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Abstract homotopy theory and homotopy theory of functor category

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Introduction

Many topologists have studied the various homotopy theories (the equivariant homotopy theory [2, 13] and the ex-homotopy theory [8, 21] etc.). Such homotopy theories have the common back-grounds. For example Puppe's theorem [19] and J. H. C. Whitehead's theorem [25] etc. hold also in these homotopy theories. Therefore these phenomena must be treated in the systematic manner. In this paper we study an axiomatization of homotopy theories mentioned above and deduce the various fundamental theorems systematically. Moreover we study the homotopy theory of the functor category in detail. Our theory enable us to treat the n-ad homotopy theory and equivariant homotopy theory in the unified manner. By introducing the cell structure in the functor category, we obtain the detailed results (e. g. the obstruction theory).

In part I we give an axiomatization of homotopy theory based on the cylinder and path functors (cf. [9, 10]). By introducing the extension condition and natural homotopy axioms 1, 2, we can obtain the various fundamental theorems (e. g. Puppe's theorem). Our axiomatization satisfies the duality principle and is closed under the constructions of the functor category and the comma category. Hence we can obtain the various fundamental theorems in the various homotopy theories in the systematic manner (cf. [8, 13, 16]).

In part II we study the homotopy theory of the functor category in detail. Let $\mathscr{D}$ be a topological small category, CGH the category of compactly generated Hausdorff spaces and continuous mappings. Let $\mathscr{F}$ = Cont Funct ($\mathscr{D}$, CGH) be the functor category whose objects are continuous contravariant functors and morphisms are natural transformations between them. The category $\mathscr{F}$ becomes an abstract homotopy category in the sense of Part I. By introducing $\mathscr{D}$-orbits D_a: $\mathscr{D} \to \mathrm{CGH}$ defined by $D_a(x) = \mathscr{D}(x, a)$ (hom-set in $\mathscr{D}$), we define the functor complex over $\mathscr{D}$ which is the natural generalization of equivariant CW complexes (cf. (2, 13)) and n-ad CW complexes. By the the natural isomorphism $\mathscr{F}(D_a \times T, X) = \mathrm{CGH}(T(a), X(a))$ where T is a constant functor and X a continuous functor, it is shown that Puppe's theorem, the celllar approximation theorem and J. H. C. Whitehead's theorem hold also in the category of functor complexes. We can also develop the obstruction theory and Postnikov systems etc. as same as the ordinary homotopy theory.

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Part I. Abstract homotopy theory

§ 1. Axioms of abstract homotopy theory

To reproduce systematically the results of D. Puppe [19] and J. H. C. Whitehead [25] in the various homotopy theories [2,8], we shall define an abstract homotopy theory. In this paper, we use the terminologies and notations on the category given in S. MacLane [12].

DEFINITION 1.1. We call a category $\mathscr{C}$ a *pre-homotopy category* if it satisfies the following axioms (A1-3):

(A1) $\mathscr{C}$ is closed under finite limits and finite colimits; hence it has the initial object ϕ and the terminal object 1, and is closed under finite (co) products, pullbacks and pushouts.

(A2) There are given covariant functors I, $P: \mathscr{C} \to \mathscr{C}$ with a natural isomorphism $\mathscr{C}(IA, B) = \mathscr{B}(A, PB)$ for any objects A, B of $\mathscr{C}$ ($\mathscr{C}(-, -)$ is the hom-set in $\mathscr{C}$); hence I(resp. P) preserves ϕ and colimits (resp. 1 and limits). We call these the *cylinder and path functors* and the *adjoint isomorphism*, respectively.

(A3) Moreover there are three natural transformations k_*: Id $\to I(k = 0, 1)$ and τ: $I \to$ Id with $\tau\, 0_* = \mathrm{Id} = \tau 1_*$. Here Id means the identity functor or identity natural transformation. 0_*, 1_* and τ are called the *top-face*, *bottom-face* and *projection transformations*, respectively.

DEFINITION 1.2. Let I^n be the n-time composed functor of I in (A2) (I^0 $= \mathrm{Id}$); and define the natural transformations

$$d_j^k = I^{n-j} k_* I^j: I^n \longrightarrow I^{n+1} \text{ and } s_j = I^{n-j} \tau I^j\colon I^{n+1} \longrightarrow I^n$$

for $(j, k) \in [n] \times [1]([m] = \{0, 1, \cdots, m\})$ using k_* and τ in (A3). We call these the *face and degeneracy operators*, respectively.

PROPOSITION 1.3. *These operators* d_j^k, s_j *satisfy the cubical simplicial relations*

$$d_i^k \cdot d_j^h = d_{j+1}^h \cdot d_i^k \,(i \leqq j); \qquad s_i s_j = s_j s_{i+1}\, (i \geqq j);$$
$$s_j \cdot d_i^k = d_i^k \cdot s_{j-1}\,(i < j), \ = \mathrm{Id}\ (i = j), = d_{i-1}^k s_j (i > j)$$

PROOF. (A3) shows the relations immediately. □

Let $(i_0, k_0) \in [n] \times [1]$. Then by patching the $2n+1$ faces $d_i^k\colon I^n \to I^{n+1}$ for $(i, k) \neq (i_0, k_0)$, or by taking the pushout of the $2n+1$ copies of I^n according to the relations in Lemma 1.3, we obtain the following

LEMMA 1.4. *There are the functors $J^n = J^n(i_0, k_0)$ and the natural transformations $\bar{d}_i^k\colon I^n \to J^n$ for $(i, k) \neq (i_0, k_0)$ and $\lambda\colon J^n \to I^{n+1}$ satisfying the following properties* (1)–(3):

(1) $\bar{d}_i^k \cdot d_j^h = \bar{d}_{j+1}^h \cdot d_i^k\colon I^{n-1} \longrightarrow J^n (i \leqq j)$ *and* $\lambda \bar{d}_i^k = d_i^k\colon I^n \longrightarrow I^{n+1}$.

(2) $f_i^k\colon I^n X \to Y$ *for* $(i, k) \neq (i_0, k_0)$ *with the compatibility conditions* $f_i^k d_j^h = f_{j+1}^h d_i^k\colon J^n X \to Y (i < j)$ *give us uniquely* $f = \{f_i^k\}\colon J^n X \to Y$ *with* $f\bar{d}_i^k = f_i^k$.

(3) *Let* $\tau_n\colon I^n \to \mathrm{Id}$ *be the composed transformation of projections* τ, and $\tau_n'\colon J^{n-1} \to \mathrm{Id}$ *the induced projection. Then these satisfy* $\tau_n \lambda = \tau_n'$.

We use the letter J^n for any (i_0, k_0). When $n = 0$, we have $J^0 = \mathrm{Id}$ and $\lambda = k_*\colon J^0 \to I$ $(k = 0, 1)$.

Now we consider the following *extension condition* and *natural homotopy axioms* for a pre-homotopy category $\mathscr{C}$, where J^n, λ, τ_n and τ_n' are the above ones:

(EC) For any morphism $f\colon J^n X \to Y$, there is a morphism $F\colon I^{n+1} X \to Y$ with $F\lambda = f$.

(NHA 1) There is a natural transformation $\mu\colon I^n \to J^{n-1}$ with $\mu\lambda = \mathrm{Id}$ for all $n > 0$, that is, (EC) holds naturally by taking $F = f\mu$ for all $n > 0$.

(NHA 2) There is a natural transformation $\mu\colon I^n \to J^{n-1}$ with $\tau_n' \mu = \tau_n$ and $\mu\lambda = \mathrm{Id}$ for all $n > 0$.

DEFINITION 1.5. Let $\mathscr{C}$ be a pre-homotopy category.

(1) We call $\mathscr{C}$ an *abstract homotopy category* if it satisfies (NHA 2).

(2) We say that two morphisms $f_0, f_1\colon X \to Y$ are *homotopic* (*relative* $j\colon A \to X$), if there is a morphism $f\colon IX \to Y$ with $f_k = fk_*$ for $k = 0, 1$ (and $fIj = f_0 j\tau$); and then we write $f_0 \simeq f_1$ (rel j) and call f a *homotopy* of f_0 and f_1.

(3) Let $f, g\colon IX \to Y$ be homotopies with $f1_* = g0_*$ and consider $h_j^k\colon IX \to Y$ given by $h_1^0 = f$, $h_0^1 = g$ and $h_1^1 = g1_*$. Then if $\mathscr{C}$ satisfies (EC), we have $H\colon I^2 X \to Y$ with $Hd_i^k = h_j^k$ for $(i, k) \neq (0, 0)$. We define $f \oplus g = Hd_0^0$ which is unique up to homotopy relative $\dot{I} = \{0_* \amalg 1_*\}$.

(4) Consider $h_i^k\colon I^2 X \to IX$ given by $h_1^0 = 1_*\tau = h_0^1$ and $h_1^1 = \mathrm{Id}$. If $\mathscr{C}$ satisfies (EC), we have $H\colon I^2 X \to IX$ with $Hd_i^k = h_i^k$ for $(i, k) \neq (0, 0)$. We define $\iota = Hd_0^0\colon IX \to IX$ which is unique up to homotopy relative $\dot{I}$.

