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Kawauchi's second duality and knotted surfaces in 4-sphere

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§1. Introduction

Let M be a compact connected oriented n-dimensional topological manifold with an integral cohomology class $\gamma \in H^1(M; \boldsymbol{Z})$ of infinite order. Then we have the infinite cyclic covering space $\tilde{M}$ over M associated with γ and the finitely generated Λ-modules

$$H_*(\tilde{M}) = H_*(\tilde{M}; \boldsymbol{Z}) \text{ and } H_*(\tilde{M}, \partial\tilde{M}) = H_*(\tilde{M}, \partial\tilde{M}; \boldsymbol{Z}).$$

Here Λ is the integral group ring of the infinite cyclic group generated by t, and t acts on these homology groups by the induced isomorphisms of the generator specified by γ of the covering transformation group.

Now, we recall the result of A. Kawauchi on these Λ-modules, by using the following notations for any finitely generated Λ-module H:

DH = the unique maximal finite Λ-submodule of H,

$\mathrm{tor}_\Lambda H$(resp. tor H) = the Λ-(resp. $\boldsymbol{Z}$-) torsion part of H,

$BH = H/\mathrm{tor}_\Lambda H$, $\mathrm{E}^i H = \mathrm{Ext}^i_\Lambda(H, \Lambda)$ = the i-th Ext-group over Λ.

THEOREM 1 (Kawauchi's second duality theorem [7]). *Let p and q be integers with $p + q = n - 2$. Then there exist t-anti Λ-epimorphisms*

$$\theta_p: DH_p(\tilde{M}) \longrightarrow \mathrm{E}^1BH_{q+1}(\tilde{M}, \partial\tilde{M}),\ \theta'_q: DH_q(\tilde{M}, \partial\tilde{M}) \longrightarrow \mathrm{E}^1BH_{p+1}(\tilde{M})$$

such that the finite Λ-submodules Ker θ_p *and* Ker θ'_q *are dual by a t-isometric, $(-1)^{pq+1}$-symmetric and non-singular pairing*

$$\ell: \mathrm{Ker}\ \theta_p \times \mathrm{Ker}\ \theta'_q \longrightarrow \boldsymbol{Q}/\boldsymbol{Z}\ (\boldsymbol{Q}\text{: the rational number field}).$$

Moreover, this pairing is a proper oriented homotopy invariant.

In this paper, we study this pairing in a geometric way under the following assumption (*), and give some applications on knotted surfaces in S^4.

(*) For M and γ of above, assume that the Poincaré dual of γ in $H_{n-1}(M, \partial M)$ can be represented by a bicollared proper oriented $(n-1)$-dimensional submanifold V of M, which may be regarded as $V \subset \tilde{M}$.

Under this assumption, we have the linking pairing

$$\ell_V\colon \mathrm{tor} H_p(V) \times \mathrm{tor} H_q(V, \partial V) \longrightarrow \boldsymbol{Q}/\boldsymbol{Z} \quad (p+q=n-2)$$

of the torsion parts of the integral homology groups $H_p(V)$ and $H_q(V, \partial V)$, and the inclusion $i\colon(V, \partial V) \subset (\tilde{M}, \partial\tilde{M})$ induces the homomorphisms

$$i_*\colon \mathrm{tor} H_p(V) \longrightarrow H_p(\tilde{M}),\ i_*\colon \mathrm{tor} H_q(V, \partial V) \longrightarrow H_q(\tilde{M}, \partial\tilde{M}).$$

Then, Theorem 1 is presented by the following

THEOREM 2. *Under the assumption* (*), *there exist subgroups*

$$\Theta_p \subset \mathrm{tor} H_p(V) \text{ and } \Theta'_q \subset \mathrm{tor} H_q(V, \partial V)$$

such that $i_*\Theta_p = \mathrm{Ker}\,\theta_p$, $i_*\Theta'_q = \mathrm{Ker}\,\theta'_q$ *and the pairing* ℓ *in Theorem* 1 *is induced from the restriction of the linking pairing* ℓ_V *via* i_*.

These theorems are a generalization of the results of Farber [5] and Levine [11] which treat the case that $\mathrm{tor} H_p(\tilde{M})$ and $\mathrm{tor} H_q(\tilde{M}, \partial\tilde{M})$ are finite and $BH_{p+1}(\tilde{M}) = BH_{q+1}(\tilde{M}, \partial\tilde{M}) = 0$. For the case of a link in S^3, Kawauchi [8] gave some applications of Theorem 1.

Now, we consider a *knotted surface* (S^4, Σ), that is, a smoothly embedded closed connected oriented surface Σ in the oriented 4-sphere S^4. Let $M = S^4 - \mathrm{Int} N(\Sigma)$ be the exterior of a closed tubular neighborfood $N(\Sigma)$. Then $H^1(M) \cong H_2(\Sigma) \cong \boldsymbol{Z}$ by the Alexander duality, and we have the infinite cyclic covering space $\tilde{M}$ over M associated with the dual class $\gamma \in H^1(M)$ of the orientation class $[\Sigma] \in H_2(\Sigma)$. Moreover, V in the assumption (*) is obtained by a Seifert manifold of (S^4, Σ).

