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Singular perturbation approach to stability properties of traveling wave solutions of reaction-diffusion systems

Hideo IKEDA

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1. Introduction

A variety of propagating phenomena for reaction-diffusion systems have been investigated in physics, chemistry, biology and other fields. Among these phenomena, there are traveling waves in bistable reaction media which correspond to propagating transition from one stable state to the other. See e.g. Ortoleva et al [29], [30], Fife[8], [9] and their references therein. These waves are basically modeled by a two-component system of the form

$$(1.1)\qquad \begin{cases} \varepsilon\tau u_t = \varepsilon^2 u_{xx} + f(u, v) \\ v_t = v_{xx} + g(u, v) \end{cases}$$

with the nonlinearities of f and g in Fig. 1. It is shown that there are three spatially constant steady states ($E_\pm$, E_0): two of them ($E_\pm$) are stable, while one (E_0) is unstable. One simple but very substantial nonlinearities of f and g is the Bonhoeffer-Van der Pol kinetics

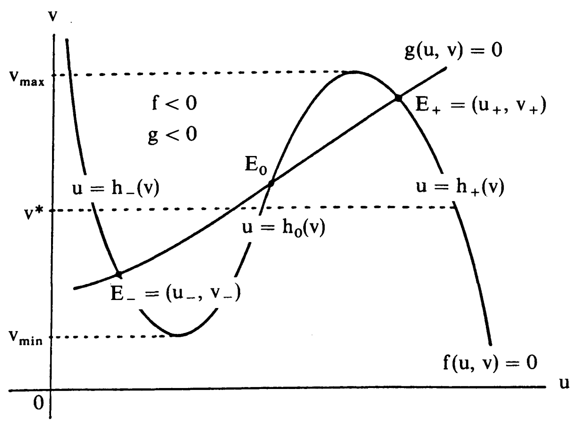

Fig. 1: Functional forms of $f = 0$ and $g = 0$ in (1.1).

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$$
(1.2) \qquad \begin{cases} f(u, v) = u(1-u)(u-a) - v \\ g(u, v) = u - \gamma v \end{cases}
$$

with positive constants a and γ. The system (1.1) includes two physical parameters ε and τ. $\varepsilon\tau$ is the ratio of the reaction rates of u, v and ε/τ is the ratio of the diffusion rates. Here we assume that one of the parameters ε is small, so that there appear internal and/or boundary layers of width $O(\varepsilon)$ in u. For this reason, we call ε a *layer* parameter. When $\tau = O(1)$ as $\varepsilon \downarrow 0$, our system (1.1) indicates that u diffuses much slower than v, while u reacts much faster than v.

In the previous papers [13] and [16], we have studied the existence of traveling wave solutions of (1.1) connecting E_- to E_+ by using singular perturbation techniques. By the traveling coordinate $z = x + ct$, a traveling wave solution with velocity c corresponds to a solution of the following equations with a parameter c:

$$
(1.3) \qquad \begin{cases} \varepsilon^2 u_{zz} - \varepsilon c\tau u_z + f(u, v) = 0 \\ v_{zz} - cv_z + g(u, v) = 0 \\ (u, v)(\pm\infty) = E_\pm. \end{cases}, \quad z \in \mathbf{R}
$$

For the system (1.3), it is shown that there exist at least three solutions with a suitable parameter c when τ is small, while there exists a unique solution when τ is large (see [16] for the details). For the special nonlinearities of (1.2), Fig. 2 shows a global structure of traveling wave solutions with respect to $\tau \in \mathbf{R}$. This picture clearly shows the appearance of critical value $\tau = \tau_c$ such that there is only one solution for $\tau_c < \tau$, while there are three solutions for $0 < \tau < \tau_c$.

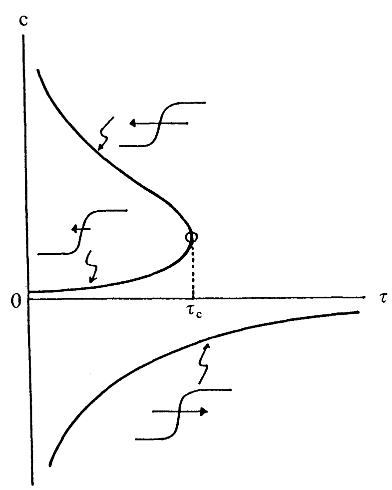

Fig. 2: Global bifurcation diagram of traveling wave solutions in (τ, c)-plane.

The stability of these traveling wave solutions can be studied by examining the spectrum of the linearized operator corresponding to (1.3) at the concerned solution (see Henry [12], for example):

$$\mathscr{L} = \begin{bmatrix} \dfrac{\varepsilon}{\tau}\cdot\dfrac{d^2}{dz^2} - c\dfrac{d}{dz} + \dfrac{1}{\varepsilon\tau}f_u^\varepsilon & \dfrac{1}{\varepsilon\tau}f_v^\varepsilon \\ g_u^\varepsilon & \dfrac{d^2}{dz^2} - c\dfrac{d}{dz} + g_v^\varepsilon \end{bmatrix}.$$

For the distribution of eigenvalues, there are two different methods: One is an analytical method of treating directly the eigenvalue problem of the linearized operator $\mathscr{L}$ (see Nishiura [25], Nishiura et al [26], [27], [28], Ito [19] and Sakamoto [31]), the other is an geometrical method of treating an equivalent four-dimensional dynamical system with a parameter c, for which eigenvalues can be found by using the geometrical properties of the orbit corresponding to the traveling wave solution in the four-dimensional phase space (see Evans [3], [4], [5], [6], Jones [20], Yanagida [33], Maginu [23], [24], Ikeda et al [15], [17], [18] and Yanagida et al [34]).

In [28], we have already discussed the stability of traveling wave solutions of (1.1) using the former method. In that place, we found that there are two critical eigenvalues: One is zero which corresponds to translation invariance, the other is real which essentially determines stability. The interesting point is that the sign of the latter corresponds exactly to that of the Jacobian of the matching condition (see 2.24)). That is, the sign of the Jacobian of the matching condition gives essential information about stability property.

The latter method was firstly proposed by Evans [3], [4], [5], [6] to show stability of traveling pulse solutions of generalized nerve axon equations which include the FitzHugh-Nagumo (FHN) equations and the Hodgkin-Huxley (HH) equations. He rewrites these as equivalent dynamical systems with velocity c, and then defines the index (which takes one of the three values $\{+1, 0, -1\}$) of the homoclinic orbit corresponding to a traveling pulse solution with velocity c^*, by the direction in which the one-dimensional unstable manifold crosses the multi-dimensional stable one with respect to the unique resting state, when the parameter c increases through c^*. It is shown that when the index takes -1, the solution is unstable, while when the index takes 1, it gives a nesessary condition for the solution to be stable. Using his method, Jones [20] and Yanagida [33] proved that the fast traveling pulse solution of the FHN equations is stable. Recently, Ikeda et al [17] and [18] have proved that the fast traveling pulse solution of the HH equations is stable, while the slow one is unstable. In Evans' index theory, it is essential that the dimension of the unstable manifold is equal to one. Maginu [24] extended Evans' index theory to a more general reaction-diffusion systems including (1.1), that is, he extended the definition of the index to the crossing manner of the multi-dimensional stable manifold and the multi-dimensional unstable one when the parameter c increases through c^*, and showed that when the index takes -1, the traveling pulse solution is unstable.

Lately, by using a distance-like function of the stable and unstable manifolds (which is different from the index theory), Kokubu et al [21] have shown that stability of traveling wave solutions is determined by a direction of strictly crossing of the stable and unstable manifolds with respect to c.

The aim of this paper is to calculate the index of the corresponding heteroclinic orbit, which gives essential information about stability property. The index can be determined as a by-product when we construct a traveling wave solution by using singular perturbation techniques. We apply it to the results obtained by Maginu [24] and show stability of traveling wave solutions connecting E_- to E_+. We shall state this more precisely. The system (1.3) can be rewritten as an equivalent four-dimensional dynamical system

$$(1.4) \qquad \begin{cases} \dfrac{d}{dz}\mathbf{V} = \mathbf{F}(\mathbf{V}; \varepsilon; \tau; c), & z \in \mathbf{R} \\ \mathbf{V}(\pm\infty) = \mathbf{P}_\pm \end{cases}$$

for $\mathbf{V} = {}^t\left(u, \varepsilon\dfrac{du}{dz}, v, \dfrac{dv}{dz}\right)$ (see (3.1), (3.2)). Here $\mathbf{P}_\pm$ are stationary points of (1.4), corresponding to $E_\pm$, in the four-dimensional phase space. (1.1) has a traveling wave solution $(u, v)(z; \varepsilon; \tau)$ connecting E_- to E_+ with velocity $c = c(\varepsilon; \tau)$ if and only if (1.4) has a heteroclinic solution $\mathscr{V}(z; \varepsilon; \tau) \equiv {}^t\left(u, \varepsilon\dfrac{du}{dz}, v, \dfrac{dv}{dz}\right)(z; \varepsilon; \tau)$ for $c = c(\varepsilon; \tau)$. Let $\gamma(\varepsilon; \tau)$ is the heteroclinic orbit of (1.4) corresponding to $\mathscr{V}(z; \varepsilon; \tau)$. In the four-dimensional phase space, it holds that

$$\gamma(\varepsilon; \tau) \subset S^+_{\varepsilon,\tau,c} \cap U^-_{\varepsilon,\tau,c}$$

for $c = c(\varepsilon; \tau)$, where $S^+_{\varepsilon,\tau,c}$ is the two-dimensional stable manifold of (1.4) with respect to $\mathbf{P}_+$ and $U^-_{\varepsilon,\tau,c}$ is the two-dimensional unstable manifold of (1.4) with respect to $\mathbf{P}_-$. In this situation, we can define the *index* of the heteroclinic orbit $\gamma(\varepsilon; \tau)$, say $\mathrm{Ind}[\gamma(\varepsilon; \tau)]$, by the *direction* in which the stable manifold $S^+_{\varepsilon,\tau,c}$ passes through the unstable one $U^-_{\varepsilon,\tau,c}$ when the parameter c increases through $c(\varepsilon; \tau)$. In general, the calculation of the index is rather difficult. However there are a few examples in nerve axon equations, for which one can calculate the index (see Langer [22] and Ikeda et al [17], [18]).

An outline of this paper is as follows: in Section 2 we give a brief summary of the existence results of traveling wave solutions studied in [13], [16]; in Section 3 we define the index of the heteroclinic orbit which corresponds to the traveling wave solution of (1.1), and show that the index corresponds in a one-to-one manner to the sign of the Jacobian of the matching condition (see Theorem 3.1); in Section 4, we conclude that when $\mathrm{Ind}[\gamma(\varepsilon;\tau)] = -1$, the traveling wave solution corresponding to $\gamma(\varepsilon;\tau)$ is unstable, while $\mathrm{Ind}[\gamma(\varepsilon;\tau)] = 1$ is a necessary condition to be stable; finally in Section 5, we remark that under the additional discussion to the condition $\mathrm{Ind}[\gamma(\varepsilon;\tau)] = 1$, the traveling wave solution is stable.

We first impose the following assumptions on the nonlinearities of f and g (see Fig. 1):

(A-1) $f = 0$ is S-shaped and consists of three branches $u = h_-(v)$, $h_0(v)$ and $h_+(v)\,(h_-(v) \le h_0(v) \le h_+(v))$, while $g = 0$ intersects each branch at once. That is, there is only one intersection point on each branch $u = h_\pm(v)$. We denote these points by $(u_\pm, v_\pm)\,(v_- < v_+)$, respectively. The signs of f and g are both negative in the upper region of the curves $f = 0$ and $g = 0$;

(A-2) $\mathscr{I}(v) = \displaystyle\int_{h_-(v)}^{h_+(v)} f(u, v)\,du$ has a unique isolated zero at $v^* \in (v_{\min}, v_{\max})$;

(A-3)
$$f_u(h_\pm(v), v) < 0 \qquad \text{for } v \in [v_-, v_+],$$
$$g(h_-(v), v) < 0 < g(h_+(v), v) \qquad \text{for } v \in (v_-, v_+)$$
and
$$\frac{d}{dv} g(h_\pm(v), v) < 0 \qquad \text{at } v = v_\pm;$$

(A-4) $f_u(u, v) < 0$ for $(u, v) \in \{(u, v) \mid h_-(v) \le u \le h_+(v),\ v_- \le v \le v_+\}$,
$g_u(u, v) > 0$ and $g_v(u, v) < 0$ at $(u, v) = (u_\pm, v_\pm)$.

Throughout this paper, we shall use the following function spaces. Let $I = \mathbf{R}_-$, $\mathbf{R}_+$ or $\mathbf{R}$, σ and μ be positive numbers, and n be an integer. Let

$$X^n_{\mu,\sigma}(I) \equiv \left\{ u \in C^n(I) \mid \ \|u\|_{X^n_{\mu,\sigma}(I)} \equiv \textstyle\sum_{i=1}^n \sup\limits_{x \in I} \left| e^{\mu|x|} \left(\sigma \dfrac{d}{dx} \right)^i u(x) \right| < +\infty \right\},$$

$$\mathring{X}^n_{\mu,\sigma}(I) \equiv \{ u \in X^n_{\mu,\sigma}(I) \mid u(0) = 0 \},$$

$$\mathring{X}_{\mu,\varepsilon}(\mathbf{R}_\pm) \equiv \mathring{X}^2_{\mu,\varepsilon}(\mathbf{R}_\pm) \times \mathring{X}^2_{\mu,1}(\mathbf{R}_\pm),$$

$BC(\mathbf{R}) \equiv \{$the set of the bounded and uniformly continuous functions defined on $\mathbf{R}\}$.

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2. Construction of traveling wave solutions

In this section, we give a short summary of the existence results of traveling wave solutions studied in the previous paper [16]. Introducing the traveling coordinate $z = x + ct$, we find that a traveling wave solution with velocity c satisfies the following equations

$$\text{(2.1)} \qquad \begin{cases} \varepsilon^2 u_{zz} - \varepsilon c\tau u_z + f(u, v) = 0 \\ v_{zz} - cv_z + g(u, v) = 0 \end{cases}, \quad z \in \mathbf{R}$$

with boundary conditions

$$\text{(2.2)} \qquad u(\pm\infty) = u_{\pm}, \quad v(\pm\infty) = v_{\pm},$$

where $0 < \varepsilon \ll 1$. Since any solution of (2.1) has translation invariance, we can normalize u by

$$\text{(2,3)} \qquad u(0) = \alpha,$$

where α is arbitrarily fixed in some interval (see Subsection 2.2). Moreover we put

$$\text{(2.4)} \qquad v(0) = \beta \in (v_-, v_+).$$

β will be determined later.

First, we shall separate the whole interval $\mathbf{R}$ into two subintervals, say $\mathbf{R}_-$ and $\mathbf{R}_+$, and consider the following boundary value problem on each subinterval:

$$\text{(2.5)}_{\pm} \qquad \begin{cases} \varepsilon^2 (u^{\pm})_{zz} - \varepsilon c\tau (u^{\pm})_z + f(u^{\pm}, v^{\pm}) = 0 \\ (v^{\pm})_{zz} - c(v^{\pm})_z + g(u^{\pm}, v^{\pm}) = 0 \\ u^{\pm}(\pm\infty) = u_{\pm}, \quad u^{\pm}(0) = \alpha \\ v^{\pm}(\pm\infty) = v_{\pm}, \quad v^{\pm}(0) = \beta. \end{cases}, \quad z \in \mathbf{R}_{\pm}$$

In Subsection 2.1, we construct *outer solutions* which approximate $(2.5)_{\pm}$ in the outside of a boundary layer region. In Subsection 2.2, we construct *inner solutions* which approximate $(2.5)_{\pm}$ in a boundary layer region. Second in Subsection 2.3, using these approximate solutions we construct *singular limit traveling wave solutions* of (2.1), (2.2) and (2.3), which give important information to construct exact solutions. Finally, in Subsection 2.4, we obtain an exact solution $(u, v)(z; \varepsilon; \tau)$ of (2.1), (2.2) and (2.3) for an appropriate $c = c(\varepsilon; \tau)$.

2.1. Outer solutions

By putting formally $\varepsilon = 0$ in $(2.5)_{\pm}$, we consider the following problems:

$$(2.6)_\pm \qquad \begin{cases} f(u^\pm, v^\pm) = 0 \\ (v^\pm)_{zz} - c(v^\pm)_z + g(u^\pm, v^\pm) = 0 \\ v^\pm(\pm\infty) = v_\pm,\ v^\pm(0) = \beta. \end{cases}, \quad z \in \mathbf{R}_\pm$$

By (A-1), we replace $f(u^\pm, v^\pm) = 0$ with $u^\pm = h_\pm(v^\pm)$. Thus, $(2.6)_\pm$ can be reduced to

$$(2.7)_\pm \qquad \begin{cases} (V^\pm)_{zz} - c(V^\pm)_z + g(h_\pm(V^\pm), V^\pm) = 0, & z \in \mathbf{R}_\pm \\ V^\pm(\pm\infty) = v_\pm,\ V^\pm(0) = \beta. \end{cases}$$

LEMMA 2.1. *For any fixed $c \in \mathbf{R}$ and $\beta \in (v_-, v_+)$, there exist unique strictly monotone increasing solutions $V_0^\pm(z; c, \beta)(z \in \mathbf{R}_\pm)$ of $(2.7)_\pm$ satisfying*

$$|V_0^\pm(z; c, \beta) - v_\pm| \in X^2_{\mu(c),1}(\mathbf{R}_\pm),$$

where $\mu(c) = \min\{\mu_-(c), \mu_+(c)\}$ and $\mu_\pm(c)$ are positive roots of $\mu_\pm^2 - c\mu_\pm + \dfrac{d}{dv} g(h_\pm(v_\pm), v_\pm) = 0$. Moreover $V_0^\pm(z; c, \beta)$ are continuous with respect to $(c, \beta) \in \mathbf{R} \times (v_-, v_+)$ in the $X^2_{\mu(c),1}(\mathbf{R}_\pm)$-topology and satisfy

$$(2.8) \qquad \frac{\partial}{\partial c}\left[\frac{d}{dz} V_0^-(0; c, \beta) - \frac{d}{dz} V_0^+(0; c, \beta)\right] > 0$$

and

$$(2.9) \qquad \frac{\partial}{\partial \beta}\left[\frac{d}{dz} V_0^-(0; c, \beta) - \frac{d}{dz} V_0^+(0; c, \beta)\right] > 0.$$

By this lemma, we directly obtain the following result which is useful for the discussion in Subsection 2.3.

LEMMA 2.2. *For any $c \in \mathbf{R}$, there uniquely exists $\beta = \beta_o(\mathbf{c}) \in C^1(\mathbf{R})$ satisfying*

$$\frac{d}{dz} V_0^-(0; c, \beta_o(c)) - \frac{d}{dz} V_0^+(0; c, \beta_o(c)) = 0.$$

The function $\beta_o(c)$ is strictly monotone decreasing and converges to $v_\pm$ as $c \to \mp\infty$, respectively. Moreover for $v^* \in (v_-, v_+)$,

$$\mathscr{J}(v^*) \gtreqqless 0 \quad \text{if and only if} \quad \beta_o(0) \gtreqqless v^*,$$

where $\mathscr{J}(\beta) = \displaystyle\int_{v_-}^{\beta} g(h_-(v), v)\, dv + \int_{\beta}^{v_+} g(h_+(v), v)\, dv.$

Using the functions $V_0^\pm(z; c, \beta)$, we define $U_0^\pm(z; c, \beta)$ by

$$U_0^\pm(z; c, \beta) = h_\pm(V_0^\pm(z; c, \beta)), \qquad z \in \mathbf{R}_\pm.$$

We thus find that $(U_0^\pm, V_0^\pm)(z; c, \beta)$ satisfy $(2.5)_\pm$ approximately in the outside of a neighborhood of $z=0$, however $U_0^{\pm}(z; c, \beta)$ do not satisfy the boundary conditions $U_0^{\pm}(0; c, \beta)=\alpha$.

2.2. Inner solutions

As being stated above, $(U_0^{\pm}, V_0^{\pm})(z; c, \beta)$ do not satisfy $(2.5)_{\pm}$ approximately in a neighborhood of $z=0$. That is, solutions $u^{\pm}$ of $(2.5)_{\pm}$ have steep gradients. Therefore we must remedy this defect by supplementing $(U_0^{\pm}, V_0^{\pm})$ with *inner solutions* $W_0^{\pm}$ in a neighborhood of $z=0$, so that $(U_0^{\pm}+W_0^{\pm}, V_0^{\pm})$ will satisfy $(2.5)_{\pm}$ approximately for all $z \in \mathbf{R}_{\pm}$.

