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# On weighted extremal length and $p$-capacity

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## 1. Introduction and statement of results

For $1 < p < \infty$, a weight $w$ (a nonnegative Lebesgue measurable function in the euclidean space $R^d$) is said to satisfy the Muckenhoupt $A_p$ condition ([4]) if

$$(A_p) \qquad \sup_Q \frac{1}{|Q|}\int_Q w dx \left(\frac{1}{|Q|}\int_Q w^{1/(1-p)}\, dx\right)^{p-1} < \infty\,,$$

where $Q$ is a cube with sides parallel to the axes and $|Q|$ stands for the volume of $Q$. By $A_p$ we denote the class of weights $w$ satisfying $(A_p)$. In this note we always assume that $w \in A_p$.

Let $\Gamma$ be a family of locally rectifiable curves in $R^d$. A nonnegative Borel measurable function $\rho$ in $R^d$ is called $\Gamma$-admissible if $\int_\gamma \rho\, ds \geqq 1$ for every $\gamma \in \Gamma$. We define the weighted module of order $p$ of $\Gamma$ by

$$M_p(\Gamma; w) = \inf\left\{\int_{R^d} \rho^p w dx;\ \rho \text{ is } \Gamma\text{-admissible}\right\}$$

and the weighted extremal length by the reciprocal of the weighted module.

Let $E$ be a compact set in $R^d$ and let $G$ be a domain containing $E$. The weighted $p$-capacity of the pair $(E, G)$ is defined to be

$$C_p^w(E; G) = \inf \int_G |\text{grad}\, u|^p w dx\,,$$

where the infimum is taken over all functions $u \in C_0^\infty(G)$ for which $u \geqq 1$ on $E$. If $G = R^d$, then we shall write $C_p^w(E)$ for $C_p^w(E; R^d)$.

Ziemer [7] gave a relation between extremal length and $p$-capacity, in case $w \equiv 1$. In this note we shall consider a similar relation between weighted extremal length and weighted $p$-capacity for $w \in A_p$. We shall first establish

THEOREM 1. *Suppose $R^d - E$ is a domain. Let $G$ be a bounded domain containing $E$ and let $\Gamma$ be the family of curves connecting $E$ and $\partial G$ in $G - E$. Then $M_p(\Gamma; w) = C_p^w(E; G)$.*

Ohtsuka [5, §6] proved Theorem 1 in a more general form in case $w$ is a positive continuous weight. The proof of Theorem 1 can be carried out along the same lines as in Ziemer [7]. Since the continuity of extremal distance with respect to $w \in A_p$ holds (Lemma 6), Theorem 1 implies

THEOREM 2. *Let $\Gamma_\infty$ be the family of curves in $R^d - E$ connecting $E$ and the point at infinity. Then $M_p(\Gamma_\infty; w) = C_p^w(E)$.*

We denote by $A_{p,1}$ (cf. [1]) the class of $w \in A_p$ satisfying the condition

$$(*) \qquad \int_{R^d} (1 + |x|)^{(1-d)p'} w(x)^{1/(1-p)}\, dx < \infty\,,$$

where $1/p + 1/p' = 1$. Using the above two theorems, we shall prove

THEOREM 3. *Let $w \in A_{p,1}$ and let $\bigwedge(E)$ be the family of curves in $R^d - E$ terminating at points of $E$. Then $M_p(\bigwedge(E); w) = 0$ if and only if $C_p^w(E) = 0$.*

In case $w \equiv 1$, the condition $(*)$ implies $p < d$. Therefore Theorem 3 is a generalization of Ziemer's result [7, Theorem 4.3]. Remark that $M_p(\bigwedge(E); w) = 0$ implies $C_p^w(E) = 0$ under the assumption $w \in A_p$. We shall give an example in which $M_p(\bigwedge(E); w) \neq C_p^w(E)$ for some $w \in A_p - A_{p,1}$.

