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A vanishing theorem of Araki-Yosimura-Bousfield-Kan spectral sequences

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§1. Statement of Results

Let X be a CW-spectrum, U a set of sub-CW-spectra of X satisfying (a) for any two elements $A, B \in U$, there is an element $C \in U$ such that $A \cup B \subset C$, and (b) $\bigcup_{A \in U} A = X$; and let h^* be a generalized cohomology functor represented by a CW-spectrum. For these data, Araki-Yosimura [2], Yosimura [9] and Bousfield-Kan [3] constructed a spectral sequence $(E_r^{p,q}, d_r^{p,q} | r \geq 2)$ with $E_2^{p,q} = \varprojlim^p_{A \in U} h^q(A) \Rightarrow h^{p+q}(X)$.

NOTATION 0. For $n \in \boldsymbol{Z}$, $\boldsymbol{Z}_n$ denotes the quotient group $\boldsymbol{Z}/n\boldsymbol{Z}$.

In this note we prove the following vanishing theorem:

MAIN THEOREM (= PROPOSITION 0). *Let* X, U, h^* *and* $(E_r^{p,q})$ *be as above. Then*:

(i) $E_3^{p,q} \cong \boldsymbol{Z}_1$ *if* $p \geq 2$.

(ii) *For* $q \in \boldsymbol{Z}$ *the following exact sequence exists*:

$$\begin{array}{ccccccccc}
\varprojlim^3_{A \in U} h^{q-2}(A) & \longrightarrow & \varprojlim^1_{A \in U} h^{q-1}(A) & \longrightarrow & h^q(X) & \xrightarrow{f} & \varprojlim^0_{A \in U} h^q(A) & \longrightarrow & \varprojlim^2_{A \in U} h^{q-1}(A) \\
\Vert & & \Vert & & & & \Vert & & \Vert \\
E_2^{3,q-2} & & E_2^{1,q-1} & & & & E_2^{0,q} & & E_2^{2,q-1}
\end{array}$$

where f *is the naturally defined map.*

(iii) *This spectral sequence finitely converges.*

If moreover any element $A \in U$ *is finite, then the following* (iv) *and* (v) *hold*:

(iv) $E_2^{p,q} = \varprojlim^p_{A \in U} h^q(A) \cong \boldsymbol{Z}_1$ *if* $p \geq 2$.

(v) *There exist exact sequences of Milnor type*

$$\boldsymbol{Z}_1 \longrightarrow \varprojlim^1_{A \in U} h^{q-1}(A) \longrightarrow h^q(X) \xrightarrow{f} \varprojlim^0_{A \in U} h^q(A) \longrightarrow \boldsymbol{Z}_1.$$

We remark that (a) the surjectivity of f in (v) is due to a theorem of Adams [1], but our proof of Main Theorem does not rely on [1]; and (b) (iv) and (v) have already been proved under the following additional assumptions:

'when h^* is of finite type' in [2], 'when U is a countable set' (cf. [2]), 'when h^* is an ordinary cohomology with an arbitrary coefficient group' in Huber-Meier [5].

In §2 we treat a general theory involving phantom maps ('V-zero' in the terminology of the present note) and show that the composition of two phantom maps is null homotopic (Proposition 11). In §3 the main theorem is proved. In §4 we give some examples.

§2. V-projective spectra and V-injective spectra

NOTATION 1. $\mathscr{S}$ denotes the category of CW-spectra and $\tilde{\mathscr{S}}$ denotes the homotopy category of $\mathscr{S}$. $\mathscr{Ab}$ denotes the category of abelian groups.

In this section we work in $\tilde{\mathscr{S}}$ unless otherwise mentioned.

NOTATION 2. Let us fix symbols V'', V', V which denote fixed one of the triples satisfying the following conditions: (i) any element of V'' is a finite spectrum, (ii) for any finite spectrum A, there is a unique element $B \in V''$ such that B is homotopy equivalent to A, (iii) $V' = \vee_{A \in V''} A$, (iv) $V = \{V'_n(-) | n \in \boldsymbol{Z}\}$.

DEFINITION 1. (i) Let X, Y be spectra. A map $f: X \to Y$ is a *V-mono* (resp. *V-epi* resp. *V-zero*) if $V(f)$ is a mono (resp. epi resp. zero homomorphism). We also use the terms *V-monic* and *V-epic.* (ii) A homotopy cofiber sequence $X \to Y \to Z$ is *V-short-exact* if the corresponding sequence $(\boldsymbol{Z}_1) \to V(X) \to V(Y) \to V(Z) \to (\boldsymbol{Z}_1)$ is exact where $(\boldsymbol{Z}_1) = (\boldsymbol{Z}_1 | n \in \boldsymbol{Z})$. (iii) A spectrum X is *V-projective* (resp. *V-injective*) if for any V-epi $f: Y \to Z$, $f_*: [X, Y] \to [X, Z]$ is epic (resp. if for any V-mono $f: Y \to Z$, $f^*: [Z, X] \to [Y, X]$ is epic). (iv) A spectrum X is *strongly V-projective* if X is homotopy equivalent to the wedge sum of finite spectra.

PROPOSITION 1. *V'' is a countable set and V' is a countable spectrum, i.e., consists of countable (stable) cells.*

PROPOSITION 2. *Let $(X_p | p \in P)$ be a collection of spectra. Then*
(i) *$\vee_{p \in P} X_p$ is V-projective iff X_p is V-projective for all $p \in P$.*
(ii) *$\prod_{p \in P} X_p$ is V-injective iff X_p is V-injective for all $p \in P$.*
(iii) *Any finite spectrum is V-projective.*
(iv) *A strongly V-projective spectrum is V-projective.*

There propositions follow directly from the definition of V.

PROPOSITION 3. (i) *For any spectrum X, there is a strongly V-projective spectrum Y and a V-epi $f: Y \to X$.* (ii) *A spectrum X is V-projective iff there is a spectrum Y such that $X \vee Y$ is strongly V-projective.*

PROOF. (i) Let $Y = \vee_{F\subset X, F:\text{finite}} F$ and $f\colon Y \to X$ be the naturally defined map. Then these are desired objects. (ii) is a corollary of (i).

