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Imaginary powers of elliptic second order differential operators in L^p-spaces

Dedicated to Professor Y. Komura on the occasion of his 60th-birthday

Jan PRÜSS and Hermann SOHR

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1. Introduction

Let Ω be an open domain in $\mathbb{R}^n$ with $n \geq 1$ and consider the second order elliptic differential operator on Ω defined by

$$(Eu)(x) = -\sum_{j,k=1}^{n} a_{jk}(x)\partial_j\partial_k u(x) + \sum_{j=1}^{n} b_j(x)\partial_j u(x) + c(x)u(x) \tag{1.1}$$

where $\partial\Omega$ is the boundary of Ω. Let $1 < p < \infty$ and let E_p denote the $L^p(\Omega)$-realization of this boundary value problem defined by $(E_p u)(x) = (Eu)(x)$ with domain of definition $\mathscr{D}(E_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. The aim of this paper is to give simple conditions for the boundedness of the purely imaginary powers E_p^{iy} and for an estimate of the form

$$\| E_p^{iy} \| \leq K e^{\theta|y|} \qquad \text{for all } y \in \mathbb{R}, \tag{1.2}$$

where $K = K(\theta) > 0$, $0 < \theta < \pi$ are constants; see Section 3 for the definition of complex powers of linear operators. We show that this holds, under some smoothness assumptions and technical restrictions on the coefficients a_{jk}, b_j, c and the domain Ω, essentially whenever the resolvent estimate

$$\|(\lambda + E_p)^{-1}\| \leq \frac{C}{|\lambda|} \qquad \text{for all } \lambda \in \Sigma_{\pi-\theta} = \{\lambda \in \mathbb{C}\colon \lambda \neq 0, |\arg \lambda| < \pi - \theta\}, \tag{1.3}$$

is satisfied. Observe that this condition for $0 < \theta < \pi/2$ implies that $-E_p$ generates an analytic semigroup e^{-tE_p} in $L^p(\Omega)$. In Prüss and Sohr [16] it has been shown that (1.3) is always necessary for (1.2); the difficult part is the proof of the converse. In the paper just mentioned it is also shown that (1.2) implies the same estimate for $\delta + E_p$ for each $\delta > 0$, with the same constants K and θ.

Estimates for the imaginary powers A^{iy} of a closed linear operator A in a Banach space X are of interest for several reasons. One of these is related to the determination of the domains $\mathscr{D}(A^\alpha)$ of the fractional powers A^α of A, where $0 < \alpha < 1$. It is known (see e.g. Triebel [22]) that uniform boundedness of A^{iy} for $|y| \leq 1$ implies $\mathscr{D}(A^\alpha) = [X, \mathscr{D}(A)]_\alpha$, the complex interpolation space between $\mathscr{D}(A)$, equipped with the graph norm of A, and X. As a consequence one then obtains via the reinterpolation theorem the identities $[\mathscr{D}(A^\beta), \mathscr{D}(A^\alpha)]_\theta = \mathscr{D}(A^\gamma)$, where $\gamma = \theta\alpha + (1-\theta)\beta$. This property in turn leads e.g. to sharp results about the solvability behavior of evolution equations of the form

$$\dot{u}(t) + Au(t) = f(t), \quad u(0) = u_0, \quad t \in [0, T); \tag{1.4}$$

see e.g. von Wahl [23] or Giga and Sohr [10]. Thus via estimate (1.2) the domains of the fractional powers of A are linked to complex interpolation theory and so its tools are available in the study of (1.4).

On the other hand, property (1.2) of the purely imaginary powers E_p^{iy} has recently become important by a new remarkable application to evolution equations; Dore and Venni [4] proved by means of (1.2) the important L^q-L^p-estimate

$$\int_0^T \left\| \frac{\partial}{\partial t} u \right\|_p^q dt + \int_0^T \|E_p u(t)\|_p^q \, dt \leq C \int_0^T \|f(t)\|_p^q \, dt \tag{1.5}$$

for evolution equations (1.4) with $A = E_p$, $u_0 = 0$, and $1 < p, q < \infty$, whenever E_p is boundedly invertible, $T < \infty$, and (1.2) is valid for some $\theta < \pi/2$. Here $C > 0$ is a constant which depends on $T > 0$, in general. Such a priori estimates are used in the theory of nonlinear equations, e.g. like

$$u_t + E_p u + N(u) = f \tag{1.6}$$

where $N(u)$ is a nonlinearity subordinate to $E_p u$. Dore and Venni's theory has been extended in [16], [10] in a direction which is especially important for unbounded domains. By means of this result estimate (1.5) is also valid for unbounded domains and with constant C independent of T. Letting $T = \infty$ in (1.5) this yields asymptotic properties of the solution $u(t)$ of (1.4) for $t \to \infty$; [16], [10].

So far there are only a few methods available for proving (1.2). In the Hilbert space case $p = 2$, one may use the functional calculus for selfadjoint linear operators and lower order perturbations to obtain (1.2). The angle θ will be optimal in this case, and the assumptions on the coefficients of E are quite weak; if E is taken in divergence form then, besides ellipticity, the coefficients are only required to be of class L^∞.

For $p \neq 2$, however, there are essentially two different methods known. The first one relies on the so-called transference method developed by Coifman and Weiss [3]; see also Stein [19]. When combined with multiplier theory for $L^p(\mathbb{R})$ it leads to the estimate

$$\|E_p^{iy}\| \leq K(1+|y|)e^{\frac{\pi}{2}|y|}, \qquad y \in \mathbb{R}, \tag{1.7}$$

whenever $-E_p$ generates a positive strongly continuous contraction semigroup $e^{-E_p t}$ in $L^p(\Omega)$. In this case the assumptions on the coefficients of E turn out to be as weak as in the case $p=2$, but in contrast to $p=2$, the angle $\pi/2$ in (1.7) is not optimal, in general. In fact, when E is of pure divergence form, i.e. $c(x) \equiv 0$ and $b(x) \equiv -\sum_{j=1}^{n} \partial_j a_{jk}(x)$, then $\sigma(E_p) \subset [0, \infty)$ and (1.2) should hold for any $\theta > 0$, with $K = K(\theta)$.

Combining (1.7) with (1.2) for $p=2$, via interpolation one obtains (1.2) also for $p \neq 2$, with some $\theta < \pi/2$, which is enough for the application of the Dore-Venni theorem to obtain the a priori estimate (1.5) mentioned above; see also the recent paper of Duong [6]. On the other hand, there are problems which require for the optimal angle θ; see the applications in the paper of the authors [16], in Clèment and Prüss [2], and in Prüss [15].

In the special case of a bounded domain Ω with $\partial\Omega$ of class C^∞ and E with C^∞-coefficients, Fujiwara [8] and Seeley [18], [17] obtained (1.2) with optimal angle θ by means of a quite different approach; see also Triebel [22]. Their method rests essentially on the theory of pseudo-differential operators and is quite involved, but it also applies to higher order equations and systems.

The motivation for this work are the following:

(i) Our proof is direct; it uses only elementary multiplier theory in $\mathbb{R}^n$ for the constant coefficient case and applies rather elementary perturbation and localization arguments.

(ii) (1.2) is proved under rather mild regularity assumptions on the coefficients and $\partial\Omega$; see Section 2 for details. This is in particular important for applications to nonlinear equations where the coefficients depend on u and ∇u; see [15].

(iii) Unbounded domains Ω are included. As mentioned above, this leads to new a priori estimates (1.5) for unbounded domains.

(iv) The estimate (1.2) we obtain is best possible concerning the value of θ.

The plan for this paper is as follows. The main results are stated and discussed in Section 2, while their proofs are carried through is Sections 4, 5 and 6, for $\Omega = \mathbb{R}^n$, $\Omega \equiv \mathbb{R}^n_+$, and $\Omega \subset \mathbb{R}^n$ arbitrary, respectively. Section 3 contains an abstract perturbation result which allows for the passage from constant to variable coefficients with small deviations of the main part. Finally, in Section 7 some resolvent estimates are sketched. Although these seem to be known we have included them here for the convenience of the reader and for the sake of future reference.

Concluding we want to mention that the method of proof presented here can be extended to arbitrary elliptic systems of order $2m$ in case $\Omega = \mathbb{R}^n$. But even for a halfspace $\Omega = \mathbb{R}^n_+$ our method is restricted to a single equation of second order, apparently due to a weakness of the perturbation result, Proposition 3.1. Therefore further research will be necessary.

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2. Main results

In the following, $\Omega \subset \mathbb{R}^n$ is either the whole space $\mathbb{R}^n$, the half space $\mathbb{R}^n_+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 > 0\}$, a bounded domain or an exterior domain in $\mathbb{R}^n$, i.e. a domain which contains a neighborhood of infinity. We always assume that the boundary $\partial\Omega$ of Ω is of class C^2.

First we explain some notations. Let $1 < p < \infty$ and let $L^p(\Omega)$ denote the usual (complex) Lebesgue space with norm $\|u\|_p = (\int_\Omega |u(x)|^p dx)^{\frac{1}{p}}$. For $m \in \mathbb{N}$, $W^{m,p}(\Omega)$ means the usual Sobolev space with norm $\|u\|_{m,p} = (\sum_{|\alpha| \le m} \|\partial^\alpha_u\|^p_p)^{\frac{1}{p}}$ where $\partial_j = \partial/\partial x_j$, $j = 1, \ldots, n$, $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. $W^{m,p}_0(\Omega)$ denotes the closure of the space of test functions $C_0^\infty(\Omega)$ under the norm of $W^{m,p}(\Omega)$, where as usual $C_0^\infty(\Omega)$ means the space of all smooth functions on Ω with compact support in Ω. We set $\nabla = (\partial_1, \partial_2, \ldots, \partial_n)$ and $\|\nabla u\|_p = (\sum_{j=1}^n \|\partial_j u\|_p^p)^{\frac{1}{p}}$, $\|\nabla^2 u\|_p = (\sum_{j,k=1}^n \|\partial_j\partial_k u\|_p^p)^{\frac{1}{p}}$. If $a(x) = (a_{jk}(x))$ is a matrix, $b(x) = (b_j(x))$ a vector and $c(x)$ a scalar, we write $a(x)\xi \cdot \xi = \sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k$ for $\xi = (\xi_1, \ldots, \xi_n)$, $a(x) : \nabla^2 u(x) = \sum_{j,k=1}^n a_{jk}(x)\partial_j\partial_k u(x)$, $b(x) \cdot \nabla u(x) = \sum_{j=1}^n b_j(x)\partial_j u(x)$.

Then the operator E_p introduced in (1.1) can be written in the form

$$(E_p u)(x) = -a(x) : \nabla^2 u(x) + b(x) \cdot \nabla u(x) + c(x)u(x), \tag{2.1}$$
$$u \in \mathscr{D}(E_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

$E_p : \mathscr{D}(E_p) \to L^p(\Omega)$ is welldefined under the assumptions (A1) $\sim$ (A3) stated below. $\mathscr{N}(E_p)$, $\mathscr{R}(E_p)$, $\sigma(E_p)$, $\rho(E_p)$ mean null space, range, spectrum, and resolvent set of the linear operator E_p. $\mathscr{L}(L^p(\Omega))$ designates the space of all bounded linear operators $T : L^p(\Omega) \to L^p(\Omega)$ with the operator norm $\|T\|$.

For $0 < \alpha < 1$, $C^\alpha(\bar{\Omega})$ denotes the space of all functions u on Ω satisfying the Hölder estimate $|u(x_2) - u(x_1)| \le C|x_2 - x_1|^\alpha$ for all $x_1, x_2 \in \bar{\Omega}$, where $\bar{\Omega}$ as usual is the closure of Ω.

Assumptions on the coefficients. Let $1 < p < \infty$.

(A1) $a(x) = (a_{jk}(x))$ is a real symmetric matrix for all $x \in \bar{\Omega}$, and there is a constant $a_0 > 0$ such that

$$a_0 \le a(x)\xi \cdot \xi \le a_0^{-1} \qquad \text{for all} \quad x \in \bar{\Omega},\ \xi \in \mathbb{R}^n,\ |\xi| = 1.$$

(A2) $a_{ij} \in C^{\alpha}(\bar{\Omega})$, for some $\alpha \in (0, 1)$; if Ω is unbounded, $a_{ij}^{\infty} = \lim_{|x| \to \infty} a_{ij}(x)$ exists and

$$|a_{ij}(x) - a_{ij}^{\infty}| \le C|x|^{-\alpha} \qquad \text{for all} \quad x \in \Omega \text{ with } |x| \ge 1,\ i, j = 1, \ldots, n,$$

where $C > 0$ is a constant.

(A3) $b_i(x), c(x)$ are measurable complex-valued functions which admit representations of the form $b_i(x) = \sum_{k=1}^{N} b_i^k(x)$, $i = 1, \ldots, n$, $c(x) = \sum_{k=1}^{N} c^k(x)$, where $b_i^k \in L^{r_k}(\Omega)$, $c^k \in L^{s_k}(\Omega)$, for some numbers r_k, s_k which are subject to the inequalities $p \le r_k \le \infty$, $p \le s_k \le \infty$, $r_k > n$, $s_k > n/2$.

(A4) $b_i \in L^{\hat{r}}(\Omega)$, $i = 1, \ldots, n$, $c \in L^{\hat{s}}(\Omega)$ for some $\hat{r}$, $\hat{s}$ with $1 \le \hat{r} < n$, $1 \le \hat{s} < n/2$.