## 2. Lemmas

Let $G$ be a domain in $R^d$. We write

$$L^{p,w}(G) = \left\{f;\ \int_G |f|^p w dx < \infty\right\} \quad \text{and} \quad \|f\|_{p,w} = \left(\int_G |f|^p w dx\right)^{1/p}.$$

Since $w^{1/(1-p)}$ is locally integrable, $f \in L^{p,w}(G)$ implies that $f$ is locally integrable.

For a locally integrable function $f$ in $G$, we define mollified functions $(f)_n$ of $f$ in $G$ by

$$(f)_n(x) = \int f\left(x + \alpha(x)\frac{\xi}{n}\right)\psi(|\xi|)\, d\xi \qquad (n = 1, 2, \ldots)\,,$$

where $\alpha(x)$ is a function in $C^\infty(G)$ such that $0 < \alpha < 1$, $|\mathrm{grad}\, \alpha| < 1/2$ and $2\alpha(x) < \mathrm{dist}(x, \partial G)$, and $\psi(r)$ is a nonnegative function on $0 \leqq r < \infty$ such that $\psi = 0$ on $1 \leqq r < \infty$, $\psi(|x|) \in C^\infty(R^d)$ and $\int_{R^d} \psi(|x|)\, dx = 1$. They are of class $C^\infty(G)$.

LEMMA 1 (cf. [6, Lemma 6]). *If $f$ belongs to $L^{p,w}(G)$, then $\|(f)_n - f\|_{p,w} \to 0$ as $n \to \infty$.*

PROOF. Let $f = 0$ on $R^d - G$. The maximal function of $f$ is defined by

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy\,.$$

Then

$$|(f)_n(x)| \leqq \left(\frac{n}{\alpha(x)}\right)^d \int |f(y)|\psi\left(\frac{n|x-y|}{\alpha(x)}\right) dy$$

$$\leqq \max|\psi| \left(\frac{n}{\alpha(x)}\right)^d \int_{|x-y|<\alpha(x)/n} |f(y)|\, dy$$

$$\leqq \text{const. } Mf(x)\,.$$

Since $(f)_n(x) \to f(x)$ as $n \to \infty$ for a.e. $x$ and $\|Mf\|_{p,w} \leqq \text{const. } \|f\|_{p,w}$ (cf. [4, Theorem 9]), the dominated convergence theorem yields that $\|(f)_n - f\|_{p,w} \to 0$ as $n \to \infty$.

We shall say that a function $f$ in $G$ is $ACL$ when $f$ is absolutely continuous on each component of the part in $G$ of almost every line parallel to each coordinate axis. If $f$ is $ACL$ in $G$, then grad $f$ exists a.e. in $G$.

LEMMA 2 (cf. [5, Theorem 4.5]). *Let $f$ be $ACL$ in $G$ and assume that* $|\text{grad}\, f|$ *belongs to* $L^{p,w}(G)$. *Then* $\|\text{grad}((f)_n - f)\|_{p,w} \to 0$ *as* $n \to \infty$.

PROOF. For the ordinary partial derivatives $\partial(f)_n/\partial x_i$ $(i = 1, 2, \ldots, d)$, we have

$$\frac{\partial(f)_n}{\partial x_i}(x) = \left(\frac{\partial f}{\partial x_i}\right)_n(x) + \int \left(\frac{\xi}{n}\cdot \text{grad}\, f(y)\right)\frac{\partial \alpha(x)}{\partial x_i}\psi(|\xi|)\, d\xi\,,$$

where $y = x + \alpha(x)\xi/n$. Set

$$A_i(x) = \int \left(\frac{\xi}{n}\cdot \text{grad}\, f(y)\right)\frac{\partial \alpha(x)}{\partial x_i}\psi(|\xi|)\, d\xi\,.$$

By Minkowski's inequality

$$\left\|\frac{\partial(f)_n}{\partial x_i} - \frac{\partial f}{\partial x_i}\right\|_{p,w} \leqq \left\|\left(\frac{\partial f}{\partial x_i}\right)_n - \frac{\partial f}{\partial x_i}\right\|_{p,w} + \|A_i\|_{p,w}\,.$$