PROPOSITION 4. *If X is a countable spectrum, then there are spectra Y, Z and maps $f\colon Y\to Z$, $g\colon Z\to X$ such that Y and Z are both strongly V-projective and the sequence $Y \xrightarrow{f} Z \xrightarrow{g} X$ is V-short-exact.*

PROOF. Let F_i *be finite subspectra of* X $(i = 0, 1, 2,\cdots)$ such that $F_i \subset F_{i+1}$ $(i = 0, 1, 2,\cdots)$ and $\bigcup_i F_i = X$, and let $f_i\colon F_i \to F_{i+1}$ be the inclusion map $(i = 0, 1, 2,\cdots)$ and put $Y = Z = \vee_{i\geq 0} F_i$, and let g be the naturally defined map as in the proof of Proposition 3, p be the composition map of $\vee_i f_i\colon \vee_{i\geq 0} F_i \to \vee_{i\geq 1} F_i$ and coprojection map $\vee_{i\geq 1} F_i \to F_0 \vee (\vee_{i\geq 1} F_i)$ and let $f = \mathrm{id}_Y - p$ (in $\tilde{\mathscr{S}}$). Then these are desired objects.

PROPOSITION 5. *For any spectrum X, there is a V-injective spectrum Y and a V-mono $f\colon X \to Y$.*

This proposition is a special case of the spectrum version of Proposition 4 in [6]. A simple proof was given by Yosimura [10]. For a proof, see Proposition 19 below.

PROPOSITION 6. *Let X be a spectrum. Then* (i) *X is V-projective iff for any spectrum Y and any V-zero $f\colon X \to Y$, '$f = 0$' holds.* (ii) *X is V-injective iff for any spectrum Y and any V-zero $f\colon Y\to X$, '$f = 0$' holds.* (The proof is easy.)

PROPOSITION 7. *Let X be a spectrum and Y, Z be subspectra of X with Z finite and $f\colon Y\to X$ be the inclusion map. If f is a V-mono then there exists a countable subspectrum G of X such that $G \supset Z$ and the inclusion maps $Y\cap G \subset G$ and $Y \cup G \subset X$ are both V-monos.*

This proposition is a special case of the spectrum version of Proposition 1 of [6] since V' is countable by Proposition 1. The following proposition was essentially proved in [6].

PROPOSITION 8. *A spectrum X is V-injective iff X satisfies the condition (C): for any countable spectrum Y and any V-zero $f\colon Y\to X$, f is null homotopic.*

PROOF. In this proof we work in the category $\mathscr{S}$. Let X be a spectrum satisfying the condition (C) as above, Y be a spectrum $f\colon Y\to X$ be a cellular map which is a V-zero, and let Z be fixed one of the mapping cones of f (note that a map between spectra is an equivalence class of functions) and $g\colon X \to Z$ be the inclusion map. Let $A = \{(H, h) | H$ is a subspectrum of Z with $H \supset X$, $h\colon H \to X$ is a cellular map with $h|_X = \mathrm{id}_X$ and the inclusion map $H \subset Z$ is a V-mono$\}$. If we introduce a partial order relation $\square$ on A defined by the condition that $(H, h) \square (H', h')$ iff $H \supset H'$ and $h|_{H'} = h'$, then, by Zorn's lemma, we select an element $(H_0, h_0) \in A$ having the property that for any element $(H, h) \in A$, $(H, h) \square (H_0, h_0)$ implies $(H, h) = (H_0, h_0)$. Let F be a finite subspectrum of Z. Then, by Proposition 7, there is a countable subspectrum of G containing F such that the inclusion maps $H_0 \cap G \subset G$ and $H_0 \cup G \subset Z$ are V-monos. From the condition (C), h_0 is extendable to $H_0 \cup G$. This implies that $F \subset H_0$ hence $H_0 = Z$, i.e., f is null homotopic. Therefore X is V-injective by Proposition 6. Conversely the V-injectivity of X leads us to the condition (C) by Proposition 6.

PROPOSITION 9. *In the following diagram of spectra and maps, we assume that two rows are homotopy cofiber sequences.*

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ & & \uparrow{\scriptstyle h} & & \\ P & \xrightarrow{k} & Q & \xrightarrow{l} & R \end{array}$$

(i) *If g, h, l are V-zeros, X is V-injective, and P, R are V-projective, then $h = 0$ (in $\tilde{\mathscr{S}}$).*

(ii) *If f, h, k are V-zeros, X, Z are V-injective and R is V-projective, then $h = 0$.*

PROOF. (i) Let X, f, etc. be as above. Since P is V-projective and $h \circ k$ is a V-zero, we obtain $h \circ k = 0$, and there is a map $m: R \to Y$ with $m \circ l = h$. Since R is V-projective and $g \circ m$ is a V-zero, we obtain $g \circ m = 0$, and there is a map $n: R \to X$ with $f \circ n = m$. Since X is V-injective and $n \circ l$ is a V-zero, we obtain $n \circ l = 0$ and consequently $h = m \circ l = f \circ n \circ l = 0$. (ii) is dually proved.

PROPOSITION 10. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a V-short-exact sequence. Then:*

(i) *If Y is V-injective, then Z is V-injective.*

(ii) *If Y is V-projective, then X is V-projective.*

PROOF. (i) Let X, f, etc. be as above. Then by Proposition 8 it suffices to prove that for any countable spectrum P and any V-zero $k: P \to Z$, '$k = 0$' holds. Let $h: Z \to \Sigma X$ be a connecting map of the above sequence. By Proposition 4, there are spectra Q, R and maps p, q such that the sequence $R \xrightarrow{q} Q \xrightarrow{p} P$ is V-short-exact and Q, R are V-projective. Let $r: P \to \Sigma R$ be a connecting map of this sequence. Then by considering the diagram: $\begin{array}{ccccc} Y & \xrightarrow{g} & Z & \xrightarrow{h} & \Sigma X \\ & & \uparrow{\scriptstyle k} & & \\ Q & \xrightarrow{p} & P & \xrightarrow{r} & \Sigma R \end{array}$, we obtain $k = 0$ by Proposition 9. Hence Z is V-injective. (ii) is dually proved by using (i).

As corollaries of this proposition, we can show the followiong three propositions.

LEMMA 1 (= PROPOSITION 11). *Let* $f: X \to Y$ *and* $g: Y \to Z$ *be V-zeros. Then* $g \circ f = 0$.

PROOF. Let X, f, etc. be as above. By Propositions 5 and 10 (i), there exists a V-short-exact sequence $Z \xrightarrow{h} Z' \xrightarrow{k} Z''$ with Z', Z'': V-injective. Let $l: \Sigma^{-1}Z'' \to Z$ be a connecting map of this sequence. Since $h \circ g = 0$ from Proposition 6, we select a map $p: Y \to \Sigma^{-1}Z''$ with $l \circ p = g$. Since $p \circ f = 0$ from Proposition 6, we obtain $g \circ f = l \circ p \circ f = 0$.