If Assumptions (A1) $\sim$ (A3) are satisfied, the operator E_p defined in (2.1) with domain of definition $\mathscr{D}(E_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is welldefined. This follows from Sobolev's embedding theorem [7]. Indeed, under these assumptions we get $\|a\|_{\infty} = \sup_{x \in \bar{\Omega}} |a(x)| < \infty$ and the following interpolation estimates. Hölder's inequality yields with $r = r_k$ and $s = s_k$ from (A3)

$$\|b^k \cdot \nabla u\|_p \le \|b^k\|_r \|\nabla u\|_{\gamma}, \quad \|c^k u\|_p \le \|c^k\|_s \|u\|_{\beta} \tag{2.2}$$

where γ, β are defined by $\dfrac{1}{p} = \dfrac{1}{r} + \dfrac{1}{\gamma}$, $\dfrac{1}{p} = \dfrac{1}{s} + \dfrac{1}{\beta}$; note that $p \le \gamma \le \infty$, $p \le \beta \le \infty$. On the other hand, the Sobolev embedding theorem implies

$$\|\nabla u\| \le C\|\nabla^2 u\|_p^{\rho} \|u\|_p^{1-\rho} \quad \text{where} \quad \rho\left(\frac{1}{p} - \frac{2}{n}\right) + (1-\rho)\frac{1}{p} = \frac{1}{\gamma} - \frac{1}{n}, \tag{2.3}$$

i.e. $\rho = (1 + n/r)/2 \in [1/2, 1)$. Similarly

$$\|u\|_{\beta} \le C\|\nabla^2 u\|_p^{\tau} \|u\|_p^{1-\tau} \quad \text{where} \quad \tau\left(\frac{1}{p} - \frac{2}{n}\right) + (1-\tau)\frac{1}{p} = \frac{1}{\beta}, \tag{2.4}$$

i.e. $\tau = n/(2s) \in [0, 1)$. Combining (2.2) with (2.3) and (2.4) we get the inequalities

$$\|b^k \cdot \nabla u\|_p \le C_1 \|\nabla^2 u\|_p^{\rho} \|u\|_p^{1-\rho} \le \varepsilon \|\nabla^2 u\|_p + C(\varepsilon)\|u\|_p, \tag{2.5}$$

$$\|c^k u\|_p \le C_2 \|\nabla^2 u\|_p^{\tau} \|u\|_p^{1-\tau} \le \varepsilon \|\nabla^2 u\|_p + C(\varepsilon)\|u\|_p, \tag{2.6}$$

where $\varepsilon > 0$ is arbitrary, and therefore summing over k we obtain

$$\| E_p u \|_p \leq C_1 \| \nabla^2 u \|_p + C_2 \| u \|_p \leq C \| u \|_{2,p}. \tag{2.7}$$

This shows that E_p is welldefined for all $u \in \mathscr{D}(E_p)$; even more, the lower order term $b \cdot \nabla u + cu$ is a strictly subordinate perturbation of the main part $- a : \nabla^2 u$.

Our main results read as follows.

THEOREM A. *Let* $1 < p < \infty$, $n \in \mathbb{N}$, *and let* $\Omega \subset \mathbb{R}^n$ *be either* $\mathbb{R}^n$, *or the half space* $\mathbb{R}^n_+$, *or a bounded domain with* C^2*-boundary or an exterior domain with* C^2*-boundary. Consider the operator* E_p *defined in* (2.1) *with Assumptions* (A1) $\sim$ (A3). *Let* $0 < \theta < \pi$, *and suppose* $\sigma(E_p) \subset \Sigma_\theta$ *and*

$$\| (\lambda + E_p)^{-1} \| \leq C(|\lambda| + 1)^{-1} \qquad \text{for all} \quad \lambda \in \Sigma_{\pi - \theta}, \tag{2.8}$$

where $C > 0$ *is a constant. Then* $E_p^{iy} \in \mathscr{L}(L^p(\Omega))$ *and*

$$\| E_p^{iy} \| \leq M e^{\theta |y|} \qquad \text{for all} \quad y \in \mathbb{R}, \tag{2.9}$$

where $M > 0$ *is a constant.*

Observe that (2.8) implies that E_p admits a bounded inverse; for unbounded domains this is frequently not the case. Then we need the additional assumption (A4).

THEOREM B. *Let* $1 < p < \infty$, $n \in \mathbb{N}$, *and let* $\Omega \subset \mathbb{R}^n$ *be as in Theorem A. Consider the operator* E_p *defined in* (2.1) *with the Assumptions* (A1) $\sim$ (A4). *Let* $0 < \theta < \pi$ *and suppose* $\mathscr{N}(E_p) = 0$, $\sigma(E_p) \subset \bar{\Sigma}_\theta$,

$$\| (\lambda + E_p)^{-1} \| \leq C_1 |\lambda|^{-1} \qquad \text{for all} \quad \lambda \in \Sigma_{\pi - \theta}, \tag{2.10}$$

and

$$\| \nabla^2 u \|_p \leq C_2 \| E_p u \|_p \qquad \text{for all} \quad u \in \mathscr{D}(E_p), \tag{2.11}$$

where $C_1, C_2 > 0$ *are constants. Then* $E_p^{iy} \in \mathscr{L}(L^p(\Omega))$ *and*

$$\| E_p^{iy} \| \leq M e^{\theta |y|} \qquad \text{for all} \quad y \in \mathbb{R}, \tag{2.12}$$

where $M > 0$ *is a constant.*

In applications the operator E is frequently encountered in divergence form, i.e. $Eu(x) = - \sum_{i,j=1}^n \partial_i (a_{ij}(x) \partial_j u(x))$, $x \in \Omega$. In this situation one can use the selfadjointness of E_2 in $L^2(\Omega)$, localization and duality arguments to verify the resolvent estimates assumed in Theorems A and B. As a consequence we have the following result.

THEOREM C. *Let* $1 < p < \infty$, $n \in \mathbb{N}$, *and* $\Omega \subset \mathbb{R}^n$ *be as in Theorem A. Suppose* E_p *has divergence form* $E_p = - \sum_{j,k=1}^n \partial_j a_{jk} \partial_k + \delta$, $\delta \geq 0$, *and let*

$a = (a_{jk})$ and $b_k = -(\sum_{j=1}^n \partial_j a_{jk})$, $k = 1, 2, \ldots, n$ fullfill the Assumptions (A1) $\sim$ (A3). *In addition, assume either of the following.*
(i) $\delta > 0$; *(ii)* Ω *is bounded*; *(iii)* $n \geq 3$ *and* (A4) *holds for* b_k, *with* $c = 0$; *(iv)* (2.11) *and* (A4) *hold for* b_k, *with* $c = 0$.
Then $\sigma(E_p) \subset [0, \infty)$, $\mathcal{N}(E_p) = 0$, $\overline{\mathcal{R}(E_p)} = L^p(\Omega)$. *For each* $0 < \theta < \pi$, *there are constants* $C_p(\theta) > 0$, $M_p(\theta) > 0$ *such that*

$$\|(\lambda + E_p)^{-1}\| \leq C_p(\theta)|\lambda|^{-1} \qquad \text{for all} \quad \lambda \in \Sigma_{\pi-\theta}, \tag{2.13}$$

and

$$E_p^{iy} \in \mathcal{L}(L^p(\Omega)), \ \|E_p^{iy}\| \leq M_p(\theta)e^{\theta|y|} \qquad \text{for all} \quad y \in \mathbb{R}. \tag{2.14}$$

If Ω *is bounded or* $\delta > 0$ *then* E_p *is boundedly invertible.*

If E is not of divergence form, $\sigma(E_p)$ need not be restricted to the positive halfline and so the resolvent estimates cannot be expected to be true for any $\theta \in (0, \pi)$. However, we show in Section 7 that away from $[0, \infty)$ the spectrum of E_p consists only of eigenvalues, and therefore the condition

$$\mathcal{N}(\lambda + E_p) = 0, \qquad \text{for all} \quad \lambda \in \Sigma_{\pi-\theta}, \tag{2.15}$$

is essentially sufficient for the resolvent estimates. Actually, for $\lambda \in \Sigma_{\pi-\theta}$, the kernels of $\lambda + E_p$ are independent of p, and therefore condition (2.17) need only be verified for one p, say for $p = 2$.

THEOREM D. *Let* $1 < p < \infty$, *and let* Ω *be as in Theorem A. Consider the operator* E_p *defined in* (2.1) *with coefficients subject to* (A1) $\sim$ (A3) *with constants* $r_k \geq p$, $r_k > n$, $s_k \geq p$, $s_k > n/2$. *Then*
(a) For each $\theta \in (0, \pi)$ *there is* $R = R_p(\theta) > 0$ *such that* $\Sigma_{\pi-\theta} \setminus B_R(0) \subset \rho(-E_p)$ *and*

$$\|(\lambda + E_p)^{-1}\| \leq C_p(\theta)|\lambda|^{-1} \qquad \text{for all} \quad \lambda \in \Sigma_{\pi-\theta}, \ |\lambda| > R_p(\theta), \tag{2.16}$$

for some constant $C_p(\theta) > 0$. *In particular,* $\delta + E_p$ *satisfies the assumptions of Theorem A for any* $\delta > R_p(\theta)$.
(b) Assume (A4). *Then* $\mathcal{N}(\lambda + E_p)$ *is increasing in p, for all* $\lambda \in \mathbb{C}$, *and for each* $\lambda \notin (-\infty, 0]$, $\mathcal{N}(\lambda + E_p)$ *is independent of p.*
(c) Assume that $-E_p$ *has no eigenvalues* $0 \neq \lambda \in \bar{\Sigma}_{\pi-\theta}$. *Then* $\sigma(E_p) \subset \Sigma_\theta \cup \{0\}$, *and for each* $\eta > 0$ *there is a constant* $C_p(\eta) > 0$, *such that*

$$\|(\lambda + E_p)^{-1}\| \leq C_p(\eta)|\lambda|^{-1} \qquad \text{for all} \quad \lambda \in \Sigma_{\pi-\theta}, \ |\lambda| > \eta. \tag{2.17}$$

Therefore $\delta + E_p$ *satisfies the assumptions of Theorem A, for each* $\delta > 0$. *If in addition* Ω *is bounded and* $\mathcal{N}(E_p) = 0$ *then* E_p *is invertible, hence satisfies the assumptions of Theorem A.*

(*d*) *Let* Ω *be unbounded, let* (A4) *hold, and assume in addition* $1 < p < n/2$ *or* (2.11); *suppose* $-E_p$ *has no eigenvalues* $\lambda \in \bar{\Sigma}_{\pi-\theta}$. *Then* $\overline{\mathscr{R}(E_p)} = L^p(\Omega)$, $\sigma(E_p) \subset \Sigma_\theta \cup \{0\}$, *and there is a constant* $C_p(\theta) > 0$, *such that*

$$\|(\lambda + E_p)^{-1}\| \leq C_p(\theta)|\lambda|^{-1} \qquad \textit{for all} \quad \lambda \in \Sigma_{\pi-\theta}, \tag{2.18}$$

and (2.11) *holds. Therefore* E_p *satisfies the assumptions of Theorem B.*

Let us remark that the additional restrictions in (d) of Theorem D are due to (2.11). Except for trivial cases, we have only been able to verify this condition in case $1 < p < n/2$; this then leads to the restriction $n \geq 3$ in (iii) of Theorem C. The difficulty arises from the generalized null space of E in the case of unbounded domains. We do not know whether this restriction is essential or a shortcoming of the methods employed here.

The proofs of these results are carried out in several steps. First we treat the case $\Omega = \mathbb{R}^n$. The crucial part in $\mathbb{R}^n$ is that of variable coefficients $a_{jk}(x)$ which have small deviations from a_{jk}^∞; the main tool here is the perturbation result stated and proved in the next section. For small variable coefficients our result is given in Proposition 4.2; Theorems A and B are then proved for the case $\Omega = \mathbb{R}^n$. Section 5 contains the proof of these theorems for the halfspace $\Omega = \mathbb{R}^n_+$ which is based on a certain variable transformation which allows for application of the reflection principle. General domains are treated in Section 6 via localization and reduction to either $\mathbb{R}^n$ or $\mathbb{R}^n_+$. Finally, in Section 7 it is indicated how the resolvent estimates can be obtained, in particular proofs for Theorem C and Theorem D are sketched. Actually, this is now of only little effort since all the technical tools have already been employed before.

To keep the technical details to a minimum we have restricted the statements and proofs to the case of Dirichlet boundary conditions on domains with boundary of class C^2 which are either bounded or exterior (as well as whole and half spaces). Actually, only minor modifications of the proofs below are needed to cover also mixed boundary conditions, where on one part $\Gamma_d \subset \partial\Omega$ Dirichlet conditions are prescribed while on the remaining part $\Gamma_n = \partial\Omega \setminus \Gamma_d$ conormal or conormal conditions of the third kind are prescribed, provided Γ_d and Γ_n are strictly separated. By means of variable transformations similar to those employed in Section 5 it is also enough to assume $\partial\Omega$ of class $C^{1+\alpha}$, for some $\alpha > 0$. Finally, the class of domains considered here, for simplicity, has been chosen such that always finite partitions of unity are sufficient. Under suitable uniformity conditions on the boundary of Ω and the coefficients a_{ij} of the main part of E it will not be difficult to extend our approach to more general domains, employing countable partions of unity.

3. Imaginary powers and perturbations

Let X be a complex Banach space with norm $|\cdot|$, and A a closed linear operator in X; then as usual $\mathscr{D}(A)$, $\mathscr{R}(A)$, $\mathscr{N}(A)$, $\rho(A)$, $\sigma(A)$ denote domain, range, kernel, resolvent set, and spectrum of A respectively. A is called *sectorial* if $\mathscr{D}(A)$ and $\mathscr{R}(A)$ are dense in X, $\mathscr{N}(A) = 0$, $(-\infty, 0) \subset \rho(A)$, and there is a constant $M \geq 1$ such that

$$\|t(t+A)^{-1}\| \leq M \qquad \text{for all } t > 0; \tag{3.1}$$

the class of sectorial operators in X will be denoted by $\mathscr{S}(X)$. It is easy to see that sectorial operators satisfy

$$\lim_{t\to\infty} t(t+A)^{-1}x = x, \quad \lim_{t\to 0+} t(t+A)^{-1}x = 0, \quad x \in X. \tag{3.2}$$

If X is reflexive, it is wellknown that (3.1) and (3.2) imply the decomposition $X = \mathscr{N}(A) \oplus \overline{\mathscr{R}(A)}$; hence in this case $\mathscr{R}(A)$ is dense in X if and only if $\mathscr{N}(A) = 0$. Let

$$\Sigma_\phi = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| < \phi\};$$

the *spectral angle* ϕ_A of $A \in \mathscr{S}(X)$ is then defined by

$$\phi_A = \inf\{\phi : \rho(-A) \supset \Sigma_{\pi-\phi},\ C_\phi := \sup_{\lambda\in\Sigma_{\pi-\phi}} \|\lambda(\lambda+A)^{-1}\| < \infty\}. \tag{3.3}$$

Obviously, we have $\phi_A \in [0, \pi)$ and $\sigma(A) \subset \bar{\Sigma}_{\phi_A}$; if $A \in \mathscr{S}(X)$ is bounded and invertible then $\phi_A = \sup_{\lambda\in\sigma(A)} |\arg\lambda|$.

