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On exterior A_n-spaces and modified projective spaces

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1. Introduction

A space X with a continuous multiplication $\mu: X \times X \to X$ with a unit is called an H-space. A typical example of H-space is a loop space. It is known that not all H-spaces have the homotopy type of loop spaces. The 7-dimensional sphere S^7 is one of such counter examples.

Sugawara [12] gave a criterion for an H-space to have the homotopy type of a loop space. His criterion is a kind of higher homotopy associativity of infinite order. Almost the same time Stasheff [9] reached the same idea, and he defined the A_n-space which is the H-space with higher homotopy associative multiplication of n-th order. In his sense A_2-spaces are H-spaces, A_3-spaces are homotopy associative H-spaces, and A_∞-spaces are spaces with the homotopy type of loop spaces.

In his paper, Stasheff defined the projective n-space $P_n(X)$ associated to a given A_n-space X, which is considered as a generalization of the n-th stage of the construction of the classifying space of a topological group or an associative H-space. In fact, $P_n(X)$ is defined inductively by $P_n(X) = P_{n-1}(X) \cup C(X^{*n})$ with $P_0(X) = *$, where X^{*n} is the n-fold join of X. Then Stasheff proved that if $X = \Omega Y$, then $P_\infty(X)$ has the homotopy type of Y, where $P_\infty(X) = \bigcup_{i=1}^\infty P_i(X)$. The name 'projective' comes from the fact that if X is the unit sphere in the real, the complex or the quaternionic numbers, then $P_n(X)$ is the usual real, complex or quaternionic projective n-space.

The projective n-space has been very useful for the study of the cohomology of A_n-spaces. In fact, we have the following fact.

THEOREM (Iwase [4]). *Let X be a simply connected A_n-space so that*

$$H^*(X; \mathbf{Z}/p) \cong \Lambda(x_1, \ldots, x_k), \qquad \dim x_i : odd,$$

where p is a fixed prime. Suppose that there are classes $y_i \in H^*(P_n(X); \mathbf{Z}/p)$ so that each y_i restricts to the suspension of x_i in $H^*(\Sigma X; \mathbf{Z}/p)$ by the homomorphism induced by the inclusion $\Sigma X \subset P_n(X)$. (This property is referred as the A_n-primitivity of x_i.) Then there is an ideal S in $H^*(P_n(X); \mathbf{Z}/p)$ closed under the action of the Steenrod operation, so that

$$H^*(P_n(X); \mathbf{Z}/p)/S \cong T_{n+1}[y_1, \dots, y_k],$$

where $T_{n+1}[y_1, \dots, y_k]$ *is the truncated polynomial algebra of height* $n+1$, *i.e., the quotient algebra of the polynomial algebra* $\mathbf{Z}/p[y_1, \dots, y_k]$ *by the ideal generated by the* $n+1$*-fold decomposable elements.*

Iwase's original theorem in [4] is on the K-ring of $P_n(X)$. We can also prove the above theorem by the same method.

The above theorem is used especially for the study of the action of the Steenrod operation on $H^*(X; \mathbf{Z}/p)$ when $n \geq p$. In fact, the unstable condition $\mathscr{P}^m y = y^p$ ($\deg y = 2m$) can be used to deduce a variety of results (cf. [2]).

On the other hand, the classes x_i are not always A_n-primitive. One can only prove the A_{n-1}-primitivity of x_i ([4]). If we don't assume the A_n-primitivity of x_i, it seems to be very difficult to give a useful structure theorem for $H^*(P_n(X); \mathbf{Z}/p)$. Iwase [5] also studied such cases. He considered the case that there exists a particular subspace of X called a generating subspace. He used this subspace to construct a modified projective n-space. Then, without assuming the A_n-primitivity of x_i, he gave a structure theorem for the K-ring of the modified space which is very similar to the one for the usual projective n-space of the A_n-primitive case.

In this paper we construct another modified projective space. Then we show that the cohomology of this space has a very similar structure to the one in the above theorem. The advantage of our construction is that we need not assume the A_n-primitivity of x_i or the existence of a particular space like a generating subspace. Our main result is as follows:

THEOREM 1.1. *Let* X *be a simply connected* A_n*-space with*

$$H^*(X; \mathbf{Z}/p) \cong \Lambda(x_1, \dots, x_k), \qquad \deg x_i : \text{odd}$$

for some odd prime p. *Then there are a space* Y, *a map* $\varepsilon: \Sigma X \to Y$, *classes* $y_i \in H^*(Y; \mathbf{Z}/p)$ $(1 \leq i \leq k)$ *and an ideal* $M \subset H^*(Y; \mathbf{Z}/p)$ *so that the following conditions are satisfied.*

(1) $\varepsilon^*(y_i) = x_i$, *where we identify* $\tilde{H}^*(\Sigma X; \mathbf{Z}/p)$ *with* $\tilde{H}^*(X; \mathbf{Z}/p)$ *via the suspension isomorphism.*

(2) $\varepsilon^*(M) = 0$.

(3) $M \cdot \tilde{H}^*(Y; \mathbf{Z}/p) = 0$.

(4) *There is a subalgebra* A^* *of* $H^*(Y; \mathbf{Z}/p)$ *isomorphic to* $T_{n+1}[y_1, \dots, y_k] \oplus M$ *as an algebra, where* $T_{n+1}[\dots]$ *is the truncated polynomial algebra of height* $n+1$.

(5) A^* *and* M *are closed under the action of the* mod p *Steenrod algebra* $\mathscr{A}_{(p)}$. *Thus* $T_{n+1}[y_1, \dots, y_k] = A^*/M$ *has a structure of an unstable* $\mathscr{A}_{(p)}$*-algebra.*

Then $R_n(X)$ is the space Y and the inclusion $\Sigma X = P_1(X) \subset R_n(X)$ is the map ε in Theorem 1.1. For the construction we consider the loop space ΩX of X. Since ΩX is an A_∞-space, we have projective spaces

$$\Sigma\Omega X = P_1(\Omega X) \subset P_2(\Omega X) \subset \cdots$$

$$P_\infty(\Omega X) = \bigcup_i P_i(\Omega X) \simeq X .$$

Hereafter, we consider $P_t(\Omega X)$ as a subspace of X by identifying $P_\infty(\Omega X)$ with X. Put

$$C_{n-1} = \bigcup_{i=1}^{n-1} (\Sigma\Omega X)^{*i-1} * P_2(\Omega X) * (\Sigma\Omega X)^{*n-1-i} \subset X^{*n-1} .$$

Take the restriction map b_{n-1} of $\beta_{n-1}\colon X^{*n-1} \to P_{n-2}(X)$ and define $R_{n-1}(X)$ as its mapping cone;

$$b_{n-1}\colon C_{n-1} \to P_{n-2}(X), \qquad R_{n-1}(X) = P_{n-2}(X) \cup_{b_{n-1}} C(C_{n-1}) .$$

Let $f_{n-1}\colon R_{n-1}(X) \to P_{n-1}(X)$ be the induced map. Put

$$C_n = (\Sigma\Omega X)^{*n} \subset X^{*n} .$$

LEMMA 2.1. *There is a map $b_n\colon C_n \to R_{n-1}(X)$ so that*

$$f_{n-1} \circ b_n \simeq \beta_n | C_n .$$

PROOF. According to Stasheff [8], there is a map $\Sigma\Omega X \times \Sigma\Omega X \to P_2(\Omega X)$ so that the following diagram is homotopy commutative, where the vertical maps are inclusions;

$$\begin{tikzcd} \Sigma\Omega X \times \Sigma\Omega X \arrow[r] \arrow[d] & P_2(\Omega X) \arrow[d] \\ X \times X \arrow[r] & X . \end{tikzcd}$$

Then the result follows immediately. q.e.d.

We study the above map $\Sigma\Omega X \times \Sigma\Omega X \to P_2(\Omega X)$ more generally in section 6.

Now we define $R_n(X)$ as the mapping cone of b_n;

$$R_n(X) = R_{n-1}(X) \cup_{b_n} C(C_n) .$$

To prove that $R_n(X)$ has the required properties, we need to know the cohomology of $P_t(X)$ and $P_t(\Omega X)$.

(6) ε^* *induces an* $\mathscr{A}_{(p)}$*-module isomorphism,*

$$Q(T_{n+1}[y_1, \dots, y_k]) \to QH^*(X; \mathbf{Z}/p),$$

where Q *denotes the indecomposable module.*

The above theorem is the odd prime version of [3].

This paper is organized as follows. The space Y in Theorem 1.1 is constructed in section 2. To prove the required properties on Y, we need to study the cohomology of projective spaces and the loop space of X. Sections 3 and 4 are devoted to it. The properties (1)–(4) are proved in section 5. In section 6 we discuss more general constructions than that of Y. Then we prove (5) and (6) in section 7. We give some applications in section 8.

2. Construction

In the rest of this paper the cohomology has a coefficient in $\mathbf{Z}/p$ for a fixed odd prime p.