PROPOSITION 12. *Let X be a spectrum. Then there exists a V-short-exact sequence* $Y \xrightarrow{g} Z \xrightarrow{f} X$ *such that Y and Z are strongly V-projective.*

PROOF. Let X be a spectrum. Then by Proposition 3, there exist strongly V-projective spectrum Z and a V-epi $f: Z \to X$. Let $g: Y \to Z$ be a homotopy fiber of f. Then by Proposition 10, Y is V-projective. Moreover by Proposition 3, there exists a V-projective spectrum W such that $Y \vee W$ is strongly V-projective. Therefore the sequence $Y \vee (\vee_{i=1}^{\infty}(W \vee Y)) \to Z \vee (\vee_{i=1}^{\infty}(W \vee Y))$ $\to X$ obtained by adding the dummy part (i.e., cancelling pair) to the original sequence $Y \to Z \to X$ is a desired object.

PROPOSITION 13. (i) *If* $f: X \to Y$ *is a V-mono* (*resp. V-epi resp. V-zero*), *then for any spectrum* $Z, f \wedge \mathrm{id}_Z$ *is also a V-mono* (*resp. V-epi resp V-mono*). (ii) *If* $f: X \to Y$ *is a V-zero, then for any map* $g: Z \to W$, $f \wedge g$ *is a V-zero.* (iii) *If* $f: X \to Y$ *and* $g: Z \to W$ *are both V-monos* (*resp. V-epis resp. V-zeros*) *then* $f \wedge g$ *is a V-mono* (*resp. a V-epi resp. zeromorphism*). (iv) *If* $X \to Y \to Z$ *is V-short-exact and W is a spectrum, then the corresponding sequence* $X \wedge W \to Y \wedge W \to Z \wedge W$ *is also V-short-exact.* (v) *If X and Y are V-projective* (*resp. strongry V-projective*), *so is* $X \wedge Y$.

PROOF. (i) is obtained by taking the direct limit of $f \wedge \mathrm{id}_F$, where F runs over on $\{A \mid A$ is a finite suspectrum of $Z\}$. (ii) and (iii) are easily obtained since $f \wedge g = (\mathrm{id}_Y \wedge g) \circ (f \wedge \mathrm{id}_Z)$ by Lemma 1. (iv) is a corollary of (i). (v) is easy.

NOTATION 3. For spectra X, Y and $n \in \boldsymbol{Z}$, we set $[X, Y]_n'' =_{\mathrm{Df}} \{f \in [X, Y]_n$ $\mid f$ is a V-zero$\}$, $[X, Y]_n' =_{\mathrm{Df}} [X, Y]_n / [X, Y]_n''$, $[X, Y]^{@} =_{\mathrm{Df}} [X, Y]_0^{@}$ $(@ \in \{', ''\}$. (Df: definition)

Then $[-, -]'$ and $[-, -]''$ become functors from $\tilde{\mathscr{S}}^{\mathrm{op}} \times \tilde{\mathscr{S}}$ to $\mathscr{Ab}$ and by definition there are natural short exact sequences $\boldsymbol{Z}_1 \to [X, Y]''_n \to [X, Y]_n \to [X, Y]'_n \to \boldsymbol{Z}_1$.

PROPOSITION 14. (i) *If* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *is V-short-exact, then a connecting map* $h: Z \to \Sigma X$ *is unique* (*in the category* $\tilde{\mathscr{S}}$). (ii) *Let* $X \xrightarrow{a} Y \xrightarrow{b} Z$ *and* $P \xrightarrow{p} Q \xrightarrow{q} R$ *be two V-short-exact sequences* $c: Z \to \Sigma X$ *and* $r: R \to \Sigma P$ *be connecting maps of them respectively, and* $f: X \to P$, $g: Y \to Q$ *and* $h: Z \to R$ *be maps satisfying that* $g \circ a = p \circ f$ *and* $h \circ b = q \circ g$. *Then* $\Sigma f \circ c = r \circ h$.

PROOF. (i) Let X, f, etc. be as above and let $P \xrightarrow{p} Q \xrightarrow{q} R \xrightarrow{r} \Sigma P$ be a (geometric) Puppe sequence and let $k: X \to P$, $l: Y \to Q$, $m: Z \to R$ be isomorphisms with $l \circ f = p \circ k$, $m \circ g = q \circ l$. Then it suffices to prove that $r \circ m = \Sigma k \circ h$. We can assume that $X = P$, $Y = Q$, $Z = R$, $f = p$, $g = q$, $h = r$, $k = \mathrm{id}_X$ and $l = \mathrm{id}_Y$ since there exists an isomorphism $n: R \to Z$ such that $n \circ q = g \circ l^{-1}$ and $(\Sigma k)^{-1} \circ r = h \circ n$. Then $n - \mathrm{id}_Z$ is a V-zero since g is V-epic, hence by Lemma 1, $h \circ m = h \circ (\mathrm{id}_Z + (m - \mathrm{id}_Z)) = h \circ \mathrm{id}_Z$ since h is a V-zero, thus $h \circ m = \Sigma \mathrm{id}_X \circ h$ as desired. (ii) is similarly proved.

THEOREM 1 (= PROPOSITION 15). *Let* T, X, Y, Z *be spectra and* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be a V-short-exact sequence and* $n \in \boldsymbol{Z}$. *Then we have following two natural exact sequences*:

$$\text{(i)} \quad \boldsymbol{Z}_1 \longrightarrow [T, X]'_n \xrightarrow{f'_*} [T, Y]'_n \xrightarrow{g'_*} [T, Z]'_n \xrightarrow{D_*} :$$

$$: \longrightarrow [T, X]''_{n-1} \xrightarrow{f''_*} [T, Y]''_{n-1} \xrightarrow{g''_*} [T, Z]''_{n-1} \longrightarrow \boldsymbol{Z}_1,$$

$$\text{(ii)} \quad \boldsymbol{Z}_1 \longleftarrow [X, T]''_n \xleftarrow{f^{*''}} [Y, T]''_n \xleftarrow{g^{*''}} [Z, T]''_n \xleftarrow{D^*} :$$

$$: \longleftarrow [X, T]'_{n+1} \xleftarrow{f^{*'}} [Y, T]'_{n+1} \xleftarrow{g^{*'}} [Z, T]'_{n+1} \longleftarrow \boldsymbol{Z}_1.$$

where D^* *and* D_* *are maps induced from the connecting maps, which are well defined by Lemma* 1, *and are unique by Proposition* 14.

