi)$ is strictly decreasing with respect to $t \in [\varphi_1(x), \varphi_2(x)]$, Theorem 3.6 is not true in general. For example, we consider the problem

$$\begin{cases} \Delta u + \lambda u^{\beta} = 0 & \text{in } \Omega = B_1\,, \quad N \geq 3\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{3.23}$$

where λ and β are positive constants. The problem (3.23) has a trivial solution $\underline{u} \equiv 0$. If $u \in W_0^{1,2}(\Omega) \cap L^{\infty}(\Omega)$ is a solution of (3.23), then we see that $u \in C^2(\bar{\Omega})$ by the regularity of elliptic equations (see e.g. [19, p. 115, Theorem 1.3; and p. 251, Theorem 2.1]). Gidas, Ni and Nirenberg [11] showed that there exists a unique positive solution $\bar{u}$ of (3.23) if $1 < \beta < (N+2)/(N-2)$. We can regard $\underline{u}$ and $\bar{u}$ as an L-subsolution and an L-supersolution of (3.23) respectively. By virtue of the maximum principle, bounded non-trivial and non-negative solutions of (3.23) are positive. Therefore we have

$$\mathscr{S} = \{u\colon u \text{ is a solution of (3.23) with } \underline{u} \leq u \leq \bar{u}\} = \{\underline{u}, \bar{u}\}\,.$$

Thus a Peano-Akô type theorem does not hold for (3.23) with $1 < \beta < (N+2)/(N-2)$. On the other hand, let $\beta = 1$ in (3.23) and let λ be the first eigenvalue of Δ under the Dirichlet condition. Then (3.23) has a positive eigenfunction $\bar{u}$, and we have

$$\mathscr{S} = \{c\bar{u}\colon c \text{ is a constant with } 0 \leq c \leq 1\}\,.$$

This shows that a Peano-Akô type theorem holds for (3.23) with $\beta = 1$.

REMARK 3.1. Theorems 3.1, 3.4 and 3.7 are related to [6, Theorem], [12, Theorem] and [3, Theorem 2.1]. We cannot prove the existence of minimal and maximal solutions and the Peano-Akô type theorem under the generalized Nagumo condition (3.21).

PROOF OF PROPOSITION 1. It suffices to prove that if $u_1, u_2 \in W^{1,p}(\Omega) \cap L^{\infty}(\Omega)$ are supersolutions (subsolutions) of (3.2), then $\min(u_1, u_2)$ ($\max(u_1, u_2)$) is a supersolution (subsolution) of (3.2). We use the method of [16, p. 42, Theorem 6.6]. We set $w = \min(u_1, u_2)$, $\tilde{u}_1 = u_1 - w$ and $\tilde{u}_2 = u_2 - w$. Let $\tilde{A}(x, \xi) = A(x, \xi + \nabla w(x))$ and $\tilde{B}(x, t) = B(x, t + w(x))$. Without loss of generality we can assume that $\tilde{A}$ satisfy (H_1), (H_2), (H_3) and (H_4). Let $T(t)$ be the truncated function

$$T(t) = \begin{cases} 0 & \text{if } t < 0 \\ t & \text{if } 0 \leq t \leq M \\ M & \text{if } M < T, \end{cases}$$

where $M = \|u_1\|_\infty + \|u_2\|_\infty$. The functions $\tilde{u}_1$ and $\tilde{u}_2$ are supersolutions of the equation

$$-\operatorname{div} \tilde{A}(x, \nabla u) + \tilde{B}(x, T(u)) = 0 \qquad \text{in } \Omega . \tag{3.24}$$

We set $V = W_0^{1,p}(\Omega)$ and

$$K = \{\varphi \in V : 0 \leq \varphi \leq M + 1 \text{ a.e. in } \Omega\} .$$

Let for $u, v \in V$

$$\langle a_1(u), v\rangle = \int_\Omega \tilde{A}(x, \nabla u) \cdot \nabla v \, dx , \qquad \langle a_2(u), v\rangle = \int_\Omega \tilde{B}(x, T(u)) v \, dx ,$$

where $T(u)(x) = T(u(x))$. It follows from the proof of Theorem 3.1 that a_1, a_2: $V \to V^*$, where V^* is the dual space of V, and that F: $V \to V^*$, $F(u) = a_1(u) + a_2(u)$, is pseudo-monotone and (3.11) holds. From Theorem 8.2 in [20, p. 247] there exists a $u \in K$ such that $\langle F(u), \varphi - u\rangle \geq 0$ for all $\varphi \in$ K, i.e.

$$\int_\Omega \{\tilde{A}(x, \nabla u) \cdot \nabla(\varphi - u) + \tilde{B}(x, T(u))(\varphi - u)\} \, dx \geq 0 \qquad \text{for all } \varphi \in K . \tag{3.25}$$

Since $\min (u, \tilde{u}_1) \in K$ and $u - \min (u, \tilde{u}_1) = (u - \tilde{u}_1)^+$, we have

$$\int_{u > \tilde{u}_1} \{\tilde{A}(x, \nabla u) \cdot \nabla(u - \tilde{u}_1) + \tilde{B}(x, T(u))(u - \tilde{u}_1)\} \, dx \leq 0 .$$

On the other hand, since $\tilde{u}_1$ is a supersolution of (3.24) and $(u - \tilde{u}_1)^+ \in W_0^{1,p}(\Omega)$, we obtain

$$\int_{u > \tilde{u}_1} \{\tilde{A}(x, \nabla \tilde{u}_1) \cdot \nabla(u - \tilde{u}_1) + \tilde{B}(x, T(\tilde{u}_1))(u - \tilde{u}_1)\} \, dx \geq 0$$