PROPOSITION 1.6. *Let $\mathscr{C}$ be a pre-homotopy category satisfying* (EC). *Then we have the following results.*

(1) *The homotopy relation* $\simeq$ (*resp.* $\simeq$ rel $\dot{I}$) *is an equivalence relation compatible with composition.*

(2) *If f and f': $IX \to Y$ (resp. g and g') are homotopic relative $\dot{I}$ and satisfy $f1_* = g0_*$, then $f \oplus g$ and $f' \oplus g'$ are homotopic relative $\dot{I}$.*

(3) *Let f, g, h: $IX \to Y$ be homotopies with $f1_* = g0_*$, $g1_* = h0_*$. Then $(f \oplus g) \oplus h$ and $f \oplus (g \oplus h)$ are homotopic relative $\dot{I}$.*

(4) *Let f, g: $IX \to Y$, h: $W \to X$, and k: $Y \to Z$ be homotopies with $f1_* = g0_*$. Then $(f \oplus g)Ih$ and $fIh \oplus gIh$ (resp. $k(f \oplus g)$ and $kf \oplus kg$) are homotopic relative $\dot{I}$.*

(5) *There is a morphism v: $I^2X \to IX$ with $vd_0^0 = \mathrm{Id} = vd_1^0$ and $vd_0^1 = 1_*\tau = vd_1^1$.*

(6) $\mathrm{Id} \oplus 1_*\tau, 0_*\tau \oplus \mathrm{Id}$ *and* Id *are homotopic relative $\dot{I}$.*

(7) $\mathrm{Id} \oplus \iota$ *and* $0_*\tau$ *(resp.* $\iota \oplus \mathrm{Id}$ *and* $1_*\tau$*) are homotopic relative $\dot{I}$.*

(8) u *and* Id *are homotopic relative $\dot{I}$.*

If $\mathscr{C}$ satisfies (NHA1), *then the above results hold also for natural transformations.*

PROOF. (1) Let f: $IX \to Y$ be a homotopy of f_0 and f_1. Then $f\iota$ gives a homotopy of f_1 and f_0. The other cases are proved analogously.

(2), (3) and (4) are proved by elementary calculations.

(5) Let $\bar{\lambda}$: $\dot{I}^2 \to I^2$ be the canonical natural transformation where $\dot{I}^2$ is the boundary of I^2 (i.e. the patching of 4 copies of I) and α: $\dot{I}^2 \to I$ the natural transformation with $\alpha\bar{d}_0^0 = \mathrm{Id} = \alpha\bar{d}_1^0$ and $\alpha\bar{d}_0^1 = 1_*\tau = \alpha\bar{d}_1^1$ where $\bar{d}_i^k$: $I \to \dot{I}^2$ is defined by $\bar{\lambda}\bar{d}_i^k = d_i^k$. Since α and $0_*\tau\alpha = 0_*\tau_2\bar{\lambda}$ are homotopic, there exists v: $I^2X \to IX$ with $v\bar{\lambda} = \alpha$ by (EC) (cf. Definition 2.1.)

(6) and (7) are proved by using the definitions of v and ι.

(8) By (6) and (7), we have $\iota \oplus u \simeq 1_*\tau \,\mathrm{rel}\, \dot{I}$. Then $\mathrm{Id} \simeq \mathrm{Id} \oplus 1_*\tau \simeq \mathrm{Id} \oplus \iota \oplus u \simeq 0_*\tau \oplus u \simeq u$ by (3), (4), (6) and (7). □

Here we note on the dual considerations.

Corresponding to Id in $\mathscr{C}(IA, IA)$ and $\mathscr{C}(PA, PA)$ by the adjoint isomorphism in (A2), we have η: $\mathrm{Id} \to PI$ (called the *unit*) and ε: $IP \to \mathrm{Id}$ (called the *counit*). Thus we have the following axiom (A3*) which is dual and equivalent to (A3) by taking $k^* = \varepsilon k_* P (k = 0, 1)$ and $\sigma = P(\tau)\eta$:

(A3*) There are three natural transformations $k^*: P \to \mathrm{Id}$ $(k = 0, 1)$ and $\sigma: \mathrm{Id} \to P$ with $0^*\sigma = \mathrm{Id} = 1^*\sigma$, called the *top-coface*, *bottom-coface* and *injection transformations*, respectively.

DEFINITION 1. 2*. For the composed functors P^n: $\mathscr{C} \to \mathscr{C}$ of P, we have the natural transformations

$$\partial_i^k = P^{n-i}(k^*)P^i;\ P^{n+1} \to P^n \text{ and } \sigma_i = P^{n-i}(\sigma)P^i:\ P^n \to P^{n+1}$$

for $(i, k) \in [n] \times [1]$, which are dual to d_i^k and s_i in Definition 1. 2.

These satisfy the cubical simplicial relations dual to the relations in Proposition 1.3.

DEFINITION 1.4*. As the adjoint to J^n with λ: $J^n \to I^{n+1}$ in Definition 1.4, we can define the functor $Q^n: \mathscr{C} \to \mathscr{C}$ with the natural transformation $\pi: P^{n+1} \to Q^n$ by the dual construction, ($Q^0 = \mathrm{Id}$ and $\pi = k^*$ ($k = 0, 1$) when $n = 0$).

These give us the axioms (EC*) and (NHA*1, 2) which are dual and equivalent to (EC) and (NHA 1, 2), respectively.

EXAMPLE 1.7. (1) The category CGH(= CGHaus in [12]) of compactly generated Hausdorff spaces and continuous mappings *is our abstract homotopy category*. Here the cylinder and path functors are given by

$$IX = X \times I \text{ (the product space) and } PX = X^I \text{ (the path space)}$$

for the closed interval $I = [0, 1]$. (A1–2) are contained in the following (a0–2) (see [12; VII §8], [22] and (A3) and (NHA) hold as usual:

(a0) *In* CGH, *the product* $X \times Y$ ($= X \square Y$ in [12]) *and the function space* Y^X *are given by the Kellyfications of the usual product topology and the compact-open topology.*

(a1) CGH *is closed under* (*infinite*) *limits and colimits, and* $\underrightarrow{\mathrm{Lim}}\,(X_i \times Y) = (\underrightarrow{\mathrm{Lim}}\,X_i) \times Y$ *holds.*

(a2) *The exponential law* $(Z^Y)^X = Z^{X \times Y}$ *holds, and the composition* $Y^X \times Z^X$ *is continuous.*

We note that so is the category Top (resp. Haus) of topological (resp. Hausdorff) spaces in the same way (see [12; V§9)).

(2) The pointed category CGH_*(cf. (12; VII §9], [22]) of CGH-spaces with base point $*$ and continuous mappings preserving $*$ *is also an abstract homotopy category*, by taking $IX = X \wedge I^+$ (the smash product with $I^+ = I \amalg *$) and $PX = X^I$ (the path space with the constant mapping $*$ as a base point), since CGH_* satisfies (a1–2) replacing $\times$ by $\wedge$ which is the product in CGH_*. (Note that CGH_* is the comma category CGH^*_* of CGH given in Theorem 1.9 below.)

(3) In the category CGH, we consider a topological group G and the category CGH_G of G-spaces and continuous G-mappings. Then for any G-space X, the above $IX = X \times I$ and $PX = X^I$ are G-spaces by the G-action on X; and CGH_G *becomes an abstract homotopy category*, as is shown by (a1–2).

(4) For a ring R with unit 1, consider the category Ch_R of chain complexes over R and chain maps. Let I be the chain complex with $I_0 = R \oplus R$ generated by e_0, e_1, $I_1 = R$ generated by f, $I_i = 0$ ($i \neq 0, 1$) and $\partial_1: I_1 \to I_0$ given by $\partial_1 f = e_0 - e_1$; and R with $R_0 = R$ and $R_i = 0$ ($i \neq 0$). Then we have I and P in (A2) by $IC = C \otimes I$ and $PC = \mathrm{Hom}\,(I, C)$ and k_* and τ in (A3) induced by the cahin maps $k_*: R \to I$, $k_*(1) = e_k$, and $\tau: I \to R$, $\tau(e_k) = 1$, respectvely, so that Ch_R *becomes an abstract homotopy category* (cf. [4]).

Let $\mathscr{C}$ be a pre-homotopy category. Consider the functor category $\mathscr{F}$

$=\mathrm{Funct}(\mathscr{D}, \mathscr{C})$ for a small category $\mathscr{D}$ of contravariant functors $\mathscr{D} \to \mathscr{C}$ and natural transformations between them. For this category, (A1) follows from (A1) for $\mathscr{C}$, I and P in (A2) are induced from I and P for $\mathscr{C}$ by composing them, and so are k_* and τ in (A3); and we see easily the following

THEOREM 1.8. *If $\mathscr{C}$ is a pre-homotopy category, then so is the functor category* Funct $(\mathscr{D}, \mathscr{C})$ $(\mathscr{D}$: *a small category*) *by I, P, k_* and τ induced from those in* (A2–3) *for $\mathscr{C}$. If $\mathscr{C}$ satisfies* (NHA 1) *or* (NHA 2) *in addition, then so does* Funct $(\mathscr{D}, \mathscr{C})$.