PROOF. (i) Let $c: Z \to \Sigma X$ be a connecting map. Since c induces maps $\partial^@: [T, Z]^@_n \to [T, X]^@_{n-1}$ ($@ \in \{(blank), ', ''\}$), and we can consider the following short exact sequence of chain complexes:

$$\begin{tikzcd}
 & & & {} \arrow[d] & \\
 & & & {[T, Z]'_{n+1}} \arrow[d, "\partial'"'] & \\
 & & {} \arrow[d, "f_*"'] \arrow[r] & {[T, X]'_n} \arrow[r] \arrow[d, "f'_*"'] & Z_1 \\
 & {} \arrow[d, "g''_*"'] & {[T, Y]_n} \arrow[r] \arrow[d, "g_*"'] & {[T, Y]'_n} \arrow[r] \arrow[d, "g'_*"'] & Z_1 \\
Z_1 \arrow[r] & {[T, Z]''_n} \arrow[r] \arrow[d, "\partial''"'] & {[T, Z]_n} \arrow[r] \arrow[d, "\partial"'] & {[T, Z]'_n} \arrow[r] \arrow[d, "\partial'"'] & Z_1 \\
Z_1 \arrow[r] & {[T, X]''_{n-1}} \arrow[r] \arrow[d, "f''_*"'] & {[T, X]_{n-1}} \arrow[r] \arrow[d, "f_*"'] & {[T, X]'_{n-1}} \arrow[r] \arrow[d, "f'_*"'] & Z_1 \\
Z_1 \arrow[r] & {[T, Y]''_{n-1}} \arrow[r] \arrow[d, "g''_*"'] & {[T, Y]_{n-1}} \arrow[r] \arrow[d, "g_*"'] & {} & \\
Z_1 \arrow[r] & {[T, Z]''_{n-1}} \arrow[r] \arrow[d, "\partial''"'] & {} & & \\
Z_1 \arrow[r] & {[T, X]''_{n-2}} \arrow[r] \arrow[d] & {} & & \\
 & {} & & &
\end{tikzcd}$$

Then it suffices to prove that (a) ∂' is a zeromorphism, (b) ∂'' is a zeromorphism, (c) f'_* is monic and (d) g''_* is epic since D_* coincides with the connecting homomorphism in the homology long exact sequence associated to the above sequence. (a) and (c) are easy. (b) follows from Lemma 1. Let us prove (d). Let $P \overset{m}{\to} Q \overset{n}{\to} T$ be a V-short-exact sequence with P, Q: V-projective (such a sequence indeed exists by Proposition 3 and 10). Let $h: T \to Z$ be a V-zero. Then by Lemma 1, we can select a map $p: T \to Y$ with $g \circ p = h$. Since $g \circ p \circ n$ is a V-zero and Q is V-projective, $g \circ p \circ n = 0$ and we can select a map $q: Q \to X$ with $f \circ q = p \circ n$. Since $q \circ m$ is a V-zero because $f \circ q \circ m = p \circ n \circ m = 0$ and f is V-monic, and since P is V-projective, $q \circ m = 0$ and we can select a map $r: T \to X$ with $r \circ n = q$. Then $p - f \circ r$ is a V-zero since $(p - f \circ r) \circ n = 0$ and n is V-epic. Hence $g \circ (p - f \circ r) = h$ i.e., g'' is epic. (ii) is dually proved.

$$\begin{tikzcd}
 & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \\
P \arrow[r, "m"'] & Q \arrow[u, "q"] \arrow[r, "n"'] & T \arrow[ul, "r"'] \arrow[u, "p"'] \arrow[ur, "h"'] &
\end{tikzcd}$$

Using Lemma 1 and Theorem 1, Main Theorem is proved in the next section. In the remaining part of this section, we describe an example of $[X, Y]^{@}$ given by [8, 9] in our context.

NOTATION 4. $\mathscr{H}om(-, -)$ and $\mathrm{Ad}[-, -, -]$ denote fixed one of the pairs of function-spectrum-functors and the natural isomorphisms (taking adjoints) $\mathrm{Ad}[X, Y, Z]: [X \wedge Y, Z] \to [X, \mathscr{H}om(Y, Z)]$ ($\mathrm{Ad}_{-,-,-}$ or simply Ad for brevity), $M(G)$ denotes fixed one of Moore spectra corresponding to a given abelian group G. For an injective (= divisible) abelian group G, $\nabla(G)$ denotes fixed one of the spectra representing the generalized cohomology theory defined by $h^*(X) = \mathrm{Hom}(\pi_{-*}(X), G)$. For spectra X and Y, $\mathrm{Dd}(X, Y)$ denotes the map $\mathrm{Ad}[X, \mathscr{H}om(X, Y), Y](f): X \to \mathscr{H}om(\mathscr{H}om(X, Y), Y)$ where $f: X \wedge \mathscr{H}om(X, Y) \to Y$ is the evaluation map.

PROPOSITION 16. (i) *Let $f: X \to Y$ be a map. Then f is a V-mono (resp. V-epi resp. V-zero) iff for any finite spectrum F, $\mathscr{H}om(\mathrm{id}_F, f)$ is a V-mono (resp. V-epi resp. V-zero).* (ii) *Let $X \to Y \to Z$ be a homotopy cofiber sequence. Then it is V-short-exact iff the corresponding sequence $\boldsymbol{Z}_1 \to [F, X] \to [F, Y] \to [F, Z] \to \boldsymbol{Z}_1$ is exact for any finite spectrum F.* (The proof is easy.)

PROPOSITION 17. *Let X, Y, Z be spectra and $f: X \wedge Y \to Z$. Then*: (i) *If $\mathrm{Ad}_{XYZ}(f)$ is a V-zero, then also is f.* (ii) *If f is a V-zero and Y is V-projective then $\mathrm{Ad}(f)$ is a V-zero.*

PROOF. Let X, f etc. be as above. (i) Let F be a finite spectrum and $h: F \to X \wedge Y$ be a map with $f \circ h = 0$. Then there exist a finite subspectrum G of X and a map $k: F \to G \wedge Y$ with $h = (g \wedge \mathrm{id}_Y) \circ k$ where $g: G \to X$ is the inclusion map. Then from the assumption $\mathrm{Ad}(f) \circ g = 0$, this implies $f \circ (g \wedge \mathrm{id}_Y) = 0$ hence $f \circ h = f \circ (g \wedge \mathrm{id}_Y) \circ k = 0$ therefore f is a V-zero. (ii) Let F be a finite spectrum and $h: F \to X$. Then since f is a V-zero and $F \wedge Y$ is V-projective, $f \circ (h \wedge \mathrm{id}_Y) = 0$ i.e. $\mathrm{Ad}(f) \circ h = 0$, hence $\mathrm{Ad}(f)$ is a V-zero.