Therefore, we have the following

COROLLARY 3. *Let* (S^4, Σ) *be a knotted surface. Then Theorems* 1 *and* 2 $(n=4,\ p=q=1)$ *are valid for the infinite cyclic covering space* $\tilde{M}$ *over its exterior and a Seifert manifold* V *of* (S^4, Σ) *by regarding* $V \subset \tilde{M}$.

In this case, tor H is finite for $H = H_1(\tilde{M})$ and $H_1(\tilde{M}, \partial\tilde{M})$ (cf. [10, Lemme II. 8]), hence $DH = \mathrm{tor}\ H$.

COROLLARY 4. *In Corollary* 3, *assume that* V *is constructed to satisfy* $\mathrm{tor}\ H_1(V) = 0$ *or* $\mathrm{tor}\ H_1(V, \partial V) = 0$. *Then* θ_1 *and* θ'_1 *in Theorem* 1 *induce the t-anti* Λ*-isomorphisms*

$$\theta_1\colon \mathrm{tor} H_1(\tilde{M}) \cong \mathrm{E}^1 BH_2(\tilde{M}, \partial\tilde{M}), \qquad \theta'_1\colon \mathrm{tor} H_1(\tilde{M}, \partial\tilde{M}) \cong \mathrm{E}^1 BH_2(\tilde{M}).$$

Moreover, we shall prove the following

THEOREM 5. *For any odd integer* p $(|p| \neq 1)$ *there exists an irreducible ribbon torus* (S^4, F) *whose group* $\pi_1(S^4 - F)$ *is isomorphic, preserving meridian, to the group of the knotted sphere called the* 2*-twist spin of the* $(2, p)$ *torus knot.*

When (S^4, Σ) is a ribbon surface, we can apply Corollary 4 because the assumption is seen by Yanagawa's method [9, 21]. So we have tor $H_1(\tilde{M}) \cong \mathrm{E}^1 BH_2(\tilde{M}, \partial\tilde{M})$. On the other hand, tor $H_1(\tilde{M})$ is calculated from the group $\pi_1(M) = \pi_1(S^4 - \Sigma)$ by using Fox's free differential calculus (cf. [19, 6.7]).

EXAMPLE 6. *For any odd integer p there is a ribbon torus* (S^4, F) *such that*

$$\pi_1(S^4 - F) = \pi \cong \langle x, a;\, xax^{-1} = a^{-1},\, a^p = 1 \rangle .$$

In Example 6, π is isomorphic to the group of the knotted sphere stated in Theorem 5 (cf. [22]), and the Λ-module $\mathrm{E}^1 BH_2(\tilde{M}, \partial\tilde{M})$ is t-anti isomorphic to $\mathrm{tor}\, H_1(\tilde{M}) \cong \Lambda/I_p$ for the ideal I_p of Λ generated by $t+1$ and p. Hence the ribbon torus in Example 6 gives the example of Theorem 5 because we can prove the following

COROLLARY 3.9. *For any reducible knotted torus,* $\mathrm{E}^1 BH_2(\tilde{M}, \partial\tilde{M}) = 0$.

The irreducibility of some knotted surfaces has been proved by several authors [1, 3, 12, 13, 14]. But their methods are not applicable to the above example because they studied only the fundamental groups.

Moreover, by using this example, we can construct knotted surfaces which have isomorphic groups and peripheral subgroups, but are not stably equivalent each other (Proposition 4.8). It is known that stably equivalent knotted surfaces have isomorphic groups and peripheral subgroups (cf. [3, Lemma 11]). Our example shows that the reverse is not true.

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§2. Proof of Theorem 2

Hereafter, the integral (co)homology group is denoted by omitting its coefficient group $\boldsymbol{Z}$.

Assume that there exists V with the inclusion $i:(V, \partial V) \subset (\tilde{M}, \partial\tilde{M})$ in (*) of §1, and consider the orientation class and its image

$$[V] \in H_{n-1}(V, \partial V) \text{ and } \mu = i_*[V] \in H_{n-1}(\tilde{M}, \partial\tilde{M}).$$