Consequently we have from (H_2)

$$\begin{aligned} 0 &\leq \int_{u > \tilde{u}_1} (\tilde{A}(x, \nabla u) - \tilde{A}(x, \nabla \tilde{u}_1)) \cdot \nabla(u - \tilde{u}_1) \, dx \\ &\leq \int_{u > \tilde{u}_1} (u - \tilde{u}_1)\{\tilde{B}(x, T(\tilde{u}_1)) - \tilde{B}(x, T(u))\} \, dx \leq 0 , \end{aligned}$$

which implies that $\nabla(u-\tilde{u}_1)^+=0$ a.e. in Ω. From Poincaré's inequality we have $u \le \tilde{u}_1$ a.e. in Ω. Similarly we have $u \le \tilde{u}_2$ a.e. in Ω and hence $0 \le u \le \min(\tilde{u}_1, \tilde{u}_2) = 0$ a.e. in Ω, i.e. $u=0$ a.e. in Ω. For any non-negative function $\psi \in C_0^\infty(\Omega)$, we can choose a positive constant δ such that $\delta\psi \in K$. By (3.25) with $\varphi = \delta\psi$ we obtain

$$\int_\Omega \{A(x, \nabla w)\cdot \nabla\psi + B(x, w)\psi\}\, dx \ge 0\,,$$

which shows that $w = \min(u_1, u_2)$ is a supersolution of (3.2). Similarly we see that $\max(u_1, u_2)$ is a subsolution of (3.2) if u_1, $u_2 \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ are subsolutions of (3.2). This completes the proof of Proposition 1.

4. Equations in unbounded domains

Throughout this section we assume that Ω is either an exterior domain in $\boldsymbol{R}^N$ or $\Omega = \boldsymbol{R}^N$ and that the conditions (H_1)–(H_3) and (H_5) hold for (1.1). We set $\Omega_R = \Omega \cap B_R$ for $R > 0$, where B_R denotes the open ball of radius R centered at the origin. In case Ω is an exterior domain we assume that there exists a positive constant a such that $\partial\Omega \subset B_a$. In case $\Omega = \boldsymbol{R}^N$ the boundary condition (1.2) is void, and the problem is to find a solution of (1.1) defined throughout $\boldsymbol{R}^N$.

DEFINITION 3. A function u is said to be a solution (subsolution, supersolution) of (1.1) in Ω if u is a solution (subsolution, supersolution) of (1.1) in Ω_R for all $R \ge a$.

A function u is said to be a W-subsolution (W-supersolution) of (1.1) in Ω if u is a W-subsolution (W-supersolution) of (1.1) in Ω_R for all $R \ge a$. L-subsolutions, L-supersolutions, C-subsolutions and C-supersolutions are defined analogously.

THEOREM 4.1. *Let φ_1 and φ_2 be a W-subsolution and a W-supersolution of* (1.1) *in Ω, respectively, such that $\varphi_1 \le \varphi_2$ a.e. in Ω and $\varphi_1 \le g \le \varphi_2$ a.e. on $\partial\Omega$ (if $\partial\Omega$ is non-empty). Suppose that for all $R \ge a$ there exist a positive constant c_R, a function $f_R \in L^q(\Omega_R)$ and a nondecreasing function $h_R\colon \bar{\boldsymbol{R}}_+ \to \bar{\boldsymbol{R}}_+$ such that $h_R(|\varphi|) \in L^q(\Omega_R)$ for $\varphi \in L^p(\Omega_R)$ and*

$$|B(x, t, \xi)| \le |f_R(x)| + h_R(|t|) + c_R|\xi|^{p-1} \tag{4.1}$$

for a.e. $x \in \Omega_R$, $^\forall\, (t, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^N$. Then the problem (1.1)–(1.2) *has a solution u such that $\varphi_1 \le u \le \varphi_2$ a.e. in Ω.*

LEMMA 4.1. *Let the hypotheses of Theorem* 4.1 *hold. Let R be a constant with $R \ge a$. If u is a solution of* (1.1) *in Ω_{2R} such that $\varphi_1 \le u \le \varphi_2$ a.e. in Ω_{2R}*

and $u = g$ a.e. on $\partial\Omega$ (if $\partial\Omega$ is non-empty), then we have the estimate

$$\|u\|_{W^{1,p}(\Omega_R)} \leq C_R ,$$

where C_R is a constant independent of u.

PROOF OF LEMMA 4.1. It suffices to prove the lemma for the case that $\partial\Omega$ is non-empty. We choose the function $\varphi \in C_0^1(B_{2R})$ so that $\varphi \equiv 1$ in B_R, $0 \leq \varphi \leq 1$ in B_{2R} and $|\nabla\varphi| \leq 4R^{-1}$ in B_{2R}. Since $\varphi^p(u-g) \in W_0^{1,p}(\Omega_{2R})$ we have

$$\int_{\Omega_{2R}} \{\varphi^p A(x, \nabla u)\cdot\nabla(u-g) + p\varphi^{p-1}(u-g)A(x, \nabla u)\cdot\nabla\varphi + \varphi^p(u-g)B(x, u, \nabla u)\}\, dx = 0 .$$

Put $\alpha_0 = \inf\{\alpha(x): x \in \Omega_{2R}\}$ and $d = \|c_0\|_{\infty;\Omega_{2R}}$. Note that $\alpha_0 > 0$. We obtain from (H_1), (H_3) and (4.1)

$$\begin{aligned}\int_{\Omega_{2R}} \alpha_0\varphi^p|\nabla u|^p\, dx \leq \int_{\Omega_{2R}} & [\varphi^p\{|f_2| + |f_1||\nabla u|^{p-1} + N|\nabla g|(|f_0| + d|\nabla u|^{p-1})\} \\ & + 4R^{-1}Np\varphi^{p-1}|u-g|(|f_0| + d|\nabla u|^{p-1}) \\ & + \varphi^p|u-g|(|f_{2R}| + h_{2R}(|u|) + c_{2R}|\nabla u|^{p-1})]\, dx .\end{aligned}$$