Lemma 1 implies that $\|(\partial f/\partial x_i)_n - \partial f/\partial x_i\|_{p,w} \to 0$ as $n \to \infty$. Since $|\text{grad}\, \alpha| \leqq 1/2$, as in the proof of Lemma 1 we have

$$|A_i(x)| \leqq \frac{1}{2n}\int |\text{grad}\, f(y)|\psi(|\xi|)\, d\xi$$

$$\leqq \frac{\text{const.}}{n} M(|\text{grad}\, f|)(x)\,.$$

Since $\|M(|\text{grad}\, f|)\|_{p,w} \leqq \text{const.}\|\text{grad}\, f\|_{p,w}$ (cf. [4, Theorem 9]), we see

$$\|A_i\|_{p,w} \leqq \frac{\text{const.}}{n} \|\text{grad}\, f\|_{p,w} \to 0 \quad (n \to \infty).$$

Hence $\|\text{grad}((f)_n - f)\|_{p,w} \to 0$ as $n \to \infty$.

LEMMA 3 (cf. [5, Theorem 2.8]). *Let $G$ be a domain and let $F_0$, $F_1$ be mutually disjoint closed sets in $\partial G$ (the boundary of $G$ in the one-point compactification of $R^d$), and denote by $\Gamma$ the family of curves in $G$ connecting $F_0$ and $F_1$. If $M_p(\Gamma; w) < \infty$, then we may restrict $\Gamma$-admissible $\rho$ to be continuous in $G$ and bounded away from zero on $G \cap K$ for any compact set $K$ in $R^d$ in defining $M_p(\Gamma; w)$.*

PROOF. Let $\rho$ be a $\Gamma$-admissible function. Consider the mollified functions $(\rho)_n$ of $\rho$ in $G$. Considering the image $\gamma_\xi$ of $\gamma \in \Gamma$ by the transformation $x \mapsto x + (\alpha(x)/n)\xi$ for $\xi \in R^d$ with $|\xi| < 1$ and noting that $|\text{grad}\, \alpha| \leqq 1/2$, we see that $(1 + 1/(2n))(\rho)_n$ is $\Gamma$-admissible. By Lemma 1, $\|(\rho)_n\|_{p,w} \to \|\rho\|_{p,w}$ as $n \to \infty$. Thus we may restrict $\Gamma$-admissible function $\rho$ to be continuous in $G$ in defining $M_p(\Gamma; w)$. Now, let $\rho$ be a $\Gamma$-admissible function which is continuous in $G$. Given $\varepsilon > 0$, choose a sequence $\{\delta_k\}_{k=1}^{\infty}$ such that $\delta_k > 0$ and

$$\delta_k^p \int_{\{k-1 \leqq |x| < k\}} w\, dx < 2^{-k}\varepsilon.$$

Set $\rho_\varepsilon(x) = \max(\rho(x), \delta_k)$ if $k - 1 \leqq |x| < k$, for each positive integer $k$. Then $\rho_\varepsilon$ is $\Gamma$-admissible and

$$\int_G (\rho_\varepsilon)^p w\, dx \to \int_G \rho^p w\, dx \qquad \text{as } \varepsilon \to 0.$$

This establishes the lemma.

A sequence $\{\gamma_n\}$ of curves is said to converge to a curve $\gamma$ in Fréchet's sense if they are represented by $x^{(n)}(t)$ and $x(t)$, $0 \leqq t \leqq 1$, such that $x^{(n)}(t)$ converges uniformly to $x(t)$. The following two lemmas are known.

