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Ascending chain conditions on special classes of ideals of Lie algebras

Falih A. M. ALDOSRAY and Ian STEWART

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0. Introduction

The class Max-$\triangleleft$ of Lie algebras with ascending chain condition on ideals, otherwise known as noetherian Lie algebras, has been studied by several authors, see Aldosray [1], Amayo and Stewart [3, 4], Kawamoto [10], Kubo and Honda [12], and Stewart [14, 15]. Less stringent chain conditions can be imposed by requiring the ascending chain condition only on certain special classes of ideals. For example Aldosray and Stewart [2] study the class Max-c of Lie algebras with ascending chain condition on centralizer ideals; and Ikeda [6], Kubo [11], and Tôgô [17] study ascending chain conditions on generalized soluble ideals.

Here we consider the classes Max-CI, Max-ESS, and Max-SMI of Lie algebras with the ascending chain condition on complement ideals, essential ideals, and small ideals, defined respectively in §§1, 2, 5 below. Our aim is to elucidate the basic properties of these classes and the relations between them.

In §1 we study the ascending chain condition on complement ideals and give a number of examples to show that most standard properties of Max-$\triangleleft$ fail for such algebras. In §2 we introduce the 'dual' concepts of essential ideals and small ideals, and the singular ideal. The main result, Theorem 2.8, shows that the Lie product of a pair of essential ideals is always essential if and only if the Lie algebra is semisimple. §3 is devoted to questions of the following kind. Suppose that every quotient L/I of L by nonzero ideals I (possibly of some special type) satisfies some ascending chain condition. Does L also satisfy this chain condition? We show that the answer is affirmative for Max-$\triangleleft$ (I any ideal); Max-CI (I a complement ideal); and Max-SMI (I a small ideal). It is *negative* for Max-c (even if I may be any nonzero ideal), answering Question 3 of Aldosray and Stewart [2]. However, it is affirmative for Max-c when L is semisimple (I any centralizer ideal).

Camillo [5] proves that a commutative ring in which every quotient is Goldie must be noetherian. In §4 we show that the natural Lie algebra analogue of Camillo's theorem is *false.* However, using results of Shock [13] we prove a substitute: L is in Max-$\lhd$ if and only if every quotient is in Max-CI and Max-SMI.

Most notation used is standard, and may be found in Amayo and Stewart [4] or Aldosray and Stewart [2]. In particular L^n and $L^{(n)}$ denote respectively the nth terms of the lower central and derived series of L, and $\zeta_1(L)$ is the centre of L. We write $I \lhd L$ if I is an ideal of L, and $I \leq L$ if I is a subalgebra. Any other notation is defined as it is needed. The end (or absence) of a proof is signalled by a box $\square$.

1. The maximal condition for complement ideals

Recall from Aldosray and Stewart [2] that an ideal K of a Lie algebra L is a *complement ideal* if there exists an ideal J of L such that $J \cap K = 0$, and if K' is any ideal of L such that $K \subsetneq K'$ then $J \cap K' \neq 0$. In fact in [2] it is further required that J be nonzero, but we relax this condition here. In particular L and 0 are always complement ideals (take $J = 0$, L respectively).

In this section we study the class Max-CI of Lie algebras satisfying the ascending chain condition on complement ideals. We begin with an obvious but useful remark:

REMARK 1.1. If L is abelian with Max-CI, then L is finite-dimensional. $\square$

A similar result trivially holds for algebras in Max-$\lhd$. However, we now show by several examples that Max-CI does *not* have several other properties analogous to Max-$\lhd$.

EXAMPLE 1.2. Let $L \in$ Max-CI and let I be an ideal of L. Then L/I need not satisfy Max-CI.