Recall that for $A \in \mathscr{S}(X)$ complex powers A^z of A can be defined by means of the formula (see Komatsu [13])

$$\begin{aligned} A^z x = -\frac{\sin \pi z}{\pi}\Big\{ \frac{x}{z} - \frac{A^{-1}x}{1+z} + \int_0^1 t^{z+1}(t+A)^{-1}A^{-1}x\,dt \\ + \int_1^\infty t^{z-1}(t+A)^{-1}Ax\,dt \Big\} \qquad \text{for } x \in \mathscr{D}(A)\cap\mathscr{R}(A),\ |\mathrm{Re}\, z| < 1. \end{aligned} \tag{3.4}$$

A^z defined by (3.4) is densely defined and closable; the closure will again be denoted by A^z. In general A^z need not be bounded; however, if $A^{is} \in \mathscr{B}(X)$ for each $s \in \mathbb{R}$ and $\sup_{|s|\leq 1} \|A^{is}\| < \infty$, then A is said to admit *bounded imaginary powers.* The class of all sectorial operators which admit bounded imaginary powers will be denoted by $\mathcal{BIP}(X)$.

If $A \in \mathcal{BIP}(X)$ then $\{A^{is} : s \in \mathbb{R}\} \subset \mathscr{B}(X)$ forms a strongly continuous group of bounded linear operators in X. The type θ_A of this group, i.e.

$$\theta_A = \overline{\lim}_{|s|\to\infty} \frac{1}{|s|} \log \| A^{is} \| \tag{3.5}$$

will be called *power angle* of A. It has been shown that the inequality $\phi_A \le \theta_A$ always holds; cp. Prüss and Sohr [16].

Observe that the class $\mathcal{BIP}(X)$ is invariant under similarity transforms. In fact, if $S \in \mathscr{L}(X, Y)$ is boundedly invertible, then $A_S = SAS^{-1}$ satisfies the relations

$$(\lambda + A_S)^{-1} = S(\lambda + A)^{-1} S^{-1} \qquad \text{for all} \quad \lambda \in \rho(-A) \tag{3.6}$$

$$(A_S)^z = SA^z S^{-1} \qquad \text{for all} \quad z \in \mathbb{C}. \tag{3.7}$$

In particular, we have $\sigma(A_S) = \sigma(A)$, $\phi_{A_S} = \phi_A$, and $\theta_{A_S} = \theta_A$. This remark will be quite useful in subsequent sections.

If $A \in \mathcal{BIP}(X)$ then $A_\varepsilon = A^\varepsilon(\varepsilon + A)^{-2\varepsilon} \in \mathscr{B}(X)$ is welldefined and $A_\varepsilon x \to x$ as $\varepsilon \to 0+$ for each $x \in X$. This can be obtained by means of the functional calculus for $\mathcal{BIP}(X)$ induced by the inverse Mellin transform; cp. Prüss and Sohr [16]. This fact will be used in the proof of Proposition 3.1 below.

It is not known whether the class $\mathcal{BIP}(X)$ is preserved under perturbations with sufficiently small relative bound. Here we present only a partial answer to this question involving a certain commutator condition. The result reads as follows.

PROPOSITION 3.1. *Let X be a complex Banach space, $A \in \mathcal{BIP}(X)$, and B a linear operator in X which is subject to*
(i) $\mathscr{D}(B) \supset \mathscr{D}(A)$ and $|Bx| \le \beta |Ax|$ for all $x \in \mathscr{D}(A)$, with some $\beta > 0$;
(ii) Let $\theta_A < \pi$ and there are $\theta \in (\theta_A, \pi)$ and $\psi \in L^1(\mathbb{R}_+)$ such that

$$|(\lambda + A)^{-1} Bx - B(\lambda + A)^{-1} x| \le \psi(|\lambda|)|Ax|, \quad x \in \mathscr{D}(A), \ \lambda \in \Sigma_{\pi-\theta};$$

(iii) $1/\beta > C_\theta := \sup_{\lambda \in \Sigma_{\pi-\theta}} |A(\lambda + A)^{-1}|$.
Then $A + B$ is closed with domain $\mathscr{D}(A + B) = \mathscr{D}(A)$, $A + B$ is sectorial and belongs to $\mathcal{BIP}(X)$, and $\theta_{A+B} \le \max(\theta_A + \arcsin \beta, \theta)$. If merely (i) *and* (iii) *hold then $A + B \in \mathscr{S}(X)$ and $\phi_{A+B} \le \theta$.*

PROOF. (a) Since $\mathscr{N}(A) = 0$ and $\mathscr{R}(A)$ is dense in X, by (i), BA^{-1} is well and densely defined and bounded by β; observe that $C_\theta \ge 1$ because of (3.2), therefore $0 < \beta < 1$. Let K denote the unique extension of BA^{-1} to all of X. With $\beta < 1$ it is obvious that $A + B$ with domain $\mathscr{D}(A + B) = \mathscr{D}(A)$ is closed, and that $\mathscr{N}(A + B) = 0$ and $\mathscr{R}(A + B)$ is dense in X. To show $(-\infty, 0) \subset \rho(A + B)$ write

$$\lambda + A + B = (1 + B(\lambda + A)^{-1})(\lambda + A),$$

and observe that by (iii)

$$\| B(\lambda + A)^{-1} \| \leq \| K \| \, \| A(\lambda + A)^{-1} \| \leq \beta C_\theta < 1, \qquad \lambda \in \Sigma_{\pi - \theta},$$

hence $\Sigma_{\pi-\theta} \subset \rho(-(A + B))$ and

$$\begin{aligned} \|(\lambda + A + B)^{-1}\| &\leq \|(\lambda + A)^{-1}\| \, \|1 + B(\lambda + A)^{-1})^{-1}\| \\ &\leq \frac{1 + C_\theta}{1 - \beta C_\theta} \cdot \frac{1}{|\lambda|}, \qquad \lambda \in \Sigma_{\pi-\theta}, \end{aligned}$$

in particular $A + B \in \mathscr{S}(X)$, and $\phi_{A+B} \leq \theta$.

(b) To show $A + B \in \mathcal{BIP}(X)$ we derive an expansion of $(\lambda + \mathrm{A} + B)^{-1}$ of the form

$$(\lambda + A + B)^{-1} = S(\lambda) + \sum_{n=0}^{\infty} A^n(\lambda + A)^{-(n+1)}(-K)^n, \tag{3.8}$$

which is appropriate for our approach. Set $R_\lambda = A(\lambda + A)^{-1}$ and $T_\lambda = KR_\lambda - R_\lambda K$; then

$$T_\lambda = KA(\lambda + A)^{-1} - A(\lambda + A)^{-1}K = \lambda(\lambda + A)^{-1}K - \lambda K(\lambda + A)^{-1},$$

hence by (ii)

$$\| T_\lambda \| \leq |\lambda| \psi(|\lambda|), \qquad \lambda \in \Sigma_{\pi-\theta}. \tag{3.9}$$

By means of a Neumann series we obtain

$$\begin{aligned} (\lambda + A + B)^{-1} &= (\lambda + A)^{-1}(1 + B(\lambda + A)^{-1})^{-1} = (\lambda + A)^{-1}(1 + KR_\lambda)^{-1} \\ &= (\lambda + A)^{-1} \sum_{n=0}^{\infty} (-1)^n (KR_\lambda)^n. \end{aligned}$$

Then induction yields the identities

$$(KR_\lambda)^n = \sum_{l=1}^{n} \sum_{j=0}^{l-1} (KR_\lambda)^{n-l} R_\lambda^j T_\lambda R_\lambda^{l-1-j} K^{l-1} + R_\lambda^n K^n. \tag{3.10}$$

For $n = 1$, (3.10) is merely the definition of T_λ. Suppose (3.10) holds for n; then

$$(KR_\lambda)^{n+1} = \sum_{l=1}^{n} \sum_{j=0}^{l-1} (KR_\lambda)^{n+1-l} R_\lambda^j T_\lambda R_\lambda^{l-1-j} K^{l-1} + KR_\lambda^{n+1} K^n,$$

and with

$$KR_\lambda^{n+1} K^n = \sum_{j=0}^{n} R_\lambda^j T_\lambda R_\lambda^{n-j} K_\lambda^n + R_\lambda^{n+1} K^{n+1},$$

(3.10) follows for $n + 1$. Thus (3.10) yields (3.8) where

$$S(\lambda) = (\lambda + A)^{-1} \sum_{n=1}^{\infty} (-1)^n \sum_{l=1}^{n} \sum_{j=0}^{l-1} (KR_\lambda)^{n-l} R_\lambda^j T_\lambda R_\lambda^{l-1-j} K^{l-1};$$

this term will be the "good" part since by (3.9)

$$\|S(\lambda)\| \le \frac{1 + C_\theta}{|\lambda|} \sum_{n=1}^{\infty} \sum_{l=1}^{n} \sum_{j=0}^{l-1} \|K\|^{n-1} \|R_\lambda\|^{n-1} \|T_\lambda\|$$

$$\le \psi(|\lambda|)(1 + C_\theta) \sum_{n=1}^{\infty} (\beta C_\theta)^{n-1} n(n+1)/2, \qquad \text{i.e.}$$

$$\|S(\lambda)\| \le \psi(|\lambda|) \frac{1 + C_\theta}{(1 - \beta C_\theta)^3}, \qquad \lambda \in \textstyle\sum_{\pi - \theta}. \tag{3.11}$$

(c) Let Γ denote the contour $(+\infty, 0] e^{-i(\pi-\theta)} \cup [0, \infty) e^{i(\pi - \theta)}$, and let $1/2 > \varepsilon > 0$ be fixed. Define $g_\varepsilon(\lambda; s) = (-\lambda)^{is+\varepsilon} (\varepsilon - \lambda)^{-2\varepsilon}$ and

$$F_\varepsilon(s) = \frac{1}{2\pi i} \int_\Gamma g_\varepsilon(\lambda; s)(\lambda + A + B)^{-1} d\lambda, \qquad s \in \mathbb{R}; \tag{3.12}$$

note that this integral is absolutely convergent. Contracting the contour to the positive real axis, Cauchy's theorem yields

$$\begin{aligned} F_\varepsilon(s) = &- \frac{\sin \pi(is + \varepsilon)}{\pi} \int_0^\varepsilon r^{is+\varepsilon} (\varepsilon - r)^{-2\varepsilon} (r + A + B)^{-1} dr \\ &- \frac{\sin \pi(is + \varepsilon)}{\pi} \int_\varepsilon^\infty r^{is+\varepsilon} (r - \varepsilon)^{-2\varepsilon} (r + A + B)^{-1} dr. \end{aligned}$$

Splitting the second integral at $r = 1$ and using the identities

$$r(r + A + B)^{-1} = 1 - (r + A + B)^{-1}(A + B) \qquad \text{for } r > 1,$$

$$\frac{1}{r}(r + A + B)^{-1} = \frac{1}{r}(A + B)^{-1} - (r + A + B)^{-1}(A + B)^{-1} \qquad \text{for } r < 1,$$

with (3.4) it is not difficult to show

$$F_\varepsilon(s) x \longrightarrow_{\varepsilon \to 0+} (A + B)^{is} x, \; x \in \mathscr{D}(A + B) \cap \mathscr{R}(A + B), \qquad s \in \mathbb{R}.$$

On the other hand, (3.8) yields

$$F_\varepsilon(s) = F_\varepsilon^1(s) + F_\varepsilon^2(s),$$

where

$$F_\varepsilon^1(s) = \frac{1}{2\pi i} \int_\Gamma g_\varepsilon(\lambda; s) S(\lambda) \, d\lambda,$$

and

$$F_\varepsilon^2(s) = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_\Gamma g_\varepsilon(\lambda; s) A^n (\lambda + A)^{-(n+1)} d\lambda \right) (-1)^n K^n.$$

Since $g_\varepsilon(\lambda; s)$ is bounded on Γ, uniformly w.r.t. $\varepsilon \in (0, 1/2)$, $s \in \mathbb{R}$ being fixed, (3.11) shows by Lebesgue's dominated convergence theorem

$$F_\varepsilon^1(s) \longrightarrow_{\varepsilon \to 0+} \frac{1}{2\pi i} \int_\Gamma (-\lambda)^{is} S(\lambda) d\lambda \tag{3.13}$$

in $\mathscr{B}(X)$. To be able to pass to the limit for $\varepsilon \to 0+$ in $F_\varepsilon^2(s)$, we employ the functional calculus for $\mathcal{BIP}(X)$ introduced in Prüss and Sohr [16], which is based on the inverse Mellin transform. Let $f_n(t) = t^n (\lambda + t)^{-(n+1)}$, $t > 0$, $n \in \mathbb{N}_0$, $\lambda \in \Sigma_\pi$; an easy computation leads to $(\mathscr{M} f_n)(\rho) = \lambda^{\rho - 1} \binom{\rho + n - 1}{n} \dfrac{\pi}{\sin \pi \rho}$ for the Mellin transform of f_n. The functional calculus then yields the representation

$$A^n (\lambda + A)^{-(n+1)} = \frac{1}{2i} \int_{-\infty}^{\infty} A^{-i\gamma} \lambda^{i\gamma - 1} \binom{i\gamma + n - 1}{n} \frac{d\gamma}{\sinh \pi \gamma}, \qquad n \in \mathbb{N};$$

observe that the integral is absolutely convergent since $\theta_A < \theta$ and $\lambda \in \Gamma$. Inserting this formula into the coefficients of the series for $F_\varepsilon^2(s)$, interchanging the order of integration and contracting Γ to the negative real axis leads to

$$\begin{aligned}
&\frac{1}{2\pi i} \int_\Gamma g_\varepsilon(\lambda; s) A^n (\lambda + A)^{-(n+1)} d\lambda \\
&= \frac{1}{2i} \int_{-\infty}^{\infty} A^{-i\gamma} \binom{i\gamma + n - 1}{n} \left(\frac{1}{2\pi i} \int_\Gamma g_\varepsilon(\lambda; s) \lambda^{i\gamma - 1} d\lambda \right) \frac{d\gamma}{\sinh \pi \gamma} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} A^{-i\gamma} \binom{i\gamma + n - 1}{n} \mathscr{M} h_\varepsilon (i\gamma + is + \varepsilon) d\gamma,
\end{aligned}$$

where $h_\varepsilon(t) = (\varepsilon + t)^{-2\varepsilon}$. The differentiation rule and the moment rule for the Mellin transform yield

$$\binom{i\gamma + n - 1}{n} \mathscr{M} h_\varepsilon (i\gamma + is + \varepsilon) = \mathscr{M} f_{\varepsilon, n}(i\gamma),$$

where

$$\begin{aligned}
f_{\varepsilon, n}(t) &= \frac{t^n}{n!} (-1)^n \left(\frac{d}{dt} \right)^n [t^{is + \varepsilon} (\varepsilon + t)^{-2\varepsilon}] \\
&= (-1)^n \sum_{k=0}^{n} \binom{is + \varepsilon}{n - k} \binom{-2\varepsilon}{k} t^{is + \varepsilon + k} (\varepsilon + t)^{-2\varepsilon - k}.
\end{aligned}$$