LEMMA 4 ([7, Lemma 3.3]; also cf. [5, Lemma 2.5]). *Let $\rho$ be nonnegative lower semicontinuous in $R^d$ and $\{\gamma_n\}$ be an infinite sequence of curves such that all $\gamma_n$ are contained in a closed ball $B$, each $\gamma_n$ connects $x_n$ and $y_n$, $x_n \to x_0$, $y_n \to y_0$ as $n \to \infty$ and the lengths of $\gamma_n$ are bounded. Then there exists a subsequence $\{\gamma_{n_k}\}$ and a curve $\gamma$ in $B$ connecting $x_0$ with $y_0$ such that $\{\gamma_{n_k}\}$ converges to $\gamma$ in Fréchet's sense and*

$$\int_\gamma \rho\, ds \leqq \liminf_{n\to\infty} \int_{\gamma_n} \rho\, ds.$$

LEMMA 5 ([2, chap. I]). *A family $\Gamma$ of curves satisfies $M_p(\Gamma; w) = 0$ if and only if there exists a nonnegative Borel measurable function $\rho \in L^{p,w}$ such that $\int_\gamma \rho\, ds = \infty$ for every $\gamma \in \Gamma$.*

## 3. Proof of Theorem 1 (cf. [7, §3])

Take any $u \in C_0^\infty(G)$ such that $u \geqq 1$ on $E$. Obviously, $|\mathrm{grad}\, u|$ is $\Gamma$-admissible. Hence we have $M_p(\Gamma; w) \leqq \int_G |\mathrm{grad}\, u|^p w\, dx$ and derive $M_p(\Gamma; w) \leqq C_p^w(E; G)$.

To prove the inverse inequality $C_p^w(E; G) \leqq M_p(\Gamma; w)$, we may assume that $M_p(\Gamma; w) < \infty$. Set $D = G - E$. By assumption, $D$ is a domain. We denote by $\mathscr{D}(D)$ the family of all $ACL$ functions $u$ in $D$ such that $|\mathrm{grad}\, u| \in L^{p,w}(D)$, $\lim_{x \to \partial E} u(x) = 1$ and $\lim_{x \to \partial G} u(x) = 0$. First we shall show

$$\inf\left\{\int_D |\mathrm{grad}\, u|^p w\, dx;\, u \in \mathscr{D}(D)\right\} \leqq M_p(\Gamma; w)\,. \tag{1}$$

Take a $\Gamma$-admissible function $\rho$ which is continuous in $D$ and satisfies $\inf\{\rho(x); x \in D\} \geqq \delta > 0$. Set $\rho_k(x) = \min\{\rho(x), k\}$ for each positive integer $k$ and extend it by $\delta$ to $R^d - D$. Given $x \in G$, denote by $\Gamma_0^x$ the family of curves in $G$ each of which starts from $x$ and tends to a point in $\partial G$. We set

$$u_k(x) = \inf\left\{\int_\gamma \rho_k\, ds;\ \gamma \in \Gamma_0^x\right\}.$$

Suppose $\{\gamma_n\}$ is a minimizing sequence in the definition of $u_k(x)$. Since $\rho_k \geqq \delta > 0$ in $G$, we may assume the lengths of $\gamma_n$ are bounded. By applying Lemma 4, we can take a curve $\gamma_k^x \in \Gamma_0^x$ such that

$$u_k(x) = \int_{\gamma_k^x} \rho_k\, ds\,.$$

This implies that

$$|u_k(x) - u_k(x')| \leqq \int_{\widetilde{xx'}} \rho_k\, ds$$

for any points $x$, $x'$ in $G$, where $\widetilde{xx'}$ is a curve connecting $x$ and $x'$ in $G$. It follows that $u_k$ is continuous $ACL$ in $G$ and that $|\mathrm{grad}\, u_k| \leqq \rho_k$ a.e. in $G$ (see, [7, Lemma 3.6]). Set

$$m_k = \min\{u_k(x); x \in E\}$$

and

$$u_k^*(x) = \min\{u_k(x), m_k\}\,.$$

The restriction of $u_k^*/m_k$ to $D$ belongs to $\mathscr{D}(D)$. By the same method as in the proof of [7, Lemma 3.7], we see that $\liminf_{k\to\infty} m_k \geqq 1$. Hence

$$\inf_{u\in\mathscr{D}(D)} \int_D |\text{grad } u|^p w\, dx \leqq \liminf_{k\to\infty} \int_D (|\text{grad } u_k^*|/m_k)^p w\, dx$$

$$\leqq \liminf_{k\to\infty} \left(\frac{1}{m_k}\right)^p \int_D \rho_k^p w\, dx \leqq \int_D \rho^p w\, dx\,.$$

From Lemma 3, (1) follows.