The space Y in Theorem 1.1 is constructed in an analogous way to the projective n-space $P_n(X)$ of X. First we recall the definition of projective spaces. The readers refer to Stasheff's original paper [9].

The projective t-space of X, denoted by $P_t(X)$, for $t \leq n$, is defined inductively by a relative homeomorphism

$$(K_{t+2} \times X^t, S_t) \to (P_t(X), P_{t-1}(X)),$$

where K_{t+2} is the Stasheff complex with $K_{t+2} \approx I^t$, $S_t = \partial K_{t+2} \times X^t \cup K_{t+2} \times X^{[t]}$ ($X^{[t]} = \{(x_1, \dots, x_t) \in X^t | x_i = * \text{ for at least one } i\}$), and the map $S_t \to P_{t-1}(X)$ is constructed from the A_n-structure of X. It is proved in Theorems 11 and 12 of [9] that $P_t(X)$ is also considered as the mapping cone of a suitable map

$$\beta_t\colon X^{*t} \to P_{t-1}(X); \qquad P_t(X) = P_{t-1}(X) \cup_{\beta_t} C(X^{*t}),$$

where X^{*t} is the t-fold join of X; $X^{*t} = X * \cdots * X$. Then by definition, we have

$$\Sigma X = P_1(X) \subset P_2(X) \subset \cdots \subset P_n(X)$$

$$P_t(X)/P_{t-1}(X) \simeq \Sigma X^{*t} \simeq \Sigma^t(X^{\wedge t}),$$

where $X^{\wedge t}$ is the t-fold smash product of X; $X^{\wedge t} = X \wedge \cdots \wedge X$.

Now we construct spaces $R_{n-1}(X)$ and $R_n(X)$ with

$$P_{n-2}(X) \subset R_{n-1}(X) \subset R_n(X).$$

3. Cohomology of projective spaces

Let X be an A_n-space. Since $P_t(X)/P_{t-1}(X) \simeq \Sigma^t(X^{\wedge t})$, we have an exact triangle for $t \leq n$;

$$\begin{tikzcd} \tilde{H}^*(P_{t-1}(X)) \arrow[dr, "\beta_t^*"'] & & \tilde{H}^*(P_t(X)) \arrow[ll, "\varepsilon_t^*"'] \\ & \tilde{H}^*(X)^{\otimes t}, \arrow[ur, "\rho_t^*"'] & \end{tikzcd}$$

where $\deg \varepsilon_t^* = 0$, $\deg \beta_t^* = 1 - t$, $\deg \rho_t^* = t$, and $R^{\otimes t} = R \otimes \cdots \otimes R$ (t-fold) for any R. Note that ρ_1^* is equal to the suspension isomorphism. Stasheff [9] introduced a mod p cohomology spectral sequence $\{E(X)_r^{*,*}, d(X)_r\}$ associated to the filtration

$$P_0(X) \subset P_1(X) \subset \cdots \subset P_n(X).$$

Then

$$E(X)_1^{t,*} = \begin{cases} \tilde{H}^*(X)^{\otimes t} & (t \leq n) \\ 0 & (t > n) \end{cases}$$

$$d(X)_1 | E(X)_1^{t,*} = \sum_{j=1}^{t} (-1)^{j-1} id^{\otimes j-1} \otimes \tilde{m}^* \otimes id^{\otimes t-j},$$

where $\tilde{m}\colon \tilde{H}^*(X) \to \tilde{H}^*(X) \otimes \tilde{H}^*(X)$ is the reduced coproduct of the H-structure $m\colon X \times X \to X$, and $id^{\otimes s}$ is the s-fold tensor product $id \otimes \cdots \otimes id$.

Suppose that

$$H^*(X) \cong \Lambda(x_1, \ldots, x_k), \qquad \det x_i : \text{odd}.$$

According to Borel [1, Theorem 4.1] we can assume that each x_i is primitive if $n \geq 3$. Let I_s be the $\mathbf{Z}/p$-submodule of $H^*(X)$ spanned by all s-fold multiplications of generators $\{x_{i_1} \ldots x_{i_s}\}$. It is clear that

$$I_s \cap I_t = 0 \qquad \text{for } s \neq t$$

$$D_t H^*(X) \cong \textstyle\bigoplus_{s \geq t} I_s,$$

where $D_t R$ is the submodule of R of all t-fold decomposables. Furthermore, $D_t H^*(X)$ are closed under the action of $\mathscr{A}_{(p)}$, and if x_i are primitive, so are I_s. Put

$$I_s^t = \bigoplus_{\substack{s_1 + \cdots + s_t = s \\ s_i \geq 1}} I_{s_1} \otimes \cdots \otimes I_{s_t} \subset \tilde{H}^*(X)^{\otimes t}.$$

Then

$$I_s^t \cap I_r^t = 0 \qquad \text{for } s \neq r$$

$$I_s^t = 0 \qquad \text{for } s < t$$

$$\tilde{H}^*(X)^{\otimes t} = \bigoplus_{s \geq t} I_s^t .$$

It is also clear that $\bigoplus_{i \geq s} I_i^t$ are $\mathscr{A}_{(p)}$ closed for any s, t, and I_s^t are $\mathscr{A}_{(p)}$ closed if x_i are primitive. By definition, $E(X)_1^{t,*} = \bigoplus_{i \geq t} I_i^t$ $(t \leq n)$, and if x_i are primitive, $d(X)_1(I_i^t) \subset I_i^{t+1}$. Put

$$M(t)_s = \rho_t^*(I_s^t)$$

$$M(t) = \sum_{s > t} M(t)_s .$$

Then

$$M(t)_i \cap M(t)_j = 0 \qquad \text{for } i \neq j$$

$$M(t)_s = 0 \qquad \text{for } s < t ,$$

and $\sum_{i \geq s} M(t)_i$ are $\mathscr{A}_{(p)}$ closed, and if x_i are primitive, so are $M(t)_s$. The following fact is immediate from the definition (cf. [4]).

LEMMA 3.1. *For the above spectral sequence, we have the following properties.*

(1) $I_i^{t-1} \xrightarrow{d(X)_1} I_i^t \xrightarrow{d(X)_1} I_i^{t+1}$ *is exact for* $t < n$ *and* $i \neq t$.

(2) $d(X)_1(I_t^t) = 0$ *for* $t \leq n$.

(3) $\rho_t^*\colon I_t^t/d(X)_1(I_t^{t-1}) \cong D_t H^*(P_t(X))$ *for* $t < n$.

(4) $\ker \beta_{t+1}^* \cap M(t) = 0$ *for* $t < n$.

(5) *If* x_i *are* A_n*-primitive,* (3) *also holds for* $t = n$. *Here* x_i *is called* A_n*-primitive if* $\rho_1^* x_i \in \varepsilon_2^* \dots \varepsilon_n^*(\tilde{H}^*(P_n(X)))$.

(6) *If* $n \geq 3$, *then*

$$E(X)_2^{*,*} \cong \cdots \cong E(X)_{n-1}^{*,*} \cong T_{n+1}[y_1, \dots, y_k] \oplus M(n) ,$$

where $y_i = [x_i] \in E(X)_1^{1,\deg x_i}$. *Furthermore, if* x_i *are* A_n*-primitive, in addition, then the spectral sequence collapses;*

$$E(X)_2^{*,*} \cong \cdots \cong E(X)_\infty^{*,*} .$$

From the above fact we have the following theorem (cf. [4]).

THEOREM 3.2. *For any* $1 \leq t \leq n-1$ *and* $1 \leq i \leq k$, *there are* $y(t)_i \in H^*(P_t(X))$ *so that the following facts hold.*

(1) $\varepsilon_t^*(y(t)_i) = y(t-1)_i$ $(2 \leq t \leq n-1)$.

(2) $\varepsilon_t^*(M(t)) = 0$ $(2 \leq t \leq n)$.

(3) $\tilde{H}^*(P_t(X))\cdot M(t) = 0$ $(1 \le t \le n)$.

(4) $D_{t+1}H^*(P_t(X)) = 0$ *for* $1 \le t \le n$. *Furthermore, if* $t \le n-1$, *then*

$$H^*(P_t(X)) \cong T_{t+1}[y(t)_1, \dots, y(t)_k] \oplus M(t)$$

as algebras.

(5) $M(t)$ *are closed under the action of* $\mathscr{A}_{(p)}$. *Thus, in particular, the quotient algebra* $H^*(P_t(X))/M(t) \cong T_{t+1}[y(t)_1, \dots, y(t)_k]$ *are unstable* $\mathscr{A}_{(p)}$ *algebras for* $t \le n-1$.

(6) $\rho_t^*(x_{i_1} \otimes \cdots \otimes x_{i_t}) = y(t)_{i_1} \cdots y(t)_{i_t}$ $(1 \le t \le n-1)$. *Thus, in particular* $\rho_1^*(x_i) = y(1)_i$.