Moreover, we consider the comma category (cf. [12; II-6) $\mathscr{C}_B^A$ for fixed objects A, B and fixed morhism a: $A \to B$ in $\mathscr{C}$, whose object is any diagram $A \xrightarrow{x} X \xrightarrow{p} B$ in $\mathscr{C}$ with $px = a$, and whose morphism f: $(A \xrightarrow{x} X \xrightarrow{p} B) \to (A \xrightarrow{y} Y \xrightarrow{q} B)$ is any morphism f: $X \to Y$ in $\mathscr{C}$ with $fx = y$ and $qf = p$. For $A \xrightarrow{x} X \xrightarrow{p} B$, I, P, k_* and τ in (A2–3) for $\mathscr{C}$ give us the diagrams

$$\begin{tikzcd}
IA \arrow[r, "Ix"] \arrow[d, "\tau"'] \arrow[dr, phantom, "\mathrm{PO}"] & IX \arrow[d, "\tau'"] & \\
A \arrow[r, "\bar{x}"'] & \bar{I}X \arrow[r, shift right, "\bar{\tau}"'] & X, \arrow[l, shift right, "\bar{k}_*"']
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, shift left, "\tilde{\sigma}"] & \tilde{P}X \arrow[l, shift left, "\tilde{k}^*"] \arrow[r, "\tilde{p}"] \arrow[d, "\sigma'"'] \arrow[dr, phantom, "\mathrm{PB}"] & B \arrow[d, "\sigma"] \\
& PX \arrow[r, "Pp"'] & PB
\end{tikzcd}$$

(PO (resp. PB) means that the square is the pushout (resp. pullback) diagram) with $\bar{\tau}\bar{x} = x$, $\bar{\tau}\tau' = \tau$, $\bar{k}_* = \tau' k_*$, $\tilde{p}\tilde{\sigma} = p$, $\sigma'\tilde{\sigma} = \sigma$ and $\tilde{k}^* = k^*\sigma'$. Hence we have $A \xrightarrow{\bar{x}} \bar{I}X \xrightarrow{\bar{p}} B (\bar{p} = p\bar{\tau})$ and $A \xrightarrow{\tilde{x}} \tilde{P}X \xrightarrow{\tilde{p}} B (\tilde{x} = \tilde{\sigma}x)$ in $\mathscr{C}_B^A$, the functors $\bar{I}$, $\tilde{P}$: $\mathscr{C}_B^A \to \mathscr{C}_B^A$ and the natural transformations $\bar{\tau}$: $\bar{I} \to \mathrm{Id}$, $\bar{k}_*$: $\mathrm{Id} \to \bar{I}$, satisfying (A2–3). For example for any f: $\bar{I}X \to Y$ with $f\bar{x} = y$ and $qf = \bar{p}$, we have $\bar{f}$: $X \to PY$ corresponding $f\tau'$ by $\mathscr{C}(IX, Y) = \mathscr{C}(X, PY)$, and $\tilde{f}$: $X \to \tilde{P}Y$ with $\sigma'\tilde{f} = \bar{f}$ and $\tilde{q}\tilde{f} = p$; and the adjoint isomorphism for $\mathscr{C}_B^A$ is given by sending f to $\tilde{f}$. Moreover, λ and μ in (NHA 2) for $\mathscr{C}$ induce those for $\mathscr{C}_B^A$, and we see the following

THEOREM 1.9. *If $\mathscr{C}$ is a pre-homotopy category, then so is the comma category $\mathscr{C}_B^A$. Moreover if $\mathscr{C}$ satisfies* NHA 2, *then so does the comma category $\mathscr{C}_B^A$.*

§ 2. Fundamental properties

Let $\mathscr{C}$ be a pre-homotopy category in Definition 1.1.

Definition 2.1. We say that $j: A \to X$ in X in $\mathscr{C}$ (resp. $p: Y \to B$) has the property HEP_n (resp. HLP_n), if any commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & P^n Z \arrow[d, "\pi"] \\ X \arrow[r] \arrow[ur, dashed] & Q^{n-1} Z \end{tikzcd} \qquad \text{(resp.} \quad \begin{tikzcd} J^{n-1} Z \arrow[r] \arrow[d, "\lambda"'] & Y \arrow[d, "p"] \\ I^n Z \arrow[r] \arrow[ur, dashed] & B \end{tikzcd} \text{)}$$

has a dotted morphism to obtain two commutative triangles; and that it has HEP (resp, HLP) if so does HEP_n (resp. HLP_n) for all $n \geqq 1$.

We note that (EC) is equivalent to HEP for any morphism $\phi \to X$ or HLP for any $X \to 1$. HEP_1 (resp. HLP_1) is known as the homotopy extension (resp. lifting) property, and coincides with HEP (resp. HLP) (cf. [22; §§, 6–7])

Lemma 2.2. *For any morphism $f: X \to Y$ in $\mathscr{C}$, consider the diagram*

$$\begin{tikzcd} & X \arrow[r, "f"] \arrow[d, "1_*"'] \arrow[dr, phantom, "\mathrm{PO}"] & Y \arrow[d, "i"] & & \\ X \arrow[r, shift left, "0_*"] & IX \arrow[l, shift left, "\tau"] \arrow[r, "h"'] & M(f) \arrow[r, "p"'] & IY \arrow[r, "\tau"'] & Y \end{tikzcd}$$

of the mapping culinder $M(f)$, where $pi = 1_*$ and $ph = If$. Then

$$f = qj: X \longrightarrow M(f) \longrightarrow Y \quad \textit{for} \quad j = h0_* \quad \textit{and} \quad q = \tau p.$$

If $\mathscr{C}$ satisfies (EC) (resp. (NHA 1)), *then i and q are homotopy equivalences with $iq \simeq \mathrm{Id}$, and j has* HEP_1 *(resp.* HEP).

Proof. Assume (EC) for $\mathscr{C}$. Then we have a morphism $\xi: I^2 X \to IX$ with $\xi d_1^0 = Id$ and $\xi d_0^1 = 1_* \tau = \xi d_1^1$ by (EC), and so $\rho: IM(f) \to M(f)$ with $\rho Ih = h\xi$ and $\rho Ii = i\tau$ by the pushout obtained by applying I to the above one (see (A2)). Now ρ is a homotopy of Id and iq; and q is a homotopy equivalence.

Consider the diagram

$$\begin{tikzcd}
J^{n-1}X \arrow[r, "J^{n-1}0_*"] \arrow[d, "\lambda_X"'] \arrow[dr, phantom, "\mathrm{PO}"] & J^{n-1}IX \arrow[r, "J^{n-1}h"] \arrow[d, "v"] \arrow[dr, phantom, "\mathrm{PO}"] & J^{n-1}M(f) \arrow[d, "v'"] \arrow[dr, "\lambda"] & I^nIX \arrow[d, "I^nh"'] \arrow[dr, phantom, "\mathrm{PO}"] & I^nX \arrow[l] \arrow[d, "I^nf"] \\
I^nX \arrow[r, "u"'] & W \arrow[r, "u'"'] & W' \arrow[r, "j'"'] & I^nM(f) & I^nY \arrow[l, "I^ni"]
\end{tikzcd}$$

with $j'u'u = I^nj$ and $j'v' = \lambda$. Then to show $j = h0_*$ has HEP, it is sufficient to construct a retraction $R\colon I^nM(f) \to W'$ with $Rj' = \mathrm{Id}$ by the above definition. Now assume (NHA 1) for $\mathscr{C}$ and consider $\mu\colon I^n \to J^{n-1}$ with $\mu\lambda = \mathrm{Id}$ and $J^{n-1}f\mu_X = \mu_Y I^nf$. Then by (EC), there exists

$$\mu'\colon I^nIX \longrightarrow W \text{ with } \mu'\lambda' = \mathrm{Id} \text{ and } \mu'I^n1_* = vJ^{n-1}1_*\mu_X$$

for $\lambda'\colon W \to I^nIX$ with $\lambda' v = \lambda_X$ and $\lambda' u = I^n0_*$. Hence we have $R\colon I^nM(f) \to W'$ with $RI^nh = u'\mu'$ and $RI^ni = vJ^{n-1}i\mu_Y$, and $Rj' = \mathrm{Id}$ is seen as desired, by the definition of pushouts and by noticing that pushouts are also preserved by J^{n-1}. When $n = 1$, this proof is valid by taking $\mu = \tau$ in (A3). □

PROPOSITION 2.3. *In $\mathscr{C}$ satisfying* (EC), *consider the pushout diagrams*

$$\begin{tikzcd}
A \arrow[r, "f_k"] \arrow[d, "j"'] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[d, "j_k"] \\
X \arrow[r, "g_k"'] & M_k
\end{tikzcd} \qquad (k = 0, 1).$$

(1) *If j has* HEP_n, *then so does j_k.*

(2) *Assume* HEP *for j. If f_0 and f_1 are homotopic, then there exists morphisms $m_k\colon M_k \to M_l$ $(l = k - 1)$ with $m_kj_k = j_l$ and $m_kg_k \simeq g_l$ and homotopies $h_k\colon IM_k \to M_k$ of $m_lm_k \simeq \mathrm{Id}$ with $h_kIj_k = j_k\tau\colon IB \to M_k$* (i.e., *$m_k$'s are cofiber homotopy equivalences*).

PROOF. (1) is easily seen by the definition of HEP_n.