PROPOSITION 18. *Let X, Y be spectra. If Y is V-injective then $\mathscr{H}om(X, Y)$ also is.*

PROOF. Let X, Y be as above, T be a spectrum, $f: T \to \mathscr{H}om(X, Y)$ a V-zero. Then since $\mathrm{Ad}_{TXY}{}^{-1}(f)$ is a V-zero by Proposition 17 and Y is V-injective, $\mathrm{Ad}^{-1}(f) = 0$ i.e. $f = 0$, hence $\mathscr{H}om(X, Y)$ is V-injective.

DEFINITION 2. Let R be a subring of $\boldsymbol{Q}$ (with unit). A spectrum X is of *R-type* (resp. *dimensionwise-R-finitely-generated-type* or *R-finite-type* for brevity) if $\pi_n(X)$ becomes an R-module (resp. finitely generated R-module) for all $n \in \boldsymbol{Z}$ (cf. [9]).

PROPOSITION 19. (i) *$\nabla(G)$ is V-injective for any divisible abelian gruop*

G. (ii) *For any spectrum X and any divisible abelian group G, $\mathscr{H}om(X, \nabla(G))$ is V-injective.* (iii) *Let R be a subring of $\boldsymbol{Q}$ and X be a spectrum of R-type. Then* $\mathrm{Dd}(X, \nabla(\boldsymbol{Q}/R))\colon X \to \mathscr{H}om(\mathscr{H}om(X, \nabla(\boldsymbol{Q}/R)), \nabla(\boldsymbol{Q}/R))$ *is V-monic if $R \neq \boldsymbol{Q}$.* (iv) *A spectrum X is V-injective iff there exist spectra Y and Z such that $X \vee Y \simeq \mathscr{H}om(Z, \nabla(\boldsymbol{Q}/\boldsymbol{Z}))$.* (v) *A spectrum of $\boldsymbol{Q}$-type is V-injective.* ((v) is just Proposition 5 of [9].)

PROOF. (i) (resp. (v)) follows from the fact that a V-mono induces a mono on π_* (resp. $H_*(-, \boldsymbol{Q})$). (ii) is derived from (i) and Proposition 18. (iii) Let F be a finite spectrum, $f\colon F \to X$, and $p\colon X \wedge \mathscr{H}om(X, \nabla(\boldsymbol{Q}/R)) \to \nabla(\boldsymbol{Q}/R)$ and $q\colon F \wedge \mathscr{H}om(F, \nabla(\boldsymbol{Q}/R)) \to \nabla(\boldsymbol{Q}/R)$ are evaluation maps. Then $\mathrm{Dd}(X, \nabla(\boldsymbol{Q}/R)) \circ f = 0$ implies $0 = p \circ (f \wedge \mathrm{id}_{\mathscr{H}om(X, \nabla(\boldsymbol{Q}/R))}) (= q \circ (\mathrm{id}_F \wedge \mathscr{H}om(f, \mathrm{id}_{\nabla(\boldsymbol{Q}/R)})))$, hence $\mathscr{H}om(f, \mathrm{id}_{\nabla(\boldsymbol{Q}/R)}) = 0$. Then for any spectrum Y, $\pi_*(f \wedge \mathrm{id}_Y) = 0$ since $X \wedge Y$ is of R-type and $R \subsetneq \boldsymbol{Q}$, and by taking $Y = \mathscr{H}om(F, M(\boldsymbol{Z}))$, we obtain $f = 0$ since F is finite. (iv) is a corollary of (ii) and (iii).

PROPOSITION 20. *Let R be a proper subring of $\boldsymbol{Q}$, X be a spectrum of R-finite-type,* $f = \mathrm{Dd}(X, \nabla(\boldsymbol{Q}/R))\colon X \to Y$ *and let* $g\colon Y \to Z$ *be a homotopy cofiber of f. Then* (i) *Z is of $\boldsymbol{Q}$-type and* (ii) $[M(\boldsymbol{Q}), Y]_n \cong [Z, Y] \cong \boldsymbol{Z}_1$ if $n \in \boldsymbol{Z}$.

PROOF. (i) Since the canonical (= double dual) map $\pi_*(X) \to \mathrm{Hom}(\mathrm{Hom}(\pi_*(X), \boldsymbol{Q}/R), \boldsymbol{Q}/R) = \pi_*(Y)$ is the profinite completion map, its cokernel becomes a $\boldsymbol{Q}$-module i.e., Z is of $\boldsymbol{Q}$-type. (ii) It suffices only to prove that $[M(\boldsymbol{Q}), Y]_* \cong \boldsymbol{Z}_1$. Since $\pi_*(\mathscr{H}om(X, \nabla(\boldsymbol{Q}/R))) = \mathrm{Hom}(\pi_{-*}(X), \boldsymbol{Q}/R)$ are torsion groups, $M(\boldsymbol{Q}) \wedge \mathscr{H}om(X, \nabla(\boldsymbol{Q}/R)) \cong M(\boldsymbol{Z}_1)$, hence $[M(\boldsymbol{Q}), \mathscr{H}om(\mathscr{H}om(X, \nabla(\boldsymbol{Q}/R)), \nabla(\boldsymbol{Q}/R))]_* \cong [M(\boldsymbol{Q}) \wedge \mathscr{H}om(X, \nabla(\boldsymbol{Q}/R)), \nabla(\boldsymbol{Q}/R)]_* \cong \boldsymbol{Z}_1$.

Above implies that f is a V-injective enveloping map in the sense of [6]. The following is due to [8, 9].

EXAMPLE 1 (= PROPOSITION 21). *Let R be a proper subring of $\boldsymbol{Q}$ and X be a spectrum and Y be a spectrum of R-finite-type. Then* (i) *$[X, Y]''$ is the largest divisible subgroup of $[X, Y]$ and* $\mathrm{Hom}(\boldsymbol{Q}, [X, Y]') \cong \boldsymbol{Z}_1$. (ii) $[X, Y]''$ *is uniquely divisible i.e., becomes a $\boldsymbol{Q}$-module.*

PROOF. Let X, Y, Z, f, g be as in Proposition 20. Note that the sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is V-short-exact, and Y and Z are V-injective. Let T be a spectrum. Then (a) $[M(\boldsymbol{Q}), \mathscr{H}om(T, Y)]_* \simeq [T, \mathscr{H}om(M(\boldsymbol{Q}), Y)]_* \simeq \boldsymbol{Z}_1$, hence $\mathrm{Hom}(\boldsymbol{Q}, [T, Y]_*) \cong \boldsymbol{Z}_1$. (b) $[T, Z]$ is uniquely divisible since Z is of $\boldsymbol{Q}$-type. These two facts show the proposition.