Next, set

$$h(x) = \min\left\{1, \max\left(0, \frac{u-\varepsilon}{1-2\varepsilon}\right)\right\}$$

for any $u \in \mathscr{D}(D)$ and sufficiently small number $\varepsilon > 0$. Let $(h)_n$ be the mollified functions of $h$ in $D$ and set $(h)_n = 1$ on $E$. Then each $(h)_n$ belongs to $C_0^\infty(G)$. Hence $C_p^w(E; G) \leqq \int_G |\text{grad }(h)_n|^p w\, dx$. By Lemma 2,

$$\|\text{grad }(h)_n\|_{p,w} \leqq \|\text{grad }((h)_n - h)\|_{p,w} + \|\text{grad } h\|_{p,w} \to \|\text{grad } h\|_{p,w}$$

as $n \to \infty$. Thus

$$C_p^w(E; G) \leqq \int_D |\text{grad } h|^p w\, dx \leqq \left(\frac{1}{1-2\varepsilon}\right)^p \int_D |\text{grad } u|^p w\, dx\,.$$

Letting $\varepsilon \to 0$, by (1) we conclude that $C_p^w(E; G) \leqq M_p(\Gamma; w)$.

## 4. Proof of Theorem 2

To prove Theorem 2, we prepare the following lemma which gives the continuity property of extremal distance in a special case. Denote by $E_0$ the union of $E$ and all bounded components of $R^d - E$. In case $R^d - E$ is a domain, $E_0 = E$. Set $G_n = \{x; |x| < n\}$. We may assume that $G_n \supset E_0$ for all $n$. Let $\Gamma_n$ (resp. $\Gamma_\infty^*$) be the family of curves in $G_n - E_0$ (resp. $R^d - E_0$) connecting $\partial E_0$ and $\partial G_n$ (resp. the point at infinity). Note that $M_p(\Gamma_\infty; w) = M_p(\Gamma_\infty^*; w)$.

LEMMA 6. $\lim_{n\to\infty} M_p(\Gamma_n; w) = M_p(\Gamma_\infty^*; w)$.

PROOF. First note that $M_p(\Gamma_\infty^*; w) \leqq M_p(\Gamma_n; w) < \infty$ for all $n$. For any $\varepsilon > 0$, by Lemma 3 we can take a $\Gamma_\infty^*$-admissible function $\rho$ which satisfies (i) $\rho$ is continuous in $R^d - E_0$, (ii) $\int \rho^p w\, dx < M_p(\Gamma_\infty^*; w) + \varepsilon$ and (iii) $\inf\{\rho(x); x \in (R^d - E_0) \cap K\} > 0$ for any compact set $K$. Set $\rho = 0$ on $E_0$. Then $\rho$ is nonnegative lower semicontinuous in $R^d$. We infer that there is $n$

such that $\int_\gamma \rho\, ds \geqq 1-\varepsilon$ for every $\gamma \in \Gamma_n$ (cf. the proof of [5, Theorem 2.6 and Lemma 2.7]). In fact, otherwise there would exist $\gamma_n \in \Gamma_n$; $n = n_0, n_0+1, \ldots,$ such that