Let $A = F[x_1, x_2, \ldots]$ be a polynomial algebra in an infinite number of indeterminates x_i. Considered as an abelian Lie algebra, A has derivations

$$\delta_i \colon f \mapsto x_i f \qquad (f \in A).$$

The δ_i commute. Let $H = \langle \delta_i : i \geq 1 \rangle$ and form the split extension $L = A \dotplus H$. Then $L \in$ Max-CI and $A \lhd L$. However, $L/A \cong H$ which is infinite-dimensional abelian, so $L/A \notin$ Max-CI by Remark 1.1. $\square$

EXAMPLE 1.3. Nilpotent Lie algebras with Max-CI need not be finite-dimensional.

Let L be the infinite Heisenberg algebra, with basis

$$\{z, x_i, y_i : i \in \mathbf{N}\}$$

such that $[x_i, y_i] = z$ and all other elements commute. Then L is nilpotent of class 2, and its centre $\zeta_1(L) = \langle z \rangle$. Every nonzero ideal contains $\langle z \rangle$, so the only complement ideals are 0 and L. Therefore $L \in$ Max-ci. $\square$

EXAMPLE 1.4. Locally nilpotent Lie algebras with Max-ci need not be soluble.

In fact we show considerably more. The algebra we use is an alternative version of the McLain algebra $\mathscr{L}_F(\mathbf{Z})$, see Amayo and Stewart [4]. We define it as follows. Begin with an associative algebra A having generators x_n $(n \in \mathbf{Z})$ and multiplication relations

$$x_m x_n = 0 \qquad \text{unless } n = m + 1\,.$$

Define monomials

$$x_{mn} = x_m x_{m+1} \dots x_n \qquad (n > m)\,.$$

Clearly the monomials form a basis for A. Since sufficiently long products in a finite set of generators must repeat an element, A is locally nilpotent. Let $L = Lie(A)$ be the Lie algebra formed by A under the commutator $[a, b] = ab - ba$. Then L is locally nilpotent.

We claim that every nonzero Lie ideal of L contains a monomial. To prove this, observe that

$$[x_m, x_n] = \begin{cases} x_m x_n & n = m + 1 \\ -x_n x_m & n = m - 1 \\ 0 & \text{otherwise}\,. \end{cases}$$

Let $I \lhd L$, containing an element

$$u = \sum \lambda_i x_{m_i n_i} \qquad (\lambda_i \neq 0)\,.$$

If $n = \max n_i$ then I contains

$$v = [u, x_{n+1}] = \sum_{j \in J} \lambda_j x_{m_j, n+1} \neq 0$$

where $J = \{j : n_j = n\}$. If $m = \min_{j \in J} m_j$ then I contains

$$[x_{m-1}, v] = \lambda_k x_{m-1, n+1}$$

where k is the unique integer such that $m_k = m$. Therefore I contains a monomial and the claim is proved.

By direct calculation $I_{mn} = \langle x_{mn} \rangle^L$ is spanned by all monomials $x_{m'n'}$ with $m' \leq m$ and $n' \geq n$. Clearly I_{mn} is abelian, so L is a Fitting algebra. Since every ideal contains a monomial, every nonzero ideal contains some I_{mn}. Moreover if $m' \leq m$ and $n' \geq n$ then $I_{mn} \supseteq I_{m'n'}$. Since

$$I_{mn} \cap I_{kl} = I_{\min(m,k),\max(n,l)}$$

it follows that any two nonzero ideals of L have nonzero intersection.

In particular the only complement ideals of L are 0 and L, whence $L \in$ Max-CI.

An easy induction shows that

$$0 \neq x_0 \dots x_{2^n-1} \in L^{(n)},$$

so L is not soluble.

Since x_{mn} does not commute with x_{n+1} it follows that $\zeta_1(L) = 0$.

On the other hand, L has many nontrivial centralizer ideals. In particular $C_L(I_{mn}) \supseteq I_{mn}$ since I_{mn} is abelian. More precisely

$$C_L(I_{mn}) = \sum_{m \leq k \leq n} I_{kk}, \tag{1}$$

which is nilpotent of class $|m - n| + 1$. Therefore every proper centralizer ideal of L is nilpotent.