§ 3. Araki-Yosimura-Bousfield-Kan Spectral Sequences and Proof of Main Theorem

We slightly modify the construction of [2] (cf. [7]).

CONSTRUCTION 1 (= NOTATION 5). X denotes a fixed spectrum and U denotes a fixed set of subspectra of X satisfying that (i) for any A, $B \in U$, there is an element $C \in U$ with $A \cup B \subset C$ and (ii) $\bigcup_{A \in U} A = X$. For these data we construct X_n, A_n ($n \in \boldsymbol{Z}$, $n \geq 0$, $A \in U$) (and check the conditions $X \subset X_n$, $A \subset A_n \subset X_n$, $(n \geq 0)$ and $A_{n-1} \subset A_n$, $X_{n-1} \subset X_n$ $(n \geq 1)$) inductively on n. $\mathrm{Cone}(Z, p)$ denotes the (abstract) cone of Z with the vertex p. Let $(p(A, n) | A \in U,\ n \geq 0)$ be a system of abstract distinct points. Firstly put $X_0 = X$ and $A_0 = A$ and define as $X_{n+1} = X_n \cup (\bigcup_{A \in U} \mathrm{Cone}(A_n,\ p(A, n)))$, $A_{n+1} = \bigcup_{B \in U, B \supset A} \mathrm{Cone}(A_n \cap B_n,\ p(B, n))$ inductively on n. We consider that $\mathrm{Cone}(A_n, p) \cap \mathrm{Cone}(B_n, q) = A_n \cap B_n \subset X_n$ if $p \neq q$ and $\mathrm{Cone}(K, p) \subset \mathrm{Cone}(L, p)$ if $K \subset L$. (We remark that the tower $(\Sigma^{-1}(X_{n+1}/X) | n \geq 0)$ is essentially equivalent to the tower $(B\mathscr{C}_n | n \geq 0)$ of [2]. If $B\mathscr{C}_n(X, S)$ denotes the $B\mathscr{C}_n$ of [2] for X and $S = \langle X_\alpha | \alpha \in I \rangle$, then A_n indeed corresponds to the cofiber of the natural map $\pi\colon B\mathscr{C}_n(A,\ \langle A_\alpha = A | \alpha \in U,\ \alpha \supset A \rangle) \to A$.) Let $f_n\colon X_n \to X_{n+1}$ be the inclusion map and consider the following commutative diagram:

$$\begin{tikzcd}
X = X_0 \arrow[r, "f_0"] & X_1 \arrow[r, "f_1"] \arrow[dl, "g_0"'] & X_2 \arrow[r, "f_2"] \arrow[dl, "g_1"'] & X_3 \arrow[r, "f_3"] \arrow[dl, "g_2"'] & X_4 \arrow[r] \arrow[dl, "g_3"'] & {} \\
Y_{-1} & Y_0 \arrow[l, "k_{-1}"] \arrow[u, "h_0"'] & Y_1 \arrow[l, "k_0"] \arrow[u, "h_1"'] & Y_2 \arrow[l, "k_1"] \arrow[u, "h_2"'] & Y_3 \arrow[l, "k_2"] \arrow[u, "h_3"'] & {} \arrow[l] \arrow[ul]
\end{tikzcd}$$

where f_n, g_n, h_n, k_n are maps of degrees $0, -n-1, n, 0$ respectively and $X_n \xrightarrow{f_n} X_{n+1} \xrightarrow{g_n} Y_n \xrightarrow{h_n} X_n$ is a Puppe sequence for $n \geq 0$, $k_n = g_n \circ h_{n+1}$ $(n \geq 1)$, $Y_{-1} = M(\boldsymbol{Z}_1)$ and $k_{-1} = 0$.

Then the following two propositions was essentially proved in [2].

PROPOSITION 22. *Let $F\colon \tilde{\mathscr{S}} \to \mathscr{A}b$ be a contravariant functor such that the corresponding (covariant) functor $F'\colon \tilde{\mathscr{S}} \to \mathscr{A}b^{\mathrm{op}}$ preserves direct sums (of arbitrary cardinality). Then* (i) *the homology groups of the cochain complex $(F(Y_*), F(k_*))$ are naturally isomorphic to $\varprojlim^*_{A \in U} F(A)$.* (ii) *Let $F, G\colon \tilde{\mathscr{S}} \to \mathscr{A}b$ be such functors and $p\colon F \to G$ be a natural transformation. Then the maps between homologies induced by p of the cochain complexes of above type coincide with $\varprojlim^*_{A \in U} p(A)$.*

PROPOSITION 23. (i) *$A_n \simeq \amalg \langle \Sigma^n A | A = E_0 \subsetneqq E_1 \subsetneqq \cdots \subsetneqq E_n,\ E_i \in U$ for all $i \rangle$,* (ii) *$X_{n+1}/X_n \simeq \amalg \langle \Sigma^{n+1} E_0 | E_0 \subsetneqq E_1 \subsetneqq \cdots \subsetneqq E_n,\ E_i \in U$ for all $i \rangle$ where $\amalg \langle - | * \rangle$ means one of the direct sums of $-$'s in the category $\tilde{\mathscr{S}}$ with suffixes*

$*$'s and $\simeq$ means 'isomorphic in $\tilde{\mathscr{S}}$', (iii) *The map f_i is a V-zero for all $i \geq 0$.*

PROOF. (i) is inductively obtained from the fact $A_{n+1} \simeq \amalg_{B \supsetneq A} \Sigma(A_n \cap B_n)$. (ii) is a corollary of (i). (iii) is obtained from the properties (i) and (ii) in the Construction 1.