Then we have the commutative diagrams

$$\begin{array}{ccccc} H^q(\tilde{M}, \partial\tilde{M}; \boldsymbol{Q}/\boldsymbol{Z}) & \xrightarrow{\delta} & H^{q+1}(\tilde{M}, \partial\tilde{M}) & \xrightarrow{\cap\mu} & H_p(\tilde{M}) \\ \downarrow i^* & & \downarrow i^* & & \uparrow i_* \\ H^q(V, \partial V; \boldsymbol{Q}/\boldsymbol{Z}) & \xrightarrow{\delta} & H^{q+1}(V, \partial V) & \underset{\cong}{\xrightarrow{\cap[V]}} & H_p(V), \end{array}$$

$$\begin{array}{ccccc} H^p(\tilde{M}; \boldsymbol{Q}/\boldsymbol{Z}) & \xrightarrow{\delta} & H^{p+1}(\tilde{M}) & \xrightarrow{\cap\mu} & H_q(\tilde{M}, \partial\tilde{M}) \\ \downarrow i^* & & \downarrow i^* & & \uparrow i_* \\ H^p(V; \boldsymbol{Q}/\boldsymbol{Z}) & \xrightarrow{\delta} & H^{p+1}(V) & \underset{\cong}{\xrightarrow{\cap[V]}} & H_q(V, \partial V) \end{array}$$

$(p+q=n-2)$, where δ denotes the Bockstein coboundary homomorphism associated with the exact sequence $0\to\boldsymbol{Z}\to\boldsymbol{Q}\to\boldsymbol{Q}/\boldsymbol{Z}\to 0$ of the coefficient groups.

Here, we recall also the following fact which is verified by A. Kawauchi in the proof of Theorem 1:

LEMMA 2.1 ([7, p.650]). *The upper compositions*

$$(\cap\mu)\circ\delta\colon H^q(\tilde{M}, \partial\tilde{M}; \boldsymbol{Q}/\boldsymbol{Z}) \longrightarrow H_p(\tilde{M}),\ H^p(\tilde{M}; \boldsymbol{Q}/\boldsymbol{Z}) \longrightarrow H_q(\tilde{M}, \partial\tilde{M})$$

in the above diagrams are t-anti Λ-homomorphisms, and their images are

$$\mathrm{Ker}\,\theta_p (\subset DH_p(\tilde{M})),\ \mathrm{Ker}\,\theta'_q (\subset DH_q(\tilde{M}, \partial\tilde{M}))$$

in Theorem 1, *respectively. Moreover, the pairing ℓ in Theorem* 1 *is given by*

$$\ell(x, y) = (-1)^{q+1}\langle x'\cup\delta y', \mu\rangle \in \boldsymbol{Q}/\boldsymbol{Z}$$

for any $x=\delta x'\cap\mu\in\mathrm{Ker}\,\theta_p$ and $y=\delta y'\cap\mu\in\mathrm{Ker}\,\theta'_q$, where $x'\in H^q(\tilde{M}, \partial\tilde{M}; \boldsymbol{Q}/\boldsymbol{Z})$, $y'\in H^p(\tilde{M}; \boldsymbol{Q}/\boldsymbol{Z})$ and $x'\cup\delta y'\in H^{n-1}(\tilde{M}, \partial\tilde{M}; \boldsymbol{Q}/\boldsymbol{Z})$.

On the other hand, in the second row of each diagram, Im δ is the torsion part of the range group by the definition of δ, and the linking pairing

$$\ell_V\colon \mathrm{tor}\,H_p(V)\times\mathrm{tor}\,H_q(V, \partial V)\longrightarrow \boldsymbol{Q}/\boldsymbol{Z}$$

is defined by $\ell_V(a, b) = (-1)^{p+1}\langle a', b\rangle$ for any $a=\delta a'\cap[V]\in\mathrm{tor}\,H_p(V)$, where $a'\in H^q(V, \partial V; \boldsymbol{Q}/\boldsymbol{Z})$, and $b\in\mathrm{tor}\,H_q(V, \partial V)$ (cf. [16, p.288]). Moreover, we have the subgroups

$$\Theta_p = \mathrm{Im}[(\cap[V])\circ i^*\circ\delta] \text{ and } \Theta'_q = \mathrm{Im}[(\cap[V])\circ i^*\circ\delta]$$

of tor $H_p(V)$ and tor $H_q(V, \partial V)$, respectively.

LEMMA 2.2. *$i_*\colon H_p(V)\to H_p(\tilde{M})$, $H_q(V, \partial V)\to H_q(\tilde{M}, \partial\tilde{M})$ induce the isomorphisms*

$$\Theta_p/(\Theta_p\cap\mathrm{Ker}\ i_*)\cong\mathrm{Ker}\ \theta_p,\ \Theta'_q/(\Theta'_q\cap\mathrm{Ker}\ i_*)\cong\mathrm{Ker}\ \theta'_q,$$

and $\ell(i_*a, i_*b) = (-1)^n \ell_V(a, b)$ *holds for any* $a \in \Theta_p$ and $b \in \Theta'_q$. *Moreover,* $\Theta_p \cap \operatorname{Ker} i_*$ *and* $\Theta'_q \cap \operatorname{Ker} i_*$ *are the annihilators of* Θ'_q *and* Θ_p, *respectively, by the restricted pairing* $\ell_V: \Theta_p \times \Theta'_q \to \boldsymbol{Q}/\boldsymbol{Z}$ *of* ℓ_V.