From (1) there exists an infinite ascending chain of centralizers, so $L \notin$ Max-c. □

EXAMPLE 1.5. Example 1.4 shows that Max-CI $\not\subset$ Max-c. Moreover, all infinite-dimensional abelian Lie algebras belong to Max-c but not to Max-CI, hence Max-c $\not\subset$ Max-CI. Thus the neither of the two classes is contained in the other, and there are no generally valid implications between Max-c and Max-CI. □

EXAMPLE 1.6. If L is hypercentral with Max-CI then L need not be soluble.

To answer this negatively it is sufficient to find an insoluble hypercentral Lie algebra L such that $\dim \zeta_1(L) = 1$. Then $\zeta_1(L)$ is contained in every nonzero ideal, so the only complement ideals are 0 and L.

We construct such an algebra as follows. Suppose that we have a sequence of Lie algebras N_n such that $\dim \zeta_1(N_n) = 1$ for all n, and whose derived length $d(n)$ (and hence also nilpotency class $c(n)$) tends to infinity with n. To be specific, let N_n be the Lie algebra of all upper triangular $n \times n$ matrices with zero diagonal, which is easily seen to satisfy the required conditions. Let $0 \neq z_n \in \zeta_1(N_n)$. Form the *direct sum with amalgamated centre*

$$L = \left(\bigoplus_{n=1}^{\infty} N_n \right) \Big/ I$$

where I is the subspace of $S = \oplus_{n=1}^{\infty} N_n$ spanned by all $z_i - z_j$, $i, j \in \mathbf{N}$. This is contained in the centre of S, so is an ideal of S.

We claim that L is hypercentral and insoluble, and that $\dim \zeta_1(L) = 1$.

Since S is a direct sum of nilpotent algebras, it is hypercentral, hence so is L. In fact $N_n \cap I = 0$, so each N_n embeds naturally in L, and the central height of L is precisely ω because $c(n) \to \infty$. In the same way L is not soluble since $d(n) \to \infty$.

Let $x \in S$ be such that its image $\bar{x}$ under the natural map $S \to S/I$ is central. We claim $x \in \zeta_1(S)$. To see this, write $x = (x_n)$ where $x_n \in N_n$ and observe that

$$[(x_n), (y_n)] \in I \qquad \text{for all } y_n \in N_n .$$

Now every element of I is of the form $\sum \mu_n z_n$ where $\sum \mu_n = 0$, therefore $[x_n, y_n] = \mu_n z_n$ where $\sum \mu_n = 0$. Replacing y_n by $2y_n$ we deduce that $\mu_n = 0$. (This follows even if char $F = 2$, when we replace y_n by 0.) Therefore $[x_n y_n] = 0$ so $x \in \zeta_1(S)$ as claimed. Thus $\bar{x} \in \zeta_1(S)/I$, which has dimension 1. □

Recall that L is *semisimple* if it has no non-zero abelian ideal.

QUESTION 1.7. If L is semisimple and I, J are complement ideals of L, then $I \cap J$ always a complement ideal of I?

If the answer is affirmative, then the following result holds: Let L be semisimple, with I a complement ideal of L, such that $I \in$ Max-CI and $L/I \in$ Max-CI. Then $L \in$ Max-CI. Note that in the circumstances described $I \cap J$ is a complement ideal of L, since in semisimple algebras complement ideals are centralizers. However, we do not know that I is semisimple.

THEOREM 1.8. *Let L be semisimple with* Max-CI. *Then every complement ideal in L is a finite intersection of maximal complement ideals in L, and these are minimal prime ideals.*

PROOF. Since I is semisimple, I is a complement ideal if and only if I is a centralizer ideal, by Aldosray and Stewart [2] Lemma 2.3. The result follows from Theorem 4.6 of Aldosray and Stewart [2]. □

2. Essential and small ideals

We define two types of ideal, that are in a sense 'dual' to each other. An ideal E of L is *essential* if it has nonzero intersection with every nonzero ideal of L. An ideal S of L is *small* if for all ideals K of L the equation $S + K = L$ implies $K = L$. In this section we establish some basic properties of essential and small ideals.

