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Behavior of bounded positive solutions of higher order differential equations

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1. Introduction

There is little known about the behavior of solutions of the differential equation of the form

$$(*)\qquad x^{(n)} + p(t)x^{(n-1)} + q(t)x^{(n-2)} + H(t, x) = 0$$

where $n \geq 3$ is an integer and $H: \mathfrak{R}^+ \times \mathfrak{R} \to \mathfrak{R}$ is continuous, decreasing in its second variable and is such that $uH(t, u) < 0$ for all $u \neq 0$. Some properties of solutions of $(*)$ are given by the author in [5] and [6]. In [7] the author gave two oscillation results for odd order equations with certain conditions on the functions p and q. This paper is a continuation of the study of differential equation $(*)$. Several results concerning bounded eventually positive solutions of $(*)$ will be proven. The nonlinear functionals which appear in the first two theorems can become very useful when studying the oscillatory behavior of solutions of $(*)$. This technique in fact was used in [7] as well as by Erbe [1], Heidel [2], Kartsatos [3], and Kartsatos & Kosmala [4].

2. Preliminaries

In what follows $\mathfrak{R}$ is used to denote the real line and $\mathfrak{R}^+$ the interval $(0, \infty)$. Also, $x(t)$, $t \in [t_x, \infty) \subset \mathfrak{R}^+$, is a solution of $(*)$ if it is n times continuously differentiable and satisfies $(*)$ on $[t_x, \infty)$. The number $t_x > 0$ depends on a particular solution $x(t)$ under consideration. We say that a function is "oscillatory" if it has an unbounded set of zeros. Moreover, a property P holds "eventually" or "for all large t" if there exists $T > 0$ such that P holds for all $t \geq T$. $C^n(I)$ denotes the space of all n times continuously differentiable functions $f: I \to \mathfrak{R}$. And we write $C(I)$ instead of $C^0(I)$. Throughout this paper we will assume that $p \in C^2[t_0, \infty)$, $q \in C^1[t_0, \infty)$ with

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$$2q(t) \leq p'(t) \tag{1}$$

for $t \geq t_0 > 0$. From [7] we quote the following lemma.

LEMMA 2.1. *If x is an eventually positive solution of $(*)$, then either $x^{(n-2)}(t) \leq 0$ or $x^{(n-2)}(t) > 0$ for all large t.*

3. Main results

THEOREM 3.1. *Suppose that n is odd, $p(t) \leq 0$, $p'(t) \leq 0$,* and *$p''(t) \geq 0$ eventually, and let*

$$F_1(x(t)) = 2x^{(n-3)}(t)x^{(n-1)}(t) + 2p(t)x^{(n-3)}(t)x^{(n-2)}(t) - p'(t)[x^{(n-3)}(t)]^2 - [x^{(n-2)}(t)]^2 .$$

If $x(t)$ is a bounded and eventually positive solution of $(*)$, then either
(a) *$x^{(n-2)}(t) \leq 0$, or*
(b) *$x^{(n-1)}(t) < 0$ and $F_1(x(t)) > 0$ eventually.*

PROOF. Suppose that all the assumptions on functions p and q are satisfied for all $t \geq t_0 \geq 0$, and that $x(t) > 0$ is a bounded solution of $(*)$ for $t \geq t_0$. By Lemma 2.1, there exists $t_1 \geq t_0$ such that $x^{(n-2)}(t) \leq 0$ or $x^{(n-2)}(t) > 0$ for all $t \geq t_1$. If $x^{(n-2)}(t) \leq 0$, then there is nothing to prove. Therefore, we assume that $x^{(n-2)}(t) > 0$ and consider three cases.

Case 1. Suppose that $x^{(n-1)}(t) > 0$. This gives a contradiction due to the boundedness of x.

Case 2. Suppose that $x^{(n-1)}(t_2) = 0$ for some $t_2 \geq t_1$. Then, from $(*)$ we have

$$x^{(n)}(t_2) = -q(t_2)x^{(n-2)}(t_2) - H(t_2, x(t_2)) > 0 .$$

Thus, $x^{(n-1)}(t)$ is increasing at any t_2, $t_2 \geq t_1$, for which it is zero. Therefore, $x^{(n-1)}(t)$ cannot have any zeros larger than t_2. This takes us to the final case.

Case 3. Suppose that $x^{(n-1)}(t) < 0$ for $t \geq t_3 \geq t_1$. Since $x(t)$ is bounded and positive and n is odd, there exist $t_4 \geq t_3$ such that $x^{(n-3)}(t) < 0$ and $x'(t) > 0$ for all $t \geq t_4$. Now we consider the nonlinear functional $F_1(x(t))$ as defined in the statement of this theorem. We will prove that $F_1(x(t)) > 0$ eventually by assuming to the contrary. So, let $t_5 \geq t_4$ be such that $F_1(x(t_5)) \leq 0$. Note that if t_5 like this does not exist, there is nothing to prove. So now, we drop the last two terms in the equation $(*)$ to obtain