(2) For $F_k\colon IA \to B$ of $f_k \simeq f_l$, there is $G_l\colon IX \to M_l$ with $G_l1_* = g_l$ and $G_lIj = j_lF_k$ by HEP for j. Then $j_lf_k = G_l0_*j$, and we have $m_k\colon M_k \to M_l$ with $m_kj_k = j_l$ and $m_kg_k = G_l0_* \simeq g_l$ by G_l. Moreover $(m_lG_l \oplus G_k)Ij = j_k(F_k \oplus F_l) \simeq j_k\tau If_k$

rel $\dot{I}$ by taking $F_l = F_k\iota$(see Proposition 1.7). Thus we have h'_k: $IX \to M_k$ with $h'_kIj = j_k\tau If_k$ and $h'_k \simeq m_lG_l \oplus G_k$ rel $\dot{I}$ by HEP for j, and so h_k: $IM_k \to M_k$ with $h_kIj_k = j_k\tau$ and $h_kIg_k = h'_k$, which is a homotopy of $m_lm_k \simeq \mathrm{Id}$. □

By taking the family of morphisms satisfying HEP (resp. homotopy relation of morphisms) as Cof $\mathscr{C}$ (resp. congruence $\simeq$) of A. Heller's h-c-category [6, 7], we can easily see that our abstract homotopy category satisfies the axioms of A. Heller's h-c-category. Hence the results in [6, 20] hold also in our abstract homotopy category.

As we showed in §§3–4 in [20], we can deduce various propositions from Lemma 2.2 and Proposition 2.3. The proof of Proposition 2.4 (resp. 2.5) below is elementary and the same as one of Lemme 3.4 and Proposition 3.5 (resp. Lemme 3.2) in [20].

Hereafter, we work in our abstract homotopy category $\mathscr{C}$.

PROPOSITION 2.4. *For the pushout diagram in Proposition* 2.3., *the following results hold.*

(1) *If j has* HEP *and f_k is a homotopy equivalence, then g_k is a homotopy equivalence.*

(2) *If j has* HEP *and is a homotopy equivalence, then so does j_k.*

(3) *If f_k has* HEP $(k = 0, 1)$ *and $f_0 \simeq f_1$, then M_0 is homotopy equivalent to M_1.*

PROPOSITION 2.5. *Let consider the commutative diagram*

$$\begin{tikzcd}
A' \arrow[rrr] \arrow[dr, "h_1"] \arrow[ddd, "j'"'] & & & B' \arrow[dl, "h_2"'] \arrow[ddd] \\
& A \arrow[r] \arrow[d, "j"] & B \arrow[d] & \\
& X \arrow[r] & Y & \\
X' \arrow[rrr] \arrow[ur, "h_3"] & & & Y' \arrow[ul, "h_4"']
\end{tikzcd}$$

where the outer and inner squares are pushouts, j and j' have HEP, *h_i $(i = 1, 2, 3)$ are homotopy equivalences. Then h_4 is a homotopy equivalence.*

DEFINITION 2.6. (1) The *cone* and *suspension functors* C, $\Sigma: \mathscr{C} \to \mathscr{C}$ are defined by the pushout diagrams (1: the constant functor to the terminal object 1)

$$\begin{tikzcd}
\mathrm{Id} \arrow[r] \arrow[d, "1_*"'] \arrow[dr, phantom, "\mathrm{PO}"] & 1 \arrow[d, "\theta"] \\
I \arrow[r, "h"'] & C,
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Id} \arrow[r] \arrow[d, "j=h0_*"'] \arrow[dr, phantom, "\mathrm{PO}"] & 1 \arrow[d] \\
C \arrow[r] & \Sigma.
\end{tikzcd}$$

(2) For $j: A \to X$, the *shrinking* $X/f\,A$, the *mapping cone* $C(f)$ and the morphisms d, f_1, f_2, m_1 are defined by the diagrams

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{PO}"] & 1 \arrow[d] \\
X \arrow[r] & X/f A,
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] \arrow[dr, phantom, "\mathrm{PO}"] & X \arrow[r] \arrow[d, "f_1"] \arrow[dr, phantom, "\mathrm{PO}"] & CX \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{PO}"] & 1 \arrow[d] \\
CA \arrow[r, "d"'] & C(f) \arrow[r, "f_2"'] & C(f_1) \arrow[r, "m_1"'] & \Sigma A.
\end{tikzcd}$$

We can define the cone functor by using the natral transformation 0_*: Id $\to I$ instead of 1_*. Though these are not isomorphic, these are homotopy equivalent to the final functor 1 by Lemma 2.2. We can also define the top-coface fixed path functor L and the loop functor Ω by the diagrams dual to Defintion 2.6, respectively. When $\mathscr{C}$ has the zero object, there are natural isomorphisms $\mathscr{C}(CA, B) = \mathscr{C}(A, LB)$ and $\mathscr{C}(\Sigma A, B) = \mathscr{C}(A, \Omega B)$ for any objects A, B of $\mathscr{C}$. If $\mathscr{C}$ has not a zero object, these don't hold.

Let Ho $\mathscr{C}$ be a quotitent category of $\mathscr{C}$ by the homotopy relation (cf. (12; II, §8)). Let $nI = I \oplus I \oplus \cdots \oplus I$ $(n \geqq 1)$ be the cylinder functor of length n obtained by patching n-copies of cylinder. Then we can define the cone functor C_n and suspension functor Σ_n of length n by using nI. The functors $\Sigma_n (n \geqq 1)$ are naturally equivalent each other on Ho$\mathscr{C}$ by Proposition 2.5. (cf. [20; §4]). Of course C_n is homotopy equivalent to the functor 1 for all $n \geqq 1$.

THEOREM 2.7. *Let $\mathscr{C}$ be an abstract homotopy category. Then we have the following*

(1) *If* $f\colon A \to X$ *has* HEP, *then* $C(f) \simeq X/fA$.
(2) *For* $f\colon A \to X$, $C(f_1) \simeq \Sigma A$ *holds.*
(3) *If* f_0 *and* f_1 *are homotopic, then* $C(f_0) \simeq C(f_1)$.
(4) *There holds the commutative diagram*

$$\begin{tikzcd}
\mathrm{M}f\colon A \arrow[r, "f"] \arrow[d, equal] & X \arrow[r, "f_1"] \arrow[d, equal] & C(f) \arrow[r, "f_2"] \arrow[d, equal] & C(f_1) \arrow[r, "f_3"] \arrow[d, "m_1"] & C(f_2) \arrow[r, "f_4"] \arrow[d, "m_2"] & C(f_3) \arrow[r] \arrow[d, "m_3"] & \cdots \\
\mathrm{P}f\colon A \arrow[r] & X \arrow[r] & C(f) \arrow[r] & \Sigma A \arrow[r, "\iota\Sigma f"'] & \Sigma X \arrow[r, "\iota\Sigma f_1"'] & \Sigma C(f) \arrow[r] & \cdots
\end{tikzcd}$$

of natural sequences, where $\mathrm{M}f$ *consists of the iterated mapping cones, each* m_i *is a homotopy equivalence and* f_i *has* HEP, *and* $\iota\colon \Sigma \to \Sigma$ *is induced by* $\iota\colon I \to I$ *in Definition* 1.5.

Proof. (1) Since $A \to X$ has HEP and CA is homotopy equivalent to 1, $C(f)$ is homotopy equivalent to X/fA by Proposition 2.4(1).

(2) and (3) are proved analogous to (1).

(4) To discuss exactly, we distinguish the cone functors C^+, C^- defined by the colimits of diagrams $I \xleftarrow{1^*} \mathrm{Id} \to 1$ and $I \xleftarrow{0^*} \mathrm{Id} \to 1$, respectively. We write $C(f_1)$, ΣX and $C(f_2)$ as $C^-X \cup C^+A$, $C^-X \cup C^+X$ and $C^-X \cup C^+C(f)$. By iterating the construction in Definition 2.6 we have shrinkings $m_1\colon C(f_1) \to \Sigma A = C^+A/A$ and $m_2\colon C(f_2) \to \Sigma X = C^-X/X$ which are homotopy equivalences. Let $k^{\mp}\colon C^-X \cup C^+X \to C^{\mp}X/X$ be shrinkings. Then $(k^-)(k^+)^{-1}\colon \Sigma X = C^+X/X \to \Sigma X = C^-X/X$ is homotopy equivalent to $\iota\colon \Sigma X \to \Sigma X$ by Definition 1.5. Hence by the above constructions, $m_2 f_3\colon C(f_1) \to C^-X \cup C^+X \xrightarrow{\simeq} C(f_2) \xrightarrow{\simeq} C^-X/X$ is homotopy equvalent to $\iota\Sigma f m_1\colon C(f) \to C^+A/A \to C^+X/X \xrightarrow{\simeq} C^-X/X$. Since f_3 has HEP, we can choose $m_2\colon C(f_2) \to \Sigma X = C^-X/X$ such that $m_2 f_3 = \iota\Sigma f_1 m_1$. □

When our abstract homotopy category $\mathscr{C}$ has the zero object 0 (i.e. $\phi = 1$), there is the natural transformations $\mu\colon \Sigma \simeq \Sigma_2 \to \Sigma \vee \Sigma$ and $\iota\colon \Sigma \to \Sigma$ on Ho $\mathscr{C}$ obtained by collapsing the equator and reflecting the cylinder, respectively. These satisfy the relations

$$(\mu \vee \mathrm{Id})\mu = (\mathrm{Id} \vee \mu)\mu,\ (\mathrm{Id} \vee \iota)\mu = 0 = (\iota \vee \mathrm{Id})\mu,\ (\mathrm{Id} \vee 0)\mu = \mathrm{Id} = (0 \vee \mathrm{Id})\mu.$$

Moreover it is proved that $\mu\Sigma$, $\Sigma\mu\colon \Sigma^2 \to \Sigma^2 \vee \Sigma^2$ are naturally equivalent each other on Ho $\mathscr{C}$. The relation $(\Sigma\mu \vee \Sigma\mu)\mu\Sigma \simeq (\mu\Sigma \vee \mu\Sigma)\Sigma\mu$ holds also as usual.