This gives the explicit formula, again employing the functional calculus

$$F_\varepsilon^2(s) = A^{is}A^{\varepsilon}(\varepsilon + A)^{-2\varepsilon} \sum_{n=0}^{\infty} P_{\varepsilon,n}(A(\varepsilon + A)^{-1})K^n,$$

where $P_{\varepsilon,n}$ denote the polynomials

$$P_{\varepsilon,n}(t) = \sum_{k=0}^{n} \binom{is+\varepsilon}{n-k}\binom{-2\varepsilon}{k} t^k, \quad n\in\mathbb{N}_0,\ t>0,\ \varepsilon\in(0, 1/2).$$

Since $\left|\binom{-2\varepsilon}{k}\right| \leq 1$ and $\left|\binom{is+\varepsilon}{m-k}\right| \leq e^{|s|}n^{|s|}$ we obtain $\| P_{\varepsilon,n}(A(\varepsilon + A)^{-1})\| \leq e^{|s|}n^{|s|+1}C_\theta^n$, and so the series for $F_\varepsilon^2(s)$ is absolutely convergent by (iii),uniformly w.r.t. $\varepsilon\in(0, 1/2)$. Passing to the limit as $\varepsilon\to 0+$ one obtains

$$F_\varepsilon^2(s) \longrightarrow_{\varepsilon\to 0} A^{is}\sum_{n=0}^{\infty}\binom{is}{n}K^n = A^{is}(I+K)^{is}.$$

We have arrived at the representation formula

$$(A+B)^{is}x = A^{is}(I+K)^{is}x + \frac{1}{2\pi i}\int_\Gamma (-\lambda)^{is}S(\lambda)x\,d\lambda, \qquad s\in\mathbb{R},\ x\in X. \tag{3.14}$$

(d) By means of (3.14) there follows $A + B\in \mathbb{BIP}(X)$ and

$$\begin{aligned}\|(A+B)^{is}\| &\leq \|A^{is}\|\,\|(I+K)^{is}\| + \frac{C}{2\pi}\int_\Gamma e^{\theta|s|}\psi(|\lambda|)|d\lambda| \\ &\leq M_\eta e^{|s|(\theta_A+\eta)}e^{|s|(\eta+\varphi)} + Ce^{\theta|s|},\end{aligned}$$

since $\psi\in L^1(\mathbb{R}_+)$ by assumption, and

$$\|(I+K)^{is}\| \leq C(\eta)\sup_{|z|=(1+\eta)\beta}|(1+z)^{is}| = C(\eta)e^{(\eta+\varphi)|s|},$$

where $\varphi = \arcsin\beta$. The proof is now complete. □

4. Proof of theorems A and B for $\Omega = \mathbb{R}^n$

(i) *Constant coefficients.*

First we consider only the main part of E_p with constant coefficients.

PROPOSITION 4.1. *Let $1 < p < \infty$, $n \geq 1$, and let $a = (a_{jk})$ be a real symmetric positive definite matrix, i.e. $a_0 \leq a\xi\cdot\xi \leq a_0^{-1}$ for all $\xi\in\mathbb{R}^n$ with $|\xi| = 1$, where $a_0 > 0$ is a constant. Then the operator $E_p = -a:\nabla^2$ with $\mathscr{D}(E_p) = W^{2,p}(\mathbb{R}^n)$ belongs to the class $\mathcal{BIP}(L^p(\mathbb{R}^n))$, in particular $\mathscr{N}(E_p) = 0$*

and $\overline{\mathscr{R}(E_p)} = L^p(\mathbb{R}^n)$. *Moreover,* $\phi_{E_p} = \theta_{E_p} = 0$, $\sigma(E_p) = [0, \infty)$ *and there are constants* $K_1 = K_1(p, a_0, n)$, $K_2 = K_2(p, a_0, n)$, $K_3 = K_3(p, a_0, n)$ *such that*

$$\|(\lambda + E_p)^{-1}\| \le K_1 \begin{cases} |\lambda|^{-1} & \text{for all } Re\,\lambda > 0 \\ (1 + |\lambda|/|Im\,\lambda|)^{1+[n/2]}/|Im\,\lambda| & \text{for all } Im\,\lambda \neq 0, \end{cases} \tag{4.1}$$

$$\|E_p^{iy}\| \le K_2(1 + |y|)^{1+[n/2]} \qquad \text{for all } y \in \mathbb{R}, \tag{4.2}$$

$$\|\nabla^2 u\|_p \le K_3 \|E_p u\|_p \qquad \text{for all } u \in \mathscr{D}(E_p). \tag{4.3}$$

PROOF. This lemma follows by applying Mikhlin's multiplier theorem [20], Theorem IV.3.2. Consider the Fourier transform

$$\tilde{u}(\xi) = \mathscr{F}(\xi) = \int_{-\infty}^{+\infty} e^{ix\cdot\xi} u(x)\,dx, \quad \xi \in \mathbb{R}^n, \quad x\cdot\xi = x_1\xi_1 + \cdots + x_n\xi_n$$

in the sense of distributions. Then we have $(\lambda + E_p)u = f$ if and only if $(\lambda + a\xi\cdot\xi)\tilde{u}(\xi) = \tilde{f}(\xi)$, i.e. $\tilde{u}(\xi) = m_\lambda(\xi)\tilde{f}(\xi)$ where $m_\lambda(\xi) = (\lambda + a\xi\cdot\xi)^{-1}$. Thus we get $-\lambda \in \rho(E_p)$ if and only if $m_\lambda(\xi)$ is an L^p-multiplier. Since boundedness of $m_\lambda(\xi)$ is necessary for the latter we obtain $[0, \infty) \subset \sigma(E_p)$. Let $\operatorname{Re}\lambda > 0$; then

$$\begin{aligned} |\lambda + a\xi\cdot\xi|^2 &= (\operatorname{Re}\lambda + a\xi\cdot\xi)^2 + (\operatorname{Im}\lambda)^2 \\ &= |\lambda|^2 + (a\xi\cdot\xi)^2 + 2\operatorname{Re}\lambda(a\xi\cdot\xi) \\ &\ge |\lambda|^2 + a_0^2|\xi|^4, \end{aligned}$$

and therefore $|m_\lambda(\xi)| \le |\lambda|^{-1}$, for $\operatorname{Re}\lambda > 0$. Moreover we also obtain

$$|\xi|^{|\alpha|}|\partial^\alpha m_\lambda(\xi)| \le C_1|\lambda|^{-1} \qquad \text{for all } \xi \in \mathbb{R}^n,\ \alpha \in \mathbb{N}_0^n,\ \operatorname{Re}\lambda > 0,$$

where $C_1 = C_1(|\alpha|, a_0) > 0$. The Mikhlin multiplier theorem now yields

$$\|(\lambda + E_p)^{-1}\| \le K_1|\lambda|^{-1} \qquad \text{for all } \operatorname{Re}\lambda > 0,$$

where $K_1 = K_1(a_0, p, n) > 0$. On the other hand, for $\operatorname{Im}\lambda \neq 0$, we have similarly

$$\begin{aligned} |\lambda + a\xi\cdot\xi| &= [(\operatorname{Re}\lambda + a\xi\cdot\xi)^2 + (\operatorname{Im}\lambda)^2]^{1/2} \\ &\ge (|\operatorname{Im}\lambda|/2)(1 + a_0|\xi|^2/|\lambda|), \end{aligned}$$

hence $|m_\lambda(\xi)| \le 2/|\operatorname{Im}\lambda|$. In addition one gets

$$|\xi|^{|\alpha|}|\partial^\alpha m_\lambda(\xi)| \le C_2(1 + |\lambda|/|\operatorname{Im}\lambda|)^{|\alpha|}/|\operatorname{Im}\lambda| \qquad \text{for all } \xi \in \mathbb{R}^n,\ \alpha \in \mathbb{N}_0^n,\ \operatorname{Im}\lambda \neq 0,$$

where $C_2 = C_2(|\alpha|, a_0) > 0$. The multiplier theorem then implies

$$\|(\lambda + E_p)^{-1}\| \le K_1(1 + |\lambda|/|\operatorname{Im}\lambda|)^{1+[n/2]}/|\operatorname{Im}\lambda| \qquad \text{for all } \operatorname{Im}\lambda \neq 0$$

where $K_1 = K_1(a_0, p, n) > 0$ possibly has to be enlarged. This proves $\sigma(E_p) = [0, \infty)$ as well as (4.1).

Since $a\xi \cdot \xi \neq 0$ for $\xi \neq 0$ we obtain $\mathcal{N}(E_p) = 0$; reflexivity of $L^p(\mathbb{R}^n)$ and (4.1) then imply also $\overline{\mathcal{R}(E_p)} = L^p(\mathbb{R}^n)$.

The multiplier for E_p^{iy} is given by $(a\xi \cdot \xi)^{iy} = e^{iy \log(a\xi \cdot \xi)}$, $\xi \in \mathbb{R}^n$, $y \in \mathbb{R}$. We have $|(a\xi \cdot \xi)^{iy}| = 1$ and $|\xi|^{|\alpha|} |\partial^\alpha (a\xi \cdot \xi)^{iy}| \leq C_3(|\alpha|, a_0)(1 + |y|)^{|\alpha|}$ where $\alpha = (\alpha_1, \dots, \alpha_n) \in N_0^n$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$. The multiplier theorem now yields (4.2).

To prove (4.3) it is enough to show that $\partial_j \partial_k E_p^{-1}$ is bounded in $L^p(\mathbb{R}^n)$ for $j, k = 1, 2, \dots, n$. This follows again by the multiplier theorem. The corresponding multiplier is $m_{jk}(\xi) = -\xi_j \xi_k / a\xi \cdot \xi$ and we obtain $|\xi|^{|\alpha|} |\partial^\alpha m_{jk}(\xi)| \leq C_4(|\alpha|, a_0)$ for all $\xi \in \mathbb{R}^n$, $\alpha \in \mathbb{N}_0^n$. This yields (4.3) and the proof is complete. $\square$

(ii) Variable coefficients of small deviation

In the next step we consider the main part of E_p when the coefficients are variable, but their deviations from $a^\infty = \lim_{|x| \to \infty} a(x)$ are small.

PROPOSITION 4.2. *Let $1 < p < \infty$, $n \geq 1$, $0 < \theta < \pi$; suppose $a(x) = (a_{jk}(x))$ is a real matrix, $x \in \mathbb{R}^n$, which satisfies* (A1) *and* (A2) *for $\Omega = \mathbb{R}^n$ with $a_0 > 0$, $0 < \alpha < 1$, $a^\infty = (a_{jk}^\infty)$, and consider the operator $E_p = -a : \nabla^2$ with $\mathcal{D}(E_p) = W^{2,p}(\mathbb{R}^n)$.*

Then there is a constant $\eta = \eta(\theta, p, a_0, n) > 0$ with the following properties. If $\| a - a^\infty \|_\infty = \sup\{ |a_{jk}(x) - a_{jk}^\infty| : x \in \mathbb{R}^n, j, k = 1, 2, \dots, n\} \leq \eta$, then E_p belongs to $\mathcal{BIP}(L^p(\mathbb{R}^n))$; in particular E_p is closed, $\mathcal{N}(E_p) = 0$, and $\overline{\mathcal{R}(E_p)} = L^p(\mathbb{R}^n)$. Moreover, $\phi_{E_p} \leq \theta$, $\theta_{E_p} \leq \theta$, $\sigma(E_p) \subset \bar{\Sigma}_\theta$, and there are constants $M_1 = M_1(\theta, p, a_0, n)$, $M_2 = M_2(\theta, p, a_0, n)$, $M_3 = M_3(p, a_0, n)$ such that

$$\| (\lambda + E_p)^{-1} \| \leq M_1 / |\lambda| \qquad \text{for all} \quad \lambda \in \Sigma_{\pi - \theta}, \tag{4.4}$$

$$\| E_p^{iy} \| \leq M_2 e^{\theta |y|} \qquad \text{for all} \quad y \in \mathbb{R}, \tag{4.5}$$

$$\| \nabla^2 u \|_p \leq M_3 \| E_p u \|_p \qquad \text{for all} \quad u \in \mathcal{D}(E_p). \tag{4.6}$$

PROOF. We write E_p in the form $E_p = A + B$ considering B as a small perturbation of A and apply Proposition 3.1. Let A, B be defined by $\mathcal{D}(A) = \mathcal{D}(B) = W^{2,p}(\mathbb{R}^n)$, $(Au)(x) = -a^\infty : \nabla^2 u(x)$, $(Bu)(x) = b(x) : \nabla^2 u(x)$, where $b(x) = -a(x) + a^\infty$. Then A has the properties of Proposition 4.1 and we may apply it to the result $\| (\lambda + A)^{-1} \| \leq K_1 |\lambda|^{-1}$ for all $\lambda \in \Sigma_{\pi - \theta}$, $\| A^{iy} \| \leq K_2 (1 + |y|)^{1 + [n/2]}$ for all $y \in \mathbb{R}$, $\| \nabla^2 u \|_p \leq K_3 \| Au \|_p$ for all $u \in \mathcal{D}(A)$, where $K_1, K_2, K_3 > 0$ are as in (4.1), (4.2), (4.3) but K_1 depends also on θ, now. In particular,

$$\|Bu\|_p \leq \|b\|_\infty \|\nabla^2 u\|_p \leq \gamma \|Au\|_p \quad \text{with} \quad \gamma = \|b\|_\infty K_3,$$

and therefore, we find a constant $\eta = \eta(\theta, p, a_0, n) > 0$ such that $\|b\|_\infty \leq \eta$ implies $\gamma < (1 + K_1)^{-1}$ and $\arcsin \gamma \leq \theta/2$. Assumptions (i) and (iii) of Proposition 3.1 are therefore satisfied.