(7) $\varepsilon_t^*(H^*(P_t(X))) \subset T_t[y(t-1)_1, \dots, y(t-1)_k]$ $(2 \le t \le n)$.

(8) *If* x_i *are* A_n*-primitive, then* (1), (4), (5) *and* (6) *also hold for* $t = n$.

4. Cohomology of ΩX

We continue to study the space X in section 3. First note that

$$H^*(\Omega X) \cong \Gamma[\sigma^*(x_1), \dots, \sigma^*(x_k)] \qquad \text{as coalgebras},$$

where the right hand side of the equation is the divided polynomial Hopf algebra over $\sigma^*(x_1)$, ..., $\sigma^*(x_k)$. (σ^* is the cohomology suspension.) This can be proved by using Eilenberg-Moore spectral sequence (cf. [7, Prop. 2.8]). In particular, the primitive module $PH^*(\Omega X)$ has $\{\sigma^*(x_1), \dots, \sigma^*(x_k)\}$ as a basis.

Now choose $\mathbf{Z}/p$-submodule J of $\tilde{H}^*(\Omega X)$ with

$$\tilde{H}^*(\Omega X) \cong J \oplus PH^*(\Omega X).$$

Put

$$J(t) = \sum_{j=1}^{t} \tilde{H}^*(\Omega X)^{\otimes j-1} \otimes J \otimes \tilde{H}^*(\Omega X)^{\otimes t-j}.$$

Note that $\tilde{H}^*(\Omega X)^{\otimes t} \cong J(t) \oplus PH^*(\Omega X)^{\otimes t}$. Since ΩX is an A_∞-space, we have the same spectral sequence as in section 3 for ΩX. Then we put

$$S(t) = \rho_t^*(J(t)) \subset \tilde{H}^*(P_t(\Omega X)) \qquad \text{and} \qquad x(t)_i = \varepsilon_{t,\infty}^*(x_i),$$

where $\varepsilon_{t,\infty}\colon P_t(\Omega X) \subset P_\infty(\Omega X) \simeq X$.

THEOREM 4.1. *Under the above notations, we have the following facts.*

(1) $\ker \varepsilon_{t,\infty}^* = D_{t+1}H^*(X)$.

(2) $x(1)_i$ *is* $\sigma^*(x_i)$ *by identifying* $\tilde{H}^*(\Sigma\Omega X)$ *with* $\tilde{H}^{*-1}(\Omega X)$.

(3) $\varepsilon_t^*(S(t)) = 0$.

(4) $\tilde{H}^*(P_t(\Omega X))\cdot S(t) = 0$.

(5) $H^*(P_t(\Omega X)) \cong T_{t+1}(\Lambda(x(t)_1, \dots, x(t)_k)) \oplus S(t)$ *as algebras, where* $T_m R$ *is the truncated algebra of height* m *of any algebra* R; $T_m R = R/D_m R$.

(6) $\rho_t^*(\sigma^*(x_{i_1}) \otimes \cdots \otimes \sigma^*(x_{i_t})) = x(t)_{i_1} \dots x(t)_{i_t}$ (*which is* 0 *if* $i_j = i_s$ *for some* $j \neq s$).

(7) $\ker \beta_{t+1}^* \cap S(t) = 0$.

(8) $\varepsilon_t^*(H^*(P_t(\Omega X))) = T_t(\Lambda(x(t-1)_1, \dots, x(t-1)_k))$

The proof of the above theorem is easy by the standard spectral sequence argument.

Put

$$P_s^t(\Omega X) = \bigcup_{\substack{s_1+\cdots+s_t=s \\ s_i \geq 1}} P_{s_1}(\Omega X) \wedge \cdots \wedge P_{s_t}(\Omega X).$$

Then

$$\begin{aligned} P_\infty^t(\Omega X) &= \bigcup_{s=1}^{\infty} P_s^t(\Omega X) = X^{\wedge t} \\ P_s^t(\Omega X) &= * \qquad \text{for } s < t \\ P_t^t(\Omega X) &= (\Sigma \Omega X)^{\wedge t} \\ P_s^t(\Omega X)/P_{s-1}^t(\Omega X) &= \bigvee_{\substack{s_1+\cdots+s_t=s \\ s_i \geq 1}} P_{s_1}(\Omega X)/P_{s_1-1}(\Omega X) \wedge \cdots \wedge P_{s_t}(\Omega X)/P_{s_t-1}(\Omega X) \\ &\simeq \bigvee_{\substack{s_1+\cdots+s_t=s \\ s_i \geq 1}} \Sigma^s(\Omega X)^{\wedge s}. \end{aligned}$$

Now we have an exact triangle of cohomology as follows;

$$\begin{tikzcd} \tilde{H}^*(P_{s-1}^t(\Omega X)) \arrow[dr, "\beta_s^{t*}"'] & & \tilde{H}^*(P_s^t(\Omega X)) \arrow[ll, "\varepsilon_s^{t*}"'] \\ & \displaystyle\bigoplus_{\substack{s_1+\cdots+s_t=s \\ s_i \geq 1}} \tilde{H}^*(\Omega X)^{\otimes s}, \arrow[ur, "\rho_s^{t*}"'] & \end{tikzcd}$$

where $\deg \varepsilon_s^{t*} = 0$, $\deg \beta_s^{t*} = 1 - s$, $\deg \rho_s^{t*} = s$.

Let $\{E^t(\Omega X)_r^{*,*}, d^t(\Omega X)_r\}$ be the spectral sequence associated to the filtration

$$P_0^t(\Omega X) \subset P_1^t(\Omega X) \subset \cdots \subset P_s^t(\Omega X) \subset \cdots.$$

Since the above filtration of $X^{\wedge t}$ is induced by the filtration $P_0(\Omega X) \subset P_1(\Omega X) \subset \cdots$, we have

$$E^t(\Omega X)_r^{*,*} \cong E(\Omega X)_r^{*,*} \otimes \cdots \otimes E(\Omega X)_r^{*,*} \qquad (t\text{-fold}).$$

Let

$$J(s)^t = \bigoplus_{\substack{s_1+\cdots+s_t=s \\ s_i \geq 1}} J(s)$$

$$S(s)^t = \rho_s^{t*}(J(s)^t)$$

$$\varepsilon_{s,\infty}^t \colon P_s^t(\Omega X) \subset P_\infty^t(\Omega X) = X^{\wedge t}.$$

The following theorem is clear by Theorem 4.1.

THEOREM 4.2. *Under the above notation, we have the following facts.*

(1) $\ker(\varepsilon_{s,\infty}^t)^* = \sum_{l_1+\cdots+l_t=s+1} D_{l_1}H^*(X) \otimes \cdots \otimes D_{l_t}H^*(X) = \sum_{i \geq s+1} I_i^t$

(2) $(\varepsilon_s^t)^*(S(s)^t) = 0.$

(3) $\tilde{H}^*(P_s^t(\Omega X)) \cdot S(s)^t = 0.$

(4) $H^*(P_s^t(\Omega X)) \cong A(s)^t \oplus S(s)^t$ *as algebras, where*

$$A(s)^t = (\varepsilon_{s,\infty}^t)^*(H^*(X^{\wedge t})).$$

(5) *For* $z \in \tilde{H}^*(P_{s_1}(\Omega X)/P_{s_1-1}(\Omega X) \wedge \cdots \wedge P_{s_t}(\Omega X)/P_{s_t-1}(\Omega X))$ *we have*

$$(\rho_s^t)^*(z) = (\varepsilon_{s,\infty}^t)^*(\bar{x}_1 \otimes \cdots \otimes \bar{x}_t),$$

where z, *which is identified with a class in* $\tilde{H}^*(\Omega X)^{\otimes s}$, *is denoted by* $\sigma^*(x_{i_1}) \otimes \cdots \otimes \sigma^*(x_{i_s})$, *and* $\bar{x}_j = x_{i_{s_{j-1}+1}} \cdots x_{i_{s_j}} \in \tilde{H}^*(X)$.

(6) $\ker(\beta_{s+1}^t)^* \cap S(s)^t = 0.$

(7) $(\varepsilon_s^t)^*(H^*(P_s^t(\Omega X))) = A(s-1)^t.$

5. Cohomology of $R_n(X)$

In this section we prove (1) $\sim$ (4) of Theorem 1.1. First we study the homomorphism $f_{n-1}^*\colon H^*(P_{n-1}(X)) \to H^*(R_{n-1}(X))$ given in section 2.

LEMMA 5.1. $\ker f_{n-1}^* = \bigoplus_{s \geq n+1} M(n-1)_s$.