LEMMA 2.1. (a) *If S and T are small ideals of L, then $S + T$ is small.*

(b) *If S is a small ideal of L and $T \subseteq S$, then T is small.*

(c) *If E and F are essential ideals then $E + F$ is essential.*

(d) *If* E *and* F *are essential ideals then* $E \cap F$ *is essential.*

(e) *If* E *is essential in* L *and* $F \supseteq E$, *then* F *is essential.* □

We also need the concept of the singular ideal, analogous to a ring-theoretic idea of Johnson [8]. The *singular ideal* of L is

$$Z(L) = \{x \in L : [x, E] = 0 \text{ for some essential ideal } E \text{ of } L\}.$$

If M is an L-module then the *singular submodule* of M is

$$Z(M) = \{x \in M : xE = 0 \text{ for some essential ideal } E \text{ of } L\}.$$

Equivalently, $x \in Z(L)$ if and only if $C_L(\langle x \rangle^L)$ is an essential ideal of L. Clearly $Z(L) \lhd L$.

LEMMA 2.2. *L is semisimple if and only if* $Z(L) = 0$.

PROOF. Suppose L is semisimple, with $a \in Z(L)$. Then $\langle a \rangle^L \cap C_L(\langle a \rangle^L) = 0$ by semisimplicity, but $C_L(\langle a \rangle^L)$ is an essential ideal of L, so $a = 0$.

Conversely suppose that $Z(L) = 0$, and let $A \lhd L$, $A^2 = 0$. We claim that $A = 0$. Let $x \in L$. Then either $[A, x] = 0$ in which case $x \in C_L(A)$, or $[A, x] \neq 0$ in which case $[A, x] \subseteq C_L(A)$. Therefore $\langle x \rangle^L \cap C_L(A) \neq 0$, so $C_L(A)$ is essential in L. Hence $A = 0$. □

LEMMA 2.3. *Let I be an ideal of L, and let $E \supseteq I$ be an ideal such that E/I is an essential ideal of L/I. Then E is an essential ideal of L.*

PROOF. Let $0 \neq J \lhd L$. Either $(I + J)/I = I/I$ or not. In the first case, $J \subseteq I$ so $J \cap I \neq 0$ so $J \cap E \neq 0$. In the second, $I \subsetneq (I + J) \cap E = I + (J \cap E)$ so $J \cap E \neq 0$. □

The dual of Lemma 2.3 is also true.

LEMMA 2.4. *If S is a small ideal of L and $I \lhd L$, then $(S + I)/I$ is small in L/I.*

PROOF. Assume $(S + I)/I + P/I = L/I$ where $P \lhd L$, $P \supseteq I$. Then $S + I + P = L$. Since S is small, $I + P = L$, so $P = L$ since $I \subseteq P$. Therefore $P/I = L/I$, so S/I is small. □

LEMMA 2.5. *Let $I, J \lhd L$ be such that $I \subseteq J$. Then J is small in L if and only if*

(a) *I is small in L,* and

(b) *J/I is small in L/I.*

PROOF. If J is small in L then Lemma 2.1(b) and Lemma 2.4 imply that (a) and (b) hold.

Conversely assume (a) and (b). Suppose that $J + K = L$, $K \lhd L$. Then $J + K + I = L$, so $J/I + (K + I)/I = L/I$. By (b) $K + I = L$, and then by (a) $K = L$. Hence J is small in L. □

PROPOSITION 2.6. (a) *If $I \lhd L$ then $I + C_L(I)$ is essential in L.*

(b) *If I is a hypercentral ideal of L then $C_L(I)$ is essential in L.*

PROOF. (a) Let $C = I + C_L(I)$ and let $0 \neq J \lhd L$. If $J \cap I \neq 0$ then $J \cap C \neq 0$. Otherwise $J \cap I = 0$ so $J \subseteq C_L(I)$ and again $J \cap C \neq 0$.