For this purpose, we introduce the stretched variable $\xi=z/\varepsilon$ in a neighborhood of $z=0$. Substituting $(U_0^{\pm}+W_0^{\pm}, V_0^{\pm})$ into $(2.5)_{\pm}$ and putting $\varepsilon=0$, we obtain the following problems:

$$(2.10)_{\pm} \quad \begin{cases} (W_0^{\pm})_{\xi\xi}-c\tau(W_0^{\pm})_{\xi}+f(h_{\pm}(\beta)+W_0^{\pm}, \beta)=0, & \xi \in \mathbf{R}_{\pm} \\ W_0^{\pm}(0)=\alpha-h_{\pm}(\beta) \\ W_0^{\pm}(\pm\infty)=0, \end{cases}$$

where β and α are fixed constants satisfying $\beta \in (v_-, v_+)$ and $\alpha \in (h_-(\beta), h_+(\beta))$, respectively. We first state the following lemma.

LEMMA 2.3 (Fife et al [11]). *For any fixed $\beta \in [v_-, v_+]$, consider the following problem*:

$$(2.11) \quad \begin{cases} W_{\xi\xi}-cW_{\xi}+f(W, \beta)=0, & \xi \in \mathbf{R} \\ W(\pm\infty)=h_{\pm}(\beta), \ W(0)=\alpha. \end{cases}$$

Then there exists $c=c_0(\beta) \in C^1([v_-, v_+])$ such that (2.11) *has a unique strictly monotone increasing solution $W(\xi; \beta)$ satisfying*

$$|W(\xi; \beta)-h_{\pm}(\beta)| \in X^2_{\sigma_{\pm}(\beta),1}(\mathbf{R}_{\pm}),$$

where

$$\sigma_{\pm}(\beta)=[\mp c_0(\beta)+((c_0(\beta))^2-4f_u(h_{\pm}(\beta), \beta))^{1/2}]/2.$$

Furthermore

$$c_0(\beta) \gtreqless 0 \quad \textit{if and only if} \quad \mathscr{J}(\beta) \gtreqless 0.$$

LEMMA 2.4. *For any fixed $\beta \in [v_-, v_+]$, let $c_I(\beta; \tau)=c_0(\beta)/\tau$. Then there exists $\delta_0>0$ such that for any fixed $(\hat{c}, \hat{\beta}) \in \Lambda_{\delta_0} \equiv \{(\hat{c}, \hat{\beta}) \mid |\hat{c}-c_I(\beta; \tau)|+|\hat{\beta}-\beta| \leq \delta_0\}$, $(2.10)_{\pm}$ have unique strictly monotone increasing solutions $W_0^{\pm}(\xi; \tau; \hat{c}, \hat{\beta})$ satisfying*

$$|W_0^{\pm}(\xi; \tau; \hat{c}, \hat{\beta})| \in X^2_{\sigma_{\pm}(\tau),1}(\mathbf{R}_{\pm}),$$

where $\sigma_\pm(\tau) = \inf_{(\hat{c},\hat{\beta})\in\Lambda_{\delta_0}} \sigma_\pm(\tau; \hat{c}, \hat{\beta})$ *and* $\sigma_\pm(\tau; c, \beta) = [\mp c\tau + ((c\tau)^2 - 4f_u(h_\pm(\beta), \beta))^{1/2}]/2$. *Furthermore* $W_0^\pm(\xi; \tau; \hat{c}, \hat{\beta})$ *are continuous with respect to* $(\hat{c}, \hat{\beta})\in\Lambda_{\delta_0}$ *in the* $X^2_{\sigma_\pm(\tau),1}(\mathbf{R}_\pm)$*-topology and satisfy*

$$\frac{d}{d\xi}W_0^-(0; \tau; c_I(\beta; \tau), \beta) - \frac{d}{d\xi}W_0^+(0; \tau; c_I(\beta; \tau), \beta) = 0, \tag{2.12}$$

$$\frac{\partial}{\partial c}\left[\frac{d}{d\xi}W_0^-(0; \tau; c_I(\beta; \tau), \beta) - \frac{d}{d\xi}W_0^+(0; \tau; c_I(\beta; \tau), \beta)\right] > 0, \tag{2.13}$$

$$\frac{\partial}{\partial \beta}\left[\frac{d}{d\xi}W_0^-(0; \tau; c_I(\beta; \tau), \beta) - \frac{d}{d\xi}W_0^+(0; \tau; c_I(\beta; \tau), \beta)\right] > 0. \tag{2.14}$$

REMARK 2.1. It follows from (2.12), (2.13) and (2.14) that $\frac{d}{d\beta}c_I(\beta; \tau)$ is strictly negative for $\beta\in[v_-, v_+]$. Then there exists an inverse function of $c = c_I(\beta; \tau)$, say $\beta = \beta_I(c; \tau)$, which is strictly decreasing for $c\in[c_I(v_+; \tau), c_I(v_-; \tau)]$.

2.3. Singular limit solutions

In the preceding subsections, we constructed the lowest order approximations $(U_0^\pm(z; c, \beta) + W_0^\pm(z/\varepsilon; \tau; c, \beta), V_0^\pm(z; c, \beta))$ of the problems $(2.5)_\pm$. It is clear that these approximations are matched at $z = 0$ in the C^0-sense. But in order that these become an approximation of (2.1), (2.2) and (2.3) uniformly in $\mathbf{R}$, their derivatives have to be matched at $z = 0$ in the C^0-sense. That is, we impose the following conditions on $(W_0^\pm, V_0^\pm)$:

$$\begin{cases} \Phi_0(\tau; c, \beta) \equiv \dfrac{d}{d\xi}W_0^-(0; \tau; c, \beta) - \dfrac{d}{d\xi}W_0^+(0; \tau; c, \beta) = 0 \\ \Psi_0(c, \beta) \equiv \dfrac{d}{dz}V_0^-(0; c, \beta) - \dfrac{d}{dz}V_0^+(0; c, \beta) = 0. \end{cases} \tag{2.15}$$

By Lemmas 2.2 and 2.4, it turns out that the above relations are equivalent to the conditions

$$\beta = \beta_o(c) \tag{2.16}$$

and

$$c = c_0(\beta)/\tau. \tag{2.17}$$

By Remark 2.1, (2.17) is identical to

$$\beta = \beta_I(c; \tau). \tag{2.18}$$

LEMMA 2.5. *The curves* (2.16) *and* (2.18)(*or*(2.17)) *have at least one intersection point for any* $\tau > 0$. *In particular, when* $v^*\in(v_-, v_+)$ *these have* *three intersection points for small τ, while only one for large τ and intersect transversally at each point, and when $v^* \in (v_{\min}, v_{\max}) \setminus (v_-, v_+)$, these have only one for small or large τ and intersect transversally* (see Fig. 3).

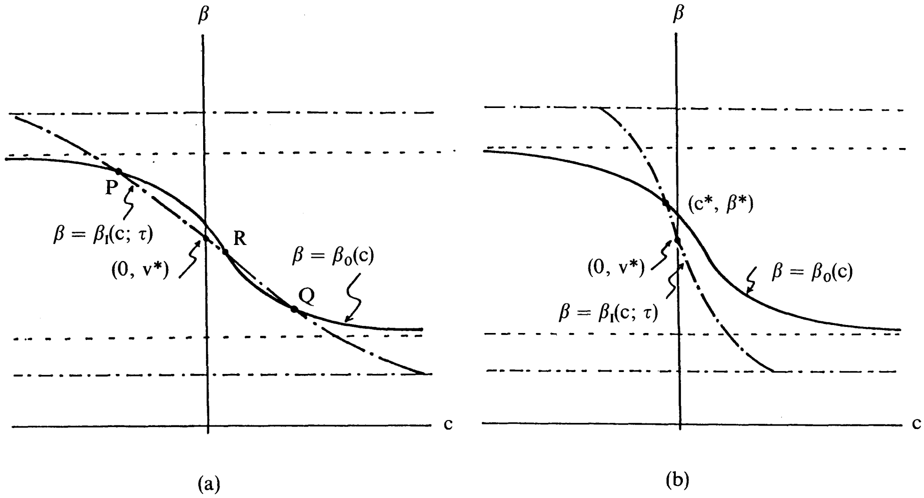

Fig. 3: The graphs of the two curves: $\beta = \beta_o(c)$ and $\beta = \beta_I(c; \tau)$.
(a) For small τ, there are three intersection points.
(b) For large τ, there is a unique intersection point.

Let $(c^*(\tau), \beta^*(\tau))$ be an arbitrary intersection point of the curves (2.16) and (2.18). Define

$$u_0(z; \varepsilon; \tau) = \begin{cases} U_0^-(z; c^*(\tau), \beta^*(\tau)) + W_0^-(z/\varepsilon; \tau; c^*(\tau), \beta^*(\tau)), & z \in \mathbf{R}_- \\ U_0^+(z; c^*(\tau), \beta^*(\tau)) + W_0^+(z/\varepsilon; \tau; c^*(\tau), \beta^*(\tau)), & z \in \mathbf{R}_+ \end{cases}$$

and

$$v_0(z; \varepsilon; \tau) = \begin{cases} V_0^-(z; c^*(\tau), \beta^*(\tau)), & z \in \mathbf{R}_- \\ V_0^+(z; c^*(\tau), \beta^*(\tau)), & z \in \mathbf{R}_+ . \end{cases}$$

We call $(u_0, v_0)(z; \varepsilon; \tau)$ a *singular limit traveling wave solution* of (2.1), (2.2) and (2.3), which becomes the lowest order approximation uniformly in $\mathbf{R}$. Also, $c^*(\tau)$ is called the *singular limit velocity.* From Lemma 2.5, it directly follows that

THEOREM 2.1. *Suppose that* (A-1) – (A-4) *hold. Then,* (2.1), (2.2) *and* (2.3) *has at least one singular limit traveling wave solution with the singular limit velocity $c^*(\tau)$ for any $\tau > 0$.*

Moreover we see that the number of the singular limit traveling wave solutions depends on τ and the location of v^*.

COROLLARY 2.1. *Suppose that* (A-1) $-$ (A-4) *hold. When* $v^* \in (v_-, v_+)$, (2.1), (2.2) *and* (2.3) *has three singular limit traveling wave solutions for small* τ *and has only one for large* τ. *On the other hand, when* $v^* \in (v_{\min}, v_{\max}) \setminus (v_-, v_+)$, *it has only one for small or large* τ.

2.4. Traveling wave solutions for $\varepsilon > 0$

In this subsection, we fix $\tau > 0$ arbitrarily. Suppose that $(c^*(\tau), \beta^*(\tau))$ is an arbitrary intersection point of the curves (2.16) and (2.18) at which these intersect *transversally*. Let $\Sigma_{\delta_1} = \{(c, \beta) | \ |c - c^*(\tau)| + |\beta - \beta^*(\tau)| \le \delta_1\}$ for some constant $\delta_1 > 0$. For any fixed $(c, \beta) \in \Sigma_{\delta_1}$, we seek exact solutions $(u^\pm, v^\pm)$ to $(2.5)_\pm$ in the following forms:

$$(2.19)_\pm \qquad \begin{cases} u^\pm(z; \varepsilon; \tau; c, \beta) = U_0^\pm(z; c, \beta) + W_0^\pm(z/\varepsilon; \tau; c, \beta) + r^\pm(z; \varepsilon; \tau; c, \beta) \\ v^\pm(z; \varepsilon; \tau; c, \beta) = V_0^\pm(z; c, \beta) + s^\pm(z; \varepsilon; \tau; c, \beta), \end{cases}$$

where $t^\pm(z; \varepsilon; \tau; c, \beta) \equiv (r^\pm, s^\pm)(z; \varepsilon; \tau; c, \beta)$ are the remainders converging to 0 uniformly as $\varepsilon \downarrow 0$. Applying the standard singular perturbation techniques, we have the following lemma.

LEMMA 2.6. *There are* $\varepsilon_1 > 0$ *and* $\delta_1 > 0$ *such that for any* $\varepsilon \in (0, \varepsilon_1)$ *and* $(c, \beta) \in \Sigma_{\delta_1}$, *there exist* $(u^\pm, v^\pm)(z; \varepsilon; \tau; c, \beta) \in \mathring{X}_{\rho,\varepsilon}(\mathbf{R}_\pm)$ *satisfying* $(2.5)_\pm$, *where* ρ *is an arbitrarily fixed constant satisfying* $0 < \rho < \mu(c^*)$. *Moreover the remainders* $t^\pm(z; \varepsilon; \tau; c, \beta)$, $\dfrac{\partial t^\pm}{\partial c}(z; \varepsilon; \tau; c, \beta)$ and $\dfrac{\partial t^\pm}{\partial \beta}(z; \varepsilon; \tau; c, \beta)$ *are uniformly continuous with respect to* $(\varepsilon, c, \beta) \in (0, \varepsilon_1) \times \Sigma_{\delta_1}$ *in the* $\mathring{X}_{\rho,\varepsilon}(\mathbf{R}_\pm)$*-topology and satisfy*

$$\begin{cases} \|t^\pm(\cdot; \varepsilon; \tau; c, \beta)\|_{\mathring{X}_{\rho,\varepsilon}(\mathbf{R}_\pm)} = o(1) \\ \|\dfrac{\partial t^\pm}{\partial c}(\cdot; \varepsilon; \tau; c, \beta)\|_{\mathring{X}_{\rho,\varepsilon}(\mathbf{R}_\pm)} = o(1) \\ \|\dfrac{\partial t^\pm}{\partial \beta}(\cdot; \varepsilon; \tau; c, \beta)\|_{\mathring{X}_{\rho,\varepsilon}(\mathbf{R}_\pm)} = o(1) \end{cases}$$

as $\varepsilon \downarrow 0$ *uniformly in* $(c, \beta) \in \Sigma_{\delta_1}$.

Finally we construct a solution of (2.1), (2.2) and (2.3) in the whole interval $\mathbf{R}$, matching $(u^-, v^-)(z; \varepsilon; \tau; c, \beta)$ and $(u^+, v^+)(z; \varepsilon; \tau; c, \beta)$ at $z = 0$ in the C^1-sense. For this purpose, we define two functions Φ and Ψ by

$$(2.20) \qquad \begin{cases} \Phi(\varepsilon; \tau; c, \beta) \equiv \varepsilon \dfrac{d}{dz} u^-(0; \varepsilon; \tau; c, \beta) - \varepsilon \dfrac{d}{dz} u^+(0; \varepsilon; \tau; c, \beta) \\ \Psi(\varepsilon; \tau; c, \beta) \equiv \dfrac{d}{dz} v^-(0; \varepsilon; \tau; c, \beta) - \dfrac{d}{dz} v^+(0; \varepsilon; \tau; c, \beta) \end{cases}$$

and determine c and β as functions of ε such that

$$\Phi(\varepsilon; \tau; c, \beta) = 0 = \Psi(\varepsilon; \tau; c, \beta) \tag{2.21}$$

hold. We call the relation (2.21) the *matching condition.* Noting that Φ and Ψ are uniformly continuous in ε, we can extend them continuously so as to be defined for $\varepsilon = 0$. Setting $\varepsilon = 0$, we can reduce (2.20) to

$$\begin{cases} \Phi(0; \tau; c, \beta) = \Phi_0(\tau; c, \beta) \\ \Psi(0; \tau; c, \beta) = \Psi_0(c, \beta). \end{cases} \tag{2.22}$$

Recall that $(c^*(\tau), \beta^*(\tau))$ is an arbitrary intersection point of the curves (2.16) and (2.18) at which these intersect *transversally*, or equivalently $(c^*(\tau), \beta^*(\tau))$ satisfies

$$\Phi_0(\tau; c^*(\tau), \beta^*(\tau)) = 0 = \Psi_0(c^*(\tau), \beta^*(\tau)) \tag{2.23}$$

and

$$\mathbf{J}(\tau; c^*(\tau), \beta^*(\tau)) \equiv \tag{2.24}$$

$$\det \begin{bmatrix} \dfrac{\partial}{\partial c}\Phi_0(\tau; c^*(\tau), \beta^*(\tau)) & \dfrac{\partial}{\partial \beta}\Phi_0(\tau; c^*(\tau), \beta^*(\tau)) \\ \dfrac{\partial}{\partial c}\Psi_0(c^*(\tau), \beta^*(\tau)) & \dfrac{\partial}{\partial \beta}\Psi_0(c^*(\tau), \beta^*(\tau)) \end{bmatrix} \neq 0.$$

We call $\mathbf{J}(\tau; c^*(\tau), \beta^*(\tau))$ the *Jacobian of the matching condition* (2.21). Thus we can apply the implicit function theorem [7: Theorem 4.3] to (2.21). That is, there is $\varepsilon_2 > 0$ such that there exist continuous functions $c(\varepsilon; \tau)$ and $\beta(\varepsilon; \tau)$ satisfying (2.21) for $\varepsilon \in [0, \varepsilon_2)$ and $\lim_{\varepsilon \downarrow 0} c(\varepsilon; \tau) = c^*(\tau)$ and $\lim_{\varepsilon \downarrow 0} \beta(\varepsilon; \tau) = \beta^*(\tau)$. Then, we have the desired results.

THEOREM 2.2. *Suppose that* (A-1) – (A-4) *hold and fix* τ *arbitrarily such that the curves* (2.16) *and* (2.18) *intersect transversally at* $(c^*(\tau), \beta^*(\tau))$. *Then, for any* $\varepsilon \in (0, \varepsilon_2)$ *there exists a traveling wave solution* $(u, v)(z; \varepsilon; \tau) \in X^2_{\rho,\varepsilon}(\mathbf{R}) \times X^2_{\rho,1}(\mathbf{R})$ *of the problem* (2.1), (2.2) *and* (2.3), *which satisfies*

$$\| u(\cdot; \varepsilon; \tau) - u_0(\cdot; \varepsilon; \tau) \|_{X^1_{\rho,\varepsilon}(\mathbf{R})} + \| v(\cdot; \varepsilon; \tau) - v_0(\cdot; \varepsilon; \tau) \|_{X^1_{\rho,1}(\mathbf{R})} \longrightarrow 0$$

as $\varepsilon \downarrow 0$. *Furthermore the velocity* $c(\varepsilon; \tau)$ *converges to the singular limit velocity* $c^*(\tau)$ *as* $\varepsilon \downarrow 0$.

COROLLARY 2.2. *Suppose that* (A-1) – (A-4) *hold. When* $v^* \in (v_-, v_+)$, (2.1), (2.2) *and* (2.3) *has three traveling wave solutions for small* τ *and has only one for large* τ. *On the other hand, when* $v^* \in (v_{\min}, v_{\max}) \setminus (v_-, v_+)$, *it has only one for small or large* τ.

3. Geometrical characteristics of traveling wave solutions

First, we rewrite the problem (2.1) as an equivalent four-dimensional dynamical system

$$(3.1)\qquad \begin{cases} \dfrac{du}{dz} = \dfrac{p}{\varepsilon} \\ \dfrac{dp}{dz} = \dfrac{c\tau}{\varepsilon} p - \dfrac{f(u, v)}{\varepsilon} \\ \dfrac{dv}{dz} = q \\ \dfrac{dq}{dz} = cq - g(u, v) \end{cases}, \quad z \in \mathbf{R},$$

where $p = \varepsilon \dfrac{du}{dz}$ and $q = \dfrac{dv}{dz}$. For $\mathbf{V} = {}^t(u, p, v, q)$, we simply write (3.1) as

$$(3.2)\qquad \frac{d\mathbf{V}}{dz} = \mathbf{F}(\mathbf{V}; \varepsilon; \tau; c).$$

Let $(u, v)(z; \varepsilon; \tau)$ be an arbitrary traveling wave solution with velocity $c = c(\varepsilon; \tau)$ of (2.1), (2.2) and (2.3). Then we find that $\mathscr{V}(z; \varepsilon; \tau) \equiv {}^t\left(u, \varepsilon \dfrac{du}{dz}, v, \dfrac{dv}{dz}\right)(z; \varepsilon; \tau)$ is a solution of (3.2) when $c = c(\varepsilon; \tau)$, which tends to $\mathbf{P}_- = {}^t(u_-, 0, v_-, 0)$ as $z \to -\infty$ and $\mathbf{P}_+ = {}^t(u_+, 0, v_+, 0)$ as $z \to +\infty$, respectively. Namely, $\mathbf{V} = \mathscr{V}(z; \varepsilon; \tau)$ corresponds to a heteroclinic orbit connecting $\mathbf{P}_-$ to $\mathbf{P}_+$, say $\gamma(\varepsilon; \tau)$, of (3.2) when $c = c(\varepsilon; \tau)$ in the four-dimensional phase space.

3.1. Stable and unstable manifolds

We consider the stable manifolds $S^{\pm}_{\varepsilon,\tau,c}$ and the unstable manifolds $U^{\pm}_{\varepsilon,\tau,c}$ of (3.2) with respect to the stationary points $\mathbf{P}_{\pm}$, which are defined by

$$S^{\pm}_{\varepsilon,\tau,c} = \{\mathbf{a} \in \mathbf{R}^4 \mid \mathbf{V}(z; \varepsilon; \tau; c; \mathbf{a}) \longrightarrow \mathbf{P}_{\pm} \quad \text{as } z \longrightarrow +\infty\}$$

and

$$U^{\pm}_{\varepsilon,\tau,c} = \{\mathbf{a} \in \mathbf{R}^4 \mid \mathbf{V}(z; \varepsilon; \tau; c; \mathbf{a}) \longrightarrow \mathbf{P}_{\pm} \quad \text{as } z \longrightarrow -\infty\},$$

respectively. Here $\mathbf{V}(z; \varepsilon; \tau; c; \mathbf{a})(\mathbf{a} \in \mathbf{R}^4)$ is a solution of (3.2) subject to the initial condition $\mathbf{V}(0; \varepsilon; \tau, c; \mathbf{a}) = \mathbf{a}$.