PROOF. Since $R_{n-1}(X)/P_{n-2}(X) = \Sigma C_{n-1} \simeq \Sigma^n P_n^{n-1}(\Omega X)$, we have the following commutative diagram;

$$\begin{array}{ccc}
\tilde{H}^*(P_{n-2}(X)) & = & \tilde{H}^*(P_{n-2}(X)) \\
\uparrow{\scriptstyle e_{n-1}^*} & & \uparrow{\scriptstyle \varepsilon_{n-1}^*} \\
\tilde{H}^*(R_{n-1}(X)) & \xleftarrow{f_{n-1}^*} & \tilde{H}^*(P_{n-1}(X)) \\
\uparrow{\scriptstyle \gamma_{n-1}^*} & & \uparrow{\scriptstyle \rho_{n-1}^*} \\
\tilde{H}^*(P_n^{n-1}(\Omega X)) & \xleftarrow[(\varepsilon_{n,\infty}^{n-1})^*]{} & \tilde{H}^*(X)^{\otimes n-1} \\
\uparrow{\scriptstyle b_{n-1}^*} & & \uparrow{\scriptstyle \beta_{n-1}^*} \\
\tilde{H}^*(P_{n-2}(X)) & = & \tilde{H}^*(P_{n-2}(X)).
\end{array}$$

Let $u \in \tilde{H}^*(P_{n-1}(X)) \cong T_n[y(n-1)_1, \dots, y(n-1)_k] \oplus M(n-1)$ with $f_{n-1}^*(u) = 0$. Then by the usual diagram chasing method (or the Mayer-Vietoris type argument), there is $v \in \tilde{H}^*(X)^{\otimes n-1}$ with

$$\rho_{n-1}^*(v) = u \qquad \text{and} \qquad (\varepsilon_{n,\infty}^{n-1})^*(v) = 0\,.$$

Since $\ker(\varepsilon_{n,\infty}^{n-1})^* = \bigoplus_{s \geq n+1} I_s^{n-1}$ by Theorem 4.2 (1),

$$u \in \rho_{n-1}^*\left(\bigoplus_{s \geq n+1} I_s^{n-1}\right) = \bigoplus_{s \geq n+1} M(n-1)_s\,.$$

It is also clear that $f_{n-1}^*(M(n-1)_s) = 0$ for $s \geq n+1$. q.e.d.

Let

$$z_i = f_{n-1}^*(y(n-1)_i) \in H^*(R_{n-1}(X))\,.$$

PROPOSITION 5.2

$$e_n^*(H^*(R_n(X))) \cap f_{n-1}^*(H^*(P_{n-1}(X))) = T_n[z_1, \dots, z_k]\,,$$

where $e_n\colon R_{n-1}(X) \subset R_n(X)$. *Thus, in particular, there is* $y_i \in H^*(R_n(X))$ *so that*

$$\varepsilon^*(y_i) = x_i\,,$$

where $\varepsilon\colon \Sigma X \subset R_n(X)$.

PROOF. Since $\tilde{H}^*(\Omega X)$ is concentrated in even dimensional, $z_i \in \ker b_n^* = e_n^*(\tilde{H}^*(R_n(X)))$ for dimensional reason. Note that $z_{i_1} \cdots z_{i_t} = f_{n-1}^*(y(n-1)_{i_1} \cdots y(n-1)_{i_t}) = 0$ if and only if $t \geq n$ by Lemma 5.1. Thus

$$T_n[z_1, \dots, z_k] \subset e_n^*(H^*(R_n(X))) \cap f_{n-1}^*(H^*(P_{n-1}(X)))\,.$$

Next choose any

$$u \in e_n^*(\tilde{H}^*(R_n(X))) \cap f_{n-1}^*(\tilde{H}^*(P_{n-1}(X))) = \ker b_n^* \cap f_{n-1}^*(\tilde{H}^*(P_{n-1}(X)))\,.$$

Then u can be written as

$$u = u_0 + f_{n-1}^*(u_1)\,, \qquad \text{where } u_0 \in T_n[z_1, \dots, z_k],\ u_1 \in M(n-1)_n\,.$$

Since $b_n^* u_0 = 0$, we have $b_n^* \circ f_{n-1}^*(u_1) = 0$. Consider the following commutative diagram

$$\begin{tikzcd}
\tilde{H}^*(R_{n-1}(X)) \arrow[r, "b_n^*"] & \tilde{H}^*(\Sigma\Omega X)^{\otimes n} \\
\tilde{H}^*(P_{n-1}(X)) \arrow[u, "f_{n-1}^*"] \arrow[r, "\beta_n^*"'] & \tilde{H}^*(X)^{\otimes n}\,. \arrow[u, "(\varepsilon_{n,\infty}^n)^*"']
\end{tikzcd}$$

Since β_n^* is mono on $M(n-1)$ by Lemma 3.1 (4), and $(\varepsilon_{n,\infty}^n)^*$ is mono on $\beta_n^*(M(n-1)_n) \subset I_n^n$ by Theorem 4.2 (1), $b_n^* \circ f_{n-1}^*$ is mono on $M(n-1)_n$. Thus $u_1 = 0$, and $u \in T_n[z_1, \dots, z_k]$. The existence of y_i is clear. q.e.d.

Put

$$P^2H^*(\Omega X) = PH^*(\Omega X) \cdot PH^*(\Omega X).$$

Then $P^2H^*(\Omega X)$ is the $\mathbf{Z}/p$-submodule of $H^*(\Omega X)$ spanned by $\{\sigma^*(x_i) \cdot \sigma^*(x_j)\}$. Since p is an odd prime, the following fact is clear by definition.

LEMMA 5.3. *$PH^*(\Omega X) \cap P^2H^*(\Omega X) = 0$, and $PH^*(\Omega X)$ and $P^2H^*(\Omega X)$ are closed under the action of $\mathscr{A}_{(p)}$.*

Choose a submodule L of $H^*(\Omega X)$ with

$$\tilde{H}^*(\Omega X) \cong PH^*(\Omega X) \oplus P^2H^*(\Omega X) \oplus L.$$

Let

$$N(t) = \sum_{i=1}^{t} PH^*(\Omega X)^{\otimes i-1} \otimes P^2H^*(\Omega X) \otimes PH^*(\Omega X)^{\otimes t-i}$$

$$\begin{aligned} T(t) = & \sum_{i=1}^{t} \tilde{H}^*(\Omega X)^{\otimes i-1} \otimes L \otimes \tilde{H}^*(\Omega X)^{\otimes t-i} \\ & + \sum_{\substack{i,j \geq 1 \\ i+j \leq t}} \tilde{H}^*(\Omega X)^{\otimes i-1} \otimes P^2H^*(\Omega X) \otimes \tilde{H}^*(\Omega X)^{\otimes j-1} \\ & \qquad \otimes P^2H^*(\Omega X) \otimes \tilde{H}^*(\Omega X)^{\otimes t-i-j}. \end{aligned}$$

Then

$$\tilde{H}^*(\Omega X)^{\otimes t} \cong PH^*(\Omega X)^{\otimes t} \oplus N(t) \oplus T(t),$$

and $PH^*(\Omega X)^{\otimes t}$ and $N(t)$ are closed under the action of $\mathscr{A}_{(p)}$.

Now since $R_n(X)/R_{n-1}(X) = \Sigma C_n \simeq \Sigma^{2n-1}(\Omega X)^{\wedge n}$, we have an exact triangle

$$\begin{tikzcd} \tilde{H}^*(R_{n-1}(X)) \arrow[dr, "b_n"'] & & \tilde{H}^*(R_n(X)) \arrow[ll, "e_n^*"'] \\ & \tilde{H}^*(\Omega X)^{\otimes n}. \arrow[ur, "r_n^*"'] & \end{tikzcd}$$

Put

$$M = r_n^*(N(n)).$$

Then we have the following fact.

PROPOSITION 5.4. *M is closed under the action of $\mathscr{A}_{(p)}$, and*

$$\varepsilon^*(M) = 0\,, \qquad M\cdot\tilde{H}^*(R_n(X)) = 0\,.$$

Moreover, for any $y_i \in \tilde{H}^*(R_n(X))$ with $\varepsilon^*(y_i) = x_i$, $T_{n+1}[y_1, \dots, y_k] \oplus M$ is a subalgebra of $H^*(R_n(X))$, and

$$y_{i_1}\cdots y_{i_n} = r_n^*(\sigma^*(x_{i_1}) \otimes \cdots \otimes \sigma^*(x_{i_n}))\,.$$

PROOF. Since $N(n)$ is $\mathscr{A}_{(p)}$ closed, so is M. Clearly we have $\varepsilon^*(M) = 0$ and $M\cdot\tilde{H}^*(R_n(X)) = 0$. The other properties are proved by the standard method and Proposition 5.2. q.e.d.

PROOF OF THEOREM 1.1 (1)–(4). (1) is proved in Proposition 5.2, and (2)–(4) are in Proposition 5.4. q.e.d.

We have shown Theorem 1.1 except for (5) and (6). (6) is a consequence of (5). Thus we need to prove (5). Furthermore, since M is $\mathscr{A}_{(p)}$ closed by Proposition 5.4, we prove that $A^* = T_{n+1}[y_1, \dots, y_k] \oplus M$ is $\mathscr{A}_{(p)}$ closed for some choice of $\{y_i\}$ hereafter.