(b) Let $0 \neq J \lhd L$. If $J \cap I = 0$ then $J \subseteq C_L(I)$. If $J \cap I \neq 0$ then $J \cap \zeta_1(I) \neq 0$, so J intersects $C_L(I)$ nontrivially. □

PROPOSITION 2.7. *Let L be semisimple and suppose that every essential ideal in L is finitely generated. Then $L \in$ Max-$\lhd$.*

PROOF. Let $I \lhd L$. Then $I + C_L(I)$ is essential in L by Proposition 2.6(a). Therefore $I + C_L(I)$ is a finitely generated ideal. Hence $(I + C_L(I))/C_L(I)$ is a finitely generated L-module. Therefore $I/(I \cap C_L(I))$ is a finitely generated L-module. Since L is semisimple $I \cap C_L(I) = 0$, so I is a finitely generated L-module, that is a finitely generated ideal. □

In contrast to Lemma 2.1(c), (d) we have:

THEOREM 2.8. *Let L be a Lie algebra. Then $[E, F]$ is essential for all essential ideals E and F if and only if L is semisimple.*

PROOF. Suppose L is semisimple and E, F are essential in L. Let $0 \neq I \lhd L$ and suppose for a contradiction that $I \cap [E, F] = 0$. Then $[I, [E, F]] = 0$. Let $B = I \cap E \cap F$ which is nonzero by Lemma 2.1(d). We have $[B^2, F] \subseteq [I, [E, F]] = 0$. Now $B^2 \neq 0$ by semisimplicity, so $C_L(F) \neq 0$. But now $F \cap C_L(F) \neq 0$ since F is essential, contradicting semisimplicity.

For the converse, suppose that L is not semisimple: we construct essential ideals E and F such that $[E, F]$ is not essential. Let $0 \neq A \lhd L$ where A is abelian, and take a maximal A with this property. If A is essential in L then $0 = [A, A]$ is not essential and we are done. If A is not essential then $A \cap K = 0$ for $0 \neq K \lhd L$. We may take K to be maximal with this property, hence a complement of A. We claim that $A + K$ is essential. Note that $A + K = A \oplus K$. If there exists an ideal $J \neq 0$ such that $(A \oplus K) \cap J = 0$ then $(A \oplus K) + J = (A \oplus K) \oplus J = A \oplus (K \oplus J)$. Then $J \oplus K \supsetneq K$ and $A \cap (J \oplus K) = 0$, contradicting K being a complement. Since $A \cap K^2 = 0$ and $A \neq 0$, we know that K^2 is not essential. Hence $[A \oplus K, A \oplus K] = K^2$ is not essential. □

Let Max-ESS be the class of Lie algebras satisfying the ascending chain condition for essential ideals. We have

LEMMA 2.9. *The class* Max-ESS is Q-*closed.*

PROOF. This is an easy consequence of Lemma 2.3. □

Recall that the *socle* soc(L) is the sum of the minimal ideals of L.

PROPOSITION 2.10. *Let $S = \mathrm{soc}(L)$. Then $S \subseteq Z(L)$ if and only if S is abelian.*

PROOF. First suppose that $S \subseteq Z(L)$. By definition S is the sum of all minimal ideals M of L. Let $0 \neq x \in M$, so that $M = \langle x \rangle^L$. By the remark immediately before Lemma 2.2, $C_L(M)$ is essential in L. Therefore $M \cap C_L(M) \neq 0$, so $M \subseteq C_L(M)$, so $M^2 = 0$. Therefore $S^2 = 0$.

Conversely suppose $S^2 = 0$. Then $C_L(S) = S + C_L(S)$ is essential by Proposition 2.6(a), hence $S \subseteq Z(L)$. □

THEOREM 2.11. *Suppose that L has socle S.*

(a) *If L is semisimple then $S^2 = S$.*

(b) *If S is essential and $S^2 = S$ then L is semisimple.*

PROOF. (a) The socle S is a direct sum of minimal ideals M of L, for which either $M^2 = M$ or $M^2 = 0$. If L is semisimple then $M^2 = M$ for all such M, hence $S^2 = S$.