From these considerations and Theorem 2.7 we obtain the following

Corollary 2.8. *If* $\mathscr{C}$ *has the zero object, then for the homotopy sets* $\mathscr{C}[-, Y]$ *with* $0 = [0]$, *the sequence* $\mathrm{P}f$ *in Throrem 2.7 induces the exact sequence*

$$\mathscr{C}[A, Y] \xleftarrow{f^*} \mathscr{C}[X, Y] \xleftarrow{f_1^*} \mathscr{C}[C(f), Y] \xleftarrow{d_1^*} \mathscr{C}[\Sigma A, Y] \longleftarrow \cdots$$

i.e., Ker $\varphi = \varphi^{-1}(0) = \mathrm{Im}\ \varphi'$ *for successive maps* $S \xleftarrow{\varphi} S' \xleftarrow{\varphi'} S''$.*Moreover this is the exact one of groups and homomo phisms except for the first three, and the group* $\mathscr{C}[\Sigma A, Y]$ *acts on* $\mathscr{C}[C(f), Y]$ *by the shrinking* $C(f) = X \cup C_2 A \to X \cup CA \vee \Sigma A$.

We also note that $\mathrm{P}f$ induces the exact sequences for a half exact functor H in [5; § 5] instead of $\mathscr{C}[-, Y]$. The above results are generalized to the Mayer-Vietoris sequence and Milnor's one.

Part II. Homotopy theory of functor categories

§ 3. Functor spaces and functor complexes

Throughout Part II, we are concerned with the category CGH of compactly generated Hausforff spaces, which is an abstract homotopy category by Example 1.7(1); and we consider spaces and their (co)limits etc. in this category unless otherwise stated, and use the properties (a0–2) in Example 1.7(1) frequently.

DEFINITION 3.1. (1) We call a category $\mathscr{D}$ a *topological category* if any set of morphisms $\mathscr{D}(a, b)$ is a CGH-space and the composition $\mathscr{D}(a, b) \times \mathscr{D}(b, c) \to \mathscr{D}(a, c)$ is continuous for any objects a, b, c in $\mathscr{C}$, and a contravariant functor $F: \mathscr{D} \to \mathscr{D}$ between them *continuous* if the induced map $F_*: \mathscr{D}(a, b) \to \mathscr{C}(F(b), F(a))$ is continuous for any objects a, b in $\mathscr{D}$.

(2) We consider always CGH as the topological category by $\mathrm{CGH}(T, S) = S^T$ in (a0). Hereafter we fix a topological small category $\mathscr{D}$, and consider the functor category

$$\mathscr{F} = \mathscr{F}(\mathscr{D}) = \mathrm{Cont\ Funct}\ (\mathscr{D}, \mathrm{CGH})$$

whose objects (called *functor spaces over* $\mathscr{D}$ or $\mathscr{F}$*-spaces* simply) are continuous contravariant funtors $X: \mathscr{D} \to \mathrm{CGH}$ and whose morphisms (called $\mathscr{F}$-morphisms) are natural transformations between them. Also we have the pointed category $\mathscr{F}_* = \mathscr{F}_*(\mathscr{D}) = \mathrm{Cont\ Funct}\ (\mathscr{D}, \mathrm{CGH}_*)$ of $\mathscr{F}$-spaces $X = (X, *)$ *with based point* $*$ (or $\mathscr{F}$*-spaces* $X \ni *$) by the pointed category CGH_* in Example 1.7(2).

(3) Any object c in $\mathscr{D}$ gives us the continuous functor $D_c: \mathscr{D} \to \mathrm{CGH}$ by $D_c(a) = \mathscr{D}(a, c)$ and $D_c(f) = f_*: \mathscr{D}(b, c) \to \mathscr{D}(a, c)$ for any a and $f: a \to b$ in $\mathscr{D}$; and we call this $\mathscr{F}$-space D_c the $\mathscr{D}$-orbit of c. We regard any CGH-space T as the $\mathscr{F}$-space T by $T(a) = T$ and $T(f) = \mathrm{Id}: T \to \mathrm{T}$.

(4) Consider the category $\Delta(n)$ of ordered objects $0 < 1 < \cdots < n$ whose

morphisms consist of unique $i \to j$ for $i \leqq j$. The category $\mathscr{F}_{n+1}$ of $(n+1)$-*ples* $X \leftarrow A_1 \leftarrow \leftarrow A_n$ *of* $\mathscr{F}$*-spaces* is considered as $\mathscr{F}_{n+1} = \text{Funct}\,(\Delta(n), \mathscr{F}) = \mathscr{F}(\mathscr{D} \times \Delta[n])$ by the product category $\mathscr{D} \times \Delta[n]$(with the usual topology). For $n = 2$, we call $\mathscr{F}_2$ the *category of pairs of* $\mathscr{F}$*-spaces*. We call $A \subset X$ *a relative pair* if $A(a)$ *is a closed subspace of* $X(a)$ for any a in $\mathscr{D}$.

THEOREM 3.2. *The categories* $\mathscr{F}$ and $\mathscr{F}_*$ *are closed under* (*co*) *limits and become abstract homotopy categories in our sense by the cylinder and path functors obtained by composing those for* CGH *and* CGH_*.

PROOF. Let $\{X_i\}$ be a direct system in $\mathscr{F}$, and consider the adjoint $\text{ad}(X_i)$: $X_i(b) \times \mathscr{D}(a, b) \to X_i(a)$ of X_i: $\mathscr{D}(a, b) \to \text{CGH}(X_i(b), X_i(a))$, which are continuous by the definition and the exponetial law in (a2). Then by taking $X(a) = \varinjlim X_i(a)$ in CGH we have $\mathscr{D}(a, b) \times X(b) = \varinjlim (\mathscr{D}(a, b) \times X_i(b)) \to X(a)$ by (a1) and its adjoint $\mathscr{D}(a, b) \to \text{CGH}(X(b), X(a))$ which are also continuous. Thus $X = \varinjlim X_i$ is well-defined in $\mathscr{F}$; and so is the inverse limit analogously. For the path functor P on CGH, P_*: $\text{CGH}(S, T) \to \text{CGH}(PS, PT)$ is continuous since the adjoint $\text{ad}\,P$: $T^S \times S^I \to T^I$ is continuous; hence we have P: $\mathscr{F} \to \mathscr{F}$ which is the continuous functor. We have I for $\mathscr{F}$ in the same way; and we see the theorem according to Theorem 1.8. Analogously we have the results for $\mathscr{F}_*$. $\square$

We now introduce the notion of functor complexes as follows, where ϕ is the empty spaces, V^n is the unit n-ball and $S^{n-1} = \dot{V}^n$ the $(n-1)$-sphere ($V^0 = *$, $S^{-1} = \phi$) in CGH.

DEFINITION 3.3. (1) We call K(*resp.* (K, L)) a *functor compex over* $\mathscr{D}$ or $\mathscr{F}$*-complex* (resp. *relative one*), if $K^{-1} = \phi$(resp. L) and $K = \varinjlim K^n$ in $\mathscr{F}$ for $\mathscr{F}$-spaces K^n with j_n: $K^{n-1} \to K^n$, which are constructed inductively by the pushout diagrams

$$(*) \qquad \begin{tikzcd} \amalg_i D_{a_i} \times S^{n-1} \arrow[r] \arrow[d, "\amalg_i f_i"'] \arrow[dr, phantom, "\text{PO}"] & \amalg_i D_{a_i} \times V^n \arrow[d] \\ K^{n-1} \arrow[r] & K^n \end{tikzcd}$$

in $\mathscr{F}$ for some objects a_i in $\mathscr{D}$, $\mathscr{F}$-morphisms f_i and the upper $\mathscr{F}$-morphism induced by $S^{n-1} \subset V^n$. Here $\amalg$ stands for the direct sum in $\mathscr{F}$ and $D_c \times T$ is the product $\mathscr{F}$-space of the $\mathscr{D}$-orbit and a CGH-space T, i.e., $(D_c \times T)(a) = \mathscr{D}(a, c) \times T$ and $(D_c \times T)(f) = \mathscr{D}(f, c) \times T$ (cf. Definition 3.1(3)). We call K^n the *n-skeleton*, $D_{a_i} \times V^n$ the *n-cells attached by* f_i, and the maximum integer of *n's appeared in* (*) *the dimension of* K (resp. (K, L)).

Note that $j_n(a)$: $K^{n-1}(a) \to K^n(a)$ satisfies the properties in the following :

(a3) Let $A \subset X$ in CGH_2 satisfy the properties that A is regular in X(i.e.,

A and $x \in X - A$ are separated by open sets in X; hence A is closed) and is a neighborhood retract of X. Then for the pushout $Y = X \cup_f B$ of $X \supset A \xrightarrow{f} B$ in CGH, we have $B \subset Y$ with the same properties.