PROOF. The first half follows from the commutativity of the above diagrams, Lemma 2.1 and the definition of ℓ_V. In fact, we see that

$$(-1)^{q+1}\ell(i_*a, i_*b) = \langle x' \cup \delta y', \mu\rangle = \langle i^*x', i^*\delta y' \cap [V]\rangle = (-1)^{p+1}\ell_V(a, b)$$

for $a = i^*\delta x' \cap [V]$ and $b = i^*\delta y' \cap [V]$, since $\mu = i_*[V]$. The second half follows from the first half, since the pairing ℓ is non-singular by Theorem 1.

Therefore, Theorem 2 is proved.

§3. Knotted surfaces in S^4

In the rest of this paper, we are concerned with a knotted surface (S^4, Σ), that is, a smoothly embedded closed connected oriented surface Σ in S^4. As stated in §1, we have the infinite cyclic covering

$$p_\Sigma: \tilde{M}_\Sigma \longrightarrow M_\Sigma \text{ over its exterior } M_\Sigma = S^4 - \operatorname{Int} N(\Sigma)$$

($N(\Sigma)$ is a closed tubular neighborfood of Σ in S^4), associated with the Alexander dual $\gamma_\Sigma \in H^1(M_\Sigma)$ of $[\Sigma]$ and the Λ-modules $H_*(\tilde{M}_\Sigma)$ and $H_*(\tilde{M}_\Sigma, \partial\tilde{M}_\Sigma)$. We study these Λ-modules by denoting

$$\Phi_*(\Sigma) = H_*(\tilde{M}_\Sigma) \text{ and } \Psi_*(\Sigma) = H_*(\tilde{M}_\Sigma, \partial\tilde{M}_\Sigma).$$

(We sometimes omit the suffix Σ.)

Two knotted surfaces (S^4, Σ) and (S^4, Σ') are *equivalent* if there exists a diffeomorphism $f: (S^4, \Sigma) \to (S^4, \Sigma')$ preserving the orientations of S^4, Σ and Σ'. In this case, f induces the diffeomorphism $f: M_\Sigma \to M_{\Sigma'}$ with $f^*\gamma_{\Sigma'} = \gamma_\Sigma$, and so the equivalence $\tilde{f}: \tilde{M}_\Sigma \to \tilde{M}_{\Sigma'}$ of the coverings; hence the Λ-modules $\Phi_*(\Sigma)$ and $\Psi_*(\Sigma)$ are isomorphic to $\Phi_*(\Sigma')$ and $\Psi_*(\Sigma')$, respectively.

For knotted surfaces (S^4, Σ) and (S^4, Σ'), consider the connected sum

$$(S^4, \Sigma) \# (S^4, \Sigma') = (S^4, \Sigma \# \Sigma')$$

(cf. [3, §3]). Then, it is easy to see that

$$\begin{aligned}
&M_{\Sigma\#\Sigma'} \approx M_\Sigma \cup M_{\Sigma'},\ M_\Sigma \cap M_{\Sigma'} \approx D^2 \times S^1,\\
&\tilde{M}_{\Sigma\#\Sigma'} \approx \tilde{M}_\Sigma \cup \tilde{M}_{\Sigma'},\ \tilde{M}_\Sigma \cap \tilde{M}_{\Sigma'} \approx D^2 \times \boldsymbol{R},\\
&\partial\tilde{M}_{\Sigma\#\Sigma'} \approx (\partial\tilde{M}_\Sigma - \operatorname{Int}(D^2 \times \boldsymbol{R})) \cup (\partial\tilde{M}_{\Sigma'} - \operatorname{Int}(D^2 \times \boldsymbol{R})),
\end{aligned}$$

where $\approx$ means a diffeomorphism. Also, for any knotted surface (S^4, Σ), the inclusion $\partial\tilde{M}_\Sigma \subset \tilde{M}_\Sigma$ induces the zero map on the second homology group by the existence of V. So we obtain the following

LEMMA 3.1. *The finitely generated Λ-modules $\Phi_*(\Sigma)$ and $\Psi_*(\Sigma)$ are determined by the equivalence class of a knotted surface (S^4, Σ). For the connected sum $(S^4, \Sigma \# \Sigma') = (S^4, \Sigma) \# (S^4, \Sigma')$ of knotted surfaces, we have the following direct sum decomposition of the Λ-modules.*

$$\Phi_i(\Sigma \# \Sigma') = \Phi_i(\Sigma) \oplus \Phi_i(\Sigma') \qquad \text{for } i \geq 1,$$
$$\Psi_j(\Sigma \# \Sigma') = \Psi_j(\Sigma) \oplus \Psi_j(\Sigma') \qquad \text{for } j = 1 \text{ and } 2.$$

Now, we consider the tensor product $\Gamma = \Lambda \otimes \boldsymbol{Q}$ over $\boldsymbol{Z}$ and prepare the following

LEMMA 3.2. *Let H be a finitely generated Γ-module, and consider $\tau: H \to H$, the multiplication by $t-1$.*

(1) *If H is a torsion Γ-module and Ker $\tau \cong \boldsymbol{Q}$, then* Coker $\tau \cong \boldsymbol{Q}$.