$$\int_{\gamma_n} \rho\, ds < 1-\varepsilon$$

for each $n \geqq n_0$. Let $\{\gamma_{1j}\}$ be a subsequence of $\{\gamma_n\}_{n=n_0}^\infty$ such that $\lim_{j\to\infty} x_{1j} = x_0 \in \partial E_0$ and $\lim_{j\to\infty} y_{1j} = y_0 \in \partial G_{n_0}$, where $x_{1j}$ is the starting point of $\gamma_{1j}$ and $y_{1j}$ is the first point of intersection of $\gamma_{1j}$ with $\partial G_{n_0}$. Let $\gamma_{1j}^*$ be the subcurve of $\gamma_{1j}$ connecting $x_{1j}$ and $y_{1j}$ in $G_{n_0}$. Since $\inf\{\rho(x); x \in (R^d - E_0) \cap G_{n_0}\} > 0$, the lengths of $\gamma_{1j}^*$ are bounded. By applying Lemma 4, we can find a curve $\gamma_1$ connecting $x_0$ and $y_0$ in $\overline{G_{n_0}}$ such that a subsequence of $\{\gamma_{1j}^*\}$ converges to $\gamma_1$ in Fréchet's sense and

$$\int_{\gamma_1} \rho\, ds \leqq \liminf_{j\to\infty} \int_{\gamma_{1j}^*} \rho\, ds\,.$$

We may assume that $\{\gamma_{1j}^*\}$ itself converges to $\gamma_1$. Next, let $\{\gamma_{2j}\}$ be a subsequence of $\{\gamma_{1j}\}$ such that $\lim_{j\to\infty} y_{2j} = y_1 \in \partial G_{n_0+1}$, where $y_{2j}$ is the first point of intersection of $\gamma_{2j}$ with $\partial G_{n_0+1}$. Note that the sequence of the starting point (which we denote by $x_{2j}$) of $\gamma_{2j}$ converges to $x_0$ in $\partial E_0$. Let $\gamma_{2j}^*$ be the subcurve of $\gamma_{2j}$ connecting $x_{2j}$ and $y_{2j}$ in $G_{n_0+1}$. Using Lemma 4 again we may assume that $\{\gamma_{2j}^*\}$ converges to a curve $\gamma_2$ connecting $x_0$ and $y_1$ in $\overline{G_{n_0+1}}$ in Fréchet's sense and

$$\int_{\gamma_2} \rho\, ds \leqq \liminf_{j\to\infty} \int_{\gamma_{2j}^*} \rho\, ds\,.$$

Since $\{\gamma_{2j}\}$ is the subsequence of $\{\gamma_{1j}\}$ and $\{\gamma_{2j}^*\}$ (resp. $\{\gamma_{1j}^*\}$) converges to $\gamma_2$ (resp. $\gamma_1$) in Fréchet's sense, we see that $\gamma_2$ contains $\gamma_1$. We continue this process and obtain a curve $\gamma$ which contains all $\gamma_k$. We have

$$\int_\gamma \rho\, ds = \lim_{k\to\infty} \int_{\gamma_k} \rho\, ds \leqq \lim_{k\to 0} \liminf_{j\to\infty} \int_{\gamma_{kj}^*} \rho\, ds \leqq 1-\varepsilon\,.$$

Since $\gamma$ contains some $\gamma^*$ in $\Gamma_\infty^*$ and $\rho$ is $\Gamma_\infty^*$-admissible, this is a contradiction. Thus $\rho/(1-\varepsilon)$ is $\Gamma_n$-admissible and hence

$$M_p(\Gamma_n; w) \leqq \left(\frac{1}{1-\varepsilon}\right)^p \int \rho^p w\, dx < \left(\frac{1}{1-\varepsilon}\right)^p (M_p(\Gamma_\infty^*; w) + \varepsilon)\,.$$

By letting $\varepsilon \to 0$ we conclude that $M_p(\Gamma_n; w) \to M_p(\Gamma_\infty^*; w)$ as $n \to \infty$.

Proof of Theorem 2. If $u \in C_0^\infty$ and $u \geqq 1$ on $E$, then $|\operatorname{grad} u|$ is $\Gamma_\infty$-admissible. Hence the inequality $M_p(\Gamma_\infty; w) \leqq C_p^w(E)$ follows.