Let $v = |\varphi_1| + |\varphi_2| + |g|$. Since $|u-g| \leq v$ and $|u| \leq v$ a.e. in Ω_{2R}, we have

$$\begin{aligned}\int_{\Omega_{2R}} \alpha_0\varphi^p|\nabla u|^p\, dx \leq \int_{\Omega_{2R}} & [|f_2| + N|\nabla g||f_0| + v(4R^{-1}Np|f_0| + |f_{2R}| + h_{2R}(v)) \\ & + \varphi^p|\nabla u|^{p-1}(|f_1| + Nd|\nabla g| + c_{2R}v) \\ & + 4R^{-1}Np\, dv\varphi^{p-1}|\nabla u|^{p-1}]\, dx .\end{aligned}$$

By virtue of Hölder's inequality, we have for $\varepsilon > 0$

$$|\nabla u|^{p-1}(|f_1| + Nd|\nabla g| + C_{2R}v) \leq \varepsilon^q|\nabla u|^p + \varepsilon^{-p}(|f_1| + Nd|\nabla g| + C_{2R}v)^p$$

and

$$4R^{-1}Np\, dv\varphi^{p-1}|\nabla u|^{p-1} \leq \varepsilon^q\varphi^p|\nabla u|^p + \varepsilon^{-p}(4R^{-1}Np\, dv)^p .$$

Lemma 4.1 then follows by choosing ε so that $\alpha_0 \geq 4\varepsilon^q$.

PROOF OF THEOREM 4.1. It suffices to prove the theorem for the case $\partial\Omega$ is non-empty. From Theorem 3.1 it follows that for $n \in N$ the problem

$$\begin{cases} -\operatorname{div} A(x, \nabla u) + B(x, u, \nabla u) = 0 & \text{in } \Omega_{n+a} , \\ u = g \quad \text{on } \partial\Omega , & u = \varphi_1 \quad \text{on } \partial B_{n+a} \end{cases}$$

has a solution $u_n \in W^{1,p}(\Omega_{n+a})$ such that $\varphi_1 \leq u_n \leq \varphi_2$ a.e. in Ω_{n+a}. By Lemma 4.1 we see that $\{u_n\}_{n\geq 4a}$ is bounded in $W^{1,p}(\Omega_{2a})$ and hence we can extract a subsequence $\{u_n^{(1)}\}_{n\in N}$ of $\{u_n\}_{n\geq 4a}$ such that for some $u^{(1)}\in W^{1,p}(\Omega_{2a})$

$$u_n^{(1)}\to u^{(1)}\qquad \text{weakly in } W^{1,p}(\Omega_{2a})\,,$$
$$u_n^{(1)}\to u^{(1)}\qquad \text{strongly in } L^p(\Omega_{2a})\,.$$

We choose the function $\varphi\in C_0^1(B_{2a})$ so that $\varphi\equiv 1$ in B_a, $0\leq\varphi\leq 1$ in B_{2a} and $|\nabla\varphi|\leq 4a^{-1}$ in B_{2a}. Since $u^{(1)}=g$ a.e. on $\partial\Omega$ and $\varphi(u_n^{(1)}-u^{(1)})\in W_0^{1,p}(\Omega_{2a})$, we have

$$\int_{\Omega_{2a}}\varphi A(x,\nabla u_n^{(1)})\cdot\nabla(u_n^{(1)}-u^{(1)})\,dx=-\int_{\Omega_{2a}}(u_n^{(1)}-u^{(1)})\{A(x,\nabla u_n^{(1)})\cdot\nabla\varphi + B(x,u_n^{(1)},\nabla u_n^{(1)})\varphi\}\,dx\,.$$

By virtue of (H_1) and (4.1), we have

$$\int_{\Omega_{2a}}|(u_n^{(1)}-u^{(1)})A(x,\nabla u_n^{(1)})\cdot\nabla\varphi|\,dx \leq 4a^{-1}N\|u_n^{(1)}-u^{(1)}\|_{p;\Omega_{2a}}\{\|f_0\|_{q;\Omega_{2a}}+d\|\nabla u_n^{(1)}\|_{p;\Omega_{2a}}^{p-1}\}\to 0\,,$$

where $d=\|c_0\|_{\infty;\Omega_{2a}}$, and

$$\int_{\Omega_{2a}}|(u_n^{(1)}-u^{(1)})B(x,u_n^{(1)},\nabla u_n^{(1)})|\,dx \leq \|u_n^{(1)}-u^{(1)}\|_{p;\Omega_{2a}}\{\|f_{2a}\|_{q;\Omega_{2a}}+\|h_{2a}(|\varphi_1|+|\varphi_2|)\|_{q;\Omega_{2a}}+c_{2a}\|\nabla u_n^{(1)}\|_{p;\Omega_{2a}}^{p-1}\} \to 0\qquad\text{as } n\to\infty\,.$$

Since $u_n^{(1)}$ converges to $u^{(1)}$ weakly in $W^{1,p}(\Omega_{2a})$, we obtain

$$\int_{\Omega_{2a}}\varphi A(x,\nabla u^{(1)})\cdot\nabla(u_n^{(1)}-u^{(1)})\,dx\to 0\qquad\text{as } n\to\infty\,.$$

Consequently we have from (H_2)

$$\int_{\Omega_a}(A(x,\nabla u_n^{(1)})-A(x,\nabla u^{(1)}))\cdot\nabla(u_n^{(1)}-u^{(1)})\,dx\to 0\qquad\text{as } n\to\infty\,.$$

By Lemmas 2.1 and 2.2 we see that $u_n^{(1)}$ converges to $u^{(1)}$ strongly in $W^{1,p}(\Omega_a)$ and hence $u^{(1)}$ is a solution of (1.1) in Ω_a such that $\varphi_1\leq u^{(1)}\leq\varphi_2$ a.e. in Ω_a and $u^{(1)}=g$ a.e. on $\partial\Omega$. By an inductive process, we can construct sequences $\{u_n^{(i)}\}_{n,i\in N}$ and $\{u^{(i)}\}_{i\in N}$ such that $\{u_n^{(i)}\}_{n\in N}$ is a subsequence of $\{u_n^{(i-1)}\}_{n\geq 4ia}$ and converges strongly in $W^{1,p}(\Omega_{ia})$ to $u^{(i)}$, which is a solution of (1.1) in Ω_{ia} such that $\varphi_1\leq u^{(i)}\leq\varphi_2$ a.e. in Ω_{ia} and $u^{(i)}=g$ a.e. on $\partial\Omega$. Since $u^{(i+1)}=u^{(i)}$ a.e. in Ω_{ia}, we can define $u\in W_{loc}^{1,p}(\Omega)$ by $u=u^{(i)}$ in Ω_{ia}. The function u is a solution of (1.1)–(1.2) such that $\varphi_1\leq u\leq\varphi_2$ a.e. in Ω. This completes the proof of Theorem 4.1.