(2) *If τ is epimorphic, then τ is isomorphic and H is a torsion Γ-module.*

(3) *If τ is monomorphic, then* Coker $\tau \cong \boldsymbol{Q}^h$ *for some $h \geq 0$, and $B_\Gamma H = H/\mathrm{tor}_\Gamma H$ is isomorphic to Γ^h.*

PROOF. Since Γ is a principal ideal domain, H is represented by a direct sum of some copies of Γ-modules

$$\Gamma,\ \Gamma/I^n (n \geq 1) \text{ and } \Gamma/J_f,$$

where I is the ideal of Γ generated by $t-1$ and J_f is the one by some $f \in \Gamma$ which is coprime to $t-1$ (cf. [2]).

By the definition, $\tau: H \to H$ is isomorphic when $H = \Gamma/J_f$,

Ker $\tau = 0$, Coker $\tau = \Gamma/I$ when $H = \Gamma$, and

Ker $\tau =$ Coker $\tau = \Gamma/I$ when $H = \Gamma/I^n (n \geq 1)$.

Thus we see the lemma since $\Gamma/I \cong \boldsymbol{Q}$ as $\boldsymbol{Z}$-module.

LEMMA 3.3. *For a knotted surface (S^4, Σ), $\mathrm{rank}_\Lambda \Psi_2(\Sigma) = 2g$ (g is the genus of Σ) and $\mathrm{rank}_\Lambda \Phi_1(\Sigma) = 0$.*

PROOF. We consider the following Wang exact sequence of the rational homology groups associated to the covering $p = p_\Sigma: (\tilde{M}, \partial\tilde{M}) \to (M, \partial M)$, where $M = M_\Sigma$ and $\tilde{M} = \tilde{M}_\Sigma$ (cf. [15]):

$$\cdots \longrightarrow H_{i+1}(M, \partial M; \boldsymbol{Q}) \xrightarrow{\partial} \Psi_i(\Sigma) \otimes \boldsymbol{Q} \xrightarrow{\tau_i}$$
$$\Psi_i(\Sigma) \otimes \boldsymbol{Q} \xrightarrow{p_*} H_i(M, \partial M; \boldsymbol{Q}) \longrightarrow \cdots$$

where $\Psi_i(\Sigma) \otimes \boldsymbol{Q} = H_i(\tilde{M}, \partial\tilde{M}; \boldsymbol{Q})$ is a finitely generated Γ-module by the Λ-module $\Psi_i(\Sigma) = H_i(\tilde{M}, \partial\tilde{M})$ and τ_i is the multiplication by $t-1$.

In this sequence, $H_i(M, \partial M; \boldsymbol{Q}) \cong H^{4-i}(M; \boldsymbol{Q})$ is isomorphic to $\boldsymbol{Q}$ if $i = 3$ or 4, $\boldsymbol{Q}^{2g}$ if $i = 2$, 0 otherwise, by the Alexander duality $\tilde{H}^{4-i}(M; \boldsymbol{Q}) \cong \tilde{H}_{i-1}(\Sigma; \boldsymbol{Q})$. Also, $\Psi_4(\Sigma) = H_4(\tilde{M}, \partial\tilde{M}) = 0$ since $\tilde{M}$ is an open 4-dimensional manifold. Therefore, by the exactness and Lemma 3.2, we see that

$$\operatorname{Ker} \tau_3 \cong \boldsymbol{Q}, \text{ hence } \operatorname{Coker} \tau_3 \cong \boldsymbol{Q} \text{ and } \operatorname{Ker} \tau_2 = 0,$$
$$\operatorname{Coker} \tau_1 = 0, \text{ hence } \operatorname{Ker} \tau_1 = 0 \text{ and } \operatorname{Coker} \tau_2 \cong \boldsymbol{Q}^{2g},$$

and these show $B_\Gamma(\Psi_2(\Sigma) \otimes \boldsymbol{Q}) \cong \Gamma^{2g}$. Hence

$$\Psi_2 \otimes_\Lambda Q(\Lambda) \cong (\Psi_2 \otimes \boldsymbol{Q}) \otimes_\Gamma Q(\Gamma) \cong B_\Gamma(\Psi_2 \otimes \boldsymbol{Q}) \otimes_\Gamma Q(\Gamma) \cong Q(\Gamma)^{2g} \cong Q(\Lambda)^{2g}$$

for $\Psi_2 = \Psi_2(\Sigma)$, where $Q(\Lambda)$ and $Q(\Gamma)$ is the quotient fields of Λ and Γ, respectively. Thus we see that $\operatorname{rank}_\Lambda \Psi_2(\Sigma) = 2g$.