(b) If L is not semisimple then it has a nonzero abelian ideal A. Since S is essential, $S \cap A \neq 0$. Then some direct summand M of S is abelian, so $S^2 \neq S$. □

PROPOSITION 2.12. *Suppose that P is a prime ideal of L but not a minimal prime ideal. Then P is an essential ideal of L.*

PROOF. Since P is not a minimal prime, there exists a prime ideal Q of L such that $Q \subsetneq P$. Let $I \lhd L$ with $I \cap P = 0$. Then $[I, P] = 0 \subseteq Q$. But $P \nsubseteq Q$, so $I \subseteq Q$ by primality. Therefore $I = 0$. Thus P is essential in L. □

PROPOSITION 2.13. *Let L be semisimple, $I \lhd L$, such that $Z(L/I) = L/I$. Then I is essential in L.*

PROOF. Let $0 \neq x \in L$. Let bars denote images modulo I. Since $Z(\bar{L}) = \bar{L}$ we have $[\langle \bar{x} \rangle^{\bar{L}}, \bar{E}] = \bar{0}$ for some essential ideal $\bar{E}$ of $\bar{L}$. Therefore $[\langle x \rangle^L, E] \subseteq I$. By Lemma 2.3 E is essential in L. Therefore $[\langle x \rangle^L, E] \neq 0$ by semisimplicity. Hence $\langle x \rangle^L \cap I \neq 0$ and I is essential in L. □

QUESTION 2.14. Is the converse of Proposition 2.13 true? If this is the case we have the following result: if L is semisimple and I is essential in L, with $J \supseteq I$, then J/I is essential in L/I.

3. Properties inherited from proper quotients

In this section we study the following type of question: if every proper quotient of a Lie algebra (perhaps by an ideal of some restricted type) has some property, does L itself inherit that property? Our first result is an abstract formulation of a general principle which we apply in several different cases.

THEOREM 3.1. *Suppose that to each Lie algebra L there is associated a set $\mathscr{X}(L)$ of ideals of L, partially ordered by inclusion. Assume that $\mathscr{X}$ preserves quotients in the following sense: if $I \in \mathscr{X}(L)$ then*

$$\mathscr{X}(L/I) = \{J/I : J \in \mathscr{X}(L) \text{ and } J \subseteq I\}. \tag{2}$$

Then $\mathscr{X}(L)$ satisfies the ascending chain condition if and only if $\mathscr{X}(L/I)$ satisfies the ascending chain condition for every nonzero ideal $I \in \mathscr{X}(L)$.

PROOF. Suppose $\mathscr{X}(L)$ satisfies the ascending chain condition. By (2), $\mathscr{X}(L/I)$ is order-isomorphic to a subset of $\mathscr{X}(L)$, hence also satisfies the ascending chain condition.

Conversely, suppose that $0 \neq I_0 \subseteq I_1 \subseteq \cdots$ is an ascending chain in $\mathscr{X}(L)$. Then $I_0/I_0 \subseteq I_1/I_0 \subseteq \cdots$ is an ascending chain in $\mathscr{X}(L/I_0)$, which must stop. Hence the chain in $\mathscr{X}(L)$ stops, so $\mathscr{X}(L)$ satisfies the ascending chain condition. □

COROLLARY 3.2. *Let L be a Lie algebra such that $L/I \in$ Max-$\lhd$ for all nonzero ideals I of L. Then $L \in$ Max-$\lhd$.* □

The analogue of Corollary 3.2 does *not* hold for Max-c. Since Max-c = Min-c this answers Question 3 of Aldosray and Stewart [2] in the negative.

EXAMPLE 3.3. If $L/I \in$ Max-c for all nonzero ideals I of L, then L need not satisfy Max-c.