THEOREM 4.2. *Let the hypotheses of Theorem* 4.1 *hold except that* (H_5) *is replaced by* (H_6). *Suppose that* $\varphi_1, \varphi_2 \in L^{\infty}_{loc}(\Omega)$ *and* $f_{R} \in L^{\infty}(\Omega_R)$ *in* (4.1) *for all* $R \geq a$. *Then the problem* (1.1)–(1.2) *has a minimal solution* $\underline{u}$ *and a maximal solution* $\bar{u}$ *such that* $\varphi_1 \leq \underline{u} \leq \bar{u} \leq \varphi_2$ *a.e. in* Ω *in the sense that if* u *is any solution of* (1.1)–(1.2) *with* $\varphi_1 \leq u \leq \varphi_2$ *a.e. in* Ω, *then* $\underline{u} \leq u \leq \bar{u}$ *a.e. in* Ω.

PROOF. It suffices to prove the theorem for the case $\partial\Omega$ is non-empty. We denote by $\mathscr{S}$ the set defined by (3.12). Let $\underline{u}$ and $\bar{u}$ be the functions defined by (3.13). It suffices to prove that $\underline{u}, \bar{u} \in \mathscr{S}$. We can construct, similarly to the proof of Theorem 3.2, a nondecreasing sequence $\{u_n\}$ of $\mathscr{S}$ such that u_n converges to $\bar{u}$ uniformly on compact subsets of Ω. Let R be an arbitrary constant with $R > a$. From Lemma 4.1 we can extract a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$u_n \to \bar{u} \qquad \text{weakly in } W^{1,p}(\Omega_{2R}),$$

$$u_n \to \bar{u} \qquad \text{strongly in } L^p(\Omega_{2R}).$$

Similarly to the proof of Theorem 4.1, since $\bar{u} = g$ a.e. on $\partial\Omega$, we see that u_n converges to $\bar{u}$ strongly in $W^{1,p}(\Omega_R)$, which shows that $\bar{u}$ is a solution of (1.1) in Ω. Thus, $\bar{u} \in \mathscr{S}$. Similarly we have $\underline{u} \in \mathscr{S}$.

THEOREM 4.3. *Let the hypotheses of Theorem* 4.2 *hold. Suppose that* $\varphi_1, \varphi_2 \in L^{\infty}(\Omega_a)$. *Moreover, suppose that* $B(x, t, \xi)$ *is nondecreasing with respect to* $t \in [\varphi_1(x), \varphi_2(x)]$ *for almost every fixed* $x \in \Omega$ *and every fixed* $\xi \in \mathbf{R}^N$. *Then we have for every* $x_0 \in \Omega$

$$\{u(x_0): u \in \mathscr{S}\} = [\underline{u}(x_0), \bar{u}(x_0)],$$

where $\mathscr{S}$ *is the set defined by* (3.12) *and* $\underline{u}$, $\bar{u}$ *are, respectively, the minimal solution and the maximal solution of* (1.1)–(1.2) *between* φ_1 *and* φ_2.

To prove Theorem 4.3 it suffices to consider the case that $\partial\Omega$ is non-empty. We set for $R > a$

$$\mathscr{S}_R = \left\{ \begin{array}{c} u\colon u \text{ is a solution of (1.1) in } \Omega_R \text{ with } \underline{u} \leq u \leq \bar{u} \\ \text{a.e. in } \Omega_R \text{ and } u = g \text{ a.e. on } \partial\Omega \end{array} \right\}.$$

We note that $\mathscr{S}_R \neq \emptyset$ because $\underline{u}$, $\bar{u} \in \mathscr{S}_R$ and that $\mathscr{S}_R \subset C(\Omega_R)$ by virtue of Lemma 2.3.

LEMMA 4.2. *Let the hypotheses of Theorem* 4.3 *hold. Then we have for all* $R > a$ *and* $x_0 \in \Omega_R$

$$\{u(x_0): u \in \mathscr{S}_R\} = [\underline{u}(x_0), \bar{u}(x_0)].$$

PROOF OF LEMMA 4.2. Suppose that $\underline{u}(x_0) < \bar{u}(x_0)$. It suffices to derive a contradiction from the assumption that there exists a $u_0 \in \boldsymbol{R}$ such that $\underline{u}(x_0) < u_0 < \bar{u}(x_0)$ and $u_0 \notin \{u(x_0): u \in \mathscr{S}_R\}$. Let $u_1 = \underline{u}$, $v_1 = \bar{u}$ and $d_1 = \|v_1 - u_1\|_{\infty;\Omega_{4R}}$. Similarly to the proof of Theorem 3.3, the functions $\underline{\lambda}_1 = \max(u_1, v_1 - d_1/2)$ and $\bar{\lambda}_1 = \min(u_1 + d_1/2, v_1)$ are respectively a W-subsolution and a W-supersolution of (3.14) in Ω_{4R} such that $\underline{\lambda}_1 \leq \bar{\lambda}_1$ in Ω_{4R} and $\underline{\lambda}_1 = g = \bar{\lambda}_1$ a.e. on $\partial\Omega$. Since $\varphi_1 \leq \underline{\lambda}_1 \leq \bar{\lambda}_1 \leq \varphi_2$ a.e. in Ω_{4R}, it follows from Theorem 3.1 that the problem