(2) A *pair* (K_0, K_1) *of* $\mathscr{F}$*-complexes*, or an $\mathscr{F}$*-subcomplex* K_1 of an $\mathscr{F}$-complex K_0 is defined to be a functor complex K over $\mathscr{D} \times \Delta[1]$ with $K_h(a) = K(a, h)$ (cf. Definition 3.1(3)). Then K_1 consists of n-cells $D_{(a_i,1)} \times V^n$ and K_0 of those together with $D_{(a_i,0)} \times V^n$. Also we have an $\mathscr{F}$*-complex* $K = (K, *)$ *with base point* $*$, which is an $\mathscr{F}_*$-space.

We see the following lemma by definition, which is basic in our theory

LEMMA 3.4. *For any* $\mathscr{F}$*-space* X *and* $D_a \times T$ *(*T*: a constant functor), the natural isomorphisms*

$$\mathscr{F}(D_a \times T, X) = \mathrm{CGH}(T(a), X(a)), \quad \mathscr{F}[D_a \times T, X] = \mathrm{CGH}[T(a), X(a)].$$

hold by corresponding $\alpha: D_a \times T \to X$ *and* $\bar{\alpha}: T(a) \to X(a)$ *satisfying* $\bar{\alpha}(t) = \alpha_a(\mathrm{Id}_a, t)$ *and* $\alpha(f, t) = X(f)(\bar{\alpha}(t))$ $(f: a \to c,\ t \in T)$. *Moreover we have the relation for* $T \supset S$ *in* CGH_2 *and* $X \supset A$ *in* $\mathscr{F}_2$, *etc. As examples,* $\mathscr{F}(D_c \times V^n, D_c \times S^{n-1}; X, A)) = \mathrm{CGH}(V^n, S^{n-1}; X(c), A(c))$; *and* $\mathscr{F}_*[D_a^+ \wedge T, X] = \mathscr{F}[D_a \times T, D_a \times *; X, *] = \mathrm{CGH}_*[T(a), X(a)]$ *for* $T \ni *$ *in* CGH_* *and* $X \ni *$ *in* $\mathscr{F}_*$, *where* $(D_a^+ \wedge T)(c) = \mathscr{D}(c, a)^+ \wedge T$ $(\mathscr{D}(c, a)^+ = \mathscr{D}(c, a) \amalg *)$.

Here (X, A) (resp. X) is *N-connected* if $\pi_n(X(a), A(a), x)$ (resp. $\pi_n(X(a), x)) = 0$ for any a in $\mathscr{D}$, $x \in A(a)$ (resp. $X(a)$) and $0 \leqq n \leqq N$.

THEOREM 3.5. *Let* (K, L) *be a relative* $\mathscr{F}$*-complex.*

(1) *The canonical morphisms* $j_n: K^{n-1} \to K^n$ *and* $L \to K$ *have* HEP; *hence Theorem* 2.7 *is valid for these morphisms.*

(2) *If a pair* (X, A) *of* $\mathscr{F}$*-space is N-connected, then any* $\mathscr{F}$*-morphism* $f:(K, L) \to (X, A)$ *is homotopic relative* L *to some* $\mathscr{F}$*-morphism* $g: K \to X$ *with* $g(K^n) \subset A$.

(3) (Cellular approximation theorem) *Let* $f: (K, L) \to (K', L')$ *be an* $\mathscr{F}$*-morphism between relative* $\mathscr{F}$*-complexes. Then there exists* $g: (K, L) \to (K', L')$ *which is homotopic relative* L *to* f *and satisfy* $g(K^n) \subset K'^n$ *for all* n.

PROOF. (1) In the pushout diagram $(*)$ in Definition 3.3 (1), the upper $\mathscr{F}$-morphism has HEP by Lemma 3.4, and so does j_n by Proposition 2.3 (1).

(2) is seen by Lemma 3.4, (1) and the induction on K^n; and (2) implies (3), because (K, K^n) is n-connected by definition and Lemma 3.4. □

THEOREM 3.6 (J. H. C. Whitehead's theorem). *Let* $f: (X, A) \to (Y, B)$ *be an* $\mathscr{F}$*-morphism between pairs of* $\mathscr{F}$*-complexes. If the induced homomorphism*

$$f_*: \pi_n(X(a), x) \longrightarrow \pi_n(Y(a), f(x)) \text{ and } f_*: \pi_n(A(a), x) \longrightarrow \pi_n(B(a), f(x))$$

are bijective for $n \leqq \max(\dim X, \dim Y)$, *any* a *in* $\mathscr{D}$ *and* $x \in A(a)$, *then* f *is* *homotopy equivalence in* $\mathscr{F}_2$.

PROOF. The assumptions imply that $(M(f), X)$ is N-connected for the mapping cylinder $M(f) \supset X$ of f; hence we have $g\colon Y \to X$ with $fg \simeq \mathrm{Id}$ by Theorem 3.5 (2). Then $g(a)_*\colon \pi_n(Y(a), y) \cong \pi_n(X(a), g(a)(y))$ for $n \leqq N$, and we have $f'\colon X \to Y$with $gf' \simeq \mathrm{Id}$ in the same way. Thus we have the absolute case, and also the relative case by the considerations in $\mathscr{F}_2$. □

We now define a coefficient system over a topological small category $\mathscr{D}$.

DEFINITION 3.7. Let $G = Ab$(resp. Gr) be the category of abelian groups (resp. groups) (for $n \geqq 2$ (resp. $n = 1$) in (2)-(3) and Lemma 3.8 below).

(1) For any $f\colon a \to b$ in $\mathscr{D}$, let $[f] \in \pi_0(\mathscr{D}(a, b))$ be its path-component. Then we call a contravariant functor $M\colon \mathscr{D} \to G$ a *coefficient system over* $\mathscr{D}$ if $M(f) = M(f')$ when $[f] = [f']$; and we have the functor category $\mathrm{CS} = \mathrm{CS}(\mathscr{D})$ of them.

(2) For a 0-connected $\mathscr{F}_*$-space X, the *homotopy group* $\pi_n(X)\colon \mathscr{D} \to Gr$ is given by $\pi_n(X)(a) = \pi_n(X(a))$ and $\pi_n(X)(f)_*$. We call X an *Eilenberg-MacLane* $\mathscr{F}$*-space of type* (M, n) if $\pi_n(X) = M$ and $\pi_i(X) = 0$ for $i \neq n$.

(3) For any c in $\mathscr{D}$, put $P_c^n = \pi_n(D_c^+ \wedge S^n)$ for $D_c^+ \wedge T$ in Lemma 3.4.

(4) Also define $P_c\colon \mathscr{D} \to \mathrm{Ab}$ by $P_c(a) = \tilde{H}_0(\mathscr{D}(a, c)^+; Z)$, and $P_c(f)[g] = [gf]$, where $[g] \in P_c(b)$ denotes the generator represented by $g \in \mathscr{D}(b, c)$.

LEMMA 3.8. (1) $P_c = P_c^n$ *when* $n \geqq 2$; *and* $P_c^1(a)$ *is the free group generated by the elements* $[f]$ *represented by* $[f^+ \wedge S^1\colon S^1 \to \mathscr{D}(a, c)^+ \wedge S^1]$ *for* $f\colon a \to c$.

(2) $\mathrm{CS}\,(P_c, M) = M(c)$ *by corresponding* $\alpha\colon P_c \to M$ *in* CS *and* $\bar{\alpha} \in M(c)$ *satisfying* $\bar{\alpha} = \alpha[\mathrm{Id}]$ *and* $\alpha[f] = M(\mathrm{f})(\bar{\alpha})(\mathrm{Id}\colon c \to c,\ f\colon a \to c)$

(3) P_c *is projective in* CS; *and* $\mathrm{CS}\,(P_c^n, \pi_n(X)) = \mathscr{F}_*[D_c^+ \wedge S^n; X]$.

This lemma is proved by definition and Lemma 3.4.

THEOREM 3.9. *For any coefficient system* $M\colon \mathscr{D} \to \mathrm{Ab}$ (*resp.* Gr) *and* $n > 1$ (*resp.* $n = 1$), *there is an Eilenberg-MacLane* $\mathscr{F}$*-complex* $K(M, n)$ *of type* (M, n), *which is unique up to homotopy type.*

PROOF. By Lemma 3.8, we have a projective resolution of M, $P_1 \xrightarrow{\phi} P_0 \to M \to 0$ where $P_0 = \bigoplus_j P_{c_j}$, $P_1 = \bigoplus_k P_{d_k}$($\bigoplus$ satnds for the sum (resp. free product)); and $\phi\colon P_1 \to P_0$ is represented by a morphism $f\colon \vee_k D_{d_k}^+ \wedge S^n \to \vee_j D_{c_j}^+ \wedge S^n$ by Lemma 3.8. Then we can construct the mapping cone $C(f)$ which satisfies $\pi_i(C(\mathrm{f})) = 0$ and M for $i < n$ and $i = n$, respectively. Iterating the above process for the higher dimensions $\pi_{n+1}(C(f))$ etc, we can obtain the result inductively. □

EXAMPLE 3.10. (1) Let G be a topological group in CGH, and consider the category CGH_G of G-spaces in Example 1.7. The quotient spaces G/H of left cosets in Haus (hence in CGH) and G-maps between them form the category $\mathcal{O}_G$ of orbit types. Then $\mathcal{O}_G$ becomes a small topological category according to the natural isomorphism $\mathcal{O}_G(G/H, G/K) = (G/K)^H$. Any G-space X can be identified with the $\mathscr{F}(\mathcal{O}_G)$-space X: $\mathcal{O}_G \to$ CGH given by $X(G/H) = X^H$ (the invariant set of H) or $X = X(e)$ (e: the unit group). A G-CW complex is defined by a colimit $\varinjlim X^n$, by taking $\amalg_j G/H_j \times S^{n-1} \to \amalg_j G/H_j \times V^n$ in place of the upper morphisms in Definition 3.3(1). The above correspondences and definitions induce the one-to-one correpondence between $\mathscr{F}(\mathcal{O}_G)$-complexes and G-CW complexes.