NOTATION 6. T denotes a fixed spectrum and $E_1^{@p,q} =_{\mathrm{Df}} [Y_p, T]_{-q}^{@}$, $E_2^{@p,q} =_{\mathrm{Df}} H_*(E_1^{@p,q}, k_p^{*@})$ ($@ \in \{(blank), ', ''\}$ and α_p: $E_1''^{p,*} \to E_1^{p,*}$, β_p: $E_1^{p,*} \to E_1'^{p,*}$ denote the naturally defined maps and induced maps $H(\alpha_p)$, $H(\beta_p)$ are also denoted by α_p, β_p for brevity. $d_2^{p,q}(d_2^{p,*}$, d^p or d^* for brevity): $E_2^{p,q} \to E_2^{p+2,q-1}$ denotes the second differential of the spectral sequence as usual. We shall define a map $\Delta_{p,q}$: $E_2'^{p,q} \to E_2''^{p-2,q+1}$ for $p \geq 3$ as follows: Let $a \in E_2'^{p,*}$ and $b \in [Y_p, T]_*'$ be a representative of a. Then there is an element $c \in [X_p, T]_*'$ with $h_p^{*\prime}(c) = b$ by Theorem 1 since b is a cycle. Then there is an element $d \in [Y_{p-2}, T]''$ such that $g_{p-2}^{*\prime\prime}(d) = D_{p-1}^*(c)$ by Theorem 1, where (and from now on) D_p^*: $[X_{p+1}, T]' \to [X_p, T]''$ denotes the boundary map of Theorem 1, and try to define as $\Delta_{p,*}(a) =_{\mathrm{Df}} \langle$the class of $d\rangle$. Well-definedness is proved in the proposition below. Moreover let us define I_*: $E_2'^{1,*} \to [X, T]_*$ and j^*: $[X, T]_* \to E_2^{0,*}$ as follows. For $a \in E_2'^{1,*}$ let $b \in [Y_1, T]_*'$ be a representative of a. Then there is an element $c \in [X_1, T]_*'$ with $h_1^{*\prime}(c) = b$ by Theorem 1 since b is a cycle, and try to define as $I_*(a) =_{\mathrm{Df}} D_0^*(c)$. Next for $a \in [X, T]_*$, try to define as $j^*(a) = h_0^*(a)$. Finally set $\delta_{p,q} =_{\mathrm{Df}} \alpha_{p-2} \circ \Delta_{p,q} \circ \beta_p$: $E_2^{p,q} \to E_2^{p-2,q+1}$ and $i_* =_{\mathrm{Df}} I_* \circ \beta_1$: $E_2^{1,*} \to [X, T]_*$.

PROPOSITION 24. (i) *$\Delta_{p,*}$ above is well defined.* (ii) *$\Delta_{p,*}$ is an isomorphism. In fact $\beta_p \circ d_2^{p-2,*} \circ \alpha_{p-2}$ is the inverse of $\Delta_{p,*}$.* (iii) *I_* and j^* are well defined and j^* coincides with the naturally defined map.* (iv) *$\delta_{p-2,*} \circ \delta_{p,*} = 0$ if $p \geq 5$.* (v) *$\delta_{p+2,*} \circ d_2^{p,*} + d_2^{p-2,*} \circ \delta_{p,*} = \langle$the identity map of $E_2^{p,*}\rangle$ if $p \geq 3$.*

(vi) $$Z_1 \longrightarrow E_2''^{p,*} \xrightarrow{\alpha_p} E_2^{p,*} \xrightarrow{\beta_p} E_2'^{p,*} \longrightarrow Z_1$$

is exact for $p \geq 0$ and is split-exact for $p \geq 1$. (vii) *The following sequence is exact*:

$$\xrightarrow{\delta_*} E_2^{5*} \xrightarrow{\delta_*} E_2^{3*} \xrightarrow{\delta_*} E_2^{1*} \xrightarrow{i_*} [X, T]_* \xrightarrow{j_*} E_2^{0*} \xrightarrow{d_*} E_2^{2*} \xrightarrow{d_*} E_2^{4*} \xrightarrow{d_*}$$

PROOF. (i) Let a, b etc. be as in Notation 6. (a) The correspondence $b \rightsquigarrow c$ is unique since $h_p^{*\prime}$ is monic by Theorem 1. (b) d is a cycle since $k_{p-2}^{*\prime\prime}(d) = h_{p-1}^{*\prime\prime}(D_{p-1}^*(c)) = 0$ by Theorem 1. (c) If d' is the another element of $[Y_{p-2}, T]_*''$ with $g_{p-2}^{*\prime\prime}(d') = g_{p-2}^{*\prime\prime}(d)$, then there is an element $e \in [X_{p-2}, T]_*''$ with $h_{p-2}^{*\prime\prime}(e) = d - d'$ by Theorem 1, and then there is an element $e' \in [Y_{p-3}, T]_*$ with $g_{p-3}^{*\prime\prime}(e') = e$ by Theorem 1. This implies that $d - d'$ is a boundary. (d) If b is a boundary i.e., if there is an element $b' \in [Y_{p-1}, T]$ with $k_{p-1}^{*\prime}(b') = b$, then

$g^{*\prime}_{p-1}(b') = c$ by (a) and $D^*_{p-1}(c) = 0$ by Theorem 1, hence we can take $d = 0$. Thus $\Delta_{p,*}$ is well defined. (ii)–(vii) are also proved by easy diagram chasings. (Note that in proving (v), we have only to prove the injectivity of α_p $(p \geq 0)$.)

(i), (ii) and (iii) of Main Theorem follow from this proposition. Moreover if any element of U is finite, we obtain (iv) and (v) since Y_n $(n \geq 0)$ is strongly V-projective by Proposition 23. Concludingly, we obtain the following theorem:

THEOREM 2 (= PROPOSITION 25). *Denote that* $h^{@p}(-) =_{\mathrm{Df}} [-, T]^{@}_{-p}$ $(@ \in \{(\mathit{blank}), ', ''\},\ p \in \boldsymbol{Z})$. *Then we have* (i) *There exist the following commutative diagrams in which all columns and rows are exact*:

(a)

$$\begin{tikzcd}
 & \boldsymbol{Z}_1 \arrow[d] & \boldsymbol{Z}_1 \arrow[d] & & \\
\boldsymbol{Z}_1 \arrow[r] & \varprojlim^1_{A\in U} h'^{n-1}(A) \arrow[r, "\mathrm{id}"] \arrow[d] & \varprojlim^1_{A\in U} h'^{n-1}(A) \arrow[r] \arrow[d] & \boldsymbol{Z}_1 \arrow[d] & \\
\boldsymbol{Z}_1 \arrow[r] & h''^n(X) \arrow[r, "\alpha"] \arrow[d, "\gamma''"] & h^n(X) \arrow[r, "\beta"] \arrow[d, "\gamma"] & h'^n(X) \arrow[r] \arrow[d, "\gamma'"] & \boldsymbol{Z}_1 \\
\boldsymbol{Z}_1 \arrow[r] & \varprojlim^0_{A\in U} h''^n(A) \arrow[r, "\alpha_0"] \arrow[d] & \varprojlim^0_{A\in U} h^n(A) \arrow[r, "\beta_0"] \arrow[d] & \varprojlim^0_{A\in U} h'^n(A) \arrow[r] \arrow[d] & \boldsymbol{Z}_1 \\
\boldsymbol{Z}_1 \arrow[r] & \varprojlim^2_{A\in U} h'^{n-1}(A) \arrow[r, "\mathrm{id}"] \arrow[d] & \varprojlim^2_{A\in U} h'^{n-1}(A) \arrow[r] \arrow[d] & \boldsymbol{Z}_1 & \\
 & \boldsymbol{Z}_1 & \boldsymbol{Z}_1 & &
\end{tikzcd}$$

for $n \in \boldsymbol{Z}$, *where* α, β, γ, γ', γ'' *are naturally defined maps*, α_0, β_0 (*and also* α_p, β_p *below*) *coincide with the limit of naturally defined maps.*