Let $A^{\pm}(\varepsilon; \tau; c)$ by the linearized matrices of $\mathbf{F}$ in (3.2) at $\mathbf{P}_{\pm}$. Then, $A^{\pm}(\varepsilon; \tau; c)$ have the following properties.

LEMMA 3.1. *For any fixed $c \in \mathbf{R}$, the matrices $A^{\pm}(\varepsilon; \tau; c)$ have two negative* *eigenvalues,* $\nu_1^\pm(\varepsilon;\tau;c)$, $\nu_2^\pm(\varepsilon;\tau;c)$ *and two positive eigenvalues* $\nu_3^\pm(\varepsilon;\tau;c)$, $\nu_4^\pm(\varepsilon;\tau;c)$ *when* $\varepsilon>0$ *is sufficiently small. More precisely,* $\nu_i^\pm(\varepsilon;\tau;c)\,(i=1,2,3,4)$ *are represented as follows:*

$$\nu_1^\pm(\varepsilon;\tau;c)=\frac{c\tau-(c^2\tau^2-4f_u^\pm)^{1/2}}{2\varepsilon}(1+o(1)),$$

$$\nu_2^\pm(\varepsilon;\tau;c)=\frac{c-(c^2-4D^\pm/f_u^\pm)^{1/2}}{2}(1+o(1)),$$

$$\nu_3^\pm(\varepsilon;\tau;c)=\frac{c+(c^2-4D^\pm/f_u^\pm)^{1/2}}{2}(1+o(1)),$$

$$\nu_4^\pm(\varepsilon;\tau;c)=\frac{c\tau+(c^2\tau^2-4f_u^\pm)^{1/2}}{2\varepsilon}(1+o(1))$$

as $\varepsilon\downarrow 0$, *where* $D^\pm=f_u^\pm g_v^\pm-f_v^\pm g_u^\pm$ *and* $f_u^\pm=f_u(u_\pm,v_\pm)$. $f_v^\pm$, $g_u^\pm$ *and* $g_v^\pm$ *are also defined similarly.*

This lemma is the special case of Lemma 4.1, so we omit the proof.

REMARK 3.1. Note that the eigenvalues of $A^\pm(\varepsilon;\tau;c)$ are classified into two cases: $\nu_2^\pm(\varepsilon;\tau;c)<0<\nu_3^\pm(\varepsilon;\tau;c)$ are of the order $O(1)$ and $\nu_1^\pm(\varepsilon;\tau;c)<0<\nu_4^\pm(\varepsilon;\tau;c)$ are of the order $O(1/\varepsilon)$ as $\varepsilon\downarrow 0$.

We define the eigenvectors $\mathbf{a}_i^\pm(\varepsilon;\tau;c)$ corresponding to the eigenvalues $\nu_i^\pm(\varepsilon;\tau;c)$ by

$$\mathbf{a}_i^\pm(\varepsilon;\tau;c)=\begin{bmatrix}1\\ \varepsilon\nu_i^\pm\\ -g_u^\pm/[(\nu_i^\pm)^2-c\nu_i^\pm+g_v^\pm]\\ -\nu_i^\pm g_u^\pm/[(\nu_i^\pm)^2-c\nu_i^\pm+g_v^\pm]\end{bmatrix}\quad(i=1,2,3,4).$$

We note the following relations:

$(3.3)_\pm$

$$\det[\mathbf{a}_1^\pm,\mathbf{a}_2^\pm,\mathbf{a}_3^\pm,\mathbf{a}_4^\pm]=\left(\frac{f_u^\pm}{f_v^\pm}\right)^2\cdot(c^2\tau^2-4f_u^\pm)^{1/2}\cdot(c^2-4D^\pm/f_u^\pm)^{1/2}+O(\varepsilon)>0$$

and

$$(3.4)\qquad \det[\mathbf{a}_1^+,\mathbf{a}_2^+,\mathbf{a}_3^-,\mathbf{a}_4^-]=\frac{f_u^-}{f_v^-}\cdot\frac{f_u^+}{f_v^+}\cdot\frac{(c^2\tau^2-4f_u^-)^{1/2}+(c^2\tau^2-4f_u^+)^{1/2}}{2}$$
$$\times\frac{(c^2-4D^-/f_u^-)^{1/2}+(c^2-4D^+/f_u^+)^{1/2}}{2}+O(\varepsilon)>0.$$

By virtue of Lemma 3.1, we know that linearly independent solutions of the linearized equations

$$\frac{d\mathbf{V}}{dz} = A^{\pm}(\varepsilon;\, \tau;\, c)\mathbf{V}$$

are given by

$$\mathbf{Q}_i^{\pm}(z;\, \varepsilon;\, \tau;\, c) = e^{\nu_i^{\pm}(\varepsilon;\tau;c)z} \cdot \mathbf{a}_i^{\pm}(\varepsilon;\, \tau;\, c) \quad (i = 1, 2, 3, 4)$$

and satisfy

$$\mathbf{Q}_i^{\pm}(z;\, \varepsilon;\, \tau;\, c) \longrightarrow \mathbf{0} \qquad \text{as} \begin{cases} z \longrightarrow +\infty & (i = 1, 2) \\ z \longrightarrow -\infty & (i = 3, 4). \end{cases}$$

According to the definitions of $S^{\pm}_{\varepsilon,\tau,c}$, solutions $\mathbf{V}^{\pm}(z;\, \varepsilon;\, \tau;\, c)$ of (3.2) lie on $S^{\pm}_{\varepsilon,\tau,c}$ if and only if these satisfy

(3.5) $$\mathbf{V}^{\pm}(z;\, \varepsilon;\, \tau;\, c) \simeq \textstyle\sum_{i=1}^{2} \alpha_i^{\pm}\, \mathbf{Q}_i^{\pm}(z;\, \varepsilon;\, \tau;\, c) \qquad \text{as } z \longrightarrow +\infty,$$

where $\alpha_i^{\pm}\,(i = 1, 2)$ are some constants. Here $a(z) \simeq b(z)$ as $z \to +\infty$ means that $\{a(z) - b(z)\}/\{\|a(z)\| + \|b(z)\|\} \to 0$ as $z \to +\infty$. Similarly, solutions $\mathbf{V}^{\pm}(z;\, \varepsilon;\, \tau;\, c)$ of (3.2) lie on $U^{\pm}_{\varepsilon,\tau,c}$ if and only if

(3.6) $$\mathbf{V}^{\pm}(z;\, \varepsilon;\, \tau;\, c) \simeq \textstyle\sum_{i=3}^{4} \alpha_i^{\pm}\, \mathbf{Q}_i^{\pm}(z;\, \varepsilon;\, \tau;\, c) \qquad \text{as } z \longrightarrow -\infty,$$

for some constants $\alpha_i^{\pm}\,(i = 3, 4)$. These conditions (3.5) and (3.6) imply that $S^{\pm}_{\varepsilon,\tau,c}$ are two-dimensional manifolds which are tangent to $\mathbf{a}_1^{\pm}$ and $\mathbf{a}_2^{\pm}$, and $U^{\pm}_{\varepsilon,\tau,c}$ are two-dimensional manifolds which are tangent to $\mathbf{a}_3^{\pm}$ and $\mathbf{a}_4^{\pm}$. That is, these manifolds pass through the stationary points $\mathbf{P}_{\pm}$, and their tangent spaces $TS^{\pm}_{\varepsilon,\tau,c}$ and $TU^{\pm}_{\varepsilon,\tau,c}$ at the points $\mathbf{V} = \mathbf{P}_{\pm}$ are spanned by the vectors $\{\mathbf{a}_1^{\pm}, \mathbf{a}_2^{\pm}\}$ and $\{\mathbf{a}_3^{\pm}, \mathbf{a}_4^{\pm}\}$, respectively. In this paper, we fix orientations to $S^{\pm}_{\varepsilon,\tau,c}$ and $U^{\pm}_{\varepsilon,\tau,c}$ by regarding the ordered pairs $(\mathbf{a}_1^{\pm}, \mathbf{a}_2^{\pm})$ and $(\mathbf{a}_3^{\pm}, \mathbf{a}_4^{\pm})$ of the basis vectors, respectively, as *positively oriented basis* of their tangent spaces $TS^{\pm}_{\varepsilon,\tau,c}$ and $TU^{\pm}_{\varepsilon,\tau,c}$ at the stationary points $\mathbf{P}_{\pm}$. Note that the conditions $(3.3)_{\pm}$ determine a relationship between the orientations of the two manifolds $S^{\pm}_{\varepsilon,\tau,c}$ and $U^{\pm}_{\varepsilon,\tau,c}$.

3.2. Positively oriented basis of tangent spaces of stable and unstable manifolds along the heteroclinic orbit

Let us consider the dynamical system (3.2) when $c = c(\varepsilon;\, \tau)$:

(3.7) $$\frac{d\mathbf{V}}{dz} = \mathbf{F}(\mathbf{V};\, \varepsilon;\, \tau;\, c(\varepsilon;\, \tau)).$$

Recalling that $\mathscr{V}(z;\, \varepsilon;\, \tau)$ is the heteroclinic solution of (3.7) connecting $\mathbf{P}_-$ to $\mathbf{P}_+$, we find that the stable manifold $S^{+}_{\varepsilon,\tau}$ ($\equiv S^{+}_{\varepsilon,\tau,c(\varepsilon;\tau)}$) and the unstable manifold $U^{-}_{\varepsilon,\tau}$ ($\equiv U^{-}_{\varepsilon,\tau,c(\varepsilon;\tau)}$) of (3.7) satisfy the condition $S^{+}_{\varepsilon,\tau} \cap U^{-}_{\varepsilon,\tau} \neq \varnothing$. That is, $\gamma(\varepsilon;\, \tau) \subset S^{+}_{\varepsilon,\tau} \cap U^{-}_{\varepsilon,\tau}$.

In order to study behavior of the flow of (3.7) along the heteroclinic orbit $\gamma(\varepsilon;\tau)$, we consider the following linearized equation of (3.7) with respect to $\mathscr{V}(z;\varepsilon;\tau)$:

$$\frac{d\mathbf{V}}{dz}=A(z;\varepsilon;\tau)\mathbf{V}, \tag{3.8}$$

where $A(z;\varepsilon;\tau)\equiv\frac{\partial\mathbf{F}}{\partial\mathbf{V}}(\mathbf{V};\varepsilon;\tau;c(\varepsilon;\tau))|_{\mathbf{V}}=\mathscr{V}(z;\varepsilon;\tau)$. Noting that

$$A(z;\varepsilon;\tau)\longrightarrow A^{\pm}(\varepsilon;\tau;c(\varepsilon;\tau))\qquad\text{as } z\longrightarrow\pm\infty,$$

we know that the equation (3.8) has just two linearly independent solutions $\mathbf{V}_i(z;\varepsilon;\tau)\,(i=1,2)$ which satisfy

$$\mathbf{V}_i(z;\varepsilon;\tau)\longrightarrow\mathbf{0}\qquad\text{as } z\longrightarrow+\infty\quad(i=1,2)$$

and just two linearly independent solutions $\mathbf{V}_i(z;\varepsilon;\tau)\,(i=3,4)$ which satisfy

$$\mathbf{V}_i(z;\varepsilon;\tau)\longrightarrow\mathbf{0}\qquad\text{as } z\longrightarrow-\infty\quad(i=3,4).$$

This implies that $\{\mathbf{V}_1(z;\varepsilon;\tau),\mathbf{V}_2(z;\varepsilon;\tau)\}$ and $\{\mathbf{V}_3(z;\varepsilon;\tau),\mathbf{V}_4(z;\varepsilon;\tau)\}$ can be regarded as basis vectors of $T_zS^+_{\varepsilon,\tau}$ and $T_zU^-_{\varepsilon,\tau}$ for all $z\in\mathbf{R}$, respectively. $T_zS^+_{\varepsilon,\tau}$(resp. $T_zU^-_{\varepsilon,\tau}$) stands for the tangent space of $S^+_{\varepsilon,\tau}$ (resp. $U^-_{\varepsilon,\tau}$) at the point $\mathscr{V}(z;\varepsilon;\tau)\in S^+_{\varepsilon,\tau}\cap U^-_{\varepsilon,\tau}$.

Let us normalize the solutions $\mathbf{V}_i(z;\varepsilon;\tau)\,(i=1,2,3,4)$ of (3.8) by assuming, without loss of generality, the condition

$$\mathbf{V}_i(z;\varepsilon;\tau)\simeq\begin{cases}\mathbf{Q}_i^+(z;\varepsilon;\tau;c(\varepsilon;\tau)) & \text{as } z\longrightarrow+\infty\quad(i=1,2)\\ \mathbf{Q}_i^-(z;\varepsilon;\tau;c(\varepsilon;\tau)) & \text{as } z\longrightarrow-\infty\quad(i=3,4).\end{cases} \tag{3.9}$$

When this condition is satisfied, the ordered pair $(\mathbf{V}_1,\mathbf{V}_2)$ can be regarded as a positively oriented basis of $T_zS^+_{\varepsilon,\tau}$ for each $z\in\mathbf{R}$ because its limit $(\mathbf{Q}_1^+,\mathbf{Q}_2^+)$ as $z\to+\infty$ is a positively oriented basis of $TS^+_{\varepsilon,\tau}$ at $\mathbf{V}=\mathbf{P}_+$. Similarly, the ordered pair $(\mathbf{V}_3,\mathbf{V}_4)$ can be regarded as a positively oriented basis of $T_zU^-_{\varepsilon,\tau}$ for each $z\in\mathbf{R}$. Therefore, the positively oriented basis of the tangent spaces of $S^+_{\varepsilon,\tau}$ and $U^-_{\varepsilon,\tau}$ are determined at each point on the heteroclinic orbit $\gamma(\varepsilon;\tau)$. Note that a relationship between the orientations of the two manifolds $S^+_{\varepsilon,\tau}$ and $U^-_{\varepsilon,\tau}$ are determined by the condition (3.4).

3.3. Index of heteroclinic orbit

In this subsection, we give the definition of the index of the heteroclinic orbit along the work by Maginu [24], which can be seen as an extension of Evans' results [6].

To study detailed properties of the manifolds $S^+_{\varepsilon,\tau}$ and $U^-_{\varepsilon,\tau}$ near the heteroclinic orbit $\gamma(\varepsilon;\tau)$, we introduce a locally transversal section M_z which passes through the point $\mathbf{V} = \mathscr{V}(z; \varepsilon; \tau)$.

DEFINITION 3.1. A *locally transversal section* M_z is an arbitrary three-dimensional smooth and small disk which meets the orbit $\gamma(\varepsilon; \tau)$ at a point $\mathbf{V} = \mathscr{V}(z; \varepsilon; \tau)$ and is transversal to the flow of (3.7).

By the definition of M_z, the manifolds $S^+_{\varepsilon,\tau}$ and $U^-_{\varepsilon,\tau}$ are transversal to M_z. Hence $S^+_{\varepsilon,\tau} \cap M_z$ and $U^-_{\varepsilon,\tau} \cap M_z$ are smooth one-dimensional manifolds, and pass through the point $\mathscr{V}(z; \varepsilon; \tau) \in \gamma(\varepsilon; \tau) \cap M_z$. Here we assume the following:

$$\text{(B)} \qquad \begin{cases} M_z \text{ is a locally transversal section, and } S^+_{\varepsilon,\tau} \cap M_z \text{ and} \\ U^-_{\varepsilon,\tau} \cap M_z \text{ cross transversally to each other at } \mathscr{V}(z; \varepsilon; \tau). \end{cases}$$

Note that the above assumption (B) does not depend on the choice of a locally transversal section M_z. Under the assumption (B), the heteroclinic orbit $\gamma(\varepsilon; \tau) \subset S^+_{\varepsilon,\tau} \cap U^-_{\varepsilon,\tau}$ is isolated.

Let $\mathbf{T}_2(z; \varepsilon; \tau)$ and $\mathbf{T}_4(z; \varepsilon; \tau)$ be tangent vectors of $S^+_{\varepsilon,\tau} \cap M_z$ and $U^-_{\varepsilon,\tau} \cap M_z$ at $\mathscr{V}(z; \varepsilon; \tau)$, respectively, that is,

$$\mathbf{T}_2(z; \varepsilon; \tau) \in T_z(S^+_{\varepsilon,\tau} \cap M_z) \quad \text{and} \quad \mathbf{T}_4(z; \varepsilon; \tau) \in T_z(U^-_{\varepsilon,\tau} \cap M_z).$$

Assumption (B) guarantees that $\mathbf{T}_2(z; \varepsilon; \tau)$ and $\mathbf{T}_4(z; \varepsilon; \tau)$ are linearly independent, that is, $\mathbf{T}_2(z; \varepsilon; \tau)$ and $\mathbf{T}_4(z; \varepsilon; \tau)$ span a two-dimensional subspace in the three-dimensional tangent space $T_z M_z$ of the section M_z. On the other hand, the tangent vector $\dfrac{d\mathscr{V}}{dz}(z; \varepsilon; \tau)$ of the orbit $\gamma(\varepsilon; \tau)$ is contained in both of the tangent spaces $T_z S^+_{\varepsilon,\tau}$ and $T_z U^-_{\varepsilon,\tau}$, but is not contained in the tangent space $T_z M_z$. Hence, defining the vectors $\mathbf{T}_1(z; \varepsilon; \tau)$ and $\mathbf{T}_3(z; \varepsilon; \tau)$ by

$$\mathbf{T}_1(z; \varepsilon; \tau) = \frac{d\mathscr{V}}{dz}(z; \varepsilon; \tau) = \mathbf{T}_3(z; \varepsilon; \tau), \tag{3.10}$$

we may regard the ordered pairs $(\mathbf{T}_1, \mathbf{T}_2)$ and $(\mathbf{T}_3, \mathbf{T}_4)$ as basis of $T_z S^+_{\varepsilon,\tau}$ and $T_z U^-_{\varepsilon,\tau}$, respectively. Without loss of generality, we may choose $\mathbf{T}_i(z; \varepsilon; \tau)$ $(i = 1, 2, 3, 4)$ such that the following relations are satisfied:

$$\mathbf{T}_i(z; \varepsilon; \tau) = \begin{cases} \sum_{j=1}^{2} \beta_{ij}(z; \varepsilon; \tau) \mathbf{V}_j(z; \varepsilon; \tau) & (i = 1, 2) \\ \sum_{j=3}^{4} \beta_{ij}(z; \varepsilon; \tau) \mathbf{V}_j(z; \varepsilon; \tau) & (i = 3, 4), \end{cases} \tag{3.11}$$

where $B_1(z; \varepsilon; \tau) \equiv \{\beta_{ij}(z; \varepsilon; \tau); 1 \le i, j \le 2\}$ and $B_2(z; \varepsilon; \tau) \equiv \{\beta_{ij}(z; \varepsilon; \tau); 3 \le i, j \le 4\}$ are some orientation preserving linear transformations, that is, $\det B_1(z; \varepsilon; \tau) > 0$ and $\det B_2(z; \varepsilon; \tau) > 0$ hold. Then, $(\mathbf{T}_1, \mathbf{T}_2)$ and $(\mathbf{T}_3, \mathbf{T}_4)$ also become positively oriented basis of $T_z S^+_{\varepsilon,\tau}$ and $T_z U^-_{\varepsilon,\tau}$, respectively. Hereafter we use these vectors as positively oriented basis in stead of $\mathbf{V}_i (i = 1, 2, 3, 4)$.

Now let us consider the *direction* in which $S^+_{\varepsilon,\tau,c} \cap M_z$ crosses through $U^-_{\varepsilon,\tau,c} \cap M_z$ when c increases through $c(\varepsilon; \tau)$. First, we note that $S^+_{\varepsilon,\tau,c} \cap M_z$ and $U^-_{\varepsilon,\tau,c} \cap M_z$ depend smoothly on c because the locally transversal section M_z is transversal to $S^+_{\varepsilon,\tau,c}$ and $U^-_{\varepsilon,\tau,c}$ when $|c - c(\varepsilon; \tau)|$ is small. Moreover, when $c = c(\varepsilon; \tau)$, these manifolds cross to each other at a point $\mathscr{V}(z; \varepsilon; \tau) \in \gamma(\varepsilon; \tau) \cap M_z$.