$$\begin{cases} -\operatorname{div} A(x, \nabla u) + B(x, u, \nabla u) = 0 & \text{in } \Omega_{4R}\,, \\ u = g \quad \text{on } \partial\Omega\,, \qquad u = \underline{\lambda}_1 & \text{on } \partial B_{4R} \end{cases}$$

has a solution $u \in W^{1,p}(\Omega_{4R})$ such that $\underline{\lambda}_1 \leq u \leq \bar{\lambda}_1$ a.e. in Ω_{4R}. Therefore we have $u \in \mathscr{S}_{4R}$. By an argument similar to that of Theorem 3.3, we can construct a nondecreasing sequence $\{u_n\}$ of $\mathscr{S}_{4R}$ such that $\tilde{u}(x_0) = u_0$, where $\tilde{u}(x) = \lim_{n\to\infty} u_n(x)$ for $x \in \Omega_{4R}$. From Lemma 4.1 we can assume that

$$\begin{aligned} u_n &\to \tilde{u} \qquad \text{weakly in } W^{1,p}(\Omega_{2R})\,, \\ u_n &\to \tilde{u} \qquad \text{strongly in } L^p(\Omega_{2R})\,. \end{aligned}$$

Similarly to the proof of Theorem 4.1, we have $\tilde{u} \in \mathscr{S}_R$. This contradicts $\tilde{u}(x_0) = u_0$.

PROOF OF THEOREM 4.3. It follows from Theorem 4.2 that

$$\mathscr{S} = \{u \in \mathscr{S} : \underline{u} \leq u \leq \bar{u} \text{ a.e. in } \Omega\}\,.$$

We can assume that $\underline{u}(x_0) < \bar{u}(x_0)$. Let R be an arbitrary constant satisfying $|x_0| < R$. Let u_0 be an arbitrary fixed constant with $\underline{u}(x_0) < u_0 < \bar{u}(x_0)$. It suffices to prove that there exists a $u \in \mathscr{S}$ such that $u(x_0) = u_0$. From Lemma 4.2, for $n \in \boldsymbol{N}$, we can choose a $u_n \in \mathscr{S}_{nR}$ such that $u_n(x_0) = u_0$. By Lemma 4.1 we see that $\{u_n\}_{n\geq 4}$ is bounded in $W^{1,p}(\Omega_{2R})$ and hence we can extract a subsequence $\{u_n^{(1)}\}_{n\in\boldsymbol{N}}$ of $\{u_n\}_{n\geq 4}$ such that for some $u^{(1)} \in W^{1,p}(\Omega_{2R})$

$$\begin{aligned} u_n^{(1)} &\to u^{(1)} \qquad \text{weakly in } W^{1,p}(\Omega_{2R})\,, \\ u_n^{(1)} &\to u^{(1)} \qquad \text{strongly in } L^p(\Omega_{2R})\,. \end{aligned}$$

Since $|u_n| \leq |\varphi_1| + |\varphi_2| \in L^\infty(\Omega_{2R})$, we can assume, from Lemma 2.3, that $u_n^{(1)}$ converges to $u^{(1)}$ uniformly on some neighborhood of x_0. Therefore we see $u^{(1)}(x_0) = u_0$. Theorem 4.3 follows from the concluding argument in Theorem 4.1.

THEOREM 4.4. *Let φ_1 and φ_2 be an L-subsolution and an L-supersolution of* (1.1) *in Ω, respectively, such that $\varphi_1 \leq \varphi_2$ a.e. in Ω and $\varphi_1 \leq g \leq \varphi_2$ a.e. on $\partial\Omega$ (if $\partial\Omega$ is non-empty). Suppose that for all $R \geq a$ there exist a positive constant*

$\varepsilon_R \in (0, 1]$, *a function* $f_R \in L^1(\Omega_R)$ *and a nondecreasing function* $h_R\colon \bar{\boldsymbol{R}}_+ \to \bar{\boldsymbol{R}}_+$ *such that*

$$|B(x, t, \xi)| \le h_R(|t|)(|f_R(x)| + |\xi|^{p-\varepsilon_R}) \tag{4.2}$$

for a.e. $x \in \Omega_R$, $\forall\, (t, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^N$. *Then the problem* (1.1)–(1.2) *has a solution* u *such that* $\varphi_1 \le u \le \varphi_2$ *a.e. in* Ω.

THEOREM 4.5. *Let the hypotheses of Theorem* 4.4 *hold except that* (H_5) *is replaced by* (H_6). *Suppose that* $f_R \in L^\infty_{loc}(\Omega_R)$ *in* (4.2) *for all* $R \ge a$. *Then the problem* (1.1)–(1.2) *has a minimal solution* $\underline{u}$ *and a maximal solution* $\bar{u}$ *such that* $\varphi_1 \le \underline{u} \le \bar{u} \le \varphi_2$ *a.e. in* Ω *in the sense that if* u *is any solution of* (1.1)–(1.2) *with* $\varphi_1 \le u \le \varphi_2$ *a.e. in* Ω, *then* $\underline{u} \le u \le \bar{u}$ *a.e. in* Ω.

THEOREM 4.6. *Let the hypotheses of Theorem* 4.5 *hold. Suppose that* $B(x, t, \xi)$ *is nondecreasing with respect to* $t \in [\varphi_1(x), \varphi_2(x)]$ *for almost every fixed* $x \in \Omega$ *and every fixed* $\xi \in \boldsymbol{R}^N$. *Then we have for every* $x_0 \in \Omega$

$$\{u(x_0)\colon u \in \mathscr{S}\} = [\underline{u}(x_0), \bar{u}(x_0)]\,,$$

where $\mathscr{S}$ *is the set defined by* (3.12) *and* $\underline{u}$, $\bar{u}$ *are, respectively, the minimal solution and the maximal solution of* (1.1)–(1.2) *between* φ_1 *and* φ_2.