$$x^{(n)}(t) > -p(t)x^{(n-1)}(t) .$$

Therefore, using this inequality when differentiating $F_1(x(t))$ we obtain

$$\frac{d}{dt}F_1(x(t)) = 2x^{(n-3)}(t)[x^{(n)}(t)] + 2x^{(n-2)}(t)x^{(n-1)}(t) + 2p(t)x^{(n-3)}(t)x^{(n-1)}(t)$$
$$+ 2p(t)[x^{(n-2)}(t)]^2 + 2p'(t)x^{(n-3)}(t)x^{(n-2)}(t) - 2p'(t)x^{(n-3)}(t)x^{(n-2)}(t)$$
$$- p''(t)[x^{(n-3)}(t)]^2 - 2x^{(n-2)}(t)x^{(n-1)}(t)$$
$$< 2p(t)[x^{(n-2)}(t)]^2 - p''(t)[x^{(n-3)}(t)]^2$$
$$\leq \text{for all } t \geq t_5 .$$

Thus, $F_1(x(t)) < 0$ for all $t > t_5$. But now, since $p(t) \leq 0$, $p'(t) \leq 0$ and F_1 is decreasing we have that

$$-[x^{(n-2)}(t)]^2 < F_1(x(t)) \leq F_1(x(t_6)) < 0$$

for $t \geq t_6 \geq t_5$. So, in view of this and the fact that $x^{(n-2)}(t)$ is decreasing and positive, there exists $m > 0$ such that $\lim_{t\to\infty} x^{(n-2)}(t) = m > 0$. This implies that $x^{(n-3)}(t)$ tends to $+\infty$ as t goes to $+\infty$, which is a contradiction. Hence, $F_1(x(t)) > 0$ for all $t \geq t_7 \geq t_4$.

THEOREM 3.2. *Suppose that n is odd, $p(t) \geq 0$, $q(t) \leq 0$, and $p(t)q(t) + q'(t) \geq 0$, eventually, and let*

$$F_2(x(t)) = \left[\exp \int_{t_0}^{t} p(s)ds\right][2x^{(n-3)}(t)x^{(n-1)}(t) - [x^{(n-2)}(t)]^2 - q(t)[x^{(n-3)}(t)]^2] .$$

for some $t_0 > 0$. If $x(t)$ is a bounded and eventually positive solution of $(*)$, then either
(a) $x^{(n-2)}(t) \leq 0$, *or*
(b) $x^{(n-1)}(t) < 0$ *and* $F_2(x(t)) > 0$ *eventually.*

PROOF. Suppose that all the assumptions on functions p and q are satisfied and that $x(t) > 0$ is a bounded solution of $(*)$ for $t \geq t_1 \geq t_0$. Also, there exists $t_2 \geq t_1$ such that $x^{(n-2)}(t) \leq 0$ or $x^{(n-2)}(t) > 0$ for all $t \geq t_2$. If $x^{(n-2)}(t) \leq 0$, then there is nothing to prove. Therefore, we assume that $x^{(n-2)}(t) > 0$ and consider three cases.

Cases 1 and 2 are the same as in the proof of Theorem 3.1.

Case 3. Suppose that $x^{(n-1)}(t) < 0$ for $t \geq t_3 \geq t_2$. Since $x(t)$ is bounded and positive and n is odd, by Lemma 2.1 there exists $t_4 \geq t_3$ such that $x^{(n-3)}(t) < 0$ and $x'(t) > 0$ for all $t \geq t_4$. Now we consider the nonlinear functional $F_2(x(t))$ as defined in the statement of this theorem. We will prove that $F_2(x(t)) > 0$ eventually by assuming to the contrary. So, let $t_5 \geq t_4$ be such that $F_2(x(t_5)) \leq 0$. Again, we drop the last two terms in the equation $(*)$ to obtain $x^{(n)}(t) > -p(t)x^{(n-1)}(t)$. Therefore, using this inequality when differentiating $F_2(x(t))$ on $[t_5, \infty)$ we obtain

$$\frac{\frac{d}{dt}F_2(x(t))}{K} = \left[\exp\int_{t_5}^{t} p(s)ds\right][2x^{(n-3)}(t)x^{(n)}(t) + 2x^{(n-2)}(t)x^{(n-1)}(t)$$

$$- 2x^{(n-2)}(t)x^{(n-1)}(t) - 2q(t)x^{(n-3)}(t)x^{(n-2)}(t) - q'(t)[x^{(n-3)}(t)]^2]$$

$$+ p(t)\left[\exp\int_{t_5}^{t} p(s)ds\right][2x^{(n-3)}(t)x^{(n-1)}(t)$$

$$- [x^{(n-2)}(t)]^2 - q(t)[x^{(n-3)}(t)]^2]$$

$$< \left[\exp\int_{t_5}^{t} p(s)ds\right][-q(t)x^{(n-3)}(t)x^{(n-2)}(t) - p(t)[x^{(n-2)}(t)]^2$$

$$- (p(t)q(t) + q'(t))[x^{(n-3)}(t)]^2]$$

$$\leq 0 \text{ for all } t \geq t_5,$$

where $K = \exp\int_{t_0}^{t_5} p(s)ds$. Thus, $F_2(x(t)) < 0$ for all $t > t_5$. Also,

$$-[x^{(n-2)}(t)]^2\left[\exp\int_{t_0}^{t} p(s)ds\right] < F_2(x(t)) \leq F_2(x(t_6)) < 0$$

for $t \geq t_6 \geq t_5$. So, in view of this and the fact that $x^{(n-2)}(t)$ is decreasing and positive, there exists $m > 0$ such that $\lim_{t\to\infty} x^{(n-2)}(t) = m > 0$. This implies that $\lim_{t\to\infty} x^{(n-3)}(t) = +\infty$, which is a contradiction. Hence, $F_2(x(t)) > 0$ for all $t \geq t_7 \geq t_4$.