To show the second equality, we consider the Wang exact sequence

$$\Phi_1(\Sigma) \otimes \boldsymbol{Q} \xrightarrow{\tau_1} \Phi_1(\Sigma) \otimes \boldsymbol{Q} \xrightarrow{p_*} H_1(M; \boldsymbol{Q}) \xrightarrow{\partial} \Phi_0(\Sigma) \otimes \boldsymbol{Q} \xrightarrow{\tau_0} \Phi_0(\Sigma) \otimes \boldsymbol{Q}$$

of the covering $p: \tilde{M} \to M$, where $\Phi_i(\Sigma) = H_i(\tilde{M})$ and τ_i is the multiplication by $t-1$. Then, $\Phi_0(\Sigma) \cong \boldsymbol{Z}$, $\tau_0 = 0$ and $H_1(M; \boldsymbol{Q}) \cong H^2(\Sigma; \boldsymbol{Q}) \cong \boldsymbol{Q}$. Hence ∂ is isomorphic and τ_1 is epimorphic. Thus $B_\Gamma(\Phi_1(\Sigma) \otimes \boldsymbol{Q}) = 0$ by Lemma 3.2 (2), which shows $\operatorname{rank}_\Lambda \Phi_1(\Sigma) = 0$ in the same way as above.

Definition 3.4. A knotted surface (S^4, Σ) is denoted frequently by (S^4, Σ_g) if the genus of Σ is g, and is called a *knotted sphere* (resp. *torus*) when $g = 0$ (resp. 1). An *unknotted torus*, denoted by (S^4, T), is a knotted torus such that T bounds a solid torus in S^4 (cf. [6]).

Lemma 3.5. *$\Psi_2(T) = H_2(\tilde{M}_T, \partial\tilde{M}_T)$ is isomorphic to Λ^2.*

Proof. Consider $(S^3, S^1) \times S^1$, where (S^3, S^1) is an unknot. Then the exterior M_T is obtained from $(S^3 - \operatorname{Int} N(S^1)) \times S^1 \approx D^2 \times S^1_0 \times S^1$ by a surgery along a curve $p \times q \times S^1$ with $p \in \operatorname{Int} D^2$ and $q \in S^1_0$. Thus $\pi_1(M_T) \cong \boldsymbol{Z}$ and $\pi_1(\tilde{M}_T) = 0$. Hence $\Psi_2(T)$ is Λ-free by [8, Lemma 2.1] and so $\Psi_2(T) \cong \Lambda^2$ by Lemma 3.3.

Definition. 3.6. For a finitely generated Λ-module H, we denote by $e(H)$ the minimum number of generators of H as Λ-module.

Proposition 3.7. *Assume that a knotted surface (S^4, Σ_g) is equivalent to the connected sum*

$$(S^4, \Sigma_h) \# (S^4, T) \# \cdots \# (S^4, T) \quad (g \geq h \geq 0)$$

of (S^4, Σ_h) and $g - h$ unknotted tori (S^4, T). Then

$$e(\mathrm{E}^2\mathrm{E}^1 B\Psi_2(\Sigma_g)) \leq 2h.$$

Moreover, if $h=0$ in addition, then

$$B\Psi_2(\Sigma_g) \cong \Lambda^{2g} \text{ and } \mathrm{E}^1 B\Psi_2(\Sigma_g) = 0.$$

(Here, $BH = H/\mathrm{tor}_\Lambda H$ and $\mathrm{E}^i H = \mathrm{Ext}^i_\Lambda(H, \Lambda)$, as stated in §1.)

PROOF. $B\Psi_2(T) \cong \Lambda^2$ by Lemma 3.5. Thus, by the assumption and Lemma 3.1, we have the direct sum decomposition

$$B\Psi_2(\Sigma_g) = B\Psi_2(\Sigma_h) \oplus \Lambda^{2(g-h)}.$$

Therefore $\mathrm{E}^1 B\Psi_2(\Sigma_g) \cong \mathrm{E}^1 B\Psi_2(\Sigma_h)$. On the other hand, by [7, Lemma 3.6], there is a natural exact sequence

$$0 \longrightarrow B\Psi_2(\Sigma_h) \longrightarrow \mathrm{E}^0\mathrm{E}^0 B\Psi_2(\Sigma_h) \longrightarrow \mathrm{E}^2\mathrm{E}^1 B\Psi_2(\Sigma_h) \longrightarrow 0$$

and $\mathrm{E}^0 B\Psi_2(\Sigma_h)$ is Λ-free. This implies that $e(\mathrm{E}^2\mathrm{E}^1 B\Psi_2(\Sigma_h)) \leq 2h$, since $\mathrm{E}^0 B\Psi_2(\Sigma_h) \cong \Lambda^{2h}$ by Lemma 3.3. In particular, if $h=0$, then $\mathrm{E}^0 B\Psi_2(\Sigma_0) = 0$, so the above exact sequence shows that $B\Psi_2(\Sigma_0) = 0$. Thus we obtain that $B\Psi_2(\Sigma_g) = \Lambda^{2g}$ and $\mathrm{E}^1 B\Psi_2(\Sigma_g) = 0$.