Let $\mathbf{X}(z; \varepsilon; \tau; c)$ and $\mathbf{Y}(z; \varepsilon; \tau; c)$ be arbitrary points on $S^+_{\varepsilon,\tau,c} \cap M_z$ and $U^-_{\varepsilon,\tau,c} \cap M_z$, respectively, satisfying

$$\mathbf{X}(z; \varepsilon; \tau; c(\varepsilon; \tau)) = \mathscr{V}(z; \varepsilon; \tau) = \mathbf{Y}(z; \varepsilon; \tau; c(\varepsilon; \tau)) \tag{3.12}$$

(see Fig. 4). We define a vector $\mathbf{R}_0(z; \varepsilon; \tau) \in T_z M_z$ by

$$\mathbf{R}_0(z; \varepsilon; \tau) \equiv \lim_{c \to c(\varepsilon, \tau)} \frac{\mathbf{X}(z; \varepsilon; \tau; c) - \mathbf{Y}(z; \varepsilon; \tau; c)}{c - c(\varepsilon; \tau)}.$$

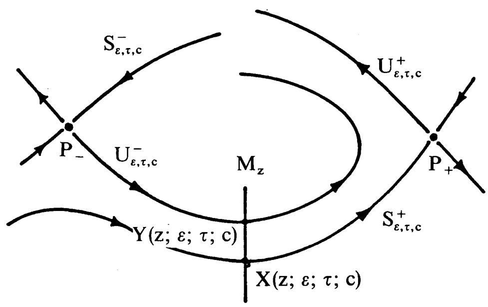

Fig. 4: Phase portrait of (3.2) and the locally transversal section M_z, and the points $\mathbf{X}(z; \varepsilon; \tau; c) \in S^+_{\varepsilon,\tau,c} \cap M_z$ and $\mathbf{Y}(z; \varepsilon; \tau; c) \in U^-_{\varepsilon,\tau,c} \cap M_z$.

Roughly speaking, this vector points to the direction in which $S^+_{\varepsilon,\tau,c} \cap M_z$ passes through $U^-_{\varepsilon,\tau,c} \cap M_z$ when c increases through $c(\varepsilon; \tau)$. Recall that $\mathbf{T}_2(z; \varepsilon; \tau)$ and $\mathbf{T}_4(z; \varepsilon; \tau)$ span a two-dimensional subspace in the three-dimensional tangent space $T_z M_z$. Thus, we may say that $S^+_{\varepsilon,\tau,c} \cap M_z$ passes through $U^-_{\varepsilon,\tau,c} \cap M_z$ with non-zero velocity when c increases through $c(\varepsilon; \tau)$ if and only if the vector $\mathbf{R}_0(z; \varepsilon; \tau)$ is not contained in this two-dimensional subspace. That is,

$$h(z; \varepsilon; \tau) \equiv \det[\mathbf{R}_0(z; \varepsilon; \tau), \mathbf{T}_2(z; \varepsilon; \tau), \mathbf{T}_3(z; \varepsilon; \tau), \mathbf{T}_4(z; \varepsilon; \tau)] \neq 0.$$

We may say that $h(z; \varepsilon; \tau)$ represents the velocity at which $S^+_{\varepsilon,\tau,c} \cap M_z$ passes through $U^-_{\varepsilon,\tau,c} \cap M_z$ when c increases through $c(\varepsilon; \tau)$.

REMARK 3.2. When $h(z; \varepsilon; \tau) = 0$, we may say that $S^+_{\varepsilon,\tau,c} \cap M_z$ touches $U^-_{\varepsilon,\tau,c} \cap M_z$ tangentially to the direction of the vector $\mathbf{R}_0(z; \varepsilon; \tau)$ when c increases through $c(\varepsilon; \tau)$.

The next lemma is very important for our purpose.

LEMMA 3.2 (Maginu [24]). *The sign of* $h(z; \varepsilon; \tau)$ *does not depend on* $z \in \mathbf{R}$. *Furthermore, the sign of* $h(z; \varepsilon; \tau)$ *does not depend on the choice of the locally transversal section* M_z, *the points* $\mathbf{X}(z; \varepsilon; \tau; c)$ *and* $\mathbf{Y}(z; \varepsilon; \tau; c)$ *in the* *definition of the vector* $\mathbf{R}_0(z; \varepsilon; \tau)$, *and the positively oriented basis* $(\mathbf{T}_1, \mathbf{T}_2)$ *and* $(\mathbf{T}_3, \mathbf{T}_4)$ *of the manifolds* $S^+_{\varepsilon,\tau}$ *and* $U^-_{\varepsilon,\tau}$ *at the point* $\mathscr{V}(z; \varepsilon; \tau)$.

This lemma guarantees that the sign of $h(z; \varepsilon; \tau)$ can be regarded as a *geometrical characteristic* of the heteroclinic orbit $\gamma(\varepsilon; \tau)$. Namely, we denote it as

$$\mathrm{Ind}[\gamma(\varepsilon; \tau)] \equiv \mathrm{sign}\{h(z; \varepsilon; \tau)\},$$

and we call it the *index of the heteroclinic orbit* $\gamma(\varepsilon; \tau)$, where sign $\{x\}$ is the function defined by

$$\mathrm{sign}\{x\} \equiv \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

REMARK 3.3. $\mathrm{Ind}[\gamma(\varepsilon; \tau)]$ represents the *direction* in which $S^+_{\varepsilon,\tau,c}$ passes through $U^-_{\varepsilon,\tau,c}$ through the orbit $\gamma(\varepsilon; \tau)$ when the parameter c increases through $c(\varepsilon; \tau)$.

3.4. Relation between index and matching condition

As we have already seen in the preceding subsections, in order to calculate the index of the heteroclinic orbit $\gamma(\varepsilon; \tau)$, we must construct the stable manifold $S^+_{\varepsilon,\tau,c}$ and the unstable one $U^-_{\varepsilon,\tau,c}$ in the concrete. Though this is very difficult in general, we shall show that it is *possible* for our method applying singular perturbation techniques.

Recall that, in Section 2, we solved the problem $(2.5)_\pm$ in each subinterval $\mathbf{R}_\pm$ in stead of the problem (2.1), (2.2) and (2.3) in the whole interval $\mathbf{R}$. This procedure is, in fact, identical with a procedure to construct parts of the stable manifold $S^+_{\varepsilon,\tau,c}$ and the unstable one $U^-_{\varepsilon,\tau,c}$ containing the heteroclinic orbit $\gamma(\varepsilon; \tau)$ when $c = c(\varepsilon; \tau)$. We shall state it more precisely. Fix a parameter c in some neighborhood of $c(\varepsilon; \tau)$. We find that

$$\hat{S}^+_{\varepsilon,\tau,c} \equiv \{\mathbf{V}^s(z; \varepsilon; \tau; c, \beta) \equiv {}^t\!\left(u^+, \varepsilon\frac{du^+}{dz}, v^+, \frac{dv^+}{dz}\right)(z; \varepsilon; \tau; c, \beta) \in \mathbf{R}^4 | \\ z \in \mathbf{R}_+, v_- \leq \beta \leq v_+\}$$

is a part of the two-dimensional stable manifold $S^+_{\varepsilon,\tau,c}$(which is parametrized by z and β) containing the heteroclinic orbit $\gamma(\varepsilon; \tau)$ when $c = c(\varepsilon; \tau)$. Similarly,

$$\hat{U}^-_{\varepsilon,\tau,c} \equiv \{\mathbf{V}^u(z; \varepsilon; \tau; c, \beta) \equiv {}^t\!\left(u^-, \varepsilon\frac{du^-}{dz}, v^-, \frac{dv^-}{dz}\right)(z; \varepsilon; \tau; c, \beta) \in \mathbf{R}^4 | \\ z \in \mathbf{R}_-, v_- \leq \beta \leq v_+\}$$

is a part of the two-dimensional unstable manifold $U^-_{\varepsilon,\tau,c}$(which is parametrized by z and β) containing the orbit $\gamma(\varepsilon;\tau)$ when $c = c(\varepsilon;\tau)$. Here $(u^\pm, v^\pm)(z;\varepsilon;\tau;c,\beta)$ are the solutions of $(2.5)_\pm$ (see $(2.19)_\pm$ and Lemma 2.6).

Let us choose a locally transversal section M_0 as follows:

$$M_0 \equiv \{{}^t(\alpha, p, v, q)\in \mathbf{R}^4|\ |p - p(0;\varepsilon;\tau)| + |v - v(0;\varepsilon;\tau)| + |q - q(0;\varepsilon;\tau)| \le \omega\},$$

where α is a fixed constant satisfying $u(0;\varepsilon;\tau) = \alpha$(see (2.3)) and ω is a small positive constant. Clearly, this section M_0 is transversal to the flow of (3.7).

Next, we construct the ordered pairs $(\mathbf{T}_1, \mathbf{T}_2)$ and $(\mathbf{T}_3, \mathbf{T}_4)$ along the heteroclinic orbit $\gamma(\varepsilon;\tau)$, which are basis of $T_z\hat{S}^+_{\varepsilon,\tau,c(\varepsilon;\tau)}(z\in\mathbf{R}_+)$ and $T_z\hat{U}^-_{\varepsilon,\tau,c(\varepsilon;\tau)}(z\in\mathbf{R}_-)$ respectively. For this purpose, we must prepare the followings: First we consider approximate manifolds of the stable manifold $\hat{S}^+_{\varepsilon,\tau,c}$ and the unstable one $\hat{U}^-_{\varepsilon,\tau,c}$. The next lemma is very important.

LEMMA 3.3. *Let* $z^\pm(\varepsilon) = \mp\varepsilon\log\varepsilon$. *Then the following assertions hold for any* $c\in\mathbf{R}$ *and* $\beta\in[v_-, v_+]$:

(i)
$$\lim_{\varepsilon\downarrow 0}\mathbf{V}^u(z^-(\varepsilon);\varepsilon;\tau;c,\beta) = {}^t(h_-(\beta), 0, \beta, (V_0^-)_z(0;c,\beta)),$$

$$\lim_{\varepsilon\downarrow 0}\mathbf{V}^s(z^+(\varepsilon);\varepsilon;\tau;c,\beta) = {}^t(h_+(\beta), 0, \beta, (V_0^+)_z(0;c,\beta)).$$

(ii)
$$\lim_{\varepsilon\downarrow 0}\mathbf{V}^u(z;\varepsilon;\tau;c,\beta)$$
$$= \begin{cases} {}^t(U_0^-(z;c,\beta), 0, V_0^-(z;c,\beta), (V_0^-)_z(z;c,\beta)) \equiv \mathbf{V}^u_s(z;c,\beta) \\ \qquad\qquad \text{on } z\in\mathbf{J}^u_s(\varepsilon) \equiv (-\infty, z^-(\varepsilon)] \\ {}^t\left(h_-(\beta) + W_0^-\left(\frac{z}{\varepsilon};\tau;c,\beta\right), (W_0^-)_\xi\left(\frac{z}{\varepsilon};\tau;c,\beta\right), \beta, (V_0^-)_z(0;c,\beta)\right) \\ \qquad\qquad \equiv \mathbf{V}^u_f(z;\varepsilon;\tau;c,\beta) \text{ on } z\in\mathbf{J}^u_f(\varepsilon) \equiv [z^-(\varepsilon), 0] \end{cases}$$

and

$$\lim_{\varepsilon\downarrow 0}\mathbf{V}^s(z;\varepsilon;\tau;c,\beta)$$
$$= \begin{cases} {}^t(U_0^+(z;c,\beta), 0, V_0^+(z;c,\beta), (V_0^+)_z(z;c,\beta)) \equiv \mathbf{V}^s_s(z;c,\beta) \\ \qquad\qquad \text{on } z\in\mathbf{J}^s_s(\varepsilon) \equiv [z^+(\varepsilon), +\infty) \\ {}^t\left(h_+(\beta) + W_0^+\left(\frac{z}{\varepsilon};\tau;c,\beta\right), (W_0^+)_\xi\left(\frac{z}{\varepsilon};\tau;c,\beta\right), \beta, (V_0^+)_z(0;c,\beta)\right) \\ \qquad\qquad \equiv \mathbf{V}^s_f(z;\varepsilon;\tau;c,\beta) \text{ on } z\in\mathbf{J}^s_f(\varepsilon) \equiv [0, z^+(\varepsilon)] \end{cases}$$

uniformly in each subinterval.

By virtue of Theorem 2.2, the proof is easily shown, so we omit it.

Let us define a *singular stable manifold* $\tilde{S}_{\varepsilon,\tau,c}$ and a *singular unstable manifold* $\tilde{U}_{\varepsilon,\tau,c}$ by

$$\begin{aligned}\tilde{S}_{\varepsilon,\tau,c} \equiv \{&\mathbf{V}_s^s(z;\, c,\, \beta) \in \mathbf{R}^4 \,|\, z \in \mathbf{J}_s^s(\varepsilon),\ v_- \le \beta \le v_+\}\\ &\cup \{\mathbf{V}_f^s(z;\, \varepsilon;\, \tau;\, c,\, \beta) \in \mathbf{R}^4 \,|\, z \in \mathbf{J}_f^s(\varepsilon),\ v_- \le \beta \le v_+\}\end{aligned}$$

and

$$\begin{aligned}\tilde{U}_{\varepsilon,\tau,c} \equiv \{&\mathbf{V}_s^u(z;\, c,\, \beta) \in \mathbf{R}^4 \,|\, z \in \mathbf{J}_s^u(\varepsilon),\ v_- \le \beta \le v_+\}\\ &\cup \{\mathbf{V}_f^u(z;\, \varepsilon;\, \tau;\, c,\, \beta) \in \mathbf{R}^4 \,|\, z \in \mathbf{J}_f^u(\varepsilon),\ v_- \le \beta \le v_+\},\end{aligned}$$

respectively. On the other hand, noting that

$$\mathscr{V}(z;\, \varepsilon;\, \tau) = \begin{cases} \mathbf{V}^u(z;\, \varepsilon;\, \tau;\, c(\varepsilon;\, \tau),\, \beta(\varepsilon;\, \tau)), & z \in \mathbf{R}_- \\ \mathbf{V}^s(z;\, \varepsilon;\, \tau;\, c(\varepsilon;\, \tau),\, \beta(\varepsilon;\, \tau)), & z \in \mathbf{R}_+, \end{cases}$$

we find by Lemma 3.3 that the heteroclinic orbit $\gamma(\varepsilon;\, \tau)$ corresponding to $\mathscr{V}(z;\, \varepsilon;\, \tau)$ can be approximated by the following *singular heteroclinic orbit* $\Gamma(\varepsilon;\, \tau) \equiv \bigcup_{i=1}^4 \Gamma_i(\varepsilon;\, \tau)$:

$$\begin{cases} \Gamma_1(\varepsilon;\, \tau)\colon\ \mathbf{V} = \mathbf{V}_s^s(z;\, c^*(\tau),\, \beta^*(\tau)) & \text{for } z \in \mathbf{J}_s^s(\varepsilon) \\ \Gamma_2(\varepsilon;\, \tau)\colon\ \mathbf{V} = \mathbf{V}_f^s(z;\, \varepsilon;\, \tau;\, c^*(\tau),\, \beta^*(\tau)) & \text{for } z \in \mathbf{J}_f^s(\varepsilon) \\ \Gamma_3(\varepsilon;\, \tau)\colon\ \mathbf{V} = \mathbf{V}_s^u(z;\, c^*(\tau),\, \beta^*(\tau)) & \text{for } z \in \mathbf{J}_s^u(\varepsilon) \\ \Gamma_4(\varepsilon;\, \tau)\colon\ \mathbf{V} = \mathbf{V}_f^u(z;\, \varepsilon;\, \tau;\, c^*(\tau),\, \beta^*(\tau)) & \text{for } z \in \mathbf{J}_f^u(\varepsilon) \end{cases}$$

(see Fig. 5). Lemma 3.3 warrants that the basis vectors of (the tangent space of the singuler stable manifold) $T_z\tilde{S}_{\varepsilon,\tau,c^*(\tau)}$ and (the tnagent space of the singular unstable manifold) $T_z\tilde{U}_{\varepsilon,\tau,c^*(\tau)}$ along the singular heteroclinic orbit $\Gamma(\varepsilon;\, \tau)$ become nice approximations to $(\mathbf{T}_1, \mathbf{T}_2)$ and $(\mathbf{T}_3, \mathbf{T}_4)$, respectively. Then we

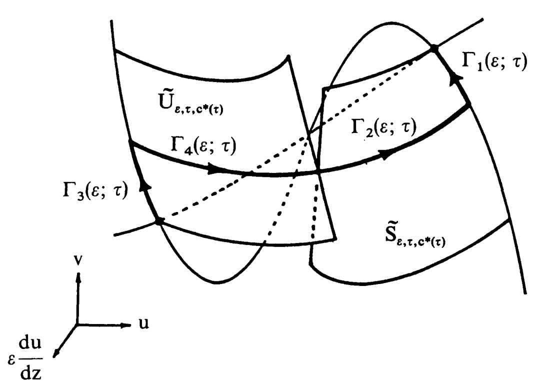

Fig. 5: A schematic picture of the singular manifolds and the singular orbit.

find that on $\Gamma_2(\varepsilon;\tau)\setminus\{z^+(\varepsilon)\}$ (that is, $z\in\mathbf{J}_f^s(\varepsilon)\setminus\{z^+(\varepsilon)\}$) $\mathbf{T}_1(z;\varepsilon;\tau)$ is represented as follows by the definition of $\mathbf{T}_1$ (see (3.10)):

$$
\begin{aligned}
\text{(3.13)}\qquad \mathbf{T}_1(z;\varepsilon;\tau) &= \frac{d}{dz}\mathbf{V}_f^s(z;\varepsilon;\tau;c^*(\tau),\beta^*(\tau)) + O(\varepsilon)\\
&= \frac{1}{\varepsilon}\begin{bmatrix} (W_0^+)_\xi\left(\dfrac{z}{\varepsilon};\tau;c^*(\tau),\beta^*(\tau)\right) + O(\varepsilon)\\ (W_0^+)_{\xi\xi}\left(\dfrac{z}{\varepsilon};\tau;c^*(\tau),\beta^*(\tau)\right) + O(\varepsilon)\\ O(\varepsilon)\\ O(\varepsilon)\end{bmatrix}.
\end{aligned}
$$

At $z=z^+(\varepsilon)$, since $T_z\tilde{S}_{\varepsilon,\tau,c^*(\tau)}$ is spanned by the both tangent spaces of the manifolds $\{\mathbf{V}_f^s(z;\varepsilon;\tau;c^*(\tau),\beta^*(\tau))\in\mathbf{R}^4\,|\,z\in\mathbf{J}_f^s(\varepsilon),\ v_-\le\beta\le v_+\}$, which is the *fast* part, and $\{\mathbf{V}_s^s(z;c^*(\tau),\beta^*(\tau))\in\mathbf{R}^4\,|\,z\in\mathbf{J}_s^s(\varepsilon),\ v_-\le\beta\le v_+\}$, which is the *slow* part, $\mathbf{T}_1(z^+(\varepsilon);\varepsilon;\tau)$ is represented as follows:

$$
\begin{aligned}
\mathbf{T}_1(z^+(\varepsilon);\varepsilon;\tau) &= \frac{d}{dz}\mathbf{V}_f^s(z^+(\varepsilon);\varepsilon;\tau;c^*(\tau),\beta^*(\tau))\\
&\quad + \frac{d}{dz}\mathbf{V}_s^s(z^+(\varepsilon);c^*(\tau),\beta^*(\tau)) + O(\varepsilon)\qquad \text{as } \varepsilon\downarrow 0.
\end{aligned}
$$

Similarly to the case at which $z=z^+(\varepsilon)$, we can construct $\mathbf{T}_1(z;\varepsilon;\tau)$ on $\Gamma_1(\varepsilon;\tau)$ (that is, $z\in\mathbf{J}_s^s(\varepsilon)$). Namely, for any fixed $z\in\mathbf{J}_s^s(\varepsilon)$ when we define $\hat{\beta}(z;\tau)$ by $V_0^+(z;c^*(\tau),\beta^*(\tau))$, $\mathbf{T}_1(z;\varepsilon;\tau)$ is represented as follows:

$$
\begin{aligned}
\text{(3.14)}\qquad \mathbf{T}_1(z;\varepsilon;\tau) &= \frac{d}{dz}\mathbf{V}_f^s(z;\varepsilon;\tau;c^*(\tau),\hat{\beta}(z;\tau))\\
&\quad + \frac{d}{dz}\mathbf{V}_s^s(z;c^*(\tau),\beta^*(\tau)) + O(\varepsilon)\qquad \text{as } \varepsilon\downarrow 0.
\end{aligned}
$$

Note that $\hat{\beta}(z;\tau)\to v_+$ as $z\to+\infty$. Using the same techniques as those in De Villiers [1: Lemma 4.1], we obtain

$$
\text{(3.15)}\quad \mathbf{T}_1(z;\varepsilon;\tau)\simeq\{-d_1\tilde{v}_1^+ + O(1)\}e^{\tilde{v}_1^+ z}\cdot\tilde{\mathbf{a}}_1^+ + \left\{d_2\cdot\frac{f_v^+}{f_u^+}\cdot\tilde{v}_2^+ + O(\varepsilon)\right\}e^{v_2^+ z}\cdot\tilde{\mathbf{a}}_2^+
$$
$$
\text{as } z\longrightarrow+\infty,
$$

where $\tilde{v}_i^+ = v_i^+(\varepsilon;\tau;c^*(\tau))$, $\tilde{\mathbf{a}}_i^+ = \mathbf{a}_i^+(\varepsilon;\tau;c^*(\tau))$ and d_i are some positive constants $(i=1,2)$. By the same ways as above, we have

$$(3.16)\qquad \mathbf{T}_3(z;\varepsilon;\tau)=\frac{1}{\varepsilon}\begin{bmatrix}(W_0^-)_\xi(z/\varepsilon;\tau;c^*(\tau),\beta^*(\tau))+O(\varepsilon)\\ (W_0^-)_{\xi\xi}(z/\varepsilon;\tau;c^*(\tau),\beta^*(\tau))+O(\varepsilon)\\ O(\varepsilon)\\ O(\varepsilon)\end{bmatrix}$$

for $z\in\mathbf{J}_f^u(\varepsilon)\setminus\{z^-(\varepsilon)\}$ and

$$(3.17)\quad \mathbf{T}_3(z;\varepsilon;\tau)\simeq\left\{-d_3\cdot\frac{f_v^-}{f_u^-}\cdot\tilde{v}_3^-+O(\varepsilon)\right\}e^{\tilde{v}_3^- z}\cdot\tilde{\mathbf{a}}_3^-+\{d_4\tilde{v}_4^-+O(1)\}e^{\tilde{v}_4^- z}\cdot\tilde{\mathbf{a}}_4^-\quad\text{as } z\longrightarrow-\infty,$$

where $\tilde{v}_i^-=v_i^-(\varepsilon;\tau;c^*(\tau))$, $\tilde{\mathbf{a}}_i^-=\mathbf{a}_i^-(\varepsilon;\tau;c^*(\tau))$ and d_i are some positive constants $(i=3,4)$.