6. General constructions

In this section we give more general constructions than in the section 2. Recall the filtration

$$\Sigma\Omega X = P_1(\Omega X) \subset P_2(\Omega X) \subset \cdots \subset X\,.$$

Put

$$F_s(X^{*t}) = \bigcup_{\substack{s_1+\cdots+s_t=s\\ s_i\geq 1}} P_{s_1}(\Omega X) * \cdots * P_{s_t}(\Omega X)\,.$$

Then

$$* = \cdots = F_{t-1}(X^{*t}) \subset F_t(X^{*t}) = (\Sigma\Omega X)^{*t} \subset F_{t+1}(X^{*t}) \subset \cdots$$
$$F_\infty(X^{*t}) = \textstyle\bigcup_{s=1}^\infty F_s(X^{*t}) = X^{*t}\,.$$

By definition

$$F_s(X^{*t})/F_{s-1}(X^{*t}) = \bigvee_{\substack{s_1+\cdots+s_t=s\\ s_i\geq 1}} V_{s_1,\dots,s_t}\,,$$

where

$$\begin{aligned} V_{s_1,\dots,s_t} &\simeq P_{s_1}(\Omega X)/P_{s_1-1}(\Omega X) * \cdots * P_{s_t}(\Omega X)/P_{s_1-1}(\Omega X)\\ &\simeq \Sigma^{t-1}(\Sigma\Omega X)^{\wedge s}\,. \end{aligned}$$

Note that C_{n-1} and C_n of section 2 are equal to $F_n(X^{*n-1})$ and $F_n(X^{*n})$, respectively.

Let $F_t(\beta_{n-1})\colon F_t(X^{*n-1}) \to P_{n-2}(X)$ be the restriction of β_{n-1}. Define $F_t(P_{n-1}(X))$ as the mapping cone of $F_t(\beta_{n-1})$;

$$F_t(P_{n-1}(X)) = P_{n-2}(X) \cup_{F_t(\beta_{n-1})} C(F_t(X^{*n-1})) .$$

Then we have

$$R_{n-1}(X) = F_n(P_{n-1}(X)) \subset F_{n+1}(P_{n-1}(X)) \subset \cdots$$

$$F_\infty(P_{n-1}(X)) = \textstyle\bigcup_{s=1}^{\infty} F_s(P_{n-1}(X)) = P_{n-1}(X) .$$

Furthermore,

$$\begin{aligned} &F_s(P_{n-1}(X))/F_{s-1}(P_{n-1}(X)) \\ &\quad \simeq \Sigma(F_s(X^{*n-1})/F_{s-1}(X^{*n-1})) \\ &\quad \simeq \bigvee_{\substack{s_1+\cdots+s_{n-1}=s \\ s_i\geq 1}} \Sigma V_{s_1,\dots,s_{n-1}} . \end{aligned}$$

The following fact is a generalization of Lemma 2.1.

PROPOSITION 6.1. *There are maps $F_t(\beta_n)\colon F_t(X^{*n}) \to F_t(P_{n-1}(X))$ with $F_\infty(\beta_n) = \beta_n$ so that the following diagram is homotopy commutative;*

$$\begin{array}{ccc} F_{t-1}(X^{*n}) & \subset & F_t(X^{*n}) \\ \Big\downarrow{\scriptstyle F_{t-1}(\beta_n)} & & \Big\downarrow{\scriptstyle F_t(\beta_n)} \\ F_{t-1}(P_{n-1}(X)) & \subset & F_t(P_{n-1}(X)) . \end{array}$$

Furthermore, if $\lambda_t\colon F_t(X^{*n})/F_{t-1}(X^{*n}) \to F_t(P_{n-1}(X))/F_{t-1}(P_{n-1}(X))$ is the induced map, then by the isomorphisms

$$\tilde{H}^*(F_t(X^{*n})/F_{t-1}(X^{*n})) \cong \bigoplus_{\substack{s_1+\cdots+s_n=t \\ s_i\geq 1}} \tilde{H}^*(\Omega X)^{\otimes t}$$

$$\tilde{H}^*(F_t(P_{n-1}(X))/F_{t-1}(P_{n-1}(X))) \cong \bigoplus_{\substack{s_1+\cdots+s_{n-1}=t \\ s_i\geq 1}} \tilde{H}^*(\Omega X)^{\otimes t} ,$$

we have that

$$\lambda_t^*\left(\bigoplus_{\substack{s_1+\cdots+s_n=t \\ s_i\geq 1}} PH^*(\Omega X)^{\otimes t}\right) \subset \bigoplus_{\substack{s_1+\cdots+s_{n-1}=t \\ s_i\geq 1}} PH^*(\Omega X)^{\otimes t} .$$

PROOF. The existence of maps $F_t(\beta_n)$ follows from the same reason as in Lemma 2.1. In fact, there are maps

$$m_{s,t}\colon P_s(\Omega X)\times P_t(\Omega X)\to P_{s+t}(\Omega X)$$

for any s and t, which are restrictions of the multiplication of X (see Stasheff [11, p. 72]).

Now the map $m_{s,t}$ induces a map

$$\begin{aligned}\bar{m}_{s,t}\colon & P_s(\Omega X)/P_{s-1}(\Omega X)\wedge P_t(\Omega X)/P_{t-1}(\Omega X)\\ &\simeq P_s(\Omega X)\times P_t(\Omega X)/(P_s(\Omega X)\times P_{t-1}(\Omega X)\cup P_{s-1}(\Omega X)\times P_t(\Omega X))\\ &\to P_{s+t}(\Omega X)/P_{s+t-1}(\Omega X).\end{aligned}$$

Since $P_k(\Omega X)/P_{k-1}(\Omega X)\simeq(\Sigma\Omega X)^{\wedge k}$, $\bar{m}_{s,t}$ is considered as a map $(\Sigma\Omega X)^{\wedge s}\wedge(\Sigma\Omega X)^{\wedge t}\to(\Sigma\Omega X)^{\wedge s+t}$. We can describe $\bar{m}_{s,t}$ by using permutations. In fact, let $\mathscr{S}(s,t)$ be the set of all (s,t)-shuffles, i.e., $\mathscr{S}(s,t)$ is a subset of $(s+t)$-th symmetric group $\mathscr{S}_{s+t}$ so that $\sigma\in\mathscr{S}(s,t)$ if and only if $\sigma(i)<\sigma(i+1)$ for $i\neq s$. For any $\sigma\in\mathscr{S}(s,t)$ we define

$$\sigma^*\colon(\Sigma\Omega X)^{\wedge s}\wedge(\Sigma\Omega X)^{\wedge t}\to(\Sigma\Omega X)^{\wedge s+t}$$

by

$$\begin{aligned}&\sigma^*((a_1,u_1,\dots,a_s,u_s),(a_{s+1},u_{s+1},\dots,a_{s+t},u_{s+t}))\\ &\quad=(a_{\sigma^{-1}(1)},u_{\sigma^{-1}(1)},\dots,a_{\sigma^{-1}(s+t)},u_{\sigma^{-1}(s+t)}),\end{aligned}$$

where $(a_i,u_i)\in S^1\wedge\Omega X=\Sigma\Omega X$. Then by definition of $m_{s,t}$

$$\bar{m}_{s,t}\simeq\sum_{\sigma\in\mathscr{S}(s,t)}\sigma^*\qquad(\text{see [11, pp. 71–72]}).$$

Now the composition $X^{*n}\to P_{n-1}(X)\to P_{n-1}(X)/P_{n-2}(X)\simeq\Sigma X^{*n-1}$ induces a homomorphism on cohomology $\tilde{H}^*(X)^{\otimes n-1}\to\tilde{H}^*(X)^{\otimes n}$ given by

$$\sum_{j=1}^{n-1}(-1)^{j-1}id^{\otimes j-1}\otimes\tilde{m}^*\otimes id^{\otimes n-1-j}.$$

In fact, this map is equivalent to the derivation $d(X)_1$ of the spectral sequence in section 3. The composition

$$F_t(X^{*n})\to F_t(P_{n-1}(X))\to F_t(P_{n-1}(X))/P_{n-2}(X)\simeq\Sigma F_t(X^{*n-1})$$

is the restriction of the above map $X^{*n}\to\Sigma X^{*n-1}$. Thus λ_t^* is described by using $\bar{m}_{s,t}$, and so it is given by appropriate shuffles. More precisely, $\lambda_t^*(\tilde{H}^*(V_{s_1,\dots,s_{n-1}}))$ is included in $\bigoplus\tilde{H}^*(V_{t_1,\dots,t_n})$ where $(t_1,\dots,t_n)$ runs all the sequences with $(t_1,\dots,t_{i-1},t_i+t_{i+1},t_{i+2},\dots,t_n)=(s_1,\dots,s_{n-1})$ for some $1\le i\le n-1$. Furthermore, if $(t_1,\dots,t_{i-1},t_i+t_{i+1},t_{i+2},\dots,t_n)=(s_1,\dots,s_{n-1})$, then for any $z\in\tilde{H}^*(V_{s_1,\dots,s_{n-1}})$, the component of $\lambda_t^*(z)$ in $\tilde{H}^*(V_{t_1,\dots,t_n})$ is given by

$$(-1)^{i-1}(id^{\otimes i-1} \otimes \bar{m}_{t_i,t_{i-1}} \otimes id^{\otimes n-1-i})(z).$$

Thus we have the result. q.e.d.