(2) Let $\Delta[n]$ be the category defined in Definition 3.1(4). Then there are $n+1$ types of $\mathscr{D}$-orbits $D_0 = \{* \leftarrow \phi \cdots \leftarrow \phi\}$, $D_1 = \{* \leftarrow * \leftarrow \cdots \leftarrow \phi\}$, $D_n = \{* \leftarrow \cdots \leftarrow *\}$, and any $\mathscr{F}(\Delta[n])$-complex can be identified with an n-ad CW complex $(X, A_1, \cdots, A_n)$ with $X \supset A_1 \supset \cdots \supset A_n$ by Definition 3.3.

(3) Let $V(n)$ be the category of the n-time products of $\Delta[1]$. Then there are 2^n types of $V(n)$-orbits and $\mathscr{F}(V(n))$-complex can be identified with an n-ad CW complex $(X; A_1, \cdots, A_n)$ with $X \supset A_i (i = 1, \cdots, n)$ by Definition 3.3(1).

§4. Cohomology theory on $\mathscr{F} = \mathscr{F}(\mathscr{D})$

Let (X, A) be a relative $\mathscr{F}$-complex with the n-skeleton X^n and the canonical morphism j_n: $X^{n-1} \to X^n$, and consider the shrinking $X^n_m = X^n/X^m$ by the composition $X^m \to X^n (m < n)$ of them. Then

$$X^n_{n-1} = X^n/X^{n-1} = (\amalg_i D_{a_i} \times V^n)/\amalg_i D_{a_i} \times S^{n-1} = \vee_i D^+_{a_i} \wedge S^n$$

and $X^{n-1} \to X^n$ induces the Puppe sequence

$$X^{n-1}_{n-2} \longrightarrow X^n_{n-2} \longrightarrow X^n_{n-1} \xrightarrow{d_n} \Sigma X^{n-1}_{n-2}.$$

Now we define $C_n(X, A)$: $\mathscr{D} \longrightarrow$ Ab in CS($\mathscr{D}$) and ∂_n: $C_n(X, A) \longrightarrow C_{n-1}(X, A)$ by

$$C_n(X, A)(a) = \bigoplus_i \tilde{H}_0(\mathscr{D}(a, a_i)^+; Z) = \tilde{H}_n(X^n_{n-1}(a); Z),$$

$$\partial_n(a) = (d_n)_*: C_n(X, A)(a) = \tilde{H}_n(X^n_{n-1}(a); Z) \longrightarrow$$

$$C_{n-1}(X, A)(a) = \tilde{H}_n(\Sigma X^{n-1}_{n-2}(a); Z)$$

and $C_n(A, A) = 0$ and $\partial_{n+1} = 0$ for $n < 0$. Since $(d_n)_*(d_{n+1})_* = 0$, we have the chain complex $C_n(X, A)$ and the homology

$$H_n(X, A): \mathscr{D} \longrightarrow \text{Ab given by } H_n(X, A)(a) = \operatorname{Ker} \partial_n(a)/\operatorname{Im} \partial_{n+1}(a).$$

Moreover any cellular $\mathscr{F}$-morphism f: $(X, A) \to (Y, B)$ with $f(X^n) \subset Y^n$ induces the $\mathscr{F}$-morphism f^n: $X^n/X^{n-1} \to Y^n/Y^{n-1}$ with $\Sigma f^{n-1}_{n-2}\, d_n = d_n f_{n-1}$ and a natural transformations

$$f_* = (f^n_{n-1})_*: C_n(X, A) \longrightarrow C_n(Y, B) \text{ and } f_*: H_n(X, A) \longrightarrow H_n(Y, B).$$

DEFINITION 4.1. For a relative $\mathscr{F}$-complex (X, A) and a coefficient system $M: \mathscr{D} \to \mathrm{Ab}$ in $\mathrm{CS}(\mathscr{D})$, we put

$$C^n(X, A; M) = \mathrm{CS}(\mathscr{D})(C_n(X, A), M), \quad \delta^n = \mathrm{CS}(\mathscr{D})(\partial_n, M)$$

and $H^n(X, A; M) = \mathrm{Ker}\ \delta^n/\mathrm{Im}\ \delta^{n-1}$ as usual. Also a cellular $\mathscr{F}$-morphism f: $(X, A) \to (Y, B)$ induces

$$f^* = \mathrm{CS}(\mathscr{D})(f_*, M): C^n(Y, B; M) \longrightarrow C^n(X, A; M) \text{ and}$$

$$f^* = H^n(Y, B; M) \longrightarrow H^n(X, A; M).$$

Thus we have the *cohomology theory* $\{H^n\}$ on the category $\mathscr{F}_2$ of paris of $\mathscr{F}$-complexes and cellular $\mathscr{F}$-morphisms.

DEFINITION 4.2. A (*generalized*) *cohomology theory* on the category $\mathscr{F}_2$ is defined to be sequence of contravariant functors h^n: $\mathscr{F}_2 \to \mathrm{Ab}$ $(-\infty < n < +\infty)$ with natural transformations δ^n: $h^n(A, B) \to h^{n+1}(X, A)$ for any triple $X \supset A \supset B$, such that the following axioms are satisfied.

(1) Homotopy axiom: If $f_0, f_1: (X, A) \to (Y, B)$ are homotopic in $\mathscr{F}_2$, then $h^n(f_0) = h^n(f_1)$.

(2) Excision axiom: The inclusion $i: (X, X \cap A) \to (X \cup A, A)$ induces an isomorphism $h^n(X \cup A, A) = h^n(X, X \cap A)$.

(3) Exactness axiom: Let $(X \supset A \supset B)$ be a triple of $\mathscr{F}$-complexes, then the following sequence is exact.

$$\cdots \longrightarrow h^n(X, A) \longrightarrow h^n(X, B) \longrightarrow h^n(A, B) \xrightarrow{\delta^n} h^{n+1}(X, A) \longrightarrow \cdots.$$

We set $h^n(X)$ for $h^n(X, \phi)$.

THEOREM 4.3. *The cohomology theory H^n defined in Definition* 4.1 *becomes a generalized cohomology theory on $\mathscr{F}_2$ and satisfies the dimension axiom* (*i.e.* $H^n(D_a; M) = 0$ *for* $n \neq 0$, *any object* a *in* $\mathscr{D}$).

PROOF. The excision axiom, exactness axiom, dimension axiom and the functorial properties are clearly satisfied by the definitions. We shall only verify the homotopy axiom. Let $f_0, f_1: (X, A) \to (Y, B)$ be two cellular $\mathscr{F}$-morphisms and F a cellular homotopy of f_0 and f_1. F induces the chain map F_n: $C_n(I \times X, I \times A; M) \to C_n(Y, B; M)$. We define D: $C_n(X, A; M) \to C_{n+1}(Y, B; M)$ by $D(a) = F_n(I \otimes a)$. Clearly it holds $\partial_{n+1}D_n + D_{n-1}\partial_n = (f_0)_* - (f_1)_*$. Hence, by taking the cohomology group, the homotopy axiom hold. $\square$

The category $\mathrm{CS}(\mathscr{D})$ becomes a Grothendieck category by Lemma 3.8 and Theorem 1 in §4.7 of [18]. Projective objects $\{P_a | a$: object in $\mathscr{D}\}$ are generators in CS($\mathscr{D}$) by Lemma 3.8. Hence by Theorem 1 in §4.9 of [18], the category CS($\mathscr{D}$) admits injective resolutions. By using the method of homological algebra, we can obtain the universal coefficient theorem (cf. [11; XI]).

THEOREM 4.4. *Let M be a coefficient system and (X, A) a relative $\mathscr{F}$-complex. Then there is a spectral sequence $\{E_r^{s,t}, d_r\}$ converging to $H^*(X, A; M)$ where $E_2^{s,t} = \mathrm{Ext}^s(H_t(X, A); M)$ and* Ext *considered in* CS($\mathscr{D}$).

We can develop the obstruction theory on the category of $\mathscr{F}$-complexes by the same way to the ordinary (equivariant) homotopy theory (cf. [2, 14, 24]). Then we have the following result.

THEOREM 4.5. *Let $K(M, n)$ be the Eilenberg-MacLane $\mathscr{F}$-complex of type (M, n) where M: $\mathscr{D} \to \mathrm{Ab}$. Then there is a natural isomorphism*

$$\mathscr{F}_*[X; K(M, n)] = H^n(X, M)$$

for any $\mathscr{F}_*$-complex X with base point, for all $n \geqq 1$. In particular there holds a natural isomorphism $\mathscr{F}_*[K(L, n); K(M, n)] = \mathrm{CS}(\mathscr{D})(L, M)$.