Put

$$\mathbf{T}_2(z;\varepsilon;\tau)=-\begin{bmatrix}\dfrac{\partial}{\partial\beta}u^+(z;\varepsilon;\tau;c(\varepsilon;\tau),\beta(\varepsilon;\tau))\\ \varepsilon\dfrac{\partial}{\partial\beta}\dfrac{du^+}{dz}(z;\varepsilon;\tau;c(\varepsilon;\tau),\beta(\varepsilon;\tau))\\ \dfrac{\partial}{\partial\beta}v^+(z;\varepsilon;\tau;c(\varepsilon;\tau),\beta(\varepsilon;\tau))\\ \dfrac{\partial}{\partial\beta}\dfrac{dv^+}{dz}(z;\varepsilon;\tau;c(\varepsilon;\tau),\beta(\varepsilon;\tau))\end{bmatrix}(z\in\mathbf{R}_+)$$

and

$$\mathbf{T}_4(z;\varepsilon;\tau)=\begin{bmatrix}\dfrac{\partial}{\partial\beta}u^-(z;\varepsilon;\tau;c(\varepsilon;\tau),\beta(\varepsilon;\tau))\\ \varepsilon\dfrac{\partial}{\partial\beta}\dfrac{du^-}{dz}(z;\varepsilon;\tau;c(\varepsilon;\tau),\beta(\varepsilon;\tau))\\ \dfrac{\partial}{\partial\beta}v^-(z;\varepsilon;\tau;c(\varepsilon;\tau),\beta(\varepsilon;\tau))\\ \dfrac{\partial}{\partial\beta}\dfrac{dv^-}{dz}(z;\varepsilon;\tau;c(\varepsilon;\tau),\beta(\varepsilon;\tau))\end{bmatrix}(z\in\mathbf{R}_-).$$

Similarly to the cases of $\mathbf{T}_1$ and $\mathbf{T}_3$, we obtain the following results:

$$(3.18)\qquad \mathbf{T}_2(z;\,\varepsilon;\,\tau)=\begin{bmatrix} 0 \\ -\dfrac{\partial}{\partial\beta}(W_0^+)_\xi\left(\dfrac{z}{\varepsilon};\,\tau;\,c^*(\tau),\,\beta^*(\tau)\right)+O(\varepsilon) \\ -1+O(\varepsilon) \\ -\dfrac{\partial}{\partial\beta}(V_0^+)_z(z;\,c^*(\tau),\,\beta^*(\tau))+O(\varepsilon) \end{bmatrix}$$

for $z\in\mathbf{J}^s_f(\varepsilon)\setminus\{z^+(\varepsilon)\}$,

$$(3.19)\qquad \mathbf{T}_2(z;\,\varepsilon;\,\tau)\simeq\left\{\frac{f_v^+}{f_u^+}\cdot\frac{\varepsilon d_1\tilde{v}_1^+}{(W_0^+)_\xi(0)}+O(\varepsilon)\right\}e^{\tilde{v}_1^+z}\cdot\tilde{\mathbf{a}}_1^+ + \left\{-d_2\cdot\frac{f_v^+}{f_u^+}\cdot\frac{\tilde{v}_2^+}{(v_0^+)_z(0)}+O(\varepsilon)\right\}e^{\tilde{v}_2^+}\cdot\tilde{\mathbf{a}}_2^+ \qquad \text{as } z\longrightarrow+\infty,$$

$$(3.20)\qquad \mathbf{T}_4(z;\,\varepsilon;\,\tau)=\begin{bmatrix} 0 \\ \dfrac{\partial}{\partial\beta}(W_0^-)_\xi\left(\dfrac{z}{\varepsilon};\,\tau;\,c^*(\tau),\,\beta^*(\tau)\right)+O(\varepsilon) \\ 1+O(\varepsilon) \\ \dfrac{\partial}{\partial\beta}(V_0^-)_z(z;\,c^*(\tau),\,\beta^*(\tau))+O(\varepsilon) \end{bmatrix}$$

for $z\in\mathbf{J}^u_f(\varepsilon)\setminus\{z^-(\varepsilon)\}$, and

$$(3.21)\qquad \mathbf{T}_4(z;\,\varepsilon;\,\tau)\simeq\left\{-d_3\cdot\frac{f_v^-}{f_u^-}\cdot\frac{\tilde{v}_3^-}{(v_0^-)_z(0)}+O(\varepsilon)\right\}e^{\tilde{v}_3^-z}\cdot\tilde{\mathbf{a}}_3^- + \left\{-\frac{f_v^-}{f_u^-}\cdot\frac{\varepsilon d_4\tilde{v}_4^-}{(W_0^-)_\xi(0)}+O(\varepsilon)\right\}e^{\tilde{v}_4^-z}\cdot\tilde{\mathbf{a}}_4^- \qquad \text{as } z\longrightarrow-\infty.$$

From the above formula, it obviously follows that

$$\mathbf{T}_2(0;\,\varepsilon;\,\tau)\in T_0(S^+_{\varepsilon,\tau}\cap M_0)\quad\text{and}\quad \mathbf{T}_4(0;\,\varepsilon;\,\tau)\in T_0(U^-_{\varepsilon,\tau}\cap M_0).$$

Furthermore, by the relations

$$(3.22)\qquad \frac{\partial}{\partial\beta}\Phi_0(\tau;\,c^*(\tau),\,\beta^*(\tau)) = \frac{\partial}{\partial\beta}(W_0^-)_\xi(0;\,\tau;\,c^*(\tau),\,\beta^*(\tau))-\frac{\partial}{\partial\beta}(W_0^+)_\xi(0;\,\tau;\,c^*(\tau),\,\beta^*(\tau))>0$$

and

(3.23) $$\frac{\partial}{\partial\beta}\Psi_0(c^*(\tau), \beta^*(\tau))$$
$$= \frac{\partial}{\partial\beta}(V_0^-)_z(0; c^*(\tau), \beta^*(\tau)) - \frac{\partial}{\partial\beta}(V_0^+)_z(0; c^*(\tau), \beta^*(\tau)) > 0$$

(see (2.9) and (2.14)), we find that $\mathbf{T}_2(0; \varepsilon; \tau)$ and $\mathbf{T}_4(0; \varepsilon; \tau)$ are linearly independent on $T_0 M_0$. These results imply that Assumption (B) in Subsection 3.3 holds for $z = 0$.

Let us show that the ordered pairs $(\mathbf{T}_1, \mathbf{T}_2)$ and $(\mathbf{T}_3, \mathbf{T}_4)$ which we now constructed become positively oriented basis of $T_0\hat{S}^+_{\varepsilon,\tau,c(\varepsilon;\tau)}$ and $T_0\hat{U}^-_{\varepsilon,\tau,c(\varepsilon;\tau)}$ when $z = 0$, respectively. Using the results (3.15), (3.17), (3.19) and (3.21), we have

$$\begin{bmatrix} \mathbf{T}_1(z; \varepsilon; \tau) \\ \mathbf{T}_2(z; \varepsilon; \tau) \end{bmatrix} \simeq B^+ \begin{bmatrix} e^{\tilde{\nu}_1^+ z} \cdot \tilde{\mathbf{a}}_1^+ \\ e^{\tilde{\nu}_2^+ z} \cdot \tilde{\mathbf{a}}_2^+ \end{bmatrix} \quad \text{as } z \longrightarrow +\infty$$

and

$$\begin{bmatrix} \mathbf{T}_3(z; \varepsilon; \tau) \\ \mathbf{T}_4(z; \varepsilon; \tau) \end{bmatrix} \simeq B^- \begin{bmatrix} e^{\tilde{\nu}_3^- z} \cdot \tilde{\mathbf{a}}_3^- \\ e^{\tilde{\nu}_4^- z} \cdot \tilde{\mathbf{a}}_4^- \end{bmatrix} \quad \text{as } z \longrightarrow -\infty$$

for sufficiently small $\varepsilon > 0$, where

$$B^+ = \begin{bmatrix} -d_1\tilde{\nu}_1^+ + O(1) & d_2 \cdot \dfrac{f_v^+}{f_u^+} \cdot \tilde{\nu}_2^+ + O(\varepsilon) \\ \dfrac{f_v^+}{f_u^+} \cdot \dfrac{\varepsilon d_1 \tilde{\nu}_1^+}{(W_0^+)_\xi(0)} + O(\varepsilon) & -\dfrac{f_v^+}{f_u^+} \cdot \dfrac{d_2 \tilde{\nu}_2^+}{(V_0^+)_z(0)} + O(\varepsilon) \end{bmatrix}$$

and

$$B^- = \begin{bmatrix} -d_3 \cdot \dfrac{f_v^-}{f_u^-} \cdot \tilde{\nu}_3^- + O(\varepsilon) & d_4\tilde{\nu}_4^- + O(1) \\ -\dfrac{f_v^-}{f_u^-} \cdot \dfrac{d_3 \tilde{\nu}_3^-}{(V_0^-)_z(0)} + O(\varepsilon) & -\dfrac{f_v^-}{f_u^-} \cdot \dfrac{\varepsilon d_4 \tilde{\nu}_4^-}{(W_0^-)_\xi(0)} + O(\varepsilon) \end{bmatrix}.$$

Comparing the above results with (3.11), we find that

$$B_1(z; \varepsilon; \tau) \longrightarrow B^+ \quad \text{as } z \longrightarrow +\infty \quad \text{and}$$
$$B_2(z; \varepsilon; \tau) \longrightarrow B^- \quad \text{as } z \longrightarrow -\infty,$$

from which we obtain

$$\det B_1(+\infty; \varepsilon; \tau) = \det B^+ = d_1 d_2 \cdot \frac{f_v^+}{f_u^+} \cdot \frac{\tilde{\nu}_1^+ \tilde{\nu}_2^+}{(V_0^+)_z(0)} + O(1) > 0$$

and

$$\det B_2(-\infty;\varepsilon;\tau) = \det B^- = d_3 d_4 \cdot \frac{f_v^-}{f_u^-} \cdot \frac{\tilde{v}_3^- \tilde{v}_4^-}{(V_0^-)_z(0)} + O(1) > 0$$

for sufficiently small $\varepsilon > 0$. Noting that the sign of $\det B_1(z;\varepsilon;\tau)$ and $\det B_2(z;\varepsilon;\tau)$ are definite for all $z \in \mathbf{R}$, we find that $\det B_1(0;\varepsilon;\tau) > 0$ and $\det B_2(0;\varepsilon;\tau) > 0$ hold, which implies that $(\mathbf{T}_1, \mathbf{T}_2)$ and $(\mathbf{T}_3, \mathbf{T}_4)$ are the positively oriented basis of $T_0\hat{S}^+_{\varepsilon,\tau,c(\varepsilon;\tau)}$ and $T_0\hat{U}^-_{\varepsilon,\tau,c(\varepsilon;\tau)}$ when $z = 0$, respectively.

Finally, we shall calculate the vector $\mathbf{R}_0(0;\varepsilon;\tau)$. Since the index does not depend on the choice of $\mathbf{X}(0;\varepsilon;\tau;c) \in \hat{S}^+_{\varepsilon,\tau,c} \cap M_0$ and $\mathbf{Y}(0;\varepsilon;\tau;c) \in \hat{U}^-_{\varepsilon,\tau,c} \cap M_0$, we fix points $\mathbf{X}(0;\varepsilon;\tau;c)$ and $\mathbf{Y}(0;\varepsilon;\tau;c)$ as follows: First we determine a function $\beta(\varepsilon;\tau;c)$ satisfying

$$\begin{aligned}(3.24)\qquad & \Psi(\varepsilon;\tau;c,\beta(\varepsilon;\tau;c)) \\ & = \frac{d}{dz}v^-(0;\varepsilon;\tau;c,\beta(\varepsilon;\tau;c)) - \frac{d}{dz}v^+(0;\varepsilon;\tau;c,\beta(\varepsilon;\tau;c)) = 0,\end{aligned}$$

which is possible by virtue of Lemmas 2.1 and 2.6. Next, using the function $\beta(\varepsilon;\tau;c)$, we define $\mathbf{X}$ and $\mathbf{Y}$ as

$$\mathbf{X}(0;\varepsilon;\tau;c) \equiv \begin{bmatrix} u^+(0;\varepsilon;\tau;c,\beta(\varepsilon;\tau;c)) \\ \varepsilon\dfrac{du^+}{dz}(0;\varepsilon;\tau;c,\beta(\varepsilon;\tau;c)) \\ v^+(0;\varepsilon;\tau;c,\beta(\varepsilon;\tau;c)) \\ \dfrac{dv^+}{dz}(0;\varepsilon;\tau;c,\beta(\varepsilon;\tau;c)) \end{bmatrix} = \begin{bmatrix} \alpha \\ \varepsilon\dfrac{du^+}{dz}(0;\varepsilon;\tau;c,\beta(\varepsilon;\tau;c)) \\ \beta(\varepsilon;\tau;c) \\ \dfrac{dv^+}{dz}(0;\varepsilon;\tau;c,\beta(\varepsilon;\tau;c)) \end{bmatrix}$$

and

$$\mathbf{Y}(0;\varepsilon;\tau;c) \equiv \begin{bmatrix} u^-(0;\varepsilon;\tau;c,\beta(\varepsilon;\tau;c)) \\ \varepsilon\dfrac{du^-}{dz}(0;\varepsilon;\tau;c,\beta(\varepsilon;\tau;c)) \\ v^-(0;\varepsilon;\tau;c,\beta(\varepsilon;\tau;c)) \\ \dfrac{dv^-}{dz}(0;\varepsilon;\tau;c,\beta(\varepsilon;\tau;c)) \end{bmatrix} = \begin{bmatrix} \alpha \\ \varepsilon\dfrac{du^-}{dz}(0;\varepsilon;\tau;c,\beta(\varepsilon;\tau;c)) \\ \beta(\varepsilon;\tau;c) \\ \dfrac{dv^-}{dz}(0;\varepsilon;\tau;c,\beta(\varepsilon;\tau;c)) \end{bmatrix},$$

respectively. From the definitions of $\mathbf{X}$ and $\mathbf{Y}$, it follows that

$$\mathbf{X}(0;\varepsilon;\tau;c) \in \hat{S}^+_{\varepsilon,\tau,c} \cap M_0,\quad \mathbf{Y}(0;\varepsilon;\tau;c) \in \hat{U}^-_{\varepsilon,\tau,c} \cap M_0,$$

and

$$\mathbf{X}(0;\varepsilon;\tau;c(\varepsilon;\tau)) = \mathscr{V}(0;\varepsilon;\tau) = \mathbf{Y}(0;\varepsilon;\tau;c(\varepsilon;\tau)).$$

Therefore, we may calculate $\mathbf{R}_0(0;\varepsilon;\tau)$ as follows:

$$(3.25)\qquad \mathbf{R}_0(0;\varepsilon;\tau) = \lim_{c\to c(\varepsilon;\tau)} \frac{\mathbf{X}(0;\varepsilon;\tau;c) - \mathbf{Y}(0;\varepsilon;\tau;c)}{c - c(\varepsilon;\tau)}$$
$$= \left[\frac{d}{dc}\mathbf{X}(0;\varepsilon;\tau;c) - \frac{d}{dc}\mathbf{Y}(0;\varepsilon;\tau;c)\right]_{c\,=\,c(\varepsilon;\tau)}$$
$$= \begin{bmatrix} 0 \\ -\dfrac{d}{dc}\Phi(\varepsilon;\tau;c,\beta(\varepsilon;\tau;c)) \\ 0 \\ 0 \end{bmatrix} \quad \text{(see (2.20))}.$$

Using the relations (3.16), (3.18), (3.20) and (3.25), we obtain

$$h(0;\varepsilon;\tau) = \det[\mathbf{R}_0(0;\varepsilon;\tau), \mathbf{T}_2(0;\varepsilon;\tau), \mathbf{T}_3(0;\varepsilon;\tau), \mathbf{T}_4(0;\varepsilon;\tau)]$$
$$= \frac{1}{\varepsilon}\{(W_0^-)_\xi(0;\tau;c^*(\tau),\beta^*(\tau)) + O(\varepsilon)\}\cdot\frac{d}{dc}\Phi(\varepsilon;\tau;c(\varepsilon;\tau),\beta(\varepsilon;\tau))$$
$$\times\left\{\frac{\partial}{\partial\beta}\Psi_0(c^*(\tau),\beta^*(\tau)) + O(\varepsilon)\right\}$$
$$= \frac{1}{\varepsilon}\{(W_0^-)_\xi(0;\tau;c^*(\tau),\beta^*(\tau)) + O(\varepsilon)\}\{\mathbf{J}(\tau;c^*(\tau);\beta^*(\tau)) + O(\varepsilon)\}.$$

Note that $(W_0^-)_\xi(0;\tau;c^*(\tau),\beta^*(\tau)) > 0$ (see Lemma 2.4). Then we obtain the desired results.

THEOREM 3.1. *Let $\gamma(\varepsilon;\tau)$ be any heteroclinic orbit of* (3.2). *Then, the index of $\gamma(\varepsilon;\tau)$ is identical to the sign of the Jacobian $\mathbf{J}(\tau;c^*(\tau);\beta^*(\tau))$ of the matching condition, namely*

$$(3.26)\qquad \mathrm{Ind}[\gamma(\varepsilon;\tau)] = \mathrm{sign}\{\mathbf{J}(\tau;c^*(\tau);\beta^*(\tau))\}.$$

4. Stability properties of traveling wave solutions

In Section 2, we have shown the existence of traveling wave solutions of (2.1), (2.2) and (2.3). That is, there exist three solutions when τ is sufficiently small and one when τ is sufficiently large. In Section 3, we calculated the index of the heteroclinic orbit which corresponds in a one-to-one manner to the traveling wave solution. That is, the index of the heteroclinic orbit is equal to the sign of the Jacobian of the matching condition. Hereafter, we fix $(u,v)(z;\varepsilon;\tau)$ as an arbitrary traveling wave solution of (2.1), (2.2) and (2.3). Recently, we have solved stability problem of $(u,v)(z;\varepsilon;\tau)$ by analyzing the *singular limit eigenvalue problem* in [28]. In this section, without solving the eigenvalue problem, we show that the index is essential to determine the stability of the traveling wave solution along the paper by Maginu [24].