Theorems 4.4, 4.5 and 4.6 are counterparts of Theorems 3.4, 3.5 and 3.6. Their proofs are omitted, since they are similar to the proofs of Theorems 4.1, 4.2 and 4.3.

THEOREM 4.7. *Let* $\Omega = \boldsymbol{R}^N$ *and let* φ_1 *and* φ_2 *be a C-subsolution and a C-supersolution of* (1.1) *in* $\boldsymbol{R}^N$, *respectively, such that* $\varphi_1 \le \varphi_2$ *in* $\boldsymbol{R}^N$. *Suppose that* $f_0 \in L^{q+\varepsilon}_{loc}(\boldsymbol{R}^N)$, $f_1 \in L^{p+\varepsilon}_{loc}(\boldsymbol{R}^N)$ *and* $f_2 \in L^{l+\varepsilon}_{loc}(\boldsymbol{R}^N)$ *for some positive constant* ε *in* (H_1) *and* (H_3). *Moreover, suppose that for all* $R > 0$ *there exist a function* $f_R \in L^q(B_R)$ *and a nondecreasing function* $h_R\colon \bar{\boldsymbol{R}}_+ \to \bar{\boldsymbol{R}}_+$ *such that*

$$|B(x, t, \xi)| \le |f_R(x)| + h_R(|t|)(1 + |\xi|^p) \tag{4.3}$$

for a.e. $x \in \Omega$, $\forall\, (t, \xi) \in \boldsymbol{R} \times \boldsymbol{R}^N$. *Then equation* (1.1) *has a solution* u *such that* $\varphi_1 \le u \le \varphi_2$ *a.e. in* $\boldsymbol{R}^N$.

LEMMA 4.3. *Let the hypotheses of Theorem* 4.7 *hold. Let* R *be a positive constant. Then there exists a constant* $p_R > p$ *such that if* u *is a solution of* (1.1) *in* B_{2R} *with* $\|u\|_{\infty;B_{2R}} \le M$, *then*

$$\|u\|_{X_R} \le C_R\,,$$

where C_R *is a constant independent of* u. *and* $X_R = W^{1,p_R}(B_R)$.

Lemma 4.3 is due to [3, Proposition 3.8].

PROOF OF THEOREM 4.7. It follows from Theorem 3.7 that for $n \in N$ the problem

$$\begin{cases} -\operatorname{div} A(x, \nabla u) + B(x, u, \nabla u) = 0 & \text{in } B_n\,, \\ u = \varphi_1 & \text{on } \partial B_n \end{cases}$$

has a solution $u_n \in W^{1,p}(B_n)$ such that $\varphi_1 \leq u_n \leq \varphi_2$ a.e. in B_n. Let R be an arbitrary positive constant. By Lemma 4.3 there exists a $p_R > p$ such that $\{u_n\}_{n \geq 4R}$ is bounded in $W^{1,p_R}(B_{2R})$. Thus we can extract a subsequence $\{u_n^{(1)}\}_{n \in N}$ of $\{u_n\}_{n \geq 4R}$ such that for some $u^{(1)} \in W^{1,p}(B_{2R})$

$$u_n^{(1)} \to u^{(1)} \qquad \text{weakly in } W^{1,p}(B_{2R})\,,$$

$$u_n^{(1)} \to u^{(1)} \qquad \text{a.e. in } B_{2R}\,.$$

Put $M = \|\varphi_1\|_{\infty; B_{2R}} + \|\varphi_2\|_{\infty; B_{2R}}$. Let $\varphi \in C_0^1(B_{2R})$ be the function satisfying $0 \leq \varphi \leq 1$, $|\nabla \varphi| \leq 4R^{-1}$ in B_{2R} and $\varphi = 1$ in B_R. Since $\varphi(u_n^{(1)} - u^{(1)}) \in W_0^{1,p}(B_{2R}) \cap L^\infty(B_{2R})$, we have

$$\int_{B_{2R}} \varphi A(x, \nabla u_n^{(1)}) \cdot \nabla (u_n^{(1)} - u^{(1)})\, dx = -\int_{B_{2R}} (u_n^{(1)} - u^{(1)}) \{A(x, \nabla u_n^{(1)}) \cdot \nabla \varphi + B(x, u_n^{(1)}, \nabla u_n^{(1)}) \varphi\}\, dx$$

By Lemma 4.3, we obtain

$$\int_{B_{2R}} |u_n^{(1)} - u^{(1)}| \, |\nabla u_n^{(1)}|^p \, dx \leq \|\nabla u_n^{(1)}\|^p_{p_R; B_{2R}} \|u_n^{(1)} - u^{(1)}\|_{C_R; B_{2R}} \to 0 \qquad \text{as } n \to \infty\,,$$

where $C_R = p_R/(p_R - p)$. Therefore we have from (4.3)

$$\int_{B_{2R}} |(u_n^{(1)} - u^{(1)}) A(x, \nabla u_n^{(1)}) \cdot \nabla \varphi| \, dx \to 0\,,$$

$$\begin{aligned} \int_{B_{2R}} |(u_n^{(1)} - u^{(1)}) B(x, u_n^{(1)}, \nabla u_n^{(1)})| \, dx &\leq \int_{B_{2R}} |u_n^{(1)} - u^{(1)}| (|f_{2R}| + h_{2R}(M))\, dx \\ &\quad + h_{2R}(M) \int_{B_{2R}} |u_n^{(1)} - u^{(1)}| \cdot |\nabla u_n^{(1)}|^p \, dx \\ &\to 0 \qquad \text{as } n \to \infty\,. \end{aligned}$$

Consequently we have by (H_2)

$$\int_{B_R} (A(x, \nabla u_n^{(1)}) - A(x, \nabla u^{(1)}) \cdot \nabla (u_n^{(1)} - u^{(1)})\, dx \to 0 \qquad \text{as } n \to \infty\,.$$

By virtue of Lemmas 2.1 and 2.2 we see that $u_n^{(1)}$ converges to $u^{(1)}$ strongly in $W^{1,p}(B_R)$ and hence $u^{(1)}$ is a solution of (1.1) in B_R such that $\varphi_1 \leq u^{(1)} \leq \varphi_2$ a.e. in B_R. Theorem 4.7 follows from the concluding argument in Theorem 4.1.