(b) $$\boldsymbol{Z}_1 \longrightarrow \varprojlim^p_{A\in U} h''^n(A) \xrightarrow{\alpha_p} \varprojlim^p_{A\in U} h^n(A) \xrightarrow{\beta_p} \varprojlim^p_{A\in U} h'^n(A) \longrightarrow \boldsymbol{Z}_1$$

for $p, n \in \boldsymbol{Z}$, $p \geq 0$, *moreover this sequence is split-exact if* $p \geq 1$,

(c) $$\boldsymbol{Z}_1 \longrightarrow \varprojlim^p_{A\in U} h''^n(A) \longrightarrow \varprojlim^{p+2}_{A\in U} h'^{n-1}(A) \longrightarrow \boldsymbol{Z}_1$$

for $p \geq 1$.

(d)

$$\longrightarrow \varprojlim^5_{A\in U} h^{q-3}(X) \longrightarrow \varprojlim^3_{A\in U} h^{q-2}(X) \longrightarrow \varprojlim^1_{A\in U} h^{q-1}(X) \longrightarrow h^q(X) \longrightarrow :$$

$$: \longrightarrow \varprojlim^0_{A\in U} h^q(X) \longrightarrow \varprojlim^2_{A\in U} h^{q-1}(X) \longrightarrow \varprojlim^4_{A\in U} h^{q-2}(X) \longrightarrow$$

(ii) *If any element of U is finite, or U is countable then* $\varprojlim^p_{A\in U} h^q(A) \cong \boldsymbol{Z}_1$ *for* $p \geq 2$. (iii) *if T is V-injective then* $\varprojlim^p_{A\in U} h^q(A) \cong \boldsymbol{Z}_1$ *for* $p \geq 1$.

The essential part of the proof has already been done. The rest is not difficult.

§4. Examples

NOTATION 6. $EM\colon \mathscr{A}\ell \to \tilde{\mathscr{S}}$ denotes fixed one of the Eilenberg-MacLane-spectrum-functors.

NOTATION 7. The definitions of terms or notions 'pure exact', 'pure injective (= algebraically compact)' and 'Pext' are the same as in Fuchs [4] (see [4]).

PROPOSITION 26. (i) *Let* $\boldsymbol{Z}_1 \to A \xrightarrow{f} B \xrightarrow{g} C \to \boldsymbol{Z}_1$ *be a short exact sequence of abelian groups. Then this is pure exact iff* $EM(A) \xrightarrow{EM(f)} EM(B) \xrightarrow{EM(g)} EM(C)$ *is V-short-exact.* (ii) *Let A, B and C be abelian groups and* $M(A) \xrightarrow{f} M(B) \xrightarrow{g} M(C)$ *be a homotopy cofiber sequence. Then this is V-short-exact iff* $\boldsymbol{Z}_1 \to \pi_0(M(A)) \xrightarrow{\pi_0(f)} \pi_0(M(B)) \xrightarrow{\pi_0(g)} \pi_0(M(C)) \to \boldsymbol{Z}_1$ *is pure exact.* (iii) *For any abelian group G, G is pure injective* $\Leftrightarrow$ *$EM(G)$ is V-injective* $\Leftrightarrow$ *$M(G)$ is V-injective.* (iv) *Let X be a spectrum and G be an abelian group. Then* $[X, EM(G)]'' \cong \mathrm{Pext}(H_{-1}(X, \boldsymbol{Z}), G)$. (The proof is easy)

This proposition (together with Theorem 1) geometrically shows Harrison's exact sequences (see [4], Theorem 53.7).

PROPOSITION 27. *Let Y be a spectrum. If Y is V-injective, then $[X, Y]$ is pure injective for any spectrum X.*

PROOF. We may assume that $X = M(\boldsymbol{Z})$ by Proposition 18. Let $f\colon M(\pi_0(Y)) \to Y$ be a map with $\pi_0(f)$: iso and $g\colon M(\pi_0(Y)) \to M(G)$ be a map such that g is V-monic and G is pure injective abelian group. Then the existence of the map $h : M(G) \to Y$ with $h \circ g = f$ implies the pure injectivity of $\pi_0(Y)$.

This proposition gives a geometric proof of the following Fuchs' theorem.

PROPOSITION 28 ([4], Theorem 47.7 and the fact (0) in 225 page). *Let A and B be abelian groups and $i \in \{0, 1\}$. Then* $\mathrm{Ext}^i(A, B)$ *is pure injective if B is pure injective.*

Finally we give two examples.

EXAMPLE 2 (= Proposition 29). *Let* $X = M\left(Z\left[\frac{1}{2}\right]/Z\right)$ *and* $Y = \Sigma EM(\bigoplus_{i\geq 0} Z_{2i})$. *Then the canonical exact sequense* $Z_1 \to [X, Y]'' \to [X, Y] \to [X, Y]' \to Z_1$ *does not split.*

PROOF. $[X, Y]''$ is a nontrivial reduced group (cf. [5], Example in 248 page). If the above sequence splits, $[X, Y]''$ is divisible since it is the 1-st Ulm subgroup of $[X, Y]$. It is a contradiction (see [4]).

EXAMPLE 3 (= PROPOSITION 30). *Let* p *be an odd prime and* $X = \bigvee_{i\geq 0}\Sigma^{-i}M(Z_p)$. *Then we have* (i) *the additive order of* id_X *in* $[X, X]$ *is* p, *therefore* $[Y, X]$ *is a* Z_p*-vector space for any spectrum* Y, *and* (ii) X *is not* V*-injective. This gives a counter example for the converse of* Proposition 27.

(i) is easy. (ii) Let Y be the mapping telescope of $M(Z_p) \xrightarrow{f} \Sigma^{2-2p}M(Z_p) \xrightarrow{\Sigma^{2-2p}(f)} \Sigma^{4-4p}M(Z_p) \to \cdots$ where f *is the* K-equivalence of Adams. Then indeed we can construct a nonzero V-zero map from Y to X. We leave the details of the proof to the reader.

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