4.1. Preliminaries for stability analysis

By the traveling coordinate system $(z, t) = (x + ct, t)$, (1.1) takes the form

$$\text{(4.1)} \qquad \begin{cases} \varepsilon\tau u_t = \varepsilon^2 u_{zz} - \varepsilon c\tau u_z + f(u, v) \\ v_t = v_{zz} - cv_z + g(u, v) \end{cases}, \ (z, t) \in \mathbf{R} \times \mathbf{R}_+.$$

Then, the traveling wave solution $(u, v)(z; \varepsilon; \tau)$ is a stationary solution of (4.1). A standard technique for determining stability is to use the linearized criterion. The linearized eigenvalue problem at $(u, v)(z; \varepsilon; \tau)(c = c(\varepsilon; \tau))$ is given by

$$\text{(4.2)} \qquad \mathscr{L}^{\varepsilon,\tau}\begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} = \lambda \begin{bmatrix} \varepsilon\tau & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix},$$

where

$$\mathscr{L}^{\varepsilon,\tau} \equiv \begin{bmatrix} \varepsilon^2 \dfrac{d^2}{dz^2} - \varepsilon c(\varepsilon; \tau)\tau \dfrac{d}{dz} + f_u^\varepsilon & f_v^\varepsilon \\ g_u^\varepsilon & \dfrac{d^2}{dz^2} - c(\varepsilon; \tau)\dfrac{d}{dz} + g_v^\varepsilon \end{bmatrix},$$

and $(\bar{u}, \bar{v})(z; \varepsilon; \tau; \lambda) \in BC(\mathbf{R}) \times BC(\mathbf{R})$. Here $f_u^\varepsilon, f_v^\varepsilon$, g_u^ε and g_v^ε denote the partial derivatives of f and g evaluated at $(u, v)(z; \varepsilon; \tau)$. The operator $\mathscr{L}^{\varepsilon,\tau}$ with the usual domain becomes a sectorial operator for $\varepsilon > 0$ and the spectral analysis of (4.2) derives the nonlinear stability or instability (for instance, see Henry [12]). Therefore our problem consists of the following two parts:
(i) Distribution of the *essential spectrum.*
(ii) Distribution of isolated *eigenvalues.*
For the problem (i), noting that $E_\pm$ are both stable constant solutions of (4.1)(see assumptions (A-1), (A-3) and (A-4)), we can conclude the following proposition.

PROPOSITION 4.1 (Nishiura et al [28]). *For any $\tau > 0$, there exists a positive constant $\delta(\tau)$ such that the essential spectrum of* (4.2) *satisfies*

$$\mathrm{Re}\{\textit{essential spectrum of } (4.2)\} \le -\delta(\tau)$$

for sufficiently small $\varepsilon < 0$.

Next, we consider the distribution of eigenvalues. The complex number λ is called an eigenvalue of (4.2) if this equation has a nontrivial solution $(\bar{u}, \bar{v})(z; \varepsilon; \tau; \lambda)$ belonging to $BC(\mathbf{R}) \times BC(\mathbf{R})$. Since $(u, v)(z; \varepsilon; \tau)$ is a solution of (2.1), (2.2) and (2.3), $\left(\frac{du}{dz}, \frac{dv}{dz}\right)(z;\, \varepsilon;\, \tau)$ satisfies the equation (4.2) when $\lambda = 0$. This implies that $\lambda = 0$ is an eigenvalue of (4.2), which corresponds to translation invariance of the traveling wave solution.

The eigenvalue problem (4.2) can be written equivalently as

$$\text{(4.3)} \qquad \frac{d}{dz}\bar{\mathbf{V}} = \{A(z;\, \varepsilon;\, \tau) + \lambda C(\tau)\}\,\bar{\mathbf{V}},$$

where $\bar{\mathbf{V}} = \bar{\mathbf{V}}(z;\, \varepsilon;\, \tau;\, \lambda) = {}^t\!\left(\bar{u},\, \varepsilon\frac{d\bar{u}}{dz},\, \bar{v},\, \frac{d\bar{v}}{dz}\right)(z;\, \varepsilon;\, \tau;\, \lambda)$, $A(z;\, \varepsilon;\, \tau)$ is the same matrix defined by (3.8) and $C(\tau)$ is a constant matrix defined by

$$C(\tau) \equiv \begin{bmatrix} 0 & 0 & 0 & 0 \\ \tau & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Since $\mathbf{V}(z;\, \varepsilon;\, \tau) \to \mathbf{P}_{\pm}$ as $z \to \pm\infty$, $\bar{\mathbf{V}}(z;\, \varepsilon;\, \tau;\, \lambda)$ obeys the following linearized equations:

$$\text{(4.4)}_{\pm} \qquad \frac{d}{dz}\bar{\mathbf{V}}^{\pm} = \{A^{\pm}(\varepsilon;\, \tau;\, c(\varepsilon;\, \tau)) + \lambda C(\tau)\}\bar{\mathbf{V}}^{\pm}$$

when z tends to $\pm\infty$, where $A^{\pm}(\varepsilon;\, \tau;\, c)$ are the same matrices stated in Lemma 3.1. Let $\mu_i^{\pm}(\varepsilon;\, \tau;\, \lambda)\,(i = 1,2,3,4)$, $\mathrm{Re}\{\mu_1^{\pm}\} \le \mathrm{Re}\{\mu_2^{\pm}\} \le \mathrm{Re}\{\mu_3^{\pm}\} \le \mathrm{Re}\{\mu_4^{\pm}\}$, denote eigenvalues of the matrices $A^{\pm}(\varepsilon;\, \tau;\, c(\varepsilon;\, \tau)) + \lambda C(\tau)$. Without loss of generality, we may assume that $\mu_i^{\pm}(\varepsilon;\, \tau;\, \lambda)$ depend analytically on λ and satisfy $\mu_i^{\pm}(\varepsilon;\, \tau;\, 0) = \nu_i^{\pm}(\varepsilon;\, \tau;\, c(\varepsilon;\, \tau))$, where $\nu_i^{\pm}(\varepsilon;\, \tau;\, c)$ are the eigenvalues of $A^{\pm}(\varepsilon;\, \tau;\, c)\,(i = 1,2,3,4)$.

LEMMA 4.1. *There exists a positive constant d independent of ε and τ such that*

$$\mathrm{Re}\{\mu_1^{\pm}(\varepsilon;\, \tau;\, \lambda)\} \le \mathrm{Re}\{\mu_2^{\pm}(\varepsilon;\, \tau;\, \lambda)\} < 0 < \mathrm{Re}\{\mu_3^{\pm}(\varepsilon;\, \tau;\, \lambda)\} \le \mathrm{Re}\{\mu_4^{\pm}(\varepsilon;\, \tau;\, \lambda)\}$$

hold for all $\lambda \in \mathbf{C}_d \equiv \{\lambda \in \mathbf{C} \,|\, \mathrm{Re}\{\lambda\} \ge -d\}$.

The proof will be stated in the appendix.

By virtue of Lemma 4.1 and $(4.4)_{\pm}$, for any $\lambda \in \mathbf{C}_d$ (4.3) has just two linearly independent solutions $\bar{\mathbf{V}}_i(z;\, \varepsilon;\, \tau;\, \lambda)\,(i = 1,2)$, which satisfy

$$\bar{\mathbf{V}}_i(z;\, \varepsilon;\, \tau;\, \lambda) \longrightarrow \mathbf{0} \qquad \text{as } z \longrightarrow +\infty \ (i = 1,2)$$

and just two linearly independent solutions $\bar{\mathbf{V}}_i(z;\, \varepsilon;\, \tau;\, \lambda)\,(i = 3,4)$, which satisfy

$$\bar{\mathbf{V}}_i(z;\ \varepsilon;\ \tau;\ \lambda) \longrightarrow \mathbf{0} \qquad \text{as } z \longrightarrow -\infty \ (i = 3, 4).$$

On the other hand, the coefficient matrix of (4.3) depends analytically on λ and coinsides with the coefficient matrix of (3.8) when $\lambda = 0$. Then we may assume, without loss of generality, that $\bar{\mathbf{V}}_i(z;\ \varepsilon;\ \tau;\ \lambda)$ depend analytically on λ and satisfy

$$\bar{\mathbf{V}}_i(z;\ \varepsilon;\ \tau;\ 0) = \mathbf{V}_i(z;\ \varepsilon;\ \tau)\,(i = 1, 2, 3, 4),$$

where $\mathbf{V}_i(z;\ \varepsilon;\ \tau)$ are the solutions of (3.8) normalized by the condition (3.9).

A nontrivial solution $\bar{\mathbf{V}}(z;\ \varepsilon;\ \tau;\ \lambda)$ of (4.3) corresponding to an eigenvalue λ must satisfy

$$\bar{\mathbf{V}}(z;\ \varepsilon;\ \tau;\ \lambda) \longrightarrow \mathbf{0} \qquad \text{as} \longrightarrow \pm\infty.$$

Hence $\bar{\mathbf{V}}(z;\ \varepsilon;\ \tau;\ \lambda)$ can be written as

$$\bar{\mathbf{V}}(z;\ \varepsilon;\ \tau;\ \lambda) = \textstyle\sum_{i=1}^{2} \alpha_i \bar{\mathbf{V}}_i(z;\ \varepsilon;\ \tau;\ \lambda) = \sum_{i=3}^{4} \alpha_i \bar{\mathbf{V}}_i(z;\ \varepsilon;\ \tau;\ \lambda)$$

for some constants α_i $(i = 1, 2, 3, 4)$. In other words, $\lambda = \lambda_0$ is an eigenvalue of (4.3) if and only if the solutions $\bar{\mathbf{V}}_i(z;\ \varepsilon;\ \tau;\ \lambda)\,(i = 1, 2, 3, 4)$ of (4.3) are not linearly independent when $\lambda = \lambda_0$.

Let $G(z;\ \varepsilon;\ \tau;\ \lambda)$ be a 4×4 matrix defined by

$$G(z;\ \varepsilon;\ \tau;\ \lambda) \equiv [\bar{\mathbf{V}}_1(z;\ \varepsilon;\ \tau;\ \lambda),\ \bar{\mathbf{V}}_2(z;\ \varepsilon;\ \tau;\ \lambda),\ \bar{\mathbf{V}}_3(z;\ \varepsilon;\ \tau;\ \lambda),\ \bar{\mathbf{V}}_4(z;\ \varepsilon;\ \tau;\ \lambda)].$$

Since $G(z;\ \varepsilon;\ \tau;\ \lambda)$ is a solution of the matrix differential equation

$$\frac{d}{dz} G = \{A(z;\ \varepsilon;\ \tau) + \lambda C(\tau)\} G,$$

we easily see that

$$\det G(z;\ \varepsilon;\ \tau;\ \lambda) = a(z;\ \varepsilon;\ \tau) \cdot \det G(0;\ \varepsilon;\ \tau;\ \lambda),$$

where $a(z;\ \varepsilon;\ \tau) = \exp\left[\int_0^z \operatorname{trace}\{A(\xi;\ \varepsilon;\ \tau)\}\, d\xi\right] > 0$. Setting

$$g(\varepsilon;\ \tau;\ \lambda) \equiv \det G(0;\ \varepsilon;\ \tau;\ \lambda), \tag{4.5}$$

we find that $g(\varepsilon;\ \tau;\ \lambda)$ is an analytic function of $\lambda \in \mathbf{C}_d$, and $\det G(z;\ \varepsilon;\ \tau;\ \lambda) = 0$ for all $z \in \mathbf{R}$ if and only if $g(\varepsilon;\ \tau;\ \lambda) = 0$. Thus, we obtain the next lemma.

LEMMA 4.2. *For any* $\lambda \in \mathbf{C}_d$, λ *is an eigenvalue of* (4.2) *if and only if* $g(\varepsilon;\ \tau;\ \lambda) = 0$ *holds.*

4.2. Relation between stability and index

From Proposition 4.1 and Lemma 4.2, it follows that stability of the solution $(u, v)(z;\ \varepsilon;\ \tau)$ is given by examining the location of the roots of

$g(\varepsilon;\tau;\lambda)=0$. In this subsection, we state the properties of $g(\varepsilon;\tau;\lambda)$ with respect to λ. Since $\lambda=0$ is an eigenvalue of (4.2), the function $g(\varepsilon;\tau;\lambda)$ satisfies

$$g(\varepsilon;\tau;0)=0. \tag{4.6}$$

For sufficiently large $|\lambda|$, we obtain the following lemma.

LEMMA 4.3. *The relation $g(\varepsilon;\tau;\lambda)\neq 0$ holds for any $\lambda\in\Omega_\varepsilon\equiv\{\lambda\in\mathbf{C}\,|\,\mathrm{Re}\{\lambda\}\geq -d$ and $\varepsilon|\lambda|\to+\infty$ as $\varepsilon\downarrow 0\}$. In particular, $g(\varepsilon;\tau;\lambda)>0$ for any real λ satisfying $\varepsilon\lambda\to+\infty$ as $\varepsilon\downarrow 0$.*

The proof is stated in the appendix.

Though the above results follow from the general theory, the distribution of eigenvalues in $\mathbf{C}_d\setminus\Omega_\varepsilon$ is not clarified without solving the singularly perturbed eigenvalue problem (4.3). But the *index* stated in Section 3 gives important information about stability.

LEMMA 4.4. $\mathrm{sign}\left\{\dfrac{\partial g}{\partial\lambda}(\varepsilon;\tau;0)\right\}=\mathrm{Ind}[\gamma(\varepsilon;\tau)]$.

PROOF. Let us define functions $\bar{\mathbf{T}}_i(z;\varepsilon;\tau;\lambda)\,(i=1,2,3,4)$ by

$$\bar{\mathbf{T}}_i(z;\varepsilon;\tau;\lambda)=\begin{cases}\sum_{j=1}^{2}\beta_{ij}(z;\varepsilon;\tau)\bar{\mathbf{V}}_j(z;\varepsilon;\tau;\lambda) & (i=1,2)\\ \sum_{j=3}^{4}\beta_{ij}(z;\varepsilon;\tau)\bar{\mathbf{V}}_j(z;\varepsilon;\tau;\lambda) & (i=3,4),\end{cases}$$

where $\beta_{ij}(z;\varepsilon;\tau)$ are the same functions appeared in (3.11). Note that $B_1\equiv\det\{\beta_{ij}(z;\varepsilon;\tau);\ 1\leq i,j\leq 2\}>0$ and $B_2\equiv\det\{\beta_{ij}(z;\varepsilon;\tau);\ 3\leq i,j\leq 4\}>0$. We find that $\bar{\mathbf{T}}_i(z;\varepsilon;\tau;\lambda)\,(i=1,2,3,4)$ satisfy the following relations:

$$\begin{aligned}&\frac{d}{dz}\bar{\mathbf{T}}_i=\{A(z;\varepsilon;\tau)+\lambda C(\tau)\}\bar{\mathbf{T}}_i,\\ &\bar{\mathbf{T}}_i(z;\varepsilon;\tau;0)=\mathbf{T}_i(z;\varepsilon;\tau)\,(i=1,2,3,4),\\ &\bar{\mathbf{T}}_i(z;\varepsilon;\tau;\lambda)\longrightarrow\mathbf{0}\quad\text{as}\ \begin{cases}z\longrightarrow+\infty & (i=1,2)\\ z\longrightarrow-\infty & (i=3,4),\end{cases}\end{aligned}\tag{4.7}$$

$$\bar{\mathbf{T}}_1(z;\varepsilon;\tau;0)=\frac{d\mathscr{V}}{dz}(z;\varepsilon;\tau)=\bar{\mathbf{T}}_3(z;\varepsilon;\tau;0).\tag{4.8}$$

According to the definition of $g(\varepsilon;\tau;\lambda)$, we obtain

$$\begin{aligned}&g(\varepsilon;\tau;\lambda)\\ &\quad=B_0\cdot\det[\bar{\mathbf{T}}_1(0;\varepsilon;\tau;\lambda),\bar{\mathbf{T}}_2(0;\varepsilon;\tau;\lambda),\bar{\mathbf{T}}_3(0;\varepsilon;\tau;\lambda),\bar{\mathbf{T}}_4(0;\varepsilon;\tau;\lambda)],\end{aligned}\tag{4.9}$$

where $B_0\equiv(B_1B_2)^{-1}>0$. Differentiating (4.9) with respect to λ at $\lambda=0$, we find that

$$(4.10)\quad \frac{\partial}{\partial\lambda}g(\varepsilon;\,\tau;\,0)$$
$$= B_0\cdot\det[\mathbf{W}_1(0;\,\varepsilon;\,\tau)-\mathbf{W}_3(0;\,\varepsilon;\,\tau),\,\mathbf{T}_2(0;\,\varepsilon;\,\tau),\,\mathbf{T}_3(0;\,\varepsilon;\,\tau),\,\mathbf{T}_4(0;\,\varepsilon;\,\tau)],$$

holds for $\mathbf{W}_i(z;\,\varepsilon;\,\tau) \equiv \frac{\partial}{\partial\lambda}\bar{\mathbf{T}}_i(z;\,\varepsilon;\,\tau;\,0)\,(i = 1, 3)$.

On the other hand, differentiating (4.7) with respect to λ at $\lambda = 0$, we have

$$\frac{d}{dz}\mathbf{W}_i = A(z;\,\varepsilon;\,\tau)\mathbf{W}_i + C(\tau)\frac{d}{dz}\mathscr{V}(z;\,\varepsilon;\,\tau) \quad (i = 1, 3)$$

and

$$\mathbf{W}_i(z;\,\varepsilon;\,\tau) \longrightarrow \mathbf{0} \qquad \text{as } \begin{cases} z \longrightarrow +\infty & (i = 1) \\ z \longrightarrow -\infty & (i = 3). \end{cases}$$

Recall the definition of the vector $\mathbf{R}_0(0;\,\varepsilon;\,\tau)$, that is,

$$\mathbf{R}_0(0;\,\varepsilon;\,\tau) = \lim_{c\to c(\varepsilon;\tau)} \frac{\mathbf{X}(0;\,\varepsilon;\,\tau;\,c) - \mathbf{Y}(0;\,\varepsilon;\,\tau;\,c)}{c - c(\varepsilon;\,\tau)},$$

where $\mathbf{X}(z;\,\varepsilon;\,\tau;\,c)$ and $\mathbf{Y}(z;\,\varepsilon;\,\tau;\,c)$ satisfy

$$(4.11)\quad \begin{cases} \dfrac{d}{dz}\mathbf{X} = \mathbf{F}(\mathbf{X};\,\varepsilon;\,\tau;\,c) \\ \mathbf{X}(z;\,\varepsilon;\,\tau;\,c) \longrightarrow \mathbf{0} \qquad \text{as } z \longrightarrow +\infty \end{cases}$$

and

$$(4.12)\quad \begin{cases} \dfrac{d}{dz}\mathbf{Y} = \mathbf{F}(\mathbf{Y};\,\varepsilon;\,\tau;\,c) \\ \mathbf{Y}(z;\,\varepsilon;\,\tau;\,c) \longrightarrow \mathbf{0} \qquad \text{as } z \longrightarrow -\infty, \end{cases}$$

respectively. Since $\mathbf{X}(z;\,\varepsilon;\,\tau;\,c)$ and $\mathbf{Y}(z;\,\varepsilon;\,\tau;\,c)$ are smooth with respect c, differentiating (4.11) and (4.12) with respect to c at $c = c(\varepsilon,\,\tau)$, we find that

$$\mathbf{X}_c(z;\,\varepsilon;\,\tau) \equiv \frac{\partial}{\partial c}\mathbf{X}(z;\,\varepsilon;\,\tau;\,c(\varepsilon;\,\tau))$$

and

$$\mathbf{Y}_c(z;\,\varepsilon;\,\tau) \equiv \frac{\partial}{\partial c}\mathbf{Y}(z;\,\varepsilon;\,\tau;\,c\varepsilon;\,\tau))$$

satisfy the following equations:

$$(4.13)\quad \begin{cases} \dfrac{d}{dz}\mathbf{X}_c = A(z;\,\varepsilon;\,\tau)\mathbf{X}_c + C(\tau)\dfrac{d}{dz}\mathscr{V}(z;\,\varepsilon;\,\tau) \\ \mathbf{X}_c(z;\,\varepsilon;\,\tau) \longrightarrow \mathbf{0} \qquad \text{as } z \longrightarrow +\infty \end{cases}$$

and

$$
(4.14) \qquad \begin{cases} \dfrac{d}{dz}\mathbf{Y}_c = A(z;\, \varepsilon;\, \tau)\mathbf{Y}_c + C(\tau)\dfrac{d}{dz}\mathscr{V}(z;\, \varepsilon;\, \tau) \\ \mathbf{Y}_c(z;\, \varepsilon;\, \tau) \longrightarrow \mathbf{0} \qquad \text{as } z \longrightarrow -\infty, \end{cases}
$$

respectively. A general solution $\mathbf{X}_c$ of (4.13) is represented as

$$\mathbf{X}_c(z;\, \varepsilon;\, \tau) = \mathbf{W}_1(z;\, \varepsilon;\, \tau) + \textstyle\sum_{i=1}^{2} \kappa_i \mathbf{T}_i(z;\, \varepsilon;\, \tau)$$

because $\mathbf{W}_1$ is a particular solution of (4.13). Similarly, a general solution $\mathbf{Y}_c$ of (4.14) is represented as

$$\mathbf{Y}_c(z;\, \varepsilon;\, \tau) = \mathbf{W}_3(z;\, \varepsilon;\, \tau) + \textstyle\sum_{i=3}^{4} \kappa_i \mathbf{T}_i(z;\, \varepsilon;\, \tau).$$

Therefore $\mathbf{R}_0(0;\, \varepsilon;\, \tau)$ is rewritten as follows:

$$
\begin{aligned}
\mathbf{R}_0(0;\, \varepsilon;\, \tau) &= \frac{\partial}{\partial c}\mathbf{X}(0;\, \varepsilon;\, \tau;\, c(\varepsilon;\, \tau)) - \frac{\partial}{\partial c}\mathbf{Y}(0;\, \varepsilon;\, \tau;\, c(\varepsilon;\, \tau)) \\
&= \mathbf{X}_c(0;\, \varepsilon;\, \tau) - \mathbf{Y}_c(0;\, \varepsilon;\, \tau) \\
&= \mathbf{W}_1(0;\, \varepsilon;\, \tau) - \mathbf{W}_3(0;\, \varepsilon;\, \tau) + \textstyle\sum_{i=1}^{2} \kappa_i \mathbf{T}_i(0;\, \varepsilon;\, \tau) - \textstyle\sum_{i=3}^{4} \kappa_i \mathbf{T}_i(0;\, \varepsilon;\, \tau).
\end{aligned}
$$

Thus, substituting this into (4.10), we conclude that

$$
\begin{aligned}
\frac{\partial}{\partial \lambda} g(\varepsilon;\, \tau;\, 0) &= B_0 \cdot \det[\mathbf{R}_0(0;\, \varepsilon;\, \tau),\, \mathbf{T}_2(0;\, \varepsilon;\, \tau),\, \mathbf{T}_3(0;\, \varepsilon;\, \tau),\, \mathbf{T}_4(0;\, \varepsilon;\, \tau)] \\
&= B_0 \cdot h(0;\, \varepsilon;\, \tau).
\end{aligned}
$$

That is, by virtue of the relation $B_0 > 0$,

$$\operatorname{sign}\left\{\frac{\partial}{\partial \lambda} g(\varepsilon;\, \tau;\, 0)\right\} = \operatorname{sign}\{h(0;\, \varepsilon;\, \tau)\} = \operatorname{Ind}[\gamma(\varepsilon;\, \tau)]. \qquad \blacksquare$$

LEMMA 4.5. *If* $\dfrac{\partial}{\partial \lambda} g(\varepsilon;\, \tau;\, 0) \neq 0$, $\lambda = 0$ *is a simple eigenvalue of* (4.2).