Consider the cofiber sequence

$$F_t(X^{*t}) \to F_{t+1}(X^{*t}) \to F_{t+1}(X^{*t})/F_t(X^{*t}) \xrightarrow{\mu} \Sigma F_t(X^{*t}).$$

Now

$$\Sigma F_t(X^{*t}) \simeq \Sigma((\Sigma\Omega X)^{*t}) \simeq \Sigma^{2t}(\Omega X)^{\wedge t}$$

$$F_{t+1}(X^{*t})/F_t(X^{*t}) = W_1 \vee \cdots \vee W_t,$$

where

$$W_i \simeq (\Sigma\Omega X)^{*i-1} * (P_2(\Omega X)/\Sigma\Omega X) * (\Sigma\Omega X)^{*t-i}$$
$$\simeq \Sigma^{2t}(\Omega X)^{\wedge t+1}.$$

(Note that $W_i = V_{1,\dots,1,2,1,\dots,1}$ where 2 is in the ith place.) Furthermore the restriction of μ on W_i is essentially the same as

$$id^{*i-1} * \Sigma\beta_2 * id^{*t-i},$$

where $\beta_2 \colon \Omega X * \Omega X \to \Sigma\Omega X$ is the map in section 2 with $P_2(\Omega X) = (\Sigma\Omega X) \cup_{\beta_2} \Sigma(\Omega X * \Omega X)$. Thus $(w_1, \dots, w_t) = \mu^*(u_1 \otimes \cdots \otimes u_t)$ $(w_i \in \tilde{H}^*(W_i) \cong \tilde{H}^*(\Omega X)^{\otimes t+1})$, for any $u_1 \otimes \cdots \otimes u_t \in \tilde{H}^*(\Omega X)^{\otimes t}$, is given by

$$w_i = u_1 \otimes \cdots \otimes u_{i-1} \otimes \tilde{m}^*(u_i) \otimes u_{i+1} \otimes \cdots \otimes u_t,$$

where $m \colon \Omega X \times \Omega X \to \Omega X$ is the loop multiplication. Then we have the following fact.

Lemma 6.2

$$(\mu^*)^{-1}(\textstyle\bigoplus_{i=1}^t PH^*(\Omega X)^{\otimes t+1})$$
$$= PH^*(\Omega X)^{\otimes t} \oplus \textstyle\sum_{i=1}^t PH^*(\Omega X)^{\otimes i-1} \otimes P^2H^*(\Omega X) \otimes PH^*(\Omega X)^{\otimes t-i}.$$

Proof. Since p is an odd prime, it is clear that

$$(\tilde{m}^*)^{-1}(PH^*(\Omega X) \otimes PH^*(\Omega X)) = PH^*(\Omega X) \oplus P^2H^*(\Omega X).$$

Thus the result follows. q.e.d.

7. Action of the Steenrod operations

In this section we prove (5) and (6) of Theorem 1.1. First we prove a technical lemma.

Consider the following homotopy commutative diagram, where e, e_0 and e_1 are natural inclusions to the mapping cones of f, φ_0 and φ_1, respectively, ρ and ρ_0 are natural projections, and h is the induced map by fixing a homotopy between $g \circ \varphi_0$ and $\varphi_1 \circ f$;

$$\begin{tikzcd}
X_0 \arrow[r, "\varphi_0"] \arrow[d, "f"'] & Y_0 \arrow[r, "e_0"] \arrow[d, "g"] & Y_0 \cup_{\varphi_0} CX_0 \arrow[r, "\rho_0"] \arrow[d, "h"] & \Sigma X_0 \\
X_1 \arrow[r, "\varphi_1"'] \arrow[d, "e"'] & Y_1 \arrow[r, "e_1"'] & Y_1 \cup_{\varphi_1} CX_1 & \\
X_1 \cup_f CX_0 \arrow[d, "\rho"'] & & & \\
\Sigma X_0 . & & &
\end{tikzcd}$$

LEMMA 7.1. *Let* $\alpha: Y_1 \to K_0$ *and* $\xi: Y_0 \cup_{\varphi_0} CX_0 \to K_0$ *be any maps with* $\xi \circ e_0 \simeq \alpha \circ g$. *Then there is a map* $\psi: X_1 \cup_f CX_0 \to K_0$ *with* $\psi \circ e \simeq \alpha \circ \varphi_1$ *so that for any* $\theta: K_0 \to K_1$ *and* $\beta: Y_1 \cup_{\varphi_1} CX_1 \to K_1$ *with* $\beta \circ e_1 \simeq \theta \circ \alpha$, *there is a map* $\lambda: \Sigma X_0 \to K_1$ *with*

$$\lambda \circ \rho \simeq \theta \circ \psi \qquad \text{and} \qquad \theta \circ \xi \simeq (\beta \circ h) * \lambda .$$

Here $(\beta \circ h) * \lambda$ *is defined by the composition*

$$Y_0 \cup_{\varphi_0} CX_0 \to (Y_0 \cup_{\varphi_0} CX_0) \vee \Sigma X_0 \xrightarrow{\beta \circ h \vee \lambda} K_1 \vee K_1 \to K_1 ,$$

where the left arrow is the natural coaction of ΣX_0 *on* $Y_0 \cup_{\varphi_0} CX_0$, *and the right one is the folding map.*

PROOF. First we note that we can assume that $\xi \circ e_0 = \alpha \circ g$ by changing ξ to a suitable homotopic map if necessary. Now the map $h: Y_0 \cup_{\varphi_0} CX_0 \to Y_1 \cup_{\varphi_1} CX_1$ is given by a homotopy H between $g \circ \varphi_0$ and $\varphi_1 \circ f$ as follows;

$$h(y) = g(y) \qquad y \in Y_0$$

$$h(t, x) = \begin{cases} H(2t, x) & 0 \le t \le 1/2, x \in X_0 \\ (2t-1, f(x)) & 1/2 \le t \le 1, x \in X_0 . \end{cases}$$

Define $\psi: X_1 \cup_f CX_0 \to K_0$ by

$$\psi(x_1) = \alpha \circ \varphi_1(x_1) \qquad x_1 \in X_1$$

$$\psi(t, x_0) = \begin{cases} \alpha \circ H(1-2t, x_0) & 0 \le t \le 1/2, x_0 \in X_0 \\ \xi(2t-1, x_0) & 1/2 \le t \le 1, x_0 \in X_0 . \end{cases}$$

It is clear that

$$\psi \circ e = \alpha \circ \varphi_1 .$$

Now this ψ satisfies the required condition. In fact, for any θ and β, we define $\lambda\colon \Sigma X_0 \to K_1$ by

$$\lambda(t, x) = \begin{cases} \beta(1-3t, f(x)) & 0 \le t \le 1/3 \\ \theta \circ \alpha \circ H(2-3t, x) & 1/3 \le t \le 2/3 \\ \theta \circ \xi(3t-2, x) & 2/3 \le t \le 1 , \end{cases}$$

where we assume $\beta \circ e_1 = \theta \circ \alpha$ by changing β to a suitable homotopic map if necessary. Then $\lambda \circ \rho\colon X_1 \cup_f CX_0 \to K_1$ is homotopic to the restriction of a map $\lambda'\colon CX_1 \cup_f CX_0 \to K_1$ defined by

$$\lambda'(t, x_1) = \beta(t, x_1) \qquad (x_1 \in X_1)$$

$$\lambda'(t, x_0) = \begin{cases} \theta \circ \alpha \circ H(1-2t, x_0) & 0 \le t \le 1/2, x_0 \in X_0 \\ \theta \circ \xi(2t-1, x_0) & 1/2 \le t \le 1, x_0 \in X_0 . \end{cases}$$

Thus $\lambda \circ \rho \simeq \theta \circ \psi$. One can also prove $\theta \circ \xi \simeq (\beta \circ h) * \lambda$ easily. q.e.d.