The essential step is the verification of the commutator relation (ii) of Proposition 3.1. It is sufficient to show it in the special case $A = -\Delta = -(\partial_1^2 + \cdots + \partial_n^2)$ since the general case can be reduced to this one by an appropriate linear coordinate transformation. Under such a transformation properties (A1), (A2) remains valid with the same constant $0 < \alpha < 1$. Supposing now $A = -\Delta$ we can express $(\lambda + A)^{-1}$ as a convolution integral; cp. [14].

$$((\lambda - \Delta)^{-1} f)(x) = \lambda^{\frac{n-2}{2}} \int_{\mathbb{R}^n} k_n(|x - y|\sqrt{\lambda}) f(y)\, dy, \qquad x \in \mathbb{R}^n, \quad f \in L^p(\mathbb{R}^n),$$

where

$$k_n(z) = C_n \int_0^\infty r^{n-2} e^{-z\sqrt{1+r^2}} \frac{dr}{\sqrt{1 + r^2}} \qquad \text{for} \quad \operatorname{Re} z > 0, \ n \geq 2,$$

$$k_n(z) = e^{-z} \qquad \text{for} \quad \operatorname{Re} z > 0, \ n = 1.$$

Observe that $k_n(t)$ is positive and nonincreasing for $t > 0$, and $|k_n(z)| \leq k_n(\operatorname{Re} z)$ for $\operatorname{Re} z > 0$, $\int_0^\infty k_n(r) r^\rho\, dr < \infty$ for all $\rho > n - 2$. Using Assumption (A2) we obtain the estimates

$$|b_{jk}(x) - b_{jk}(x - y)| \leq C|y|^\alpha, \quad |b_{jk}(x)| \leq C|x|^{-\alpha} \qquad \text{for all} \quad x, y \in \mathbb{R}^n,$$

where $C > 0$ is a constant. A direct calculation yields for $f \in W^{2,p}(\mathbb{R}^n)$

$$\begin{aligned}&(B(\lambda + A)^{-1} f)(x) - ((\lambda + A)^{-1} B f)(x) \\ &= \lambda^{\frac{n}{2}-1} \int_{\mathbb{R}^n} k_n(|y| \operatorname{Re}\sqrt{\lambda}) \Big(\sum_{j,k=1}^n (b_{jk}(x) - b_{jk}(x - y))\, \partial_j \partial_k f(x - y) \Big)\, dy,\end{aligned}$$

and using Young's inequality for convolutions we get

$$\|B(\lambda+A)^{-1} f - (\lambda+A)^{-1} B f\|_p \leq C_1 \|\nabla^2 f\|_p |\lambda|^{\frac{n}{2}-1} \int_{\mathbb{R}^n} k_n\Big(|y|\sqrt{|\lambda|}\cos\frac{\theta}{2}\Big) |y|^\alpha\, dy$$

$$\leq C_2 |\lambda|^{-1} \|Af\|_p \int_0^\infty k_n(r) r^{n-1} r^\alpha |\lambda|^{-\frac{\alpha}{2}}\, dr \leq C_2 |\lambda|^{-1-\frac{\alpha}{2}} \|Af\|_p \quad \text{for all} \quad \lambda \in \Sigma_{\pi-\theta},$$

where $C_1, C_2, C_3 > 0$ are constants. This yields (ii) of Proposition 3.1 for $|\lambda| > 1$.

Next we prove the commutator relation for $0 < |\lambda| \leq 1$, $\lambda \in \Sigma_{\pi-\theta}$. We obtain $\|B(\lambda+A)^{-1}f-(\lambda+A)^{-1}Bf\|_p \leq \|B(\lambda+A)^{-1}\|_p + \|(\lambda+A)^{-1}Bf\|_p$ and use interpolation inequalities, similarly as in (2.2) $\sim$ (2.4). Here we choose $p_1 > p$, $p_2 > p$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $p_2 > \frac{n}{\alpha}$; setting $\delta = \frac{n}{2p_2}$ we obtain $0 < \delta < \frac{\alpha}{2}$ and $\delta\left(\frac{1}{p} - \frac{2}{n}\right) + (1-\delta)\frac{1}{p} = \frac{1}{p_1}$. Using $p_2 > \frac{n}{\alpha}$, (A2) yields $\|b\|_{p_2} < \infty$, hence by Hölder's inequality and the interpolation inequality [7]

$$\begin{aligned}\|B(\lambda+A)^{-1}f\|_p &\leq C_1\|b\|_{p_2}\|(\lambda+A)^{-1}\nabla^2 f\|_{p_1}\\ &\leq C_2\|\nabla^2(\lambda+A)^{-1}\nabla^2 f\|_p^\delta\|(\lambda+A)^{-1}\nabla^2 f\|_p^{1-\delta},\end{aligned}$$

and by (4.3)

$$\begin{aligned}\|\nabla^2(\lambda+A)^{-1}\nabla^2 f\|_p &\leq C_3\|A(\lambda+A)^{-1}\nabla^2 f\|_p \leq C_4\|\nabla^2 f\|_p \leq C_5\|Af\|_p,\\ \|(\lambda+A)^{-1}\nabla^2 f\|_p &\leq C_6|\lambda|^{-1}\|\nabla^2 f\|_p \leq C_7|\lambda|^{-1}\|Af\|_p,\end{aligned}$$

and therefore

$$\|B(\lambda+A)^{-1}f\|_p \leq C_8|\lambda|^{\delta-1}\|Af\|_p.$$

A similar estimate holds for $\|(\lambda+A)^{-1}Bf\|_p$. In this case we choose $p_2 > p_1 > 1$, $0 < \delta < \frac{\alpha}{2}$ such that $\frac{1}{p_1} = \frac{1}{p} + \frac{1}{p_2}$, $p_1 < p$, $p_2 > \frac{n}{\alpha}$, $\delta\left(\frac{1}{p_1} - \frac{2}{n}\right) + (1-\delta)\frac{1}{p_1} = \frac{1}{p}$, i.e. again $\delta = \frac{n}{2p_2}$. Then

$$\begin{aligned}\|(\lambda+A)^{-1}Bf\|_p &\leq C_1\|\nabla^2(\lambda+A)^{-1}Bf\|_{p_1}^\delta\|(\lambda+A)^{-1}Bf\|_{p_1}^{1-\delta}\\ &\leq C_2\|Bf\|_{p_1}^\delta|\lambda|^{\delta-1}\|Bf\|_{p_1}^{1-\delta}\\ &= C_2|\lambda|^{\delta-1}\|Bf\|_{p_1} \leq C_3|\lambda|^{\delta-1}\|b\|_{p_2}\|\nabla^2 f\|_p\\ &\leq C_4|\lambda|^{\delta-1}\|Af\|_p.\end{aligned}$$

This implies (ii) of Proposition 3.1 for $0 < |\lambda| < 1$, and so we conclude that (4.4) and (4.5) are valid.

To prove (4.6) note that with $\|Bu\|_p \leq \gamma\|Au\|_p$ for all $u \in W^{2,p}(\mathbb{R}^n)$ and $\gamma < (1+K_1)^{-1} < 1$ we also obtain

$$\begin{aligned}\|Au+Bu\|_p &\geq \|Au\|_p - \|Bu\|_p \geq \|Au\|_p - \gamma\|Au\|_p\\ &= (1-\gamma)\|Au\|_p \geq (1-\gamma)K_3^{-1}\|\nabla^2 u\|_p.\end{aligned}$$

Proposition 4.2 is proved. $\square$

REMARK 4.3. Proposition 4.2 remains valid if $|a(x)-a^\infty| \leq C|x|^{-\alpha}(|x| \geq 1)$

in Assumption (A2) is replaced by the following weaker condition: There exists some $\gamma \in (1, \infty)$ with $\gamma > p$, $\gamma > p' = \dfrac{p}{p-1}$, $\gamma > \dfrac{n}{2}$ and $a(\cdot) - a^\infty \in L^\gamma(\mathbb{R}^n)$. Then we can choose $p_2 = \gamma$ in the above proof and all conclusions remain true.

(iii) The general case. Proof of Theorems A and B for $\Omega = \mathbb{R}^n$

Let $E_p = -a : \nabla^2 + b \cdot \nabla + c$ be the general elliptic 2^{nd}-order differential operator for $\Omega = \mathbb{R}^n$, let $0 < \theta < \pi$, assume (A1), (A2), (A3), and the resolvent estimate (2.8), for the proof of Theorem A. We carry out a localization procedure and apply Proposition 4.2 locally. Let $\eta = \eta(\theta, p, a_0, n) > 0$ be the constant of Proposition 4.2, and $B_R(x) = \{y \in \mathbb{R}^n : |y - x| < R\}$.

(a) First choose a large ball $\overline{B_R(0)}$ such that $|a(x) - a^\infty| \le \eta/2$ for $|x| \ge R$ and let U_0 be its complement. Since $\overline{B_R(0)}$ is compact there are finitely many balls $B_{r_1}(x_1), B_{r_2}(x_2), \dots, B_{r_N}(x_N)$ such that $\mathbb{R}^n = U_0 \cup B_{r_1}(x_1) \cup \dots \cup B_{r_N}(x_N)$ and $|a(x) - a(x_j)| \le \eta/2$ for all $x \in B_{r_j}(x_j)$, $j = 1, \dots, N$. We set $U_j = B_{r_j}(x_j)$ and choose functions $\varphi_j \in C^\infty(\mathbb{R}^n)$ such that $0 \le \varphi_j \le 1$, supp $\varphi_j \subseteq U_j$ for $j = 0, 1, \dots, N$ and $\sum_{j=0}^N \varphi_j(x) = 1$ for all $x \in \mathbb{R}^n$. Observe that $\varphi_0(x) = 1$ for large $|x|$, say for $|x| \ge R_0 > R$. Within each of the sets $U_0, \dots, U_N$ we extend $a(x)$ to all of $\mathbb{R}^n$ by reflection at the boundary; these extensions will be denoted by a^j. Then each a^j satisfies (A1), (A2) with the same values of $\alpha \in (0, 1)$ and a_0, and $a^{j,\infty} = \lim_{|x| \to \infty} a^j(x) = a(x_j)$. Define the local operators E_p^i for $j = 0, 1, \dots, N$ by means of

$$(E_p^i u)(x) = -a^j(x) : \nabla^2 u(x), \ x \in \mathbb{R}^n, \qquad \text{for} \quad u \in \mathscr{D}(E_p^j) = W^{2,p}(\mathbb{R}^n).$$

Then the assertions of Proposition 4.2 are valid for each E_p^j, $j = 0, 1, \dots, N$.

(b) Now we use (2.8) and consider the equation

$$\lambda u + E_p u = f$$

with $f \in L^p(\mathbb{R}^n)$, $\lambda \in \Sigma_{\pi-\theta}$. We write this equation in the form

$$\lambda u - a : \nabla^2 u = f - b \cdot \nabla u - cu$$

and multiply with φ_j, $j = 0, 1, \dots, N$. Setting $u_j = \varphi_j u$ we get $a : \nabla^2 u_j = a^j : \nabla^2 u_j = a^j : \nabla^2(\varphi_j u) = (a : \nabla^2 \varphi_j)u + 2a(\nabla\varphi_j) \cdot \nabla u + \varphi_j a : \nabla^2 u$ and obtain the local equations

$$\lambda u_j + \delta u_j + E_p^j u_j = \varphi_j f + F_j^\delta(a, b, c, \nabla)u, \qquad j = 0, \dots, N, \tag{4.7}$$

where $\delta \in [0, 1]$, $F_j^\delta(a, b, c, \nabla)u = -\varphi_j b \cdot \nabla u - \varphi_j cu - 2a(\nabla\varphi_j) \cdot (\nabla u) \ -(a : \nabla^2 \varphi_j)u + \delta\varphi_j u$. Observe that supp $\varphi_j \subset U_j$ implies $E_p^j u_j = -a : \nabla^2 u_j$. Proposition 4.2 yields $\Sigma_{\pi-\theta} \subset \rho(-E_p^j)$, hence applying $(\lambda + \delta + E_p^j)^{-1}$ to (4.7) and summing over j leads to the identity

$$(\lambda + E_p)^{-1} f = \sum_{j=0}^{N} (\lambda + \delta + E_p^j)^{-1} \{\varphi_j f + F_j^\delta(a, b, c, \nabla)(\lambda + E_p)^{-1} f\}. \tag{4.8}$$

By (4.4) we have $\|(\lambda + \delta + E_p^j)^{-1}\| \le C^j(|\lambda| + \delta)^{-1}$ for $\lambda \in \Sigma_{\pi-\theta}$ and some constants $C^j > 0$. In order to estimate $F_j^\delta(a, b, c, \nabla)(\lambda + E_p)^{-1} f$ we apply (2.2), (2.3), (2.4) and $2a(\nabla\varphi_j) \in L^r(\mathbb{R}^n)$, $(a : \nabla^2\varphi_j) \in L^s(\mathbb{R}^n)$ with $r = r_k$ and $s = s_k$ in (A3); here we used $\|a\|_\infty < \infty$ and compactness of supp $\nabla\varphi_j$.

(c) Next (2.8) implies the estimate

$$|\lambda| \, \|u\|_p + \|u\|_p + \|\nabla^2 u\|_p \le C \|(\lambda + E_p)u\|_p, \qquad u \in \mathscr{D}(E_p),\ \lambda \in \Sigma_{\pi-\theta}, \tag{4.9}$$

for some constant $C > 0$. With ρ, τ as in (2.2) $\sim$ (2.4) this leads to

$$\begin{aligned} \|F_j^\delta(a, b, c, \nabla)u\|_p &\le \|(\varphi_j b + 2a(\nabla\varphi_j)) \cdot \nabla u\|_p + \|\varphi_j(c + \delta)u + (a : \nabla^2\varphi_j)u\|_p \\ &\le C_1 \|\nabla^2 u\|_p^\rho \|u\|_p^{1-\rho} + C_2 \|\nabla^2 u\|_p^\tau \|u\|_p^{1-\tau}, \end{aligned}$$

hence by (4.9)

$$\|F_j^\delta(a, b, c, \nabla)(\lambda + E_p)^{-1} f\|_p \le C_3(\delta + |\lambda|)^{\rho-1} \|f\|_p + C_4(\delta + |\lambda|)^{\tau-1} \|f\|_p. \tag{4.10}$$

With $F_\lambda^\delta f = \sum_{j=0}^N (\lambda + E_p^j)^{-1} F_j^\delta(a, b, c, \nabla)(\lambda + E_p)^{-1} f$, and (4.4) for each j, and (4.10), we obtain

$$\|F_\lambda^\delta f\|_p \le C(|\lambda| + \delta)^{-1}(\delta + |\lambda|)^{\rho-1} + (\delta + |\lambda|)^{\tau-1}) \|f\|_p, \quad \lambda \in \Sigma_{\pi-\theta}, \quad f \in L^p(\mathbb{R}^n), \tag{4.11}$$

where $0 < \rho$, $\tau < 1$.