Let L be the infinite Heisenberg algebra as in Example 1.3. Then L is nilpotent of class 2, and its centre $\zeta_1(L) = \langle z \rangle$. Every nonzero ideal I contains $\langle z \rangle$, so L/I is abelian, hence satisfies Max-c. However,

$$\langle z \rangle \subsetneqq \langle z, x_0 \rangle \subsetneqq \langle z, x_0, x_1 \rangle \subsetneqq \cdots$$

is a strictly ascending chain of centralizer ideals. Indeed

$$\langle z, x_0, \ldots, x_k \rangle = C_L(\langle y_{k+1}, y_{k+2}, \ldots \rangle^L).$$ □

However, an analogue of Corollary 3.2 *does* hold for Max-c if L is semisimple, and indeed the statement can be strengthened:

THEOREM 3.4. *If L is semisimple and $L/I \in$ Max-c for all nonzero centralizer ideals I of L, then $L \in$ Max-c.*

For the proof, we first require:

LEMMA 3.5. *Let L be a semisimple Lie algebra with $A \lhd L$. Let $I \supseteq A$ be a centralizer ideal in L. Then I/A is a centralizer ideal in L/A.*

PROOF. Let bars denote images modulo A. Suppose that $I = C_L(J)$ where $J \lhd L$. Then $\bar{I} \subseteq C_{\bar{L}}(\bar{J})$. We claim that $\bar{I} = C_{\bar{L}}(\bar{J})$. If not, choose $\bar{x} \in C_{\bar{L}}(\bar{J}) \setminus \bar{I}$. Then $[\bar{x}, \bar{J}] = 0$ so $[x, J] \subseteq A$, and $x \notin I$. Thus $[x, J] \subseteq A \cap J \subseteq I \cap J = 0$ since L is semisimple, which is a contradiction. □

PROOF OF THEOREM 3.4. Let $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n \subseteq \cdots$ be an ascending chain of centralizer ideals of L. Then $I_1/I_0 \subseteq I_2/I_0 \subseteq \cdots$ is an ascending chain of centralizer ideals of L/I_0, by Lemma 3.5. But $L/I_0 \in$ Max-c, so the chain stops. □

Next we consider complement ideals.

LEMMA 3.6. *Let L be a Lie algebra and let I be a complement ideal in L. Then the complement ideals in L/I are in bijective correspondence with the complement ideals of L that contain I.*

PROOF. Let $\bar{L} = L/I$ and let bars denote images modulo I. Let $J \supseteq I$ be a complement ideal in L. Then $J \cap K = 0$ for some $K \lhd L$. Therefore $\bar{J} \cap \bar{K} = \bar{0}$. Let $\bar{H}$ be a complement of $\bar{K}$ in $\bar{L}$ such that $\bar{H} \supseteq \bar{J}$. We have $\bar{H} \cap \bar{K} = \bar{0}$, so that $H \cap K \subseteq I \cap K \subseteq J \cap K = 0$. Since J is a complement of K and $H \supseteq J$ it follows that $H = J$. Therefore $\bar{H} = \bar{J}$ and $\bar{J}$ is a complement ideal in $\bar{L}$.

Conversely, suppose that $\bar{J}$ is a complement of $\bar{K}$ in $\bar{L}$. Let I be a complement of H in L. Then clearly $J \cap K \cap H = 0$. Let R be a complement of $K \cap H$ such that $R \supseteq J$. Then $R \cap K \cap H = 0$ implies that $R \cap K = I$, which implies that $\bar{R} \cap \bar{K} = \bar{0}$. Therefore $\bar{R} = \bar{J}$. Thus $R = J$ and J is a complement ideal in L. □

THEOREM 3.7. *$L \in$ Max-CI if and only if $L/I \in$ Max-CI for any nonzero complement ideal I of L.*

PROOF. Let $\mathscr{X}(L)$ be the set of all complement ideals of L and apply Theorem 3.1, using Lemma 3.6 to verify condition (2). □

Finally we turn to small ideals. Say that a Lie algebra $L \in$ Max-SMI if L satisfies the ascending chain condition for small ideals of L.