Put

$$Q_{n-1}(X) = F_{n+1}(P_{n-1}(X)) = P_{n-2}(X) \cup C(F_{n+1}(X^{*n-1})) .$$

Let

$$g\colon R_{n-1}(X) \to Q_{n-1}(X) \qquad \text{and} \qquad h\colon Q_{n-1}(X) \to P_{n-1}(X)$$

be inclusions with $h \circ g = f_{n-1}$. Define

$$\alpha\colon P_{n-1}(X) \to K_0 = \prod_{i=1}^{k} K(\mathbf{Z}/p, \deg y(n-1)_i)$$

by $\alpha(w_i) = y(n-1)_i$ where $w_i \in H^*(K_0)$ correspond to the fundamental classes in $H^*(K(\mathbf{Z}/p, \deg y(n-1)_i))$. Then we have the following homotopy commutative diagram;

$$\begin{tikzcd}
F_n(X^{*n}) \arrow[r] \arrow[d, "b_n"'] & F_{n+1}(X^{*n}) \arrow[r, "\pi'"] \arrow[d, "\gamma_n"'] & F_{n+1}(X^{*n})/F_n(X^{*n}) \arrow[r, "\mu"] \arrow[d, "\lambda_{n+1}"] & \Sigma F_n(X^{*n}) \\
R_{n-1}(X) \arrow[r, "g"'] \arrow[d, "e_n"'] & Q_{n-1}(X) \arrow[r, "\pi"'] \arrow[d, "h"'] & Q_{n-1}(X)/R_{n-1}(X) & \\
R_n(X) \arrow[d, "r_n"'] & P_{n-1}(X) \arrow[d, "\alpha"'] & & \\
\Sigma F_n(X^{*n}) & K_0 , & &
\end{tikzcd}$$

where the left vertical and the upper two horizontal sequences are cofiber sequences. We note that the cohomology homomorphism induced by inclusions $F_n(X^{*n}) \subset F_{n+1}(X^{*n}) \subset X^{*n}$ is equivalent to the ones by $\varepsilon^n_{n+1}: (\Sigma\Omega X)^{*n} = P^n_n(\Omega X) \to P^n_{n+1}(\Omega X)$ and $\varepsilon^n_{n+1,\infty}: P^n_{n+1}(\Omega X) \to X^{\wedge n}$.

LEMMA 7.2. *There is a map* $\xi: F_{n+1}(X^{*n})/F_n(X^{*n}) \to K_0$ *so that* $\xi \circ \pi' \simeq \alpha \circ h \circ \gamma_n$ *and*

$$\xi^*(\tilde{H}^*(K_0)) \subset \bigoplus PH^*(\Omega X)^{\otimes n+1},$$

where $\tilde{H}^*(F_{n+1}(X^{*n})/F_n(X^{*n}))$ *is identified with* $\bigoplus \tilde{H}^*(\Omega X)^{\otimes n+1}$ *as described in section 6. Moreover, there is a map* $\psi: R_n(X) \to K_0$ *with* $\psi \circ e_n \simeq \alpha \circ h \circ g$ *so that, for any maps* $\theta: K_0 \to K_1$ *and* $\beta: Q_{n-1}(X)/R_{n-1}(X) \to K_1$ *with* $\beta \circ \pi \simeq \theta \circ \alpha \circ h$, *there is* $\lambda: \Sigma F_n(X^{*n}) \to K_1$ *with*

$$\lambda \circ r_n \simeq \theta \circ \psi \qquad \text{and} \qquad \theta \circ \xi \simeq (\beta \circ \lambda_{n+1}) * \lambda .$$

Furthermore $\psi^*(H^*(K_0))$ *is an* $\mathscr{A}_{(p)}$ *subalgebra of* $H^*(R_n(X))$ *generated by some* y_i *with* $e^*_n(y_i) = z_i$.

PROOF. Since $\deg w_i$ is even, $(\varepsilon^n_{n+1})^* \circ \gamma^*_n \circ h^* \circ \alpha^*(w_i) = 0 \in \tilde{H}^*(\Omega X)^{\otimes n}$ for dimensional reason. On the other hand, $h \circ \gamma_n$ is a restriction of $\beta_n: X^{*n} \to P_{n-1}(X)$. Thus, $\gamma_n \circ h^* \circ \alpha^*(w_i) \in (\varepsilon^n_{n+1,\infty})^*(\tilde{H}^*(X)^{\otimes n})$, and so by Theorem 4.2 (1) (5),

$$\begin{aligned} \gamma^*_n \circ h^* \circ \alpha^*(w_i) &\in \ker (\varepsilon^n_{n+1})^* \cap (\varepsilon^n_{n+1,\infty})^*(\tilde{H}^*(X)^{\otimes n}) \\ &= (\varepsilon^n_{n+1,\infty})^* \left(\sum_{i=1}^{n} PH^*(X)^{\otimes i-1} \otimes P^2H^*(X) \otimes PH^*(X)^{\otimes n-i} \right) \\ &\subset (\rho^n_{n+1})^*(\bigoplus PH^*(\Omega X)^{\otimes n+1}), \end{aligned}$$

where the map $\pi': \tilde{H}^*(F_{n+1}(X^{*n})/F_n(X^{*n})) \to \tilde{H}^*(F_{n+1}(X^{*n}))$ is identified with $(\rho^n_{n+1})^*: \bigoplus \tilde{H}^*(\Omega X)^{\otimes n+1} \to \tilde{H}^*(P^n_{n+1}(\Omega X))$. Thus there is a map

$$\xi: F_{n+1}(X^{*n})/F_n(X^{*n}) \to K_0$$

so that

$$\xi^*(w_i) \in \bigoplus PH^*(\Omega X)^{\otimes n+1} \qquad \text{and} \qquad \xi \circ \pi' \simeq \alpha \circ h \circ \gamma_n .$$

Furthermore $DH^*(F_{n+1}(X^{*n})/F_n(X^{*n})) = 0$ since $F_{n+1}(X^{*n})/F_n(X^{*n})$ is a suspension. Thus

$$\xi^*(w) \in \bigoplus PH^*(\Omega X)^{\otimes n+1} \qquad \text{for all } w \in \tilde{H}^*(K_0) .$$

This proves the first part. For the second one, we can use Lemma 7.1. The last one is clear since $e^*_n \circ \psi^*(w_i) = g^* \circ h^* \circ \alpha^*(w_i) = z_i$. q.e.d.

Now we prove Theorem 1.1 (5), (6).

Proof of Theorem 1.1 (5), (6). By Proposition 5.4, M is closed under the action of $\mathscr{A}_{(p)}$. Thus we prove that $\tau(u) \in T_{n+1}[y_1, \ldots, y_k] \oplus M$ for any $u \in T_{n+1}[y_1, \ldots, y_k]$ and $\tau \in \mathscr{A}_{(p)}$.

Take any $u \in T_{n+1}[y_1, \ldots, y_k]$ and $\tau \in \mathscr{A}_{(p)}$. Now $e_n^*(u) \in T_n[z_1, \ldots, z_k] \subset f_{n-1}^*(H^*(P_{n-1}(X)))$, and so by using Proposition 5.2 we have

$$e_n^*(\tau u) \in e_n^*(H^*(R_n(X))) \cap f_{n-1}^*(H^*(P_{n-1}(X))) = T_n[z_1, \ldots, z_k].$$

Thus there is $v \in T_{n+1}[y_1, \ldots, y_k]$ so that

$$e_n^*(\tau u - v) = 0.$$

Let $\theta: K_0 \to K_1 = K(\mathbf{Z}/p, \deg(\tau u - v))$ be a map so that $\theta \circ \psi$ represents $\tau u - v$. Then

$$\theta \circ \psi \circ e_n \simeq *. \tag{1}$$

Consider the following homotopy commutative diagram, where horizontal sequences are cofiber sequences;

$$\begin{tikzcd}
P_{n-2}(X) \arrow[r, "e_{n-1}"] \arrow[d, equal] & R_{n-1}(X) \arrow[r, "r"] \arrow[d, "g"] & \Sigma^{n-1}P_n^{n-1}(\Omega X) \arrow[r, "\Sigma b_{n-1}"] \arrow[d, "\Sigma^{n-1}\varepsilon_{n+1}^{n-1}"] & \Sigma P_{n-2}(X) \arrow[d, equal] \\
P_{n-2}(X) \arrow[r] \arrow[d, equal] & Q_{n-1}(X) \arrow[r, "\rho'"] \arrow[d, "h"] & \Sigma^{n-1}P_{n+1}^{n-1}(\Omega X) \arrow[r, "\Sigma^{n-1}\varepsilon_{n+1}^{n-1}"] \arrow[d, "\Sigma^{n-1}\varepsilon_{n+1,\infty}^{n-1}"] & \Sigma P_{n-2}(X) \arrow[d, equal] \\
P_{n-2}(X) \arrow[r, "\varepsilon_{n-1}"'] & P_{n-1}(X) \arrow[r, "\rho_{n-1}"'] \arrow[d, "\alpha"] & \Sigma^{n-1}X^{\wedge n-1} \arrow[r, "\Sigma\beta_{n-1}"'] & \Sigma P_{n-2}(X) \\
& K_0 \arrow[r, "\theta"'] & K_1 .
\end{tikzcd}$$