(d) For the proof of Theorem A, (4.11) with $\delta = 1$ is sufficient to estimate E_p^{iy}; for the proof of Theorem B we obtain (4.11) with $\delta = 0$ for $|\lambda| \ge 1$, replacing (2.8) by (2.10) and (2.11). Near $\lambda = 0$ we have to estimate $F_\lambda^0 f$ differently, here assumption (A4) comes in.

From (A2), (A3), (A4), and compactness of supp $\nabla\varphi_j$ we get $b_0 = b + 2a\nabla\varphi_j \in L^r(\mathbb{R}^n) \cap L^{\hat{r}}(\mathbb{R}^n)$, for some numbers $\hat{r} < n < r$, and therefore $b_0 \in L^\gamma(\mathbb{R}^n)$ for all $\gamma \in [\hat{r}, r]$. In virtue of $\hat{r} < n < r$ it is possible to choose $\kappa \in (0, 1)$, $\mu \in (\frac{1}{2}, 1)$, such that $\kappa + \mu > 1$, $\gamma \in [\hat{r}, r]$, $p_1, p_2 \in (1, \infty)$ such that $1/p = \kappa(1/p_1 - 2/n) + (1 - \kappa)/p_1$, $1/p_1 = 1/\gamma + 1/p_2$, $1/p_2 = 1/n + \mu(1/p - 2/n) + (1 - \mu)/p$; the interpolation inequality then yields with $u = (\lambda + E_p)^{-1} f$

$$\begin{aligned} \|(\lambda + E_p^j)^{-1} b_0 \cdot \nabla u\|_p &\le C_1 \|\nabla^2(\lambda + E_p^j)^{-1} b_0 \cdot \nabla u\|_{p_1}^\kappa \|(\lambda + E_p^j)^{-1} b_0 \cdot \nabla u\|_{p_1}^{1-\kappa} \\ &\le C_2 |\lambda|^{\kappa-1} \|b_0 \cdot \nabla u\|_{p_1} \le C_3 |\lambda|^{\kappa-1} \|b_0\|_\gamma \|\nabla u\|_{p_2} \\ &\le C_4 |\lambda|^{\kappa-1} \|\nabla^2(\lambda + E_p)^{-1} f\|_p^\mu \|(\lambda + E_p)^{-1} f\|_p^{1-\mu} \\ &\le C_5 |\lambda|^{\kappa+\mu-2} \|f\|_p. \end{aligned}$$

A similar estimate holds for $\varphi_j cu(a\nabla \cdot \nabla\varphi_j)u$, and so we obtain with (A4)

$$\|F_\lambda^0 f\|_p \leq C|\lambda|^{-\beta}\|f\|_p, \qquad \lambda\in\Sigma_{\pi-\theta},\ f\in L^p(\mathbb{R}^n), \tag{4.12}$$

where $\beta\in(0, 1)$ and $C > 0$ are constants.

(e) Now we use the decomposition

$$(\lambda + E_p)^{-1}f = \sum_{j=0}^{N} (\lambda + \delta + E_p^j)^{-1}(\varphi_j f) + F_\lambda^\delta f, \qquad \lambda\in\Sigma_{\pi-\theta}, \tag{4.13}$$

to obtain the desired estimate for the imaginary powers of E_p. In fact, (3.12) for $B = 0$ yields

$$E_p^{iy} f = \sum_{j=0}^{N} (\delta + E_p^j)^{iy}(\varphi_j f) + \lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_\Gamma g_\varepsilon(\lambda; y)F_\lambda^\delta f\, d\lambda, \qquad y\in\mathbb{R}, \tag{4.14}$$

where $g_\varepsilon(\lambda, y) = (-\lambda)^{iy+\varepsilon}(\varepsilon - \lambda)^{-2\varepsilon}$ and $\Gamma = (\infty, 0]e^{-i(\pi-\theta)}\cup[0, \infty)e^{i(\pi-\theta)}$ as in Section 3. By Proposition 4.2 we have $\|(\delta + E_p^j)^{iy}\| \leq M_j e^{\theta|y|}$, $y\in\mathbb{R}$, hence the first term of (4.14) is estimated. For the second term observe that $\|F_\lambda^\delta\|$ is integrable along Γ by (4.12) and (4.11), hence by the dominated convergence theorem

$$\begin{aligned}\left\|\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_\Gamma g_\varepsilon(\lambda; y)F_\lambda^\delta f\, d\lambda\right\|_p &= \left\|\frac{1}{2\pi i}\int_\Gamma (-\lambda)^{iy}F_\lambda^\delta f\, d\lambda\right\|_p \\ &\leq Ce^{\theta|y|}\left(\int_\Gamma \|F_\lambda^\delta\|\,|d\lambda|\right)\|f\|_p,\end{aligned}$$

and so Theorems A and B are proved for $\Omega = \mathbb{R}^n$. $\square$

REMARK 4.4. The above proof shows that for Theorem A to be valid, the lower order term $b\cdot\nabla u + cu$ instead of (A3) needs only satisfy an estimate of the form

$$\|b\nabla u + cu\|_p \leq C\|\nabla^2 u\|_p^\gamma\|u\|_p^{1-\gamma}, \qquad u\in W^{2,p}(\mathbb{R}^n), \tag{4.15}$$

for some $\gamma\in[0, 1)$. Theorem B remains true if $b\cdot\nabla u + cu$ satisfies (4.15) instead of (A3), (A4) and in addition

$$\|b\nabla u + cu\|_{p_1} \leq C\|\nabla^2 u\|_p^\beta\|u\|_p^{1-\beta}, \qquad u\in W^{2,p}(\mathbb{R}^n), \tag{4.16}$$

for some $\beta\in(0, 1]$, $p_1 < p$ such that $\beta + \frac{n}{2}\left(\frac{1}{p_1} - \frac{1}{p}\right) > 1$ and $\frac{n}{2}\left(\frac{1}{p_1} - \frac{1}{p}\right) < 1$.

5. Proof of Theorems A and B for $\Omega = \mathbb{R}^n_+$

This case can be reduced to the previous one by the reflection principle. Let $\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n)\colon x_1 > 0\}$, $1 < p < \infty$, $u \in L^p(\mathbb{R}^n_+)$. An odd extension $u_0 \in L^p(\mathbb{R}^n)$ of u is defined by $u_0(x_1, x_2, \dots, x_n) = u(x)$ for $x_1 > 0$, $u_0(x_1, x_2, \dots, x_n) = -u(-x_1, x_2, \dots, x_n)$ for $x_1 < 0$; an even extension $u_e \in L^p(\mathbb{R}^n)$ of u is defined by $u_e(x_1, \dots, x_n) = u(x)$ for $x_1 > 0$, $u_e(x_1, x_2, \dots, x_n) = u(-x_1, x_2, \dots, x_n)$ for $x_1 < 0$. We call u_0 an odd and u_e an even function of $L^p(\mathbb{R}^n)$. Let $L^p_{odd}(\mathbb{R}^n)$, $W^{2,p}_{odd}(\mathbb{R}^n)$ be the subspaces of odd and $L^p_{even}(\mathbb{R}^n)$, $W^{2,p}_{even}(\mathbb{R}^n)$ the subspaces of even functions of $L^p(\mathbb{R}^n)$, $W^{2,p}(\mathbb{R}^n)$, respectively; they are isomorphic to $L^p(\mathbb{R}^n_+)$, $W^{2,p}(\mathbb{R}^n_+)$, respectively. Observe that $u \in W^{2,p}_{odd}(\mathbb{R}^n)$ yields $u = 0$ on $\partial\mathbb{R}^n_+$ while $u \in W^{2,p}_{even}(\mathbb{R}^n)$ implies $u_{x_1} = 0$ on $\mathbb{R}^n_+$; hence odd resp. even extension to $\mathbb{R}^n$ reflect Dirichlet resp. Neumann boundary conditions.

Consider the operator $E_p = -a\colon \nabla^2 + b \cdot \nabla + c$ in Theorem A for $\Omega = \mathbb{R}^n_+$, assume (A1) $\sim$ (A3) with constants α, r_k, s_k such that $0 < \alpha < 1$, $r_k \geq p$, $s_k \geq p$, $r_k > n$, $s_k > n/2$ and let (2.8) be satisfied. We construct an extension $E^e_p = -a^e\colon \nabla^2 + b^e \cdot \nabla + c^e$ of E_p in such a way that $\mathscr{D}(E^e_p) = W^{2,p}_{odd}(\mathbb{R}^n)$ is mapped into $L^p_{odd}(\mathbb{R}^n)$ and the Assumptions (A1) $\sim$ (A3) are preserved in $\mathbb{R}^n$. For this purpose we let a^e_{jk} for $j = k = 1$, and $j, k = 2, 3, \dots, n$, $b^e_2, b^e_3, \dots, b^e_n$ and c^e be the even extensions of a_{jk}, $b_2, \dots, b_n$ and c. The mixed coefficients $a_{1k} = a_{k1} (k = 2, \dots, n)$ and b_1 have to be extended oddly. For continuity reasons, a^e will then fulfill (A2) if and only if the following condition holds

$$a_{1k}(x) = a_{k1}(x) = 0 \qquad \text{for all} \quad x \in \partial\mathbb{R}^n_+ = \{x \in \mathbb{R}^n\colon x_1 = 0\},\ k = 2, \dots, n. \tag{5.1}$$

Under this restriction on $a(x)$, (A1) $\sim$ (A3) remain valid for a^e, and so Theorem A follows for $\Omega = \mathbb{R}^n_+$ from the corresponding result for $\mathbb{R}^n$. In the same way we see that Theorem B follows for $\Omega = \mathbb{R}^n_+$ under this restriction.

To remove (5.1) we construct a diffeormorphism of $\mathbb{R}^n_+$ onto itself leaving the boundary invariant, which induces a similarity transform S of E_p in such a way that the coefficients of SE_pS^{-1} are subject to (5.1). This is the main task of this section.

(a) Let us first recall some general properties of variable transformations. So let $\Omega \subset \mathbb{R}^n$ be a region with $\partial\Omega \in C^2$, and let $g \in C^2(\bar{\Omega}; \mathbb{R}^n)$ be injective and such that its Jacobian $Dg(x)$ satisfies $\eta \leq |\det Dg(x)| \leq \eta^{-1}$, $x \in \Omega$, for some $\eta > 0$; if Ω is unbounded we require in addition the existence of $Dg(\infty) = \lim_{|x| \to \infty} Dg(x)$, and boundedness of $D^2 g(x)$. Such transformations will be called admissable in the sequel. If g is admissable let $\Omega^g = g(\Omega)$; then $\overline{\Omega^g} = g(\bar{\Omega})$, $\partial\Omega^g = g(\partial\Omega)$, g is invertible and g^{-1} enjoys the same properties as g, i.e. is admissable as well.

Given a function $v\colon \overline{\Omega^g} \to \mathbb{R}^n$ define $(Gv)(x) = v(g(x))$; then $Gv\colon \Omega \to \mathbb{R}^n$ and $u = Gv$ satisfies

$$u_{x_i}(x) = \sum_{k=1}^{n} v_{y_k}(g(x))g_{kx_i}(x), \qquad x \in \Omega, \tag{5.2}$$

$$u_{x_i x_j}(x) = \sum_{k=1}^{n} v_{yk}(g(x))g_{kx_i x_j}(x) + \sum_{k,l=1}^{n} v_{y_k y_l}(g(x))g_{kx_i}(x)g_{lx_j}(x), \qquad x \in \Omega.$$

Therefore we obtain

$$\begin{aligned}(Eu)(x) = {} & - Dg(x) \cdot a(x) \cdot Dg^T(x)\colon \nabla_y^2 v(g(x)) \\ & + (Dg(x)b(x) - D^2 g(x)\colon a(x)) \cdot \nabla_y v(g(x)) + c(x)v(g(x)), \qquad x \in \Omega,\end{aligned}$$

i.e. operator G transforms a second order differential operator E on Ω to $E^g = G^{-1}EG$ defined on Ω^g with coefficients

$$\begin{aligned}& a^g(y) = (Dg \cdot a \cdot Dg^T) \circ g^{-1}(y), \; b^g(y) = (Dg \cdot b - D^2 g\colon a) \circ g^{-1}(y) \\ & c^g(y) = c(g^{-1}(y)), \qquad y \in \Omega^g.\end{aligned} \tag{5.3}$$

Obviously, Dirichlet boundary conditions are transformed to Dirichlet conditions since $\partial\Omega^g = g(\partial\Omega)$. But also a conormal derivative of u at $\partial\Omega$ is transferred to a conormal derivative of v at $\partial\Omega^g$, since an outer normal $\vartheta(x)$ of Ω at $x \in \partial\Omega$ is transformed to an outer normal $\vartheta^g(y)$ of Ω^g at $y = g(x) \in \partial\Omega^g$ via the rule

$$\vartheta^g(g(x)) = [Dg(x)]^T \vartheta(x), \qquad x \in \partial\Omega. \tag{5.4}$$

Equation (5.3) shows that the ellipticity constant a_0 appearing in (A1) remains unchanged under an admissable variable transformation g. Also, since Dg, Dg^{-1}, det g, det g^{-1}, $D^2 g$, $D^2 g^{-1}$ are bounded, the change of variable formula for the Lebesgue integral shows that G induces isomorphisms $G_p\colon W^{j,p}(\Omega^g) \to W^{j,p}(\Omega)$ for each $p \in [1, \infty]$, $j = 0, 1, 2$. Thus for an admissable variable transformation g, the L^p-realizations E_p of E, resp. E_p^g of E^g are similiarity transforms of each other, i.e.

$$E_p^g = G_p^{-1} E_p G_p. \tag{5.5}$$

Therefore spectrum and resolvent set, in particular the spectral angle remain unchanged and

$$(\lambda - E_p^g)^{-1} = G_p^{-1}(\lambda - E_p)^{-1} G_p, \qquad \lambda \in \rho(E_p) \tag{5.6}$$

shows that the resolvent estimates (2.8) or (2.10) are satisfied simultaneously, only the constants C, C_1 may be different. From (5.6) we then obtain

$$(E_p^g)^{iy} = G_p^{-1} E_p^{iy} G_p, \qquad y \in \mathbb{R}, \tag{5.7}$$

and so E_p and E_p^g admit bounded imaginary powers simultaneously, and their power angles are equal.