REMARK 4.1. In Theorems 4.3 and 4.6 we assumed that $B(x, t, \xi)$ is nondecreasing with respect to $t \in [\varphi_1(x), \varphi_2(x)]$. The following example shows that Theorems 4.3 and 4.6 are not true in general when $B(x, t, \xi)$ is strictly decreasing with respect to $t \in [\varphi_1(x), \varphi_2(x)]$. We consider the equation

$$\Delta u + c(x)u^\beta = 0 \qquad \text{in } \boldsymbol{R}^N, \tag{4.4}$$

where $c \in C^1(\boldsymbol{R}^N)$ is positive, $0 < \beta < 1$ is constant and $N \geq 3$. Equation (4.4) has a trivial solution $\underline{u} \equiv 0$. Fukagai [9] showed that (4.4) has a unique positive solution $\bar{u}$ such that $\bar{u}(x) \to 0$ as $|x| \to \infty$ if

$$\int_0^\infty r^{1+(1-\beta)(N-2)}(\max_{|x|=r} c(x))\, dx < \infty .$$

Therefore we see that under the above condition

$$\{u : u \text{ is a solution of (4.4) with } \underline{u} \leq u \leq \bar{u} \text{ in } \boldsymbol{R}^N\} = \{\underline{u}, \bar{u}\} .$$

5. Application

In this section we shall establish the existence of positive solutions of the equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + B(x, u, \nabla u) = 0 \qquad \text{in } \boldsymbol{R}^N, \tag{5.1}$$

where $1 < p \leq 2$, $N \geq 3$ and $B(x, t, \xi)$ is as in Theorem 4.7.

THEOREM 5.1. *Suppose that there exist a continuous function* $\phi: \bar{\boldsymbol{R}}_+ \to \bar{\boldsymbol{R}}_+$ *and a continuous nondecreasing function* $F: \bar{\boldsymbol{R}}_+ \times \bar{\boldsymbol{R}}_+ \to \bar{\boldsymbol{R}}_+$ *such that*

$$|B(x, t, \xi)| \leq \phi(|x|)F(t, |\xi|) \tag{5.2}$$

for a.e. $x \in \boldsymbol{R}^N$, $\forall\, (t, \xi) \in \bar{\boldsymbol{R}}_+ \times \boldsymbol{R}^N$. *Moreover, suppose that*

$$\int_0^\infty r^{1/(p-1)}\phi(r)^{1/(p-1)}\, dr < \infty , \tag{5.3}$$

and that one of the following conditions is satisfied:

$$(\mathrm{F}_1) \qquad \lim_{t\to 0} t^{-1}F(t, s)^{1/(p-1)} = 0 \qquad \textit{for each fixed } s \geq 0 ;$$

$$(\mathrm{F}_2) \qquad \lim_{t\to\infty} t^{-1}F(t, t)^{1/(p-1)} = 0 .$$

Then equation (5.1) *possesses infinitely many positive solutions in* $W^{1,p}_{loc}(\boldsymbol{R}^N)$ *which are bounded and bounded away from zero in* $\boldsymbol{R}^N$.

PROOF. The proof is similar to that of Theorem 1 in [18]. From Jensen's inequality we have for $s > 0$

$$\left(\int_0^s (t/s)^{N-1}\phi(t)\,dt\right)^{1/(p-1)} \le s^{-1-(N-2)/(p-1)} \int_0^s t^{(N-1)/(p-1)}\phi(t)^{1/(p-1)}\,dt \tag{5.4}$$
$$\le \int_0^s t^{(2-p)/(p-1)}\phi(t)^{1/(p-1)}\,dt$$

and hence for $r > 0$

$$\int_0^r \left(\int_0^s (t/s)^{N-1}\phi(t)\,dt\right)^{1/(p-1)} ds \tag{5.5}$$
$$\le [(p-1)/(N-2)] \int_0^r [1-(t/r)^{(N-2)/(p-1)}]\,t^{1/(p-1)}\phi(t)^{1/(p-1)}\,dt\,.$$

Let $C^1(\bar{\boldsymbol{R}}_+)$ denote the locally convex space of continuously differentiable functions on $\bar{\boldsymbol{R}}_+$ with the topology of uniform convergence on compact subsets of $\bar{\boldsymbol{R}}_+$.

We first consider the case (F_1) holds. Let $\alpha > 0$ be small so that

$$[(p-1)/(N-2)]F(\alpha, 1)^{1/(p-1)} \int_0^\infty t^{1/(p-1)}\phi(t)^{1/(p-1)}\,dt \le \alpha/2$$

and

$$F(\alpha, 1)^{1/(p-1)} \int_0^\infty t^{(2-p)/(p-1)}\phi(t)^{1/(p-1)}\,dt \le 1\,.$$

Consider the set

$$Y = \{y \in C^1(\bar{\boldsymbol{R}}_+) : \alpha/2 \le y(r) \le \alpha,\ |y'(r)| \le 1 \text{ for } r \ge 0\}\,,$$

where "$'$" $= d/dr$. Define the operator $\mathscr{F}$ in $C^1(\bar{\boldsymbol{R}}_+)$ by

$$\mathscr{F}y(r) = \alpha - \int_0^r \left(\int_0^s (t/s)^{N-1}\phi(t)F(y(t), |y'(t)|)\,dt\right)^{1/(p-1)} ds\,. \tag{5.6}$$