DEFINITION 3.8. A knotted surface (S^4, Σ) is *irreducible* if it is not equivalent to the connected sum $(S^4, \Sigma')\#(S^4, T)$ of any knotted surface (S^4, Σ') and an unknotted torus (S^4, T). Otherwise, it is *reducible.*

COROLLARY 3.9. $\mathrm{E}^1 B\Psi_2(\Sigma) = 0$ *for a reducible torus* (S^4, Σ).

§4. Theorem 5 and applications.

DEFINITION. 4.1. A knotted surface (S^4, Σ) is said to be a *ribbon surface* if there is an immersion $i\colon W^3 \to S^4$, of a solid handlebody W^3, satisfying the following three properties (cf. [9,12]);

(1) $i|\partial W$ is a diffeomorphism onto Σ.

(2) i has no triple points.

(3) The singular set of i consists of disjoint 2-disks $D_1, \cdots, D_n, D'_1, \cdots, D'_n$, where D_i is properly embedded in W, $D'_j \subset \mathrm{Int}\,(W)$ and $i^{-1}(i(D_j)) = D_j \cup D'_j$ for $j = 1, \cdots, n$.

PROPOSITION 4.2. *For a ribbon surface (S^4, Σ_g), we have* $\mathrm{E}^1 B\Psi_2(\Sigma_g) \cong \mathrm{tor}\, \Phi_1(\Sigma_g)$ *and* $e(\mathrm{E}^2\Phi_1(\Sigma_g)) \leq g$.

PROOF. By Yanagawa's method (cf. [21, Th. 2.3], [9, Lemma 4.2]), we can construct a Seifert manifold V^3 of (S^4, Σ_g) such that $\mathrm{tor}\, H_1(V) = 0$, where a Seifert manifold means a smoothly embedded compact connected oriented 3-dimensional manifold in S^4 whose boundary is Σ_g. According to Corollary 4, $\mathrm{tor}\, \Phi_1(\Sigma_g)$ and $\mathrm{E}^1 B\Psi_2(\Sigma_g)$ are t-anti isomorphic.

On the other hand $\pi_1(S^4 - \Sigma_g)$ has a Wirtinger presentation with n generators and $n-1+g$ relations for some positive integer n, because (S^4, Σ_g) is a ribbon surface (cf. [9, Cor. 4.13]). Then using a pre-abelian presentation, Fox's free differential calculus deduces the following exact sequence of Λ-modules (cf. [19, 6.7]):

$$\Lambda^{n-1+g} \xrightarrow{A} \Lambda^{n-1} \longrightarrow \Phi_1(\Sigma_g) \longrightarrow 0,$$

where A is a presentation matrix. Since Λ has the global dimension 2, Ker A is Λ-projective and so it is Λ-free (cf. [4, VI, Prop. 2.1], [18]). Therefore we see that Ker $A \cong \Lambda^g$ by taking the tensor products $\otimes_\Lambda Q(\Lambda)$ of the above exact sequence, because $\operatorname{rank}_\Lambda \Phi_1(\Sigma_g) = 0$ by Lemma 3.3. Thus, we have a Λ-free resolution

$$0 \longrightarrow \Lambda^g \longrightarrow \Lambda^{n-1+g} \longrightarrow \Lambda^{n-1} \longrightarrow \Phi_1(\Sigma_g) \longrightarrow 0$$

of $\Phi_1(\Sigma_g)$. Therefore by the definition of $\mathrm{E}^2 = \mathrm{Ext}^2_\Lambda(-, \Lambda)$, we have the Λ-epimorphism $\Lambda^g \to \mathrm{E}^2\Phi_1(\Sigma_g)$, which implies $e(\mathrm{E}^2\Phi_1(\Sigma_g)) \leq g$.

REMARK 4.3. Let I_p be an ideal of Λ generated by $t+1$ and p, where p is an odd prime. Then $\Phi_1(\Sigma_g)$ is not isomorphic to Λ/I_p^n for all $n \geq g+1$. This follows from Proposition 4.2, because $e(\mathrm{E}^2(\Lambda/I_p^n)) = n$ by [17, Lemma 3].

Now, we construct a ribbon torus (S^4, F) in Example 6 and prove Theorem 5.