To prove the inverse inequality, we may assume that $M_p(\Gamma_\infty; w) < \infty$. By Theorem 1, $C_p^w(E_0; G_n) = M_p(\Gamma_n; w)$. Obviously, $C_p^w(E) \leqq C_p^w(E_0) \leqq C_p^w(E_0; G_n)$. Hence $C_p^w(E) \leqq M_p(\Gamma_n; w)$. By Lemma 6 we have, $C_p^w(E) \leqq M_p(\Gamma_\infty^*; w) = M_p(\Gamma_\infty; w)$.

## 5. Proof of Theorem 3

To prove Theorem 3 we prepare two lemmas, the first of which follows from [1, Theorem 2] and [3, Theorems 3 and 4].

LEMMA 7. *Let $w \in A_{p,1}$ and $\{g_n\}$ be a sequence such that $\|g_n\|_{p,w} \to 0$. Then $\int |x-y|^{1-d} g_n(y)\, dy \to 0$ in measure in any bounded domain.*

LEMMA 8. *Let $w \in A_{p,1}$. If $C_p^w(E) = 0$, then $C_p^w(E; G) = 0$ for any bounded domain $G$ containing $E$.*

PROOF. Take a sequence $\{u_n\}$ of $C_0^\infty$ functions such that $u_n \geqq 1$ on $E$ and $\|\operatorname{grad} u_n\|_{p,w} \to 0$. We may assume that $0 \leqq u_n \leqq 1$ for all $n$. Take any $\varphi \in C_0^\infty(G)$ with $\varphi = 1$ on $E$. Then $\varphi u_n$ is admissible in the definition of $C_p^w(E; G)$, and satisfies

$$\|\operatorname{grad}(\varphi u_n)\|_{p,w} \leqq \|\varphi(\operatorname{grad} u_n)\|_{p,w} + \|u_n(\operatorname{grad} \varphi)\|_{p,w} .$$

Since $u_n \in C_0^\infty$, it is well known that

$$|u_n(x)| \leqq \text{const.} \int |x-y|^{1-d} |\operatorname{grad} u_n(y)|\, dy .$$

By Lemma 7, there is a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that

$$\int |x-y|^{1-d} |\operatorname{grad} u_{n_j}(y)|\, dy \to 0 \qquad \text{for a.e. } x \text{ in } G .$$

Hence we see that $u_{n_j}(x) \to 0$ for a.e. $x$ in $G$. Since $0 \leqq u_{n_j} \leqq 1$ and $w$ is integrable on $G$, the dominated convergence theorem yields that $\|u_{n_j}(\operatorname{grad} \varphi)\|_{p,w} \to 0$ as $j \to \infty$. Obviously, $\|\varphi(\operatorname{grad} u_{n_j})\|_{p,w} \to 0$ and therefore $\|\operatorname{grad}(\varphi u_{n_j})\|_{p,w} \to 0$ as $j \to \infty$. Thus we conclude that $C_p^w(E; G) = 0$.

PROOF OF THEOREM 3. Suppose that $M_p(\bigwedge(E); w) = 0$. Since $\Gamma_\infty \subset \bigwedge(E)$, $M_p(\Gamma_\infty; w) \leqq M_p(\bigwedge(E); w)$, so that $M_p(\Gamma_\infty; w) = 0$. From Theorem 2, $C_p^w(E) = 0$ follows.