(b) Now let $\Omega = \mathbb{R}_+^n = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ be the right halfspace; it is convenient to write (t, x) for the variable in Ω, and to split the coefficient matrix $a(t, x)$ as

$$a(t, x) = \begin{pmatrix} a_0(t, x), & a_1(t, x)^T \\ a_1(t, x), & a_2(t, x) \end{pmatrix}, \quad \text{where } a_0 \in \mathbb{R},\ a_1 \in \mathbb{R}^{n-1},\ a_2 \in \mathbb{R}^{(n-1)^2}$$

We want to construct a variable transformation $g(t, x)$ such that a^g satisfies $a_1^g(0, x) \equiv 0$. For this purpose we let $g(t, x)$ be of the special form

$$g(t, x) = \begin{pmatrix} t \\ x - t\dfrac{a_1^\infty}{a_0^\infty} - \psi(t)h(t, x) \end{pmatrix}, \ t \geq 0,\ x \in \mathbb{R}^{n-1}, \tag{5.8}$$

where a_1^∞, a_0^∞ are the limits of a_1, a_0 as $|(t, x)| \to \infty$; here $\psi \in C_0^\infty(\mathbb{R})$ is such that $\psi(t) \equiv 1$ for of $|t| < t_0$, $\psi(t) \equiv 0$ for $|t| \geq 2t_0$, $0 \leq \psi(t) \leq 1$, and t_0 will be fixed later. With

$$Dg(t, x) = \begin{pmatrix} 1 & 0 \\ -\dfrac{a_1^\infty}{a_0^\infty} - \dot{\psi} h - \psi h_t, & I - \psi \nabla h \end{pmatrix} \tag{5.9}$$

and $g_0(0, x) \equiv 0$, it is easily verified that a^g satisfies $a_1^g(0, y) \equiv 0$ if

$$a_0(0, x)h_{kt}(0, x) + a_1(0, x) \cdot \nabla h_k(0, x) = a_{1k}(0, x) - a_0(0, x)\frac{a_{1k}^\infty}{a_0^\infty},$$
$$\text{for all } x \in \mathbb{R}^{n-1},\ k = 1,\dots,n-1. \tag{5.10}$$

We shall construct the functions h_k in such a way that

$$h_k(0, x) = 0,\ h_{kt}(0, x) = d_k(x), \quad x \in \mathbb{R}^{n-1},\ k = 1,\dots,n-1, \tag{5.11}$$

where

$$d_k(x) = \frac{a_{1k}(0, x)}{a_0(0, x)} - \frac{a_{1k}^\infty}{a_0^\infty}, \quad x \in \mathbb{R}^{n-1},\ k = 1,\dots,n-1. \tag{5.12}$$

If functions $h_k \in C^2(\mathbb{R}_+^n)$ can be found such that (5.11) holds, then g will be admissable provided t_0 is chosen small enough, since $\det Dg = \det(I - \psi \nabla h) = 1$ for $t = 0$.

(c) Let $\{P(t)\}_{t \geq 0}$ denote the Poisson semigroup defined by

$$(P(t)f)(x) = \int_{\mathbb{R}^{n-1}} p(t, x-y) f(y)\, dy, \qquad x \in \mathbb{R}^{n-1},\ t > 0, \tag{5.13}$$

where the Poisson kernel $p(t, x)$ is given by

$$p(t, x) = \Gamma(n/2)\pi^{-n/2} t(t^2 + |x|^2)^{-n/2}, \qquad x \in \mathbb{R}^{n-1},\ t > 0.$$

The Fourier multiplier corresponding to $P(t)$ is

$$\tilde{p}(t, \xi) = e^{-|\xi| t}, \qquad t \geq 0,\ \xi \in \mathbb{R}^{n-1}.$$

It is wellknown that $P(t)$ is a bounded analytic C_0-semigroup in $C_0(\mathbb{R}^{n-1})$, the space of continuous functions $f\colon \mathbb{R}^{n-1} \to \mathbb{C}$ vanishing at infinity, and that the following estimates hold.

$$|P(t)f|_\infty \leq |f|_\infty, \qquad t > 0,\ f \in C_0(\mathbb{R}^{n-1}) \tag{5.14}$$

$$|\nabla P(t)f|_\infty + \left|\frac{\partial}{\partial t} P(t)f\right|_\infty \leq ct^{-1}|f|_\infty, \qquad t > 0,\ f \in C_0(\mathbb{R}^{n-1}) \tag{5.15}$$

$$|\nabla P(t)f|_\infty + \left|\frac{\partial}{\partial t} P(t)f\right|_\infty \leq ct^{\alpha-1}|f|_{\alpha,\infty}, \qquad t > 0,\ f \in C_0^\alpha(\mathbb{R}^{n-1}). \tag{5.16}$$

Here and below we use the notation $|\cdot|_p$ for the norm in $L^p(\mathbb{R}^{n-1})$, $1 \leq p \leq \infty$, and $|\cdot|_{\alpha,\infty}$ for the norm in $C_0^\alpha(\mathbb{R}^{n-1})$. (5.14) $\sim$ (5.16) can be obtained directly by estimating the Poisson kernel.

(d) We are now in position to define the functions h_k we are looking for. Note that $d_k \in C_0^\alpha(\mathbb{R}^n)$ by (A2). We set

$$h_k(t, x) = t(P(t)d_k)(x), \qquad t \geq 0,\ x \in \mathbb{R}^{n-1},\ k = 1, \ldots, n-1. \tag{5.17}$$

Obviously $h_k(0, x) \equiv 0$, and $h_{kt}(t, 0) = t(\partial/\partial t)P(t)d_k + P(t)d_k$ shows $h_{kt}(0, x) = d_k(x)$ by (5.16). Moreover, we have the following estimates which are implied by (5.14) to (5.16).

$$|\nabla h_k(t, \cdot)|_\infty + |h_k(t, \cdot) - d_k|_\infty \leq ct^\alpha, \qquad t > 0; \tag{5.18}$$

$$|\nabla^2 h_k(t, \cdot)|_\infty + |\nabla h_{kt}(t, \cdot)|_\infty + |h_{ktt}(t, \cdot)|_\infty \leq ct^{\alpha-1}, \qquad t > 0. \tag{5.19}$$

Unfortunately, (5.18) and (5.19) are not enough to ensure that g defined by (5.8) with h given by (5.17) is an admissable transformation in the sense of (a) of this section. All conditions for admissability are fulfilled, except for $g \in C^2(\mathbb{R}^n_+)$. In fact, in general g cannot be C^2 since this would require $d_k \in C_0^1$. However, as shown below, the arguments presented in (a) can be modified in such a way that the similarity transform G still works, and so enables the reduction of the case $a_1(0, x) \not\equiv 0$ to $a_1^g(0, y) \equiv 0$.

(e) The similarity G will in general not leave $W^{2,p}(\mathbb{R}^n_+)$ invariant, unless $\alpha > 1 - 1/p$. However, we still have $G \in \mathscr{L}(W^{2,p}(\mathbb{R}^n_+) \cap W_0^{1,p}(\mathbb{R}^n_+))$, as the following estimate for $u(t, x) = v(g(t, x))$ shows. By (5.19) we obtain

$$|\nabla v(t, \cdot) D^2 g(t, \cdot)|_p \leq |\nabla v(t, \cdot)|_p \cdot |D^2 g(t, \cdot)|_\infty$$
$$\leq c t^{\alpha - 1} |\nabla v(t, \cdot)|_p, \qquad t \geq t_0, \tag{5.20}$$

and with $v(0, y) = 0$,

$$|\nabla v(t, \cdot)| \leq |\nabla v(0, \cdot)|_p + \int_0^t |\nabla v_t(s, \cdot)|_p \, ds$$
$$\leq t^{1/p'} \| \nabla v_t \|_p, \qquad t > 0. \tag{5.21}$$

Hence for each $\alpha > \gamma \geq 0$ we obtain

$$\begin{aligned}
\| \nabla v D^2 d \|_p &= \left(\int_0^{2t_0} |\nabla v(t, \cdot) D^2 g(t, \cdot)|_p^p \, dt \right)^{1/p} \\
&\leq C \left(\int_0^{2t_0} |\nabla v(t, \cdot)|_p^{\gamma p} |\nabla v(t, \cdot)|_p^{(1-\gamma)p} \cdot t^{(\alpha - 1)p} \, dt \right)^{1/p} \\
&\leq C \left(\int_0^{2t_0} |\nabla v(t, \cdot)|_p^{\gamma p} t^{(\alpha - 1 + (1-\gamma)/p')} \, dt \right)^{1/p} \| \nabla v_t \|_p^{1-\gamma} \\
&\leq C \| \nabla v_t \|_p^{1-\gamma} \| \nabla v \|_p^{\gamma} \cdot \left(\int_0^{2t_0} t^{(\alpha - 1 + (1-\gamma)/p')p/(1-\gamma)} \, dt \right)^{1/p(1-\gamma)},
\end{aligned}$$

and therefore

$$\| \nabla v D^2 g \|_p \leq C \| \nabla v_t \|_p^{1-\gamma} \| \nabla v \|_p^{\gamma}, \qquad \text{for each } 0 \leq \gamma < \alpha, \tag{5.22}$$

since $(\alpha - 1 + (1 - \gamma)/p')p/(1 - \gamma) > -1$. This estimate shows that G maps the domain $D(E_p) = W^{2,p}(\mathbb{R}^n_+) \cap W_0^{1,p}(\mathbb{R}^n_+)$ onto $W^{2,p}(\mathbb{R}^n_+) \cap W_0^{1,p}(\mathbb{R}^n_+)$, i.e. $\mathscr{D}(E_p^g) = W^{2,p}(\mathbb{R}^n_+) \cap W_0^{1,p}(\mathbb{R}^n_+)$ again. Moreover the choice $\gamma = 0$ in (5.22) shows also that estimate (2.11) is preserved under G; recall $\eta \leq \det g \leq \eta^{-1}$ for some $\eta > 0$ by choice of $t_0 > 0$.

By the properties of h it becomes apparent now that (A1) to (A4) are preserved under G, as are the resolvent estimates (2.8) and (2.10), as well as (2.11), provided we exclude the perturbation term $-(D^2 g \cdot a) \circ g^{-1}(v)$ arising in E_p^g by the chain rule; cp. (5.2). This term cannot be shown to be subject to (A3) and (A4) without additional restrictions on p and α. However, (5.22) with $\gamma > 0$ shows that this term is still subordinate to the main part of E_p^g, namely $-a^g \nabla^2$, and so (2.6) is still valid for this perturbation term; cp. Remark 4.4. Since (2.6) was the only property of the lower order terms needed in the proof of Theorem A for $\Omega = \mathbb{R}^n$ in Section 4, we see by the reflection principle that Theorem A is also valid for $\mathbb{R}^n_+$.

Concerning Theorem B, observe that we only need to know that the following estimate holds, by Remark 4.4.

$$\|\nabla v D^2 g\|_{p_1} \leq C \|\nabla^2 v\|_p, \qquad v \in W^{2,p}(\mathbb{R}^n_+) \cap W_0^{1,p}(\mathbb{R}^n_+), \tag{5.23}$$

$p_1 < p$. Since $d_k \in L^q(\mathbb{R}^{n-1})$ for all $q > \dfrac{n-1}{\alpha}$ by (A2), by interpolation we obtain $d_k \in D(Q_q^{\alpha_q})$, $\alpha_q < \alpha - \dfrac{n-1}{q}$, where Q_q denotes the negative generator of the Poisson semigroup in $L_q(\mathbb{R}^{n-1})$. This implies $|D^2 g(t, \cdot)|_q \leq C t^{\alpha_q - 1}$, for all $q > \dfrac{n-1}{\alpha}$. With $1/p_1 = 1/p + 1/q$ we therefore obtain by (5.12)

$$\begin{aligned} |\nabla v(t, \cdot) D^2 g(t, \cdot)|_{p_1} &\leq |\nabla v(t, \cdot)|_p |D^2 g(t, \cdot)|_q \leq C t^{\alpha_q - 1} |\nabla v(t, \cdot)|_p \\ &\leq C \|\nabla v_t\|_p t^{\alpha_q - 1 + 1/p'}, \end{aligned}$$

hence (5.23) is valid provided $p_1(\alpha - 1 + 1/p') > -1$, which means $q > (n-2)/\alpha$. Thus Theorem B is proved for the halfspace $\mathbb{R}^n_+$ as well. In particular, we have extended Proposition 4.2 to the half space case.

PROPOSITION 5.1. *Let $1 < p < \infty$, $n \geq 1$, $0 < \theta < \pi$; suppose $a(x) = (a_{jk}(x))$ is a real matrix, $x \in \mathbb{R}^n_+$, which satisfies* (A1) *and* (A2) *for $\Omega = \mathbb{R}^n_+$ with $a_0 > 0$, $0 < \alpha < 1$, $a^\infty = (a_{jk}^\infty)$, and consider the operator $E_p = -a : \nabla^2$ with $\mathscr{D}(E_p) = W^{2,p}(\mathbb{R}^n) \cap W_0^{1,p}(\mathbb{R}^n_+)$.*

Then there is a constant $\eta = \eta(\theta, p, a_0, n) > 0$ with the following properties. If $\|a - a^\infty\|_\infty = \sup\{|a_{jk}(x) - a_{jk}^\infty| : x \in \mathbb{R}^n, j, k = 1, 2, \ldots, n\} \leq \eta$, then E_p belongs to $\mathbb{BIP}(L^p(\mathbb{R}^n_+))$; in particular E_p is closed, $\mathscr{N}(E_p) = 0$, and $\overline{\mathscr{R}(E_p)} = L^p(\mathbb{R}^n_+)$. Moreover, $\phi_{E_p} \leq \theta_{E_p} \leq \theta$, $\sigma(E_p) \subset \bar{\Sigma}_\theta$, and there are constants $M_1 = M_1(\theta, p, a_0, n)$, $M_2 = M_2(\theta, p, a_0, n)$, $M_3 = M_3(p, a_0, n)$ such that

$$\|(\lambda + E_p)^{-1}\| \leq M_1/|\lambda| \qquad \textit{for all } \lambda \in \Sigma_{\pi - \theta}, \tag{5.24}$$

$$\|E_p^{iy}\| \leq M_2 e^{\theta|y|} \qquad \textit{for all } y \in \mathbb{R}, \tag{5.25}$$

$$\|\nabla^2 u\|_p \leq M_3 \|E_p u\|_p \qquad \textit{for all } u \in \mathscr{D}(E_p). \tag{5.26}$$

6. Bounded and exterior domains

Before we prove Theorems A and B for bounded and exterior domains, we apply Proposition 5.1 and an admissable variable transform to derive the corresponding result for curved halfplanes.