REMARK 3.3. Functions $p(t) = \frac{1}{t^2}$ and $q(t) = \frac{-1}{t^3}$ satisfy all the conditions in Theorem 3.2.

REMARK 3.4. Suppose that in Theorem 3.2 we further assume that $\int^{\infty} H(t, k)dt = -\infty$ for any positive constant k. Then, if $x(t)$ is a bounded and eventually positive solution of $(*)$, then $x^{(n-2)}(t) \leq 0$.

PROOF. Suppose that all the assumptions on functions p and q are satisfied and that $x(t) > 0$ is a bounded solution of $(*)$ for $t \geq t_0 \geq 0$. By Lemma 2.1, there exists $t_1 \geq t_0$ such that $x^{(n-2)}(t) \leq 0$ or $x^{(n-2)}(t) > 0$ for all $t \geq t_1$. If $x^{(n-2)}(t) \leq 0$, then there is nothing to prove. Therefore, we assume that $x^{(n-2)}(t) > 0$ and consider three cases.

Cases 1 and 2 are the same as in the proof of Theorem 3.1.

Case 3. Suppose that $x^{(n-1)}(t) < 0$ for $t \geq t_3 \geq t_1$. Observe that the differential equation $(*)$ can be written as

$$\left\{x^{(n-1)}(t)\exp\left[\int_{t_3}^{t}p(s)ds\right]\right\}' + q(t)x^{(n-2)}(t)\exp\left[\int_{t_3}^{t}p(s)ds\right]$$
$$+ H(t,x(t))\exp\left[\int_{t_3}^{t}p(s)ds\right] = 0\,.$$

Dropping the second term we get

$$\left\{x^{(n-1)}(t)\exp\left[\int_{t_3}^{t}p(s)ds\right]\right\}' + H(t,x(t))\exp\left[\int_{t_3}^{t}p(s)ds\right] \geq 0\,.$$

Since $x'(t) > 0$ we have $0 < k \equiv x(t_3) \leq x(t)$ for all $t \geq t_3$, and the above line can be rewritten as

$$\left\{x^{(n-1)}(t)\exp\left[\int_{t_3}^{t}p(s)ds\right]\right\}' + H(t,k)\exp\left[\int_{t_3}^{t}p(s)ds\right] \geq 0\,.$$

Integrating this inequality from t_3 to t, $t \geq t_3$, we get

$$-x^{(n-1)}(t)\exp\left[\int_{t_3}^{t}p(s)ds\right] \leq -x^{(n-1)}(t_3) + \int_{t_3}^{t}\left[H(s,k)\exp\left(\int_{t_3}^{s}p(u)du\right)\right]ds\,.$$

Due to the integral condition on H, the right hand side tends to $-\infty$, and thus, so does the left-hand side. Therefore, $\lim_{t\to\infty} x^{(n-1)}(t) = +\infty$, which contradicts the fact that $x^{(n-1)}(t) < 0$. Hence, $x^{(n-2)}(t) > 0$ eventually prevents $x^{(n-1)}(t)$ from existing. This proves Remark 3.4.

THEOREM 3.5. *Suppose that n is odd, $p(t) \leq 0$, $q(t) \leq 0$, and*

$$q(t) \leq p'(t) \tag{3}$$

eventually, and suppose that

$$\int^{\infty} H(t,k)dt = -\infty$$

for any positive constant k. If $x(t)$ is a bounded and eventually positive solution of $(*)$, then $x^{(n-2)}(t) \leq 0$ eventually.

Note that condition (3) implies condition (1) but condition (3) is not implied by condition (1). For example, two eventually nonpositive functions p and q with $p'(t) = \dfrac{-2}{t}$ and $q(t) = \dfrac{-3}{2t}$ satisfy condition (1) but not condition (3).

PROOF. Suppose that all the assumptions on functions p and q are satisfied and that $x(t) > 0$ is a bounded solution of $(*)$ for $t \geq t_0$. By Lemma

2.1, there exists $t_1 \geq t_0$ such that $x^{(n-2)}(t) \leq 0$ or $x^{(n-2)}(t) > 0$ for all $t \geq t_1$. If $x^{(n-2)}(t) \leq 0$, then there is nothing to prove. Therefore, we assume that $x^{(n-2)}(t) > 0$ and consider three cases.

Cases 1 and 2 are the same as in the proof of Theorem 3.1.

Case 3. Suppose that $x^{(