EXAMPLE 4.6. (1) Let $\mathcal{O}_G$ be the category defined in Definition 3.10(1). Then the cohomology theory $\{H_n\}$ defined in Definition 4.10 is the same as the classical equivariant cohomology theory defined in §3 of [14].

(2) Let $\Delta[2]$ be the category defined in Definition 3.10(2). For the coefficient systems M_1, M_2 and M_3 given by $0 \to \pi$, $\pi \to \pi$ and $\pi \to 0$, respectively, the cohomology groups $H^n(\tilde{X}; M_i)(i = 1, 2, 3)$ of $\mathscr{F}$-complexes $\tilde{X} = (A \to X)$ are isomorphic to the ordinary cohomology groups $H^n(X, A; \pi)$, $H^n(X; \pi)$, and $H^n(A; \pi)$, respectively. Since the sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact, we have the long exact sequence induced by the coefflcient systems

$$H^n(X, A; \pi) \longrightarrow H^n(X: \pi) \longrightarrow H^n(A; \pi) \longrightarrow H^{n+1}(X, A; \pi) \longrightarrow \cdots$$

(3) Let $V(2)$ be the category defined in Definition 3.10(3). For the cefficient systems M_1, M_2', M_2'', M_3 given by the following diagrams

$$\begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 0 \arrow[dr] & & 0 \arrow[dl] \\ & \pi & \end{tikzcd} \qquad \begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ \pi \arrow[dr] & & 0 \arrow[dl] \\ & \pi & \end{tikzcd} \qquad \begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ 0 \arrow[dr] & & \pi \arrow[dl] \\ & \pi & \end{tikzcd} \qquad \begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ \pi \arrow[dr] & & \pi \arrow[dl] \\ & \pi & \end{tikzcd}$$

$$M_1 \qquad\qquad M_2' \qquad\qquad M_2'' \qquad\qquad M_3$$

the $\mathscr{F}$-cohomology groups $H^n(\tilde{X}, M_1)$, $H^n(\tilde{X}, M_2')$, $H^n(\tilde{X}, M_2'')$ and $H^n(\tilde{X}, M_3)$ of the $\mathscr{F}$-complex $\tilde{X} = (X, A, B)$ are isomorphic to the ordinary cohomology

groups $H^n(X, A \cup B; \pi)$, $H^n(X, B; \pi)$, $H^n(X, A \cap B; \pi)$ and $H^n(X, A \cap B; \pi)$, respectively. For $M_2 = M_2' \oplus M_2''$, we have $H^n(\tilde{X}; M_2) = H^n(X, A; \pi) \oplus H^n(X, B; \pi)$.
Since $0 \to M_1 \to M_2 \to M_3 \to 0$ is exact, we have the long exact sequence induced by the coefficient systems which is the same as the ordinary Mayer-Vietoris exact sequence.

Hereafter we work in the category $\mathscr{F}_*$ of (0-connected) $\mathscr{F}_*$-spaces.

DEFINITION 4.7. Let p: $X \to B$ be a fibration with a fiber F in $\mathscr{F}_*$ and X, B, F 0-connected $\mathscr{F}_*$-spaces. Then we say that p admits an *$\mathscr{F}_*$-Postnikov system* if there exists the following system:

(P1) There is a sequence of fibrations p_n: $X_n \to X_{n-1}$ with a fiber $K_n = K(\pi_n(F), n)$ for $n \geqq 1$ $(X_0 = B)$.

(P2) There is a sequence of $\mathscr{F}$-morphisms p^n: $X \to X_n$ which is $(n+1)$-connected $(n > 0)$. These satisfy $p_n\, p^n = p^{n-1}$ for all $n \geqq 1$.

Moreover when p_n: $X_n \to X_{n-1}$ is induced by k_n: $X_{n-1} \to \bar{K}_n = K(\pi_n(X), n+1)$ from the canonical fibration over $\bar{K}_n$, we say that p_n is a principal fibration. When p_n is a principal fibration for all $n > 0$, it is called that p: $X \to B$ admits a *principal $\mathscr{F}_*$-Postnikov system.*

By the standard arguments (cf. [23; 5, 24]), we have the following

THEOREM 4.8. *Let p: $X \to B$ be a fibration with a fiber F and X, B, F 0-connected $\mathscr{F}_*$-complexes. If $\pi_1(X)$ acts simply on $\pi_n(F) = \pi_{n+1}(B, X)$ for all $n \geqq 1$, then p admits a principal $\mathscr{F}_*$-Postnikov system.*

Let X be a 0-connected $\mathscr{F}$-complex with base point and Y a 0-connected simple $\mathscr{F}$-space. We assume that 0-skeleton X^0 of X is $*$. By using the filtration by skeletons of X, we can construct the homotopy spectral sequence as §1 in [17].

$$
(4.9)\qquad
\begin{aligned}
D_1^{s,t} &= \mathscr{F}_*[\Sigma^{t-s}X^s; Y] && (t \geqq s \geqq 0)\\
E_1^{s,t} &= \mathscr{F}_*[\Sigma^{t-s}X^sX^s/X^{s-1}; Y] && (t \geqq s \geqq 0)\\
E_1^{s,s-1} &= \mathscr{F}_*[\vee_j D_{a_j}^+ \wedge S^{s-1}; Y] && (s \geqq 1)
\end{aligned}
$$

where $\mathscr{F}_*[-; -]$ denotes the homotopy set in $\mathscr{F}_*$. Then the homotopy exact sequence yields an exact $\{D_r^{s,t}, E_r^{s,t}, \alpha_r, \beta_r, \gamma_r\}$ were α_1, β_1 and γ_1 are induced by j_s: $X^{s-1} \to X^s$, $\amalg_i f_i$: $\vee_i D_{a_i}^+ \wedge S^{s-1} \to X^s$ and k_s: $X^s \to X^s/X^{s-1}$, respectively. The E_2 term is described by the formula $E_2^{s,t} = H^s(X; \pi_t(Y))(t+1 \geqq s \geqq 0)$.

For the $\mathscr{F}_*$-Postnikov system of Y, we can construct the homotopy spectral sequence as §1 in [17] (cf. [15]).

$$
(4.10)\qquad
\begin{aligned}
D_2^{s,t} &= \mathscr{F}_*[X; \Omega^{t-s}Y_t]\ (t \geqq s \geqq 0)\\
E_2^{s,t} &= \mathscr{F}_* = \mathscr{F}_*[X; \Omega^{t-s+1}K_t] = H^s\,(X; \pi_t(Y))\ (t+1 \geqq s \geqq 0).
\end{aligned}
$$

Then the homotopy exact sequence yields an exact couple $\{\bar{D}_r^{s,t}, \bar{E}_r^{s,t}, \bar{\alpha}_r, \bar{\beta}_r, \bar{\gamma}_r\}$. where $\bar{\alpha}_1$, $\bar{\beta}_1$ and $\bar{\gamma}_1$ are induced by p_s: $Y_s \to Y_{s-1}$, k_s: $Y_s \to BK_{s+1}$ and i_s: $K_s \to Y_s$, respectively.

Now we can prove the next theorem by the same way as §1 in [17].

THEOREM 4.11. *Let X be a 0-connected $\mathscr{F}$-complex with base point and Y a 0-connected simple $\mathscr{F}_*$-space. Then there exist isomorphisms*

$$\phi\colon D_r^{s,t} \longrightarrow \bar{D}_r^{s,t}, \quad \psi\colon E_r^{s,t} \longrightarrow \bar{E}_r^{s,t}$$

which commutes with α_r, β_r, γ_r and $\bar{\alpha}_r$, $\bar{\beta}$, $\bar{\gamma}_r$.

Let $\mathscr{F}_*^0$ be the category of 0-connected $\mathscr{F}$-complexes with base point and cellular $\mathscr{F}_*$-morphisms, and $\mathrm{F}\mathscr{F}_*^0$ the full subcategory of $\mathscr{F}_*^0$ consisting of $\mathscr{F}$-complexes with finite cells, and $\mathrm{Ho}(\mathscr{F}_*^0)$ and $\mathrm{Ho}(\mathrm{F}\mathscr{F}_*^0)$ the quotient categories of $\mathscr{F}_*^0$ and $\mathrm{F}\ \mathscr{F}_*^0$ by the homotopy relation, respectively. Then we see easily the following

THEOREM 4.12. *A pair of the categories* $(\mathrm{Ho}(\mathscr{F}_*^0), \mathrm{Ho}(\mathrm{F}\mathscr{F}_*^0))$ *becomes an abstract homotopy category in the sense of* §2 *in* [3].

A functor K from $\mathrm{Ho}(\mathscr{F}_*^0)$ to the category of sets which satisfies (2.6) and (2.7) in [3] called a *Brown functor*. By the same way as [3], we can prove the representability theorem analogous to Theorem 2.8 in [3] (cf. [1]).

THEOREM 4.13. *A Brown functor $\mathscr{K}$ defined on* $\mathrm{Ho}(\mathscr{F}_*^0)$ *is representable, that is, there is a unique $\mathscr{Y}$ in $\mathscr{F}_*$ and a natural isomorphism T: $\mathscr{F}_*[X, Y] = K(X)$ for all X in $\mathscr{F}_*$.*

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