PROPOSITION 6.1. *Let $1 < p < \infty$, $n \geq 1$, $0 < \theta < \pi$, and assume $h \in$*

$C^2(\mathbb{R}^{n-1})$, ∇h, $\nabla^2 h$ bounded, and $\lim_{|y|\to\infty} \nabla h(y)$ exists. Define $\Omega \subset \mathbb{R}^n$ by $\Omega = \{x = (t, y) \in \mathbb{R} \times \mathbb{R}^{n-1} : t \geq h(y),\ y \in \mathbb{R}^{n-1}\}$. Suppose $a(x) = (a_{jk}(x))$ is a real matrix, $x \in \Omega$, which satisfies (A1) *and* (A2) *with $a_0 > 0$, $0 < \alpha < 1$, $a^\infty = (a_{jk}^\infty) = (\lim_{|x|\to\infty} a_{jk})$, and consider the operator $E_p = -a : \nabla^2$ with $\mathscr{D}(E_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$.*

Then there is a constant $\eta = \eta(\theta, p, a_0, n) > 0$ with the following properties. If $\|a - a^\infty\|_\infty = \sup\{|a_{jk}(x) - a_{jk}^\infty| : x \in \Omega, j, k = 1, 2, \ldots, n\} \leq \eta$, then E_p belongs to $\mathcal{BIP}(L^p(\Omega))$; is particular E_p is closed, $\mathcal{N}(E_p) = 0$, and $\overline{\mathscr{R}(E_p)} = L^p(\Omega)$. Moreover, $\phi_{E_p} \leq \theta$, $\theta_{E_p} \leq \theta$, $\sigma(E_p) \subset \bar{\Sigma}_\theta$, and there are constants $M_1 = M_1(\theta, p, a_0, n, h)$, $M_2 = M_2(\theta, p, a_0, n, h)$, $M_3 = M_3(p, a_0, n, h)$ such that

$$\|(\lambda + E_p)^{-1}\| \leq M_1/|\lambda| \qquad \text{for all } \lambda \in \Sigma_{\pi-\theta}, \tag{6.1}$$

$$\|E_p^{iy}\| \leq M_2 e^{\theta|y|} \qquad \text{for all } y \in \mathbb{R}, \tag{6.2}$$

$$\|\nabla^2 u\|_p \leq M_3 \|E_p u\|_p \qquad \text{for all } u \in \mathscr{D}(E_p). \tag{6.3}$$

PROOF. The map $g(t, y) = (t + h(y), y)$ defines an admissable variable transformation of $\mathbb{R}_+^n$ onto Ω in the sense of (a) of Section 5. Therefore the result follows from Proposition 5.1. □

After this additional preparation we consider now the operator E_p defined by (2.1) on a bounded or exterior domain Ω under the assumptions of Theorem A or B. Let $0 < \theta < \pi$ be fixed such that (2.8) resp. (2.10) and (2.11) are satisfied. The localization procedure is carried out as in Section 4 (iii) with some modifications which are due to the presence of the boundary of Ω. As there, a decomposition $\Omega = U_0 \cup U_1 \cup \cdots \cup U_N$ is obtained, where $U_1, \ldots, U_N$ are the intersections of balls with Ω and where U_0 is absent if Ω is a bounded domain; some of these sets intersect the boundary. As in Section 4 we choose cut-off functions $\varphi_j \in C^\infty(\mathbb{R}^n)$ for $j = 0, 1, \ldots, N$ such that $0 \leq \varphi_j \leq 1$, supp $\varphi_j \in U_j$ and $\sum_{j=1}^N \varphi_j(x) = 1$ for all $x \in \Omega$; observe that $\varphi_0 = 1$ for say $|x| > R$. Then consider the equation

$$\lambda u + E_p u = f$$

where $\lambda \in \Sigma_{\theta-\pi}$, $f \in L^p(\Omega)$, $u \in \mathscr{D}(E_p)$, i.e. $u = (\lambda + E_p)^{-1} f$. Multiplying by φ_j and setting $u_j = \varphi_j u$, $F_j(a, b, c, \nabla)u = -\varphi_j b \cdot \nabla u - \varphi_j c u - 2a(\nabla \varphi_j)(\nabla u) - (a\nabla^2 \varphi_j)u$, we obtain as in Section 4 the local equations

$$\lambda u_j + E_p^j u_j = \varphi_j f + F_j(a, b, c, \nabla)u, \qquad j = 0, \ldots, N, \tag{6.4}$$

where $E_p^j u_j = -a : \nabla^2 u_j$. The local operators E_p^j are defined as in Section 4 in case U_j does not meet the boundary $\partial\Omega$; otherwise they are defined by $E_p^j = G_j E_p^{g_j} G_j^{-1}$, where G_j denotes the similiarity transform induced by the C^2-regularity of the boundary $\partial\Omega$, i.e. for these indices j, E_p^j are the operators corresponding to curved halfplanes; cp. Proposition 6.1.

This leads as before to the following representation of $(\lambda + E_p)^{-1}$.

$$(\lambda + E_p)^{-1} f = \sum_{j=0}^{N} (\lambda + \delta + E_p^j)^{-1} \{\varphi_j f + F_j^\delta(a, b, c, \nabla)(\lambda + E_p)^{-1} f\} \tag{6.5}$$

Since according to Propositions 4.2 and 6.1 the essential estimates for the local operators are still valid we may now follow the proofs for Theorems A and B given in Section 4 for the case $\Omega = \mathbb{R}^n$.

7. Resolvent estimates

PROOF OF THEOREM D.

The proof relies on the localization procedure and a compactness argument which is wellknown. Consider E_p as defined in (2.1) with coefficients subject to (A1) $\sim$ (A3), with constants $r_k \geq p$, $r_k > n$, $s_k \geq p$, $s_k > n/2$. We use the localization described in Section 4 and also employ the notation used there.

(a) If $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is a solution of $(\lambda + E_p)u = f \in L^p(\Omega)$, then

$$u = \sum_{j=0}^{N} (\lambda + E_p^j)^{-1} \{\varphi_j f + F_j u\}, \tag{7.1}$$

where $F_j = F_j(a, b, c, \nabla)$ is as in Section 4.

By means of (2.2) $\sim$ (2.6) we obtain for $j = 0, 1, \ldots, N$ the inequality

$$\| F_j u \|_p \leq \varepsilon \| \nabla^2 u \|_p + C(\varepsilon) \| u \|_p, \tag{7.2}$$

where $\varepsilon > 0$ is arbitrary. For the local operators E_p^j, defined in Section 4, we use estimate (4.4) resp. its analog for the halfplane and for the curved halfplane. Combining this estimate with (7.2) yields the following a priori estimate for the solution u of $\lambda u + E_p u = f$.

$$|\lambda| \, \| u \|_p + \| \nabla^2 u \|_p \leq C(\| f \|_p + \| u \|_p), \qquad \lambda \in \Sigma_{\pi - \theta}, \tag{7.3}$$

where C does not depend on f and λ. For $|\lambda|$ large enough, this implies injectivity and closed range of $\lambda + E_p$. On the other hand, (7.1) can also be used to prove that $\lambda + E_p$ is surjective for λ large enough. In fact, define $T \in \mathscr{L}(W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$ by means of $Tu = \sum_0^N (\lambda + E_p^j)^{-1} F_j u$; then (7.2) implies the estimates

$$|\lambda| \, \| Tu \|_p + \| \nabla^2 Tu \|_p \leq \varepsilon \| \nabla^2 u \|_p + C(\varepsilon) \| u \|_p, \qquad \text{for all } u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

Choosing the equivalent norm $\| u \| = \eta \| \nabla^2 u \|_p + \| u \|_p$ on $W^{2,p}(\Omega)$, where $\eta > 0$ is sufficiently small, for λ sufficiently large, T becomes a contraction, and therefore (7.1) admits a solution. Therefore $\lambda + E_p$ is invertible for

$\lambda \in \Sigma_{\pi-\theta}$, $|\lambda| > R(\theta)$, for some $R(\theta) \geq 0$. The a priori estimate (7.3) then yields the resolvent estimate (2.16), and so (a) of Theorem D is proved.

(b) Let $\lambda \in \mathbb{C}$ and $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$ be a solution of $\lambda u + E_p u = 0$; then with some $\omega > 0$ we have $(\omega + E_p)u = (\omega - \lambda)u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \subset L^q(\Omega)$ for all $q \in (1, \infty)$ such that $1/p \geq 1/q \geq 1/p - 2/n$. Choosing ω large enough, by (a) we conclude $u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega))$; this shows that the null spaces $\mathcal{N}(\lambda + E_p)$ are increasing with p.

Conversely, we show that the null spaces are also decreasing with p. This is obvious if Ω is bounded; if Ω is unbounded this is not true for $\lambda < 0$, as the example of the Laplacian on $\mathbb{R}^n$ shows. So let us assume that $\lambda \notin (-\infty, 0]$ and (A4) holds. Suppose u belongs to the null space of $\lambda + E_p$. Then u belongs locally to $W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega))$, for any $q < p$, so we need only consider $v = \psi u$, where $\psi \in C^\infty(\mathbb{R}^n)$ denotes a cut-off function which is one in a neighborhood of infinity. v then satisfies $(\lambda + E)v = [E, \psi]u = g$, where the bracket indicates the commutator between E and ψ; observe that g has compact support, hence belongs to $L^q(\Omega)$, for any $q \leq p$. But for $|x|$ very large, E is a small perturbation of $A^\infty = -a^\infty : \nabla^2$, thanks to (A1) $\sim$ (A4), hence has spectrum contained in say $\bar{\Sigma}_\theta$, θ small; see Proposition 3.1. This then implies $v = \psi u \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega))$, and so u belongs to the null space of $\lambda + E_q$, $q \leq p$, as well. This proves (b).

(c) Next we apply a well known compactness argument to prove (c) in Theorem D. Fix any small $\eta > 0$; we claim that the second term on the right of (7.3) can be dropped, i.e. that the following estimate holds.

$$|\lambda| \, \|u\|_p + \|\nabla^2 u\|_p \leq C(\eta) \|f\|_p, \qquad \lambda \in \Sigma_{\pi-\theta}, \ |\lambda| > \eta. \tag{7.4}$$

Suppose the contrary and choose $\lambda_n \in \Sigma_{\pi-\theta}$, $|\lambda_n| > \eta$, $u_n \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$, $f_n = \lambda_n u_n + E_p u_n$ such that $|\lambda_n| \, \|u_n\|_p + \|\nabla^2 u\|_p = 1$ for $n \in \mathbb{N}$, and $\|f_n\|_p \to 0$ as $n \to \infty$. Then there is a subsequence (w.l.o.g. the same sequence) such that $\lambda_n \to \lambda$, and u_n tends to some $u \in \mathcal{D}(E_p)$ weakly in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega))$ as $n \to \infty$; (7.3) yields $u \neq 0$. Since E_p is weakly closed we may conclude $\lambda u + E_p u = 0$. This gives a contradiction to $\mathcal{N}(\lambda + E_p) = 0$.

Thus by (7.4), the range of $\lambda + E_p$ is closed for any $\lambda \in \Sigma_{\pi-\theta}$, and since this operator is also injective it is semi-Fredholm. The continuity of the Fredholm index (see e.g. Kato [12]) together with (a) then yield surjectivity. If Ω is bounded we need not exclude $\lambda = 0$, by Poincaré's inequality. This proves (c).

(d) Assume in addition (A4), and suppose first that (2.11) holds. Then we can show again by an interpolation argument as in Section 4 that (7.4) is also valid near $\lambda = 0$. In fact by (4.12) which remains valid for general domains we obtain

$$\|\lambda(\lambda+E_p)^{-1}\|_p \leq C(1+|\lambda|^\beta \|\lambda(\lambda+E_p)^{-1}\|_p),$$

for some $\beta > 0$. This together with (2.11) yields the resolvent estimate (2.18) near $\lambda = 0$.

On the other hand, if $1 < p < n/2$, then the compactness argument from (c) of this proof still works near $\lambda = 0$. In fact, let λ_n, u_n, f_n be as in (c), except $\lambda_n \to 0$, now. Then, by the Sobolev embedding theorem, we obtain $u_n \to u$ weakly in $L^q(\Omega)$, where $1/q = 1/p - 2/n$. Again by (7.3) one obtains $u \neq 0$ and $Eu = 0$. By assumption, $\mathcal{N}(E_p) = 0$, hence as in (b), by Proposition 4.2 and the Sobolev embedding one can deduce $\mathcal{N}(E_q) = 0$, which as before gives a contradiction. Thus (7.4) in this case also holds for $\eta = 0$. Finally, by reflexivity of $L^p(\Omega)$ this yields also $\overline{\mathcal{R}(E_p)} = L^p(\Omega)$. This proves (d) of Theorem D. □

PROOF OF THEOREM C. If $E = -\sum_{j,k=1}^n \partial_j a_{jk} \partial_k = -a : \nabla^2 + b \cdot \nabla$ is of divergence form then it is wellknown that E_2 is a positive selfadjoint operator with $\mathcal{N}(E_2) = 0$, and $\sigma(E_2) \subset [0, \infty)$. Therefore Theorems A, B, and D yields the assertions of Theorem C for the cases (i), (ii), and (iv).

In case (iii) we apply a duality argument. So let first $1 < p < n/2$ and apply Theorems B and D. Consider next the case of very large p, say $p > n/(n-2)$. Then the adjoint exponent p' satisfies $1 < p' < n/2$, hence Theorems B and C apply to $E_{p'}$. But due to the divergence form of E, we have the duality relation $E_{p'}^* = E_p$; this proves the result for $p > n/(n-2)$. Finally, the Riesz-Thorin interpolation theorem yields the claims for all intermediate values of p. The proof is complete. □

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Fachbereich 17 *Mathematik*
Universität-GH Paderborn
Warburger Str. 100
4790 *Paderborn*
Federal Republic of Germany