Conversely, suppose that $w \in A_{p,1}$ and $C_p^w(E) = 0$. First we shall show that $R^d - E$ is a domain. Assume that $R^d - E$ is not a domain. Let $x^0$ be a point in a bounded component of $R^d - E$. Take a ring domain

$R = \{x; a < |x - x^0| < b\}$ such that $R \supset E$. Set $G = \{x; |x - x^0| < b\}$. Take any $u \in C_0^\infty(G)$ such that $u \geqq 1$ on $E$. For any ray $\gamma_\theta$: $x^0 + r\theta$ $(a < r < b)$, $|\theta| = 1$, we have

$$1 \leqq \int_{\gamma_\theta} |\mathrm{grad}\, u|\, ds\,.$$

Hence

$$\begin{aligned}\int_{|\theta|=1} d\theta &\leqq \int_R |\mathrm{grad}\, u|\, |x - x^0|^{1-d}\, dx \\ &\leqq a^{1-d}\left(\int_R |\mathrm{grad}\, u|^p w\, dx\right)^{1/p}\left(\int_R w^{1/(1-p)}\, dx\right)^{1/p'}.\end{aligned}$$

Since $w^{1/(1-p)}$ is locally integrable,

$$\begin{aligned}\int_G |\mathrm{grad}\, u|^p w\, dx &\geqq \int_R |\mathrm{grad}\, u|^p w\, dx \\ &\geqq a^{(d-1)p}\left(\int_{|\theta|=1} d\theta\right)^p \left(\int_R w^{1/(1-p)}\, dx\right)^{-p/p'} > 0\,.\end{aligned}$$

Hence $C_p^w(E; G) > 0$. On the other hand, $C_p^w(E; G) = 0$ by Lemma 8. Thus we obtain a contradiction. Therefore $R^d - E$ is a domain.

Let $\{G_n\}$ be a sequence of relatively compact open sets such that $\overline{G_{n+1}} \subset G_n$ for each $n$, $\bigcap_{n=1}^\infty G_n = E$ and every $G_n$ consists of a finite number of components $G_{n,i}$ $(i = 1, \ldots, i(n))$ each of which meets $E$. Set $E_{n,i} = G_{n,i} \cap E$. Denote by $\Gamma_{n,i}$ the family of curves connecting $E_{n,i}$ and $\partial G_{n,i}$ in $G_{n,i} - E$. By assumption, $C_p^w(E_{n,i}) = 0$. From Lemma 8 and Theorem 1, it follows that $M_p(\Gamma_{n,i}; w) = 0$. Let $\Gamma_n = \bigcup_{i=1}^{i(n)} \Gamma_{n,i}$. Then we see that $M_p(\Gamma_n; w) = 0$. By Lemma 5, there exists a sequence $\{\rho_n\}$ of nonnegative Borel measurable functions such that $\|\rho_n\|_{p,w} < 2^{-n}$ and $\int_\gamma \rho_n\, ds = \infty$ for every $\gamma \in \Gamma_n$ for each $n$. We set $\rho_0 = \Sigma\, \rho_n$. Then $\|\rho_0\|_{p,w} < \infty$. For each $\gamma \in \bigwedge(E)$, there exists a curve $\gamma_n \in \Gamma_n$ such that $\gamma_n \subset \gamma$. Hence

$$\int_\gamma \rho_0\, ds \geqq \int_{\gamma_n} \rho_0\, ds \geqq \int_{\gamma_n} \rho_n\, ds = \infty\,.$$

Using Lemma 5 again we conclude that $M_p(\bigwedge(E); w) = 0$.

Remark. Let $E = \{0\}$. We show by example that $C_p^w(\{0\}) = 0 < M_p(\bigwedge\{0\}; w)$ for some $w \in A_p - A_{p,1}$.

Let $0 < \beta < p$ and let $w(x) = |x|^{\beta - d}$. Then $w \in A_p - A_{p,1}$. For some $\alpha$ with $\beta/p < \alpha < 1$, we set

$$\rho(x) = \begin{cases} 1 & \text{if } |x| \leqq 1 \\ |x|^{-\alpha} & \text{if } |x| > 1 . \end{cases}$$

Then $\rho \in L^{p,w}$ and $\int_\gamma \rho\, ds = \infty$ for every $\gamma \in \Gamma_\infty$. From Lemma 5 and Theorem 2, $C_p^w(\{0\}) = 0$ follows. On the other hand, by Ohtsuka [6, Corollary], we see that $M_p(\bigwedge\{0\}; w) \neq 0$.

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