Now by Lemma 7.2, $\alpha \circ h \circ g \simeq \psi \circ e_n$. Thus $\theta \circ \alpha \circ \varepsilon_{n-1} \simeq \theta \circ \alpha \circ h \circ g \circ e_{n-1} \simeq \theta \circ \psi \circ e_n \circ e_{n-1} \simeq *$ by (1), and there is a map $\tilde{\beta}: \Sigma^{n-1}X^{\wedge n} \to K_1$ so that

$$\tilde{\beta} \circ \rho_{n-1} \simeq \theta \circ \alpha. \tag{2}$$

Since $\tilde{\beta} \circ \Sigma^{n-1}\varepsilon_{n+1,\infty}^{n-1} \circ \Sigma^{n-1}\varepsilon_{n+1}^{n-1} \circ r \simeq \theta \circ \alpha \circ h \circ g \simeq \theta \circ \psi \circ e_n \simeq *$ also by (1), there is $\eta: \Sigma P_{n-2}(X) \to K_1$ so that

$$\eta \circ \Sigma b_{n-1} \simeq \tilde{\beta} \circ \Sigma^{n-1}\varepsilon_{n+1,\infty}^{n-1} \circ \Sigma^{n-1}\varepsilon_{n+1}^{n-1}.$$

Put $\bar{\beta} = \tilde{\beta} - \eta \circ \Sigma\beta_{n-1}$. Then

$$\bar{\beta} \circ \Sigma^{n-1}\varepsilon_{n+1,\infty}^{n-1} \circ \Sigma^{n-1}\varepsilon_{n+1}^{n-1} \simeq *.$$

Now by Theorem 4.2 (5),

$$(\varepsilon_{n+1,\infty}^{n-1})^*(\tilde{H}^*(X)^{\otimes n-1})\cap\ker(\varepsilon_{n+1}^{n-1})^*\subset(\rho_{n+1}^{n-1})^*(\bigoplus PH^*(\Omega X)^{\otimes n+1}).$$

Here $(\rho_{n+1}^{n-1})^*$ is induced by the following natural map

$$\begin{aligned}\nu\colon \Sigma^{n-1}P_{n+1}^{n-1}(\Omega X)&\to\Sigma^{n-1}P_{n+1}^{n-1}(\Omega X)/\Sigma^{n-1}P_n^{n-1}(\Omega X)\\&\simeq Q_{n-1}(X)/R_{n-1}(X).\end{aligned}$$

Then we have a map $\beta\colon Q_{n-1}(X)/R_{n-1}(X)\to K_1$ so that

$$\beta\circ\nu\simeq\bar{\beta}\circ\Sigma^{n-1}\varepsilon_{n+1,\infty}^{n-1}\qquad\text{and}\qquad\beta^*\ w)\notin\bigoplus PH^*(\Omega X)^{\otimes n+1},$$

where $w\in H^*(K_1)$ is the fundamental class. This shows that

$$\begin{aligned}\beta\circ\pi&\simeq\beta\circ\nu\circ\rho'\\&\simeq\bar{\beta}\circ\Sigma^{n-1}\varepsilon_{n+1,\infty}^{n-1}\circ\rho'\\&\simeq\tilde{\beta}\circ\rho_{n-1}\circ h-\eta\circ\Sigma\beta_{n-1}\circ\rho_{n-1}\circ h\\&\simeq\theta\circ\alpha\circ h\end{aligned}$$

by (2). Now we can apply Lemma 7.2, to get a map $\lambda\colon\Sigma F_n(X^{*n})\to K_1$ with

$$\lambda\circ r_n\simeq\theta\circ\psi\qquad\text{and}\qquad\theta\circ\xi\simeq(\beta\circ\lambda_{n+1})*\lambda.$$

On the other hand, we have by Proposition 6.1 that

$$\lambda_{n+1}^*(\bigoplus PH^*(\Omega X)^{\otimes n+1})\subset\bigoplus PH^*(\Omega X)^{\otimes n+1}.$$

Thus

$$\mu^*\circ\lambda^*(w)=\xi^*\circ\theta^*(w)-\lambda_{n+1}^*\circ\beta^*(w)\in PH^*(\Omega X)^{\otimes n+1}$$

by Lemma 7.2. Then by Lemma 6.2

$$\lambda^*(w)\in PH^*(\Omega X)^{\otimes n}\oplus\sum_{i=1}^{n}PH^*(\Omega X)^{\otimes i-1}\otimes P^2H^*(\Omega X)\otimes PH^*(\Omega X)^{\otimes n-i}.$$

Thus

$$r_n^*\circ\lambda^*(w)\in D_n(T_{n+1}[y_1,\dots,y_k])\oplus M,$$

and so

$$\tau u=v+\psi^*\circ\theta^*(w)=v+r_n^*\circ\lambda^*(w)\in T_{n+1}[y_1,\dots,y_k]\oplus M.$$

This proves Theorem 1.1 (5). Since (6) is a direct consequence of (5), this completes the proof of Theorem 1.1. q.e.d.

8. Application

Theorem 1.1 can be used to deduce variety of results on the action of Steenrod operations on the cohomology of A_p-spaces. For example, the main result in [2] is still valid without the hypothesis of the A_p primitivity of generators. In this section we give some more applications. First we prove the following fact.

THEOREM 8.1. *Let $H^* = T_{p+1}[y_1, \dots, y_k]$ be an unstable algebra over the* mod *p Steenrod algebra $\mathscr{A}_{(p)}$. Let* deg $y_i = 2n_i$ $(n_1 \leq \cdots \leq n_k)$. *Define non negative integers a, b with $b \not\equiv 0$* mod *p by $n_k = p^a b$. If $b > p$, then y_k is detected by primary operations modulo decomposable elements, that is, there exist operations $\theta_i \in \mathscr{A}_{(p)}$ $(1 \leq i \leq k-1)$ so that*

$$y_k - \sum_{i=1}^{k-1} \theta_i y_i \in DH^* .$$

PROOF. We prove by contradiction. Suppose y_k is not detected by primary operations. Let I be the ideal of H^* generated by $\{y_1, \dots, y_{k-1}\}$. Then for any operation $\theta \in \mathscr{A}_{(p)}$ with $\deg \theta > 0$, we have $\theta(H^*) \subset I + DH^*$. Then the inductive argument implies

$$\theta(D_t H^*) \subset I + D_{t+1} H^* .$$

Now

$$y_k^p = \mathscr{P}^{n_k} y_k = \sum_{i=0}^{a} \mathscr{P}^{p^i} \alpha_i y_k$$

for some $\alpha_i \in \mathscr{A}_{(p)}$. Here $\alpha_i y_k \in D_p H^*$ for dimensional reasons, and then $\mathscr{P}^{p^i} \alpha_i y_k \in I$ since $D_{p+1} H^* = 0$. This is a contradiction, and the theorem is proved. q.e.d.

The following theorem follows from the above theorem by Theorem 1.1.

THEOREM 8.2. *Let X be a simply connected A_n-space with*

$$H^*(X; \mathbf{Z}/p) \cong \Lambda(x_1, \dots, x_k) \qquad \deg x_i = 2n_i - 1 \qquad (n_1 \leq \cdots \leq n_k) .$$

Let $n_k = p^a b$ with $b \not\equiv 0$ mod *p. If $b > p$, then there exist operations $\theta_i \in \mathscr{A}_{(p)}$ $(1 \leq i \leq k-1)$ so that*

$$x_k = \sum_{i=1}^{k-1} \theta_i x_i .$$

Let $(G(n), d) = (SU(n), 2)$ or $(Sp(n), 4)$. Let M_λ be the total space of principal $G(n-1)$-bundle over $G(n)/G(n-1) = S^{dn-1}$ induced by a degree λ map on S^{dn-1} from the principal bundle $G(n-1) \to G(n) \to G(n)/G(n-1)$.

THEOREM 8.3. *Let $dn/2 = p^a b$ with $b \not\equiv 0 \bmod p$. If $b > p$, then the following conditions are equivalent.*

(1) *M_λ is a $\bmod p$ A_p-space.*
(2) *M_λ is a $\bmod p$ loop space.*
(3) *$\lambda \not\equiv 0 \bmod p$.*

PROOF. We have only to prove that if $\lambda \equiv 0 \bmod p$, M_λ is not a $\bmod p$ A_p-space. But this follows immediately from Theorem 8.2. In fact, let $f: M_\lambda \to G(n)$ be the induced map. Then

$$H^*(G(n); \mathbf{Z}/p) \cong \Lambda(x_1, \ldots, x_k)$$

$$H^*(M_\lambda: \mathbf{Z}/p) \cong \Lambda(x'_1, \ldots, x'_k)$$

where $k = n$ or $n-1$, $\deg x_1 < \cdots < \deg x_k = dn-1$, and $f^* x_i = x'_i$ for $i < k$, and $f^* x_k = 0$. This shows that x'_k is not detected by primary operations. q.e.d.

The above theorem strengthens Iwase's results [6].

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