If $y \in Y$, we see, from (5.4) and (5.5) that for $r > 0$

$$\alpha \ge \mathscr{F}y(r) \ge \alpha - F(\alpha, 1)^{1/(p-1)} \int_0^r \left(\int_0^s (t/s)^{N-1}\phi(t)\,dt\right)^{1/(p-1)} ds$$
$$\ge \alpha - [(p-1)/(N-2)]F(\alpha, 1)^{1/(p-1)} \int_0^\infty t^{1/(p-1)}\phi(t)^{1/(p-1)}\,dt \ge \alpha/2$$

and

$$|(\mathscr{F}y)'(r)| = \left(\int_0^r (t/r)^{N-1}\phi(t)F(y(t), |y'(t)|)\,dt\right)^{1/(p-1)}$$
$$\le F(\alpha, 1)^{1/(p-1)} \int_0^\infty t^{(2-p)/(p-1)}\phi(t)^{1/(p-1)}\,dt \le 1\,,$$

which shows that $\mathscr{F}: Y \to Y$. Let $\{y_n\}$ be a sequence in Y converging to $y \in Y$ as $n \to \infty$ in the topology of $C^1(\bar{\boldsymbol{R}}_+)$. We have for $r > 0$

$$\begin{aligned}|(\mathscr{F}y_n)'(r) - (\mathscr{F}y)'(r)| &\\ \le (p-1)^{-1}\Big[F(\alpha, 1)\int_0^r \phi(t)\,dt\Big]^{(2-p)/(p-1)} &\int_0^r \phi(t)|F(y_n(t), |y_n'(t)|)\\ - F(y(t), |y'(t)|)|\,dt\,,&\end{aligned}$$

which implies that $\mathscr{F}: Y \to Y$ is continuous. We have for $y \in Y$ and $r > 0$

$$\begin{aligned}|(\mathscr{F}y)''(r)| \le (p-1)^{-1}F(\alpha, 1)^{1/(p-1)}\Bigg[(N-1)r^{(2-p)/(p-1)}\left(\frac{1}{r}\int_0^r \phi(t)\,dr\right)^{1/(p-1)}\\ + \phi(r)\left(\int_0^r t^{N-1}\phi(t)\,dt\right)^{(2-p)/(p-1)}\Bigg].\end{aligned}$$

Therefore we see that $\mathscr{F}Y$ is relatively compact in $C^1(\bar{\boldsymbol{R}}_+)$. Thus we are able to apply the Schauder-Tychonoff fixed point theorem and conclude that $\mathscr{F}$ has a fixed point $y \in Y$. The function $v(x) = y(|x|)$ is a solution of the equation

$$-\operatorname{div}(|\nabla v|^{p-2}\nabla v) + \phi(|x|)F(v, |\nabla v|) = 0 \qquad \text{in } \boldsymbol{R}^N,$$

so that it is a C-supersolution of (5.1). Similarly we can show that the operator $\mathscr{G}$ defined by

$$\mathscr{G}z(r) = \beta + \int_0^r \left(\int_0^s (t/s)^{N-1}\phi(t)F(z(t), |z'(t)|)\,dt\right)^{1/(p-1)} ds \tag{5.7}$$

has a fixed point z in the set

$$Z = \{z \in C^1(\bar{\boldsymbol{R}}_+) : \beta \le z(r) \le 2\beta,\ |z'(r)| \le 1 \text{ for } r \ge 0\}\,,$$

provided $\beta > 0$ is chosen small enough so that

$$[(p-1)/(N-2)]F(2\beta, 1)^{1/(p-1)}\int_0^\infty t^{1/(p-1)}\phi(t)^{1/(p-1)}\,dt \le \beta$$

and

$$F(2\beta, 1)^{1/(p-1)}\int_0^\infty t^{(2-p)/(p-1)}\phi(t)^{1/(p-1)}\,dt \le 1\,.$$

The function $w(x) = z(|x|)$ is a C-subsolution of (5.1). If $4\beta \le \alpha$, then $w \le v$ in $\boldsymbol{R}^N$ and hence it follows from Theorem 4.7 that (5.1) has a solution u such that $w \le u \le v$ a.e. in $\boldsymbol{R}^N$.

Next, we consider the case (F_2) holds. We take positive constants α and β so large that

$$[(p-1)/(N-2)]F(\alpha, \alpha)^{1/(p-1)} \int_0^\infty t^{1/(p-1)}\phi(t)^{1/(p-1)}\,dt \le \frac{\alpha}{2},$$

$$F(\alpha, \alpha)^{1/(p-1)} \int_0^\infty t^{(2-p)/(p-1)}\phi(t)^{1/(p-1)}\,dt \le \alpha,$$

and

$$[(p-1)/(N-2)]F(2\beta, 2\beta)^{1/(p-1)} \int_0^\infty t^{1/(p-1)}\phi(t)^{1/(p-1)}\,dt \le \beta,$$

$$F(2\beta, 2\beta)^{1/(p-1)} \int_0^\infty t^{(2-p)/(p-1)}\phi(t)^{1/(p-1)}\,dt \le 2\beta.$$

Arguing as in the case of (F_1), we can verify that the operators $\mathscr{F}$ and $\mathscr{G}$ defined by (5.6) and (5.7) have fixed points y and z in the sets

$$\{y \in C^1(\bar{\boldsymbol{R}}_+) : \alpha/2 \le y(r) \le \alpha,\ |y'(r)| \le \alpha \text{ for } r \ge 0\}$$

and

$$\{z \in C^1(\bar{\boldsymbol{R}}_+) : \beta \le z(r) \le 2\beta,\ |z'(r)| \le 2\beta \text{ for } r \ge 0\},$$

respectively. The functions $v(x) = y(|x|)$ and $w(x) = z(|x|)$ then give respectively a C-supersolution and a C-subsolution of (5.1), which ensure the existence of the desired solution of (5.1) provided $4\beta \le \alpha$. The proof of Theorem 5.1 is thus complete.

The particular case ($p = 2$) of the above problem has been considered by numerous authors including Kawano [15], Kusano and Oharu [18], and Furusho [10]. The condition (5.3) generalizes the one given by Kawano [15] for the case $p = 2$.

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