PROOF. It follows from (4.6) that $\lambda = 0$ is an eigenvalue of (4.2). If $\lambda = 0$ is not simple, there exists a bounded solution $\mathbf{U}(z;\, \varepsilon;\, \tau)$ satisfying

$$\frac{d}{dz}\mathbf{U} = A(z;\, \varepsilon;\, \tau)\mathbf{U} + C(\tau)\frac{d}{dz}\mathscr{V}(z;\, \varepsilon;\, \tau).$$

Namely, $\mathbf{U}(z;\, \varepsilon;\, \tau)$ is represented as follows:

$$
(4.15) \qquad \begin{aligned}
\mathbf{U}(z;\, \varepsilon;\, \tau) &= \mathbf{W}_1(z;\, \varepsilon;\, \tau) + \textstyle\sum_{i=1}^{2} \tilde{\kappa}_i \mathbf{T}_i(z;\, \varepsilon;\, \tau), \\
&= \mathbf{W}_3(z;\, \varepsilon;\, \tau) + \textstyle\sum_{i=3}^{4} \tilde{\kappa}_i \mathbf{T}_i(z;\, \varepsilon;\, \tau),
\end{aligned}
$$

where $\tilde{\kappa}_i (i = 1, 2, 3, 4)$ are appropriate constants. By the relations (4.8), (4.10) and (4.15), we obtain

$$\frac{\partial}{\partial \lambda} g(\varepsilon; \tau; 0)$$

$$= B_0 \cdot \det[\mathbf{W}_1(0; \varepsilon; \tau) - \mathbf{W}_3(0; \varepsilon; \tau), \mathbf{T}_2(0; \varepsilon; \tau), \mathbf{T}_3(0; \varepsilon; \tau), \mathbf{T}_4(0; \varepsilon; \tau)] = 0,$$

which contradicts the assumption of Lemma 4.5. ■

From Lemmas 4.3 and 4.4, it follows that

THEOREM 4.1 (*instability*). *The traveling wave solution is unstable if the index of the corresponding heteroclinic orbit is equal to* -1.

Combining this results with Theorem 3.1, we have

COROLLARY 4.1. *If the sign of the Jacobian of the matching condition is negative, the traveling wave solution constructed from this matching condition is unstable.*

On the other hand, when the index of the heteroclinic orbit is equal to 1, we know that zero is a simple eigenvalue by virtue of Lemma 4.5. Thus we obtain the following theorem for stability of the traveling wave solution.

THEOREM 4.2 (*stability*). *If the index of the corresponding heteroclinic orbit is equal to* 1, *it gives a necessary condition for the traveling wave solution to be stable.*

COROLLARY 4.2. *When the sign of the Jacobian of the matching condition is positive, it gives a necessary condition for the traveling wave solution to be stable.*

5. Concluding remarks

In our previous paper [28], we have concluded by using the SLEP method that the sign of the Jacobian of the matching condition corresponds in a one-to-one manner to stability of the traveling wave solution. That is, the traveling wave solution is stable (resp. unstable) if and only if the sign of the Jacobian is positive (resp. nagative). But our results in Section 4 is incomplete for stability because an information from the index only determine the distribution of eigenvalues in some small neighborhood, say $\mathcal{N}$, of the origin (see Theorem 4.2).

In order to examine the distribution of eigenvalues in $\mathbf{C}_d \setminus \{\Omega_\varepsilon \cup \mathcal{N}\} (\equiv \Omega_r)$, for any fixed $\lambda \in \Omega_r$ we must construct solutions $\bar{\mathbf{V}}_i(z; \varepsilon; \tau; \lambda) (i = 1, 2, 3, 4)$ satisfying

$$\frac{d}{dz}\bar{\mathbf{V}}_i = \{A(z; \varepsilon; \tau) + \lambda C(\tau)\}\bar{\mathbf{V}}_i, \tag{5.1}$$

$$\bar{\mathbf{V}}_i(z;\varepsilon;\tau;0)=\mathbf{V}_i(z;\varepsilon;\tau)(i=1,2,3,4),$$

$$\bar{\mathbf{V}}_i(z;\varepsilon;\tau;\lambda)\longrightarrow \mathbf{0} \quad \text{as } z\longrightarrow +\infty \ (i=1,2) \tag{5.2}$$

and

$$\bar{\mathbf{V}}_i(z;\varepsilon;\tau;\lambda)\longrightarrow \mathbf{0} \quad \text{as } z\longrightarrow -\infty \ (i=3,4). \tag{5.3}$$

And using these solutions $\bar{\mathbf{V}}_i$ $(i=1,2,3,4)$, we must examine the location of roots of $g(\varepsilon;\tau;\lambda)=0$ with respect to λ. Indeed, this is possible. In the following, we shall give a brief sketch.

First, we consider the following singularly perturbed eigenvalue problem:

$$\begin{cases} \varepsilon^2\bar{u}_{zz}-\varepsilon c(\varepsilon;\tau)\tau\bar{u}_z+f_u^\varepsilon\bar{u}+f_v^\varepsilon\bar{v}=\varepsilon\tau\lambda\bar{u} \\ \bar{v}_{zz}-c(\varepsilon;\tau)\bar{v}_z+g_u^\varepsilon\bar{u}+g_v^\varepsilon\bar{v}=\lambda\bar{v} \end{cases}, \ z\in\mathbf{R}, \tag{5.4}$$

where $\lambda\in\Omega_r$ and $(\bar{u},\bar{v})(z;\varepsilon;\tau;\lambda)\in BC(\mathbf{R})\times BC(\mathbf{R})$ (see (4.2)). By Lemma 4.1, $(\bar{u},\bar{v})(z;\varepsilon;\tau;\lambda)(\in BC(\mathbf{R})\times BC(\mathbf{R}))$ must satisfy the condition $(\bar{u},\bar{v})(\pm\infty;\varepsilon;\tau;\lambda)=(0,0)$. Then we impose these boundary conditions

$$\bar{u}(\pm\infty;\varepsilon;\tau;\lambda)=0 \quad \text{and} \quad \bar{v}(\pm\infty;\varepsilon;\tau;\lambda)=0 \tag{5.5}$$

on (5.4). The problem (5.4), (5.5) is the same singularly perturbed problem as we have already treated in Section 2. Similarly, we divide the whole interval $\mathbf{R}$ into two subintervals $\mathbf{R}_\pm$ and consider the following problems:

$$\begin{cases} \varepsilon^2\varphi^\pm_{zz}-\varepsilon c(\varepsilon;\tau)\tau\varphi^\pm_z+f_u^\varepsilon\varphi^\pm+f_v^\varepsilon\psi^\pm=\varepsilon\tau\lambda\varphi^\pm & \\ \psi^\pm_{zz}-c(\varepsilon;\tau)\psi^\pm_z+g_u^\varepsilon\varphi^\pm+g_v^\varepsilon\psi^\pm=\lambda\psi^\pm & , \ z\in\mathbf{R}_\pm \\ \varphi^\pm(\pm\infty)=0,\ \varphi^\pm(0)=a & \\ \psi^\pm(\pm\infty)=0,\ \psi^\pm(0)=b, & \end{cases} \tag{5.6$_\pm$}$$

where a and b are arbitrary fixed constants. By using the same techniques as we used in Section 2, we can construct the solutions $(\varphi^\pm,\psi^\pm)(z;\varepsilon;\tau;\lambda;a,b)$ of $(5.6)_\pm$. Put

$$\mathbf{U}_1(z;\varepsilon;\tau;\lambda)=\begin{bmatrix} \varphi^+(z;\varepsilon;\tau;\lambda;1,0) \\ \varepsilon\dfrac{d}{dz}\varphi^+(z;\varepsilon;\tau;\lambda;1,0) \\ \psi^+(z;\varepsilon;\tau;\lambda;1,0) \\ \dfrac{d}{dz}\psi^+(z;\varepsilon;\tau;\lambda;1,0) \end{bmatrix},$$

$$\mathbf{U}_2(z;\, \varepsilon;\, \tau;\, \lambda) = \begin{bmatrix} \varphi^+(z;\, \varepsilon;\, \tau;\, \lambda;\, 0,\, 1) \\ \varepsilon \dfrac{d}{dz}\varphi^+(z;\, \varepsilon;\, \tau;\, \lambda;\, 0,\, 1) \\ \psi^+(z;\, \varepsilon;\, \tau;\, \lambda;\, 0,\, 1) \\ \dfrac{d}{dz}\psi^+(z;\, \varepsilon;\, \tau;\, \lambda;\, 0,\, 1) \end{bmatrix} (z \in \mathbf{R}_+),$$

$$\mathbf{U}_3(z;\, \varepsilon;\, \tau;\, \lambda) = \begin{bmatrix} \varphi^-(z;\, \varepsilon;\, \tau;\, \lambda;\, 1,\, 0) \\ \varepsilon \dfrac{d}{dz}\varphi^-(z;\, \varepsilon;\, \tau;\, \lambda;\, 1,\, 0) \\ \psi^-(z;\, \varepsilon;\, \tau;\, \lambda;\, 1,\, 0) \\ \dfrac{d}{dz}\psi^-(z;\, \varepsilon;\, \tau;\, \lambda;\, 1,\, 0) \end{bmatrix}$$

and

$$\mathbf{U}_4(z;\, \varepsilon;\, \tau;\, \lambda) = \begin{bmatrix} \varphi^-(z;\, \varepsilon;\, \tau;\, \lambda;\, 0,\, 1) \\ \varepsilon \dfrac{d}{dz}\varphi^-(z;\, \varepsilon;\, \tau;\, \lambda;\, 0,\, 1) \\ \psi^-(z;\, \varepsilon;\, \tau;\, \lambda;\, 0,\, 1) \\ \dfrac{d}{dz}\psi^-(z;\, \varepsilon;\, \tau;\, \lambda;\, 0,\, 1) \end{bmatrix} (z \in \mathbf{R}_-).$$

We find that $\mathbf{U}_i(z;\, \varepsilon;\, \tau;\, \lambda)(i = 1, 2)$ are linearly independent solutions of (5.1), (5.2) and $\mathbf{U}_i(z;\, \varepsilon;\, \tau;\, \lambda)(i = 3, 4)$ are linearly independent solutions of (5.1), (5.3) for all $\lambda \in \Omega_r$. Then $\bar{\mathbf{V}}_i(z;\, \varepsilon;\, \tau;\, \lambda)(i = 1, 2, 3, 4)$(which were defined in Section 3) are represented as

$$\text{(5.7)} \qquad \begin{bmatrix} \bar{\mathbf{V}}_1(z;\, \varepsilon;\, \tau;\, \lambda) \\ \bar{\mathbf{V}}_2(z;\, \varepsilon;\, \tau;\, \lambda) \end{bmatrix} = K_1(z;\, \varepsilon;\, \tau;\, \lambda) \begin{bmatrix} \mathbf{U}_1(z;\, \varepsilon;\, \tau;\, \lambda) \\ \mathbf{U}_2(z;\, \varepsilon;\, \tau;\, \lambda) \end{bmatrix} (z \in \mathbf{R}_+)$$

and

$$\text{(5.8)} \qquad \begin{bmatrix} \bar{\mathbf{V}}_3(z;\, \varepsilon;\, \tau;\, \lambda) \\ \bar{\mathbf{V}}_4(z;\, \varepsilon;\, \tau;\, \lambda) \end{bmatrix} = K_2(z;\, \varepsilon;\, \tau;\, \lambda) \begin{bmatrix} \mathbf{U}_3(z;\, \varepsilon;\, \tau;\, \lambda) \\ \mathbf{U}_4(z;\, \varepsilon;\, \tau;\, \lambda) \end{bmatrix} (z \in \mathbf{R}_-),$$

where $K_i(z;\, \varepsilon;\, \tau;\, \lambda)$ are some linear transformations satisfying $\det K_i(z;\, \varepsilon;\, \tau;\, \lambda) \neq 0 (i = 1, 2)$. Using these relations, $g(\varepsilon;\, \tau;\, \lambda)$ takes the following form:

$$\text{(5.9)} \qquad g(\varepsilon;\, \tau;\, \lambda) = (K_1 K_2)^{-1}$$
$$\det[\mathbf{U}_1(0;\, \varepsilon;\, \tau;\, \lambda),\, \mathbf{U}_2(0;\, \varepsilon;\, \tau;\, \lambda),\, \mathbf{U}_3(0;\, \varepsilon;\, \tau;\, \lambda),\, \mathbf{U}_4(0;\, \varepsilon;\, \tau;\, \lambda)],$$

where $K_i = \det K_i(0; \varepsilon; \tau; \lambda) \neq 0$ $(i = 1, 2)$ for all $\lambda \in \Omega_r$. For any fixed $\lambda \in \Omega_r$, using the functions $\mathbf{U}_i$ $(i = 1, 2, 3, 4)$ constructed above, we can conclude that $g(\varepsilon; \tau; \lambda) \neq 0$. That is, there is no eigenvalue of (4.2) in Ω_r. This procedure is a new technique to solve the singularly perturbed eigenvalue problems such as (5.4). Since it is very complicated, we shall state the detailed proof in the forthcoming paper [14]. In a two-component system such as (1.1), when the diffusion coefficient of v is zore (for example, nerve axon equations) or degenerate as $\varepsilon \downarrow 0$ (for example, the Belousov-Zhabotinskii reaction equations, see Fife [10] and Tyson et al [32]), it is difficult to apply the SLEP-method. But our method is well applicable to such problems.

Appendix

Proof of Lemma 4.1.

The eigenvalues $\mu_j^\pm(\varepsilon; \tau; \lambda)(j = 1, 2, 3, 4)$ of the matrices $A^\pm(\varepsilon; \tau; c(\varepsilon; \tau)) + \lambda C(\tau)$ are obtained as the roots of the characteristic polynomials

$$F^\pm(\mu; \varepsilon; \tau; \lambda) \equiv \det[\mu I - A^\pm(\varepsilon; \tau; c(\varepsilon; \tau)) - \lambda C(\tau)] = 0,$$

where I is the 4×4 unit matrix. By the definition of $A^\pm(\varepsilon; \tau; c(\varepsilon; \tau))$ and $C(\tau)$, the polynomials $F^\pm(\mu; \varepsilon; \tau; \lambda)$ can be written equivalently as

$$\tilde{F}^\pm(\mu; \varepsilon; \tau; \lambda) = \det[\mu^2 D - \mu c(\varepsilon; \tau)B + N^\pm - \lambda B],$$

where

$$D = \begin{bmatrix} \varepsilon^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad N^\pm = \begin{bmatrix} f_u^\pm & f_v^\pm \\ g_u^\pm & g_v^\pm \end{bmatrix}, \quad B = \begin{bmatrix} \varepsilon\tau & 0 \\ 0 & 1 \end{bmatrix}.$$

Let us seek the eigenvalues $\mu_j^\pm(j = 1, 2, 3, 4)$ by using the polynomials $\tilde{F}^\pm(\mu; \varepsilon; \tau; \lambda) = 0$ in stead of $F^\pm(\mu; \varepsilon; \tau; \lambda) = 0$.

First for any $\omega \in \mathbf{R}$, we consider the following polynomials with respect to σ

$$\det[-\omega^2 D + N^\pm - \sigma B] = 0. \tag{A.1$_\pm$}$$

Solving (A.1)$_\pm$ directly, we find that

$$\sigma_1^\pm(\omega; \varepsilon; \tau) = \frac{1}{2\varepsilon\tau}[-(\varepsilon\tau + \varepsilon^2)\omega^2 + (f_u^\pm + \varepsilon\tau g_v^\pm) + [\{(\varepsilon\tau - \varepsilon^2)\omega^2 + (f_u^\pm - \varepsilon\tau g_v^\pm)\}^2 + 4\varepsilon\tau f_v^\pm g_u^\pm]^{1/2}]$$

and

$$\sigma_2^\pm(\omega;\varepsilon;\tau)=\frac{1}{2\varepsilon\tau}[-(\varepsilon\tau+\varepsilon^2)\omega^2+(f_u^\pm+\varepsilon\tau g_v^\pm)$$
$$-[\{(\varepsilon\tau-\varepsilon^2)\omega^2+(f_u^\pm-\varepsilon\tau g_v^\pm)\}^2+4\varepsilon\tau f_v^\pm g_u^\pm]^{1/2}].$$

When ε is sufficiently small, $\sigma_j^\pm (j=1,2)$ are real numbers and have the following estimates:

$$\sigma_j^\pm(\omega;\varepsilon;\tau)\le -d\equiv \max\{f_u^\pm,\, g_v^\pm\}<0 \qquad (j=1,2)$$

for any $\omega\in\mathbf{R}$. That is, $(\mathrm{A}.1)_\pm$ have two negative roots $\sigma_j^\pm(\omega;\varepsilon;\tau)(j=1,2)$.

Next we consider the increment of $\arg\{\tilde{F}^\pm(\mu;\varepsilon;\tau;\lambda)\}$ when the complex number μ moves along the imaginary axis from $-i\infty$ to $+i\infty$. Substituting $\mu=i\omega$ $(\omega\in\mathbf{R})$ in $\tilde{F}^\pm(\mu;\varepsilon;\tau;\lambda)$, we obtain

$$\tilde{F}^\pm(i\omega;\varepsilon;\tau;\lambda)=\det[-\omega^2D+N^\pm-(\lambda+i\omega c(\varepsilon;\tau))B]$$
$$=\{\lambda+i\omega c(\varepsilon;\tau)-\sigma_1^\pm(\omega;\varepsilon;\tau)\}\{\lambda+i\omega c(\varepsilon;\tau)-\sigma_2^\pm(\omega;\varepsilon;\tau)\}.$$

For any $\lambda\in\mathbf{C}_d\equiv\{\lambda\in\mathbf{C}\,|\,\mathrm{Re}\lambda\ge -d\}$, we put

$$k_j^\pm(\omega;\varepsilon;\tau;\lambda)\equiv\lambda+i\omega c(\varepsilon;\tau)-\sigma_j^\pm(\omega;\varepsilon;\tau)(j=1,2).$$

By the negativity of $\sigma_j^\pm(\omega;\varepsilon;\tau)(j=1,2)$, $k_j^\pm(j=1,2)$ have the following properties:

$$k_j^\pm(\omega;\varepsilon;\tau;\lambda)\ne 0 \quad\text{and}\quad -\frac{\pi}{2}<\arg\{k_j^\pm(\omega;\varepsilon;\tau;\lambda)\}<\frac{\pi}{2}\ (j=1,2)$$

for all $\omega\in\mathbf{R}$.