Let p be an odd integer, and consider the group

$$\pi = \langle x, y; x = vxv^{-1}, y = wxw^{-1} \rangle$$

where $v = a^p$, $w = a^{(1-p)/2}$ and $a = yx^{-1}$. Then, by Yajima's method [20], we can construct a ribbon torus (S^4, F) whose group $\pi_1(S^4 - F)$ is π as follows: Take the trivial 2-link in S^4 with two components whose meridians are x and y. Then we obtain (S^4, F) by attaching two embedded 1-handles to it corresponding to the elements v and w. Eliminating y from the above presentation of π, we obtain

$$\pi = \langle x, a; xax^{-1} = a^{-1}, a^p = 1 \rangle.$$

This is the group of the 2-twist spin of the $(2, p)$ torus knot (cf. [22]) and note that x is a meridian element. On the other hand, Fox's free differential calculus (cf. [19, 6.7]) on the above presentation of $\pi_1(S^4 - F) = \pi$ shows that $\Phi_1(F) = \Lambda/I_p$, where I_p is an ideal of Λ generated by $t+1$ and p. Therefore $\mathrm{E}^1 B\Psi_2(F) \cong \operatorname{tor} \Phi_1(F) \cong \Lambda/I_p$ by Proposition 4.2; hence (S^4, F) is irreducible for $|p| \neq 1$ by Corollary 3.9. Thus Theorem 5 is proved by this ribbon torus (S^4, F).

Hereafter, we study the following subgroup of $\pi_1(S^4 - \Sigma) \cong \pi_1(M_\Sigma)$:

DEFINITION 4.4. We call the image of $i_\sharp: \pi_1(\partial M_\Sigma) \to \pi_1(M_\Sigma)$ induced by $i: \partial M_\Sigma \subset M_\Sigma$ the *peripheral subgroup of a knotted surface* (S^4, Σ). If this subgroup is isomorphic to $\boldsymbol{Z}$, then we say that it is *trivial*. For example if (S^4, Σ) is a knotted sphere or an unknotted torus, then the peripheral subgroup is trivial.

LEMMA 4.5. *If knotted surfaces (S^4, Σ) and (S^4, Σ') have trivial peripheral subgroups, then so does $(S^4, \Sigma \# \Sigma')$.*

PROOF. We see the lemma because $\pi_1(M_{\Sigma\#\Sigma'})$ is the amalgamated product of $\pi_1(M_\Sigma)$ and $\pi_1(M_{\Sigma'})$ by the infinite cyclic group generated by meridian.

LEMMA 4.6. *The peripheral subgroup of the ribbon torus (S^4, F) constructed above is trivial.*

PROOF. Consider a closed tubular neighborhood $N(F)$ as a normal bundle over F, which is a trivial bundle. Then by [12, Lemma 1], there is a section $s: F \to \partial N(F) = \partial M_F$ such that $s: F \to \partial M_F \subset M_F$ induces $s_* = 0: H_1(F) \to H_1(M_F)$, and such s is unique up to homotopy.

Now, we take generators α_1, α_2 and α_3 of $\pi_1(\partial M_F) \cong \boldsymbol{Z} \times \boldsymbol{Z} \times \boldsymbol{Z}$ as follows: $\alpha_i = s(\alpha_i')$ $(i = 1, 2)$, where α_1' is a belt circle (resp. α_2' is a loop which goes round once) on the embedded 1-handle in S^4 corresponding to v in the construction of F, and α_3 is a meridian, that is, a fiber of the normal circle bundle $\partial N(F) \to F$.

Consider $i_\sharp: \pi_1(\partial M_F) \to \pi_1(M_F)$, then $i_\sharp \alpha_1 = 1$ by definition. Also $i_\sharp \alpha_2 = 1$ by the relation $v = a^p = 1$ in the second presentation of $\pi = \pi_1(M_F)$. Thus $i_\sharp \pi_1(\partial M_F) \cong \boldsymbol{Z}$ as desired.

DEFINITION 4.7. Knotted surfaces (S^4, Σ) and (S^4, Σ') are said to be *stably equivalent* if $(S^4, \Sigma)\#(\overset{m}{\#}(S^4, T))$ and $(S^4, \Sigma')\#(\overset{n}{\#}(S^4, T))$ are equivalent for some m and n.

PROPOSITION 4.8. *For an arbitrary positive integer n, there exist n knotted surfaces in S^4 which have isomorphic groups and peripheral subgroups, but are not stably equivalent each other.*

PROOF. Let (S^4, F) be the same one in Lemma 4.6 and (S^4, K) be the 2-twist spin of the $(2, p)$ torus knot. For $i = 0, \cdots, n-1$, we consider the connected sum

$$(S^4, F_i) = \overset{i}{\#}(S^4, F)\#(\overset{n-i-1}{\#}(S^4, K)).$$

Since $\mathrm{E}^1 B\Psi_2(K) = 0$ by Proposition 3.7 and $\mathrm{E}^1 B\Psi_2(F) \cong \Lambda/I_p$ by the proof of Theorem 5, we obtain by Lemma 3.1, $\mathrm{E}^1 B\Psi_2(F_i) \cong (\Lambda/I_p)^i$. By Lemmas 3.1 and 3.5, $\mathrm{E}^1 B\Psi_2$ does not change by the connected sum with an unknotted torus. Thus these knotted surfaces (S^4, F_i) are not stably equivalent each other. Moreover, (S^4, K) and (S^4, F) have isomorphic groups and trivial peripheral subgroups by Theorem 5 and Lemma 4.6. Therefore the proof is completed by Lemma 4.5.

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