THEOREM 3.8. *$L/I \in$ Max-SMI for all nonzero small ideals I of L if and only if $L \in$ Max-SMI.*

PROOF. In one direction let $\mathscr{X}(L)$ be the set of all small ideals of L and apply Theorem 3.1, using Lemma 2.4 to verify condition (2). For the converse use Lemma 2.5. □

4. Camillo's theorem and Shock's theorems

Camillo [5] proves that a commutative ring R is Noetherian if and only if R/I is a Goldie ring for all ideals I. A natural analogue of this result for Lie algebras is the following: $L \in \text{Max-}\lhd$ if and only if $L/I \in \text{Max-CI} \cap \text{Max-c}$ for every ideal I. We show that this analogue is *false*, even if L is semisimple.

EXAMPLE 4.1. Let V be a vector space over F of infinite dimension d, where d is a limit cardinal. For example, well-order the cardinals as $\aleph_\mu$ for ordinals μ and let $d = \aleph_\omega$. Define $\mathscr{L}(V, V)$ to be the Lie algebra of all linear maps $V \to V$, and let $Z = \{cI : c \in F\}$. By Stewart [18] $L = \mathscr{L}(V, V)/Z$ has a unique ascending chain of ideals

$$I_0/Z \subseteq I_1/Z \subseteq \cdots$$

where

$$I_\mu = \{A \in \mathscr{L}(V, V) : \text{rank}(A) < \aleph_\mu\} + Z .$$

Each factor in the chain is simple and non-abelian, so L is semisimple. In each quotient L/I_μ the only centralizer ideals are I_μ/I_μ and L/I_μ, and these are also the only complement ideals. Therefore $L/I_\mu \in \text{Max-CI} \cap \text{Max-c}$ for all μ. But clearly $L \notin \text{Max-}\lhd$. □

In view of this example we may ask whether some alternative characterization of Lie algebras with $\text{Max-}\lhd$ exists, in terms of weaker chain conditions on all quotient algebras. We establish such a result in Theorem 5.6 below. To state and prove it we first observe that Lie algebra analogues of some module-theoretic results due to Shock [15] are true. The statements and proofs are easily obtained from the following observation. Let L be a Lie algebra and let M be an L-module. Then M has a natural structure as $U(L)$-module, where $U(L)$ is the universal enveloping algebra of L, Jacobson [7]. This construction preserves submodules, quotient modules, and module generating sets. We may therefore transfer ring-theoretic results about general R-modules, where R is an associative ring, to L-modules. In particular the results of Shock [13] carry over to L-modules without extra effort. In consequence we obtain the following substitute for Camillo's Theorem:

THEOREM 4.2. *A Lie algebra* $L \in \text{Max-}\lhd$ *if and only if* L/I *belongs to* $\text{Max-SMI} \cap \text{Max-CI}$ *for any ideal* I *of* L.

PROOF. If $L \in$ Max-$\lhd$ then every quotient algebra L/I satisfies Max-$\lhd$, hence certainly lies in Max-SMI $\cap$ Max-CI.

For the converse, suppose that every quotient algebra L/I belongs to Max-SMI $\cap$ Max-CI. The sum of two small ideals is small, hence by Max-SMI the Jacobson radical $rad(L/I)$ of L/I is small for all ideals I of L. (See Kamiya [9] for the definition and basic properties of $rad(L)$.) Since $L/I \in$ Max-CI the socle $s(L/I)$ is finitely generated, by Aldosray and Stewart [2] Lemma 2.2. Now apply the Lie analogue of Theorem 3.8 of Shock [13]. $\square$

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Department of Mathematics,
Umm Al-Qura University,
Makkah, PO Box 3711,
Saudi Arabia
and
Mathematics Institute,
University of Warwick,
*Coventry CV*4 7*AL,*
England