On the other hand, from $(\mathrm{A}.1)_\pm$ it follows that

$$\sigma_1^\pm(\omega;\varepsilon;\tau)=-\frac{\varepsilon\omega^2}{\tau}+O(1)$$

and

$$\sigma_2^\pm(\omega;\varepsilon;\tau)=-\omega^2+O(1)$$

for sufficiently large $|\omega|$. Hence $k_j^\pm(\omega;\varepsilon;\tau)$ satisfy

$$\arg\{k_j^\pm(\omega;\varepsilon;\tau;\lambda)\}\longrightarrow 0 \qquad \text{as } |\omega|\longrightarrow\infty \quad (j=1,2).$$

This implies that

$$\triangle_j^\pm\equiv\arg\{k_j^\pm(+\infty;\varepsilon;\tau;\lambda)\}-\arg\{k_j^\pm(-\infty;\varepsilon;\tau;\lambda)\}=0\quad(j=1,2).$$

Therefore we obtain

$$\triangle^\pm\equiv\arg\{\tilde{F}^\pm(+i\infty;\varepsilon;\tau;\lambda)\}-\arg\{\tilde{F}^\pm(-i\infty;\varepsilon;\tau;\lambda)\}=0.$$

That is, the increment $\triangle^\pm$ of $\arg\{\tilde{F}^\pm(\mu;\varepsilon;\tau;\lambda)\}$ is equal to 0 when μ moves along the imaginary axis from $-i\infty$ to $+i\infty$. Applying the argument principle to $\tilde{F}^{\pm}(\mu;\, \varepsilon;\, \tau;\, \lambda) = 0$, we find that the characteristic polynomials $\tilde{F}^{\pm}(\mu;\, \varepsilon;\, \tau;\, \lambda) = 0$ have just two roots $\mu_j^{\pm}$ $(j = 1, 2)$ in the left half plane and just two roots $\mu_j^{\pm}$ $(j = 3, 4)$ in the right half plane. ■

Proof of Lemma 4.3.

Here, we use the technique stated in Section 5. That is, we use the relation (5.9). Though $g(\varepsilon;\, \tau;\, \lambda)$ in (5.9) is defined only for $\lambda \in \Omega_r$, it is easily seen that this relation

$$\text{(A.2)} \qquad g(\varepsilon;\, \tau;\, \lambda) = (K_1 K_2)^{-1} \det[\mathbf{U}_1(0;\, \varepsilon;\, \tau;\, \lambda),\, \mathbf{U}_2(0;\, \varepsilon;\, \tau;\, \lambda),\, \mathbf{U}_3(0;\, \varepsilon;\, \tau;\, \lambda),\, \mathbf{U}_4(0;\, \varepsilon;\, \tau;\, \lambda)]$$

also holds for any $\lambda \in \mathbf{C}_d$.

First we determine the sign of $(K_1 K_2)^{-1}$, which is definite for all $\lambda \in \mathbf{C}_d$. Let $\lambda = 0$ in (5.7) and (5.8), we have the following relations:

$$\text{(A.3)} \qquad \begin{bmatrix} \mathbf{V}_1(z;\, \varepsilon;\, \tau) \\ \mathbf{V}_2(z;\, \varepsilon;\, \tau) \end{bmatrix} = K_1(z;\, \varepsilon;\, \tau;\, 0) \begin{bmatrix} \mathbf{U}_1(z;\, \varepsilon;\, \tau;\, 0) \\ \mathbf{U}_2(z;\, \varepsilon;\, \tau;\, 0) \end{bmatrix}$$

and

$$\text{(A.4)} \qquad \begin{bmatrix} \mathbf{V}_3(z;\, \varepsilon;\, \tau) \\ \mathbf{V}_4(z;\, \varepsilon;\, \tau) \end{bmatrix} = K_2(z;\, \varepsilon;\, \tau;\, 0) \begin{bmatrix} \mathbf{U}_3(z;\, \varepsilon;\, \tau;\, 0) \\ \mathbf{U}_4(z;\, \varepsilon;\, \tau;\, 0) \end{bmatrix}.$$

Since both $\mathbf{U}_i(z;\, \varepsilon;\, \tau;\, 0)$ and $\mathbf{T}_i(z;\, \varepsilon;\, \tau)$ satisfy the equation (3.8), comparing the values of $\mathbf{U}_i(0;\, \varepsilon;\, \tau;\, 0)$ in Section 5 with $\mathbf{T}_i(0;\, \varepsilon;\, \tau)$ in Subsection 3.4 $(i = 1, 2, 3, 4)$, we find that

$$\mathbf{U}_1(z;\, \varepsilon;\, \tau;\, 0) = \frac{\varepsilon \mathbf{T}_1(z;\, \varepsilon;\, \tau)}{(W_0^+)_\xi(0;\, \tau;\, c^*(\tau),\, \beta^*(\tau)) + O(\varepsilon)},$$

$$\mathbf{U}_2(z;\, \varepsilon;\, \tau;\, 0) = -\mathbf{T}_2(z;\, \varepsilon;\, \tau),$$

$$\mathbf{U}_3(z;\, \varepsilon;\, \tau;\, 0) = \frac{\varepsilon \mathbf{T}_3(z;\, \varepsilon;\, \tau)}{(W_0^-)_\xi(0;\, \tau;\, c^*(\tau),\, \beta^*(\tau)) + O(\varepsilon)}$$

and

$$\mathbf{U}_4(z;\, \varepsilon;\, \tau;\, 0) = \mathbf{T}_4(z;\, \varepsilon;\, \tau).$$

Using these relations and (3.15), (3.17), (3.19) and (3.21), we easily see that

$$\begin{bmatrix} \mathbf{U}_1(z;\, \varepsilon;\, \tau;\, 0) \\ \mathbf{U}_2(z;\, \varepsilon;\, \tau;\, 0) \end{bmatrix} \simeq \hat{B}^+ \begin{bmatrix} e^{\tilde{v}_1^+ z} \cdot \tilde{\mathbf{a}}_1^+ \\ e^{\tilde{v}_2^+ z} \cdot \tilde{\mathbf{a}}_2^+ \end{bmatrix} \qquad \text{as } z \longrightarrow +\infty$$

and

$$\begin{bmatrix} \mathbf{U}_3(z;\varepsilon;\tau;0) \\ \mathbf{U}_4(z;\varepsilon;\tau;0) \end{bmatrix} \simeq \hat{B}^- \begin{bmatrix} e^{\tilde{v}_3^- z}\cdot\tilde{\mathbf{a}}_3^- \\ e^{\tilde{v}_4^- z}\cdot\tilde{\mathbf{a}}_4^- \end{bmatrix} \qquad \text{as } z \longrightarrow -\infty$$

holds, where

$$\hat{B}^+ = \begin{bmatrix} -\dfrac{\varepsilon d_1 \tilde{v}_1^+}{(W_0^+)_\xi(0)} + O(\varepsilon) & \dfrac{f_v^+}{f_u^+}\cdot\dfrac{\varepsilon d_2 \tilde{v}_2^+}{(W_0^+)_\xi(0)} + O(\varepsilon^2) \\ -\dfrac{f_v^+}{f_u^+}\cdot\dfrac{\varepsilon d_1 \tilde{v}_1^+}{(W_0^+)_\xi(0)} + O(\varepsilon) & \dfrac{f_v^+}{f_u^+}\cdot\dfrac{d_2 \tilde{v}_2^+}{(V_0^+)_z(0)} + O(\varepsilon) \end{bmatrix}$$

and

$$\hat{B}^- = \begin{bmatrix} -\dfrac{f_v^-}{f_u^+}\cdot\dfrac{\varepsilon d_3 \tilde{v}_3^-}{(W_0^-)_\xi(0)} + O(\varepsilon^2) & \dfrac{\varepsilon d_4 \tilde{v}_4^-}{(W_0^-)_\xi(0)} + O(\varepsilon) \\ -\dfrac{f_v^-}{f_u^-}\cdot\dfrac{d_3 \tilde{v}_3^-}{(V_0^-)_z(0)} + O(\varepsilon) & -\dfrac{f_v^-}{f_u^-}\cdot\dfrac{\varepsilon d_4 \tilde{v}_4^-}{(W_0^-)_\xi(0)} + O(\varepsilon) \end{bmatrix}.$$

On the other hand, by virtue of (3.9) we have

$$\begin{bmatrix} \mathbf{V}_1(z;\varepsilon;\tau) \\ \mathbf{V}_2(z;\varepsilon;\tau) \end{bmatrix} \simeq \begin{bmatrix} e^{\tilde{v}_1^+ z}\cdot\tilde{\mathbf{a}}_1^+ \\ e^{\tilde{v}_2^+ z}\cdot\tilde{\mathbf{a}}_2^+ \end{bmatrix} \qquad \text{as } z \longrightarrow +\infty$$

and

$$\begin{bmatrix} \mathbf{V}_3(z;\varepsilon;\tau) \\ \mathbf{V}_4(z;\varepsilon;\tau) \end{bmatrix} \simeq \begin{bmatrix} e^{\tilde{v}_3^- z}\cdot\tilde{\mathbf{a}}_3^- \\ e^{\tilde{v}_4^- z}\cdot\tilde{\mathbf{a}}_4^- \end{bmatrix} \qquad \text{as } z \longrightarrow -\infty.$$

Substitute the above results into (A.3) and (A.4), and let $z \to +\infty$ and $z \to -\infty$, we obtain

$$I = K_1(+\infty;\varepsilon;\tau;0)\hat{B}^+ \quad \text{and} \quad I = K_2(-\infty;\varepsilon;\tau;0)\hat{B}^-,$$

respectively, from which it follows that

$$\det K_1(+\infty;\varepsilon;\tau;0) < 0 \quad \text{and} \quad \det K_2(-\infty;\varepsilon;\tau;0) > 0.$$

Noting that the sign of $\det K_1(z;\varepsilon;\tau;0)$ (resp. $\det K_2(z;\varepsilon;\tau;0)$) is definite for all $z \in \mathbf{R}_+$ (resp. $z \in \mathbf{R}_-$), we may conclude that

(A.5) $$(K_1 K_2)^{-1} = (\det K_1(0;\varepsilon;\tau;\lambda)\cdot\det K_2(0;\varepsilon;\tau;\lambda))^{-1} < 0.$$

Next we construct $\mathbf{U}_i(z;\varepsilon;\tau;\lambda)\,(i = 1,2,3,4)$ for any fixed $\lambda \in \Omega_\varepsilon \equiv \{\lambda \in \mathbf{C}_d \mid \varepsilon|\lambda| \to \infty \text{ as } \varepsilon \downarrow 0\}$. To do so, we consider the problems:

(A.6)$_\pm$ $$\begin{cases} \varepsilon^2\varphi_{zz}^\pm - \varepsilon c(\varepsilon;\tau)\tau\varphi_z^\pm + f_u^\varepsilon\varphi^\pm + f_v^\varepsilon\psi^\pm = \varepsilon\tau\lambda\varphi^\pm \\ \psi_{zz}^\pm - c(\varepsilon;\tau)\psi_z^\pm + g_u^\varepsilon\varphi^\pm + g_v^\varepsilon\psi^\pm = \lambda\psi^\pm \\ \varphi^\pm(\pm\infty) = 0,\ \varphi^\pm(0) = a \\ \psi^\pm(\pm\infty) = 0,\ \psi^\pm(0) = b \end{cases} , z \in \mathbf{R}_\pm$$

(see $(5.6)_\pm$). Since $\varepsilon|\lambda| \to +\infty$ as $\varepsilon \downarrow 0$, there exists a real, positive and continuous function $\omega(\varepsilon)$ such that $\varepsilon\omega(\varepsilon) \to +\infty$ as $\varepsilon \downarrow 0$ and $\lambda(\varepsilon)$ is represented as

$$\lambda(\varepsilon) = \mu(\varepsilon)\omega(\varepsilon), \tag{A.7}$$

where $\mu(\varepsilon)$ satisfies $|\mu(0)| \neq 0$ (see Eckhaus [2: Lemma 1.1.1]). Using the relation (A.7), we rewrite $(A.6)_\pm$ as

$$(A.8)_\pm \quad \begin{cases} \dfrac{\varepsilon}{\omega(\varepsilon)}\varphi^\pm_{zz} - \dfrac{c(\varepsilon;\tau)}{\omega(\varepsilon)}\tau\varphi^\pm_z + \dfrac{f^\varepsilon_u}{\varepsilon\omega(\varepsilon)}\varphi^\pm + \dfrac{f^\varepsilon_v}{\varepsilon\omega(\varepsilon)}\psi^\pm = \tau\mu(\varepsilon)\varphi^\pm \\ \dfrac{1}{\omega(\varepsilon)}\psi^\pm_{zz} - \dfrac{c(\varepsilon;\tau)}{\omega(\varepsilon)}\psi^\pm_z + \dfrac{g^\varepsilon_u}{\omega(\varepsilon)}\varphi^\pm + \dfrac{g^\varepsilon_v}{\omega(\varepsilon)}\psi^\pm = \mu(\varepsilon)\psi^\pm \end{cases}, z \in \mathbf{R}_\pm$$
$$\varphi^\pm(\pm\infty) = 0,\ \varphi^\pm(0) = a$$
$$\psi^\pm(\pm\infty) = 0,\ \psi^\pm(0) = b.$$

By the transformation $y = \omega(\varepsilon)^{1/2}z$, $(A.8)_\pm$ can be rewritten as the following regularly perturbed problems:

$$(A.9)_\pm \quad \begin{cases} \varepsilon\varphi^\pm_{yy} - \dfrac{c(\varepsilon;\tau)}{\omega(\varepsilon)^{1/2}}\tau\varphi^\pm_y + \dfrac{\tilde{f}^\varepsilon_u}{\varepsilon\omega(\varepsilon)}\varphi^\pm + \dfrac{\tilde{f}^\varepsilon_v}{\varepsilon\omega(\varepsilon)}\psi^\pm = \tau\mu(\varepsilon)\varphi^\pm \\ \psi^\pm_{yy} - \dfrac{c(\varepsilon;\tau)}{\omega(\varepsilon)^{1/2}}\psi^\pm_y + \dfrac{\tilde{g}^\varepsilon_u}{\omega(\varepsilon)}\varphi^\pm + \dfrac{\tilde{g}^\varepsilon_v}{\omega(\varepsilon)}\psi^\pm = \mu(\varepsilon)\psi^\pm \end{cases}, y \in \mathbf{R}_\pm$$
$$\varphi^\pm(\pm\infty) = 0,\ \varphi^\pm(0) = a$$
$$\psi^\pm(\pm\infty) = 0,\ \psi^\pm(0) = b,$$

where $\tilde{f}^\varepsilon_u = f_u\left(u\left(\dfrac{y}{\omega(\varepsilon)^{1/2}};\varepsilon;\tau\right), v\left(\dfrac{y}{\omega(\varepsilon)^{1/2}};\varepsilon;\tau\right)\right)$ and $\tilde{f}^\varepsilon_v$, $\tilde{g}^\varepsilon_u$, $\tilde{g}^\varepsilon_v$ are defined similarly. In order to construct approximate solutions of $(A.9)_\pm$, we consider the following reduced problems:

$$(A.10)_\pm \quad \begin{cases} \varepsilon(\varphi^\pm_0)_{yy} = \tau\mu(0)\varphi^\pm_0 \\ (\psi^\pm_0)_{yy} = \mu(0)\psi^\pm_0 \end{cases}, y \in \mathbf{R}_\pm$$
$$\varphi^\pm_0(\pm\infty) = 0,\ \varphi^\pm_0(0) = a$$
$$\psi^\pm_0(\pm\infty) = 0,\ \psi^\pm_0(0) = b.$$

Note that $|\mu(0)| \neq 0$. We can obtain solutions of $(A.10)_\pm$ explicitly as follows:

$$\begin{cases} \varphi^\pm_0(y;\varepsilon;\tau;\mu(0);a,b) = a\cdot\exp[\mp(\tau\mu(0)/\varepsilon)^{1/2}y] \\ \psi^\pm_0(y;\mu(0);a,b) = b\cdot\exp[\mp\mu(0)^{1/2}y] \end{cases}, y \in \mathbf{R}_\pm.$$

Using a standard technique of (regular) perturbation method, we can construct exact solutions $(\varphi^{\pm}, \psi^{\pm})(y; \varepsilon; \tau; \mu(\varepsilon); a, b)$ of $(\mathrm{A.9})_{\pm}$ satisfying

$$\begin{cases} \varepsilon \| (\varphi^{\pm} - \varphi_0^{\pm})_{yy} \|_{C(\mathbf{R}_{\pm})} + \varepsilon^{1/2} \| (\varphi^{\pm} - \varphi_0^{\pm})_y \|_{C(\mathbf{R}_{\pm})} \\ \qquad\qquad + \| \varphi^{\pm} - \varphi_0^{\pm} \|_{C(\mathbf{R}_{\pm})} \longrightarrow 0 \qquad \text{as } \varepsilon \downarrow 0 \\ \| (\psi^{\pm} - \psi_0^{\pm})_{yy} \|_{C(\mathbf{R}_{\pm})} + \| (\psi^{\pm} - \psi_0^{\pm})_y \|_{C(\mathbf{R}_{\pm})} \\ \qquad\qquad + \| \psi^{\pm} - \psi_0^{\pm} \|_{C(\mathbf{R}_{\pm})} \longrightarrow 0 \qquad \text{as } \varepsilon \downarrow 0. \end{cases}$$

Then, $(\hat{\varphi}^{\pm}, \hat{\psi}^{\pm})(z; \varepsilon; \tau; \lambda; a, b) \equiv (\varphi^{\pm}, \psi^{\pm})(\omega(\varepsilon)^{1/2} z; \varepsilon; \tau; \mu(\varepsilon); a, b)$ are solutions of $(\mathrm{A.6})_{\pm}$. Therefore we obtain the following relations:

$$\mathbf{U}_1(0; \varepsilon; \tau; \lambda) = \begin{bmatrix} \hat{\varphi}^{+}(0; \varepsilon; \tau; \lambda; 1, 0) \\ \varepsilon \hat{\varphi}_z^{+}(0; \varepsilon; \tau; \lambda; 1, 0) \\ \hat{\psi}^{+}(0; \varepsilon; \tau; \lambda; 1, 0) \\ \hat{\psi}_z^{+}(0; \varepsilon; \tau; \lambda; 1, 0) \end{bmatrix} = \begin{bmatrix} 1 + o(1) \\ -(\tau \mu(0) \varepsilon \omega(\varepsilon))^{1/2} (1 + o(1)) \\ o(1) \\ o(1) \end{bmatrix},$$

$$\mathbf{U}_2(0; \varepsilon; \tau; \lambda) = \begin{bmatrix} \hat{\varphi}^{+}(0; \varepsilon; \tau; \lambda; 0, 1) \\ \varepsilon \hat{\varphi}_z^{+}(0; \varepsilon; \tau; \lambda; 0, 1) \\ \hat{\psi}^{+}(0; \varepsilon; \tau; \lambda; 0, 1) \\ \hat{\psi}_z^{+}(0; \varepsilon; \tau; \lambda; 0, 1) \end{bmatrix} = \begin{bmatrix} o(1) \\ o(1) \\ 1 + o(1) \\ -(\mu(0) \omega(\varepsilon))^{1/2} (1 + o(1)) \end{bmatrix},$$

$$\mathbf{U}_3(0; \varepsilon; \tau; \lambda) = \begin{bmatrix} \hat{\varphi}^{-}(0; \varepsilon; \tau; \lambda; 1, 0) \\ \varepsilon \hat{\varphi}_z^{-}(0; \varepsilon; \tau; \lambda; 1, 0) \\ \hat{\psi}^{-}(0; \varepsilon; \tau; \lambda; 1, 0) \\ \hat{\psi}_z^{-}(0; \varepsilon; \tau; \lambda; 1, 0) \end{bmatrix} = \begin{bmatrix} 1 + o(1) \\ (\tau \mu(0) \varepsilon \omega(\varepsilon))^{1/2} (1 + o(1)) \\ o(1) \\ o(1) \end{bmatrix}$$

and

$$\mathbf{U}_4(0; \varepsilon; \tau; \lambda) = \begin{bmatrix} \hat{\varphi}^{-}(0; \varepsilon; \tau; \lambda; 0, 1) \\ \varepsilon \hat{\varphi}_z^{-}(0; \varepsilon; \tau; \lambda; 0, 1) \\ \hat{\psi}^{-}(0; \varepsilon; \tau; \lambda; 0, 1) \\ \hat{\psi}_z^{-}(0; \varepsilon; \tau; \lambda; 0, 1) \end{bmatrix} = \begin{bmatrix} o(1) \\ o(1) \\ 1 + o(1) \\ (\mu(0) \omega(\varepsilon))^{1/2} (1 + o(1)) \end{bmatrix}$$

as $\varepsilon \downarrow 0$. By virtue of (A.2), we have

$$\begin{aligned} \text{(A.11)} \qquad g(\varepsilon; \tau; \lambda) &= (K_1 K_2)^{-1} \{ -4(\tau \mu(0) \varepsilon \omega(\varepsilon))^{1/2} \cdot (\mu(0) \omega(\varepsilon))^{1/2} \cdot (1 + o(1)) \} \\ &= -4(K_1 K_2)^{-1} (\varepsilon \tau \lambda(\varepsilon))^{1/2} \cdot \lambda(\varepsilon)^{1/2} \cdot (1 + o(1)) \neq 0 \end{aligned}$$

for sufficiently small ε. In particular, when λ is real, (A.11) implies that, by (A.5), $g(\varepsilon; \tau; \lambda) > 0$ holds. This completes the proof. ■

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Department of Mathematics,
Faculty of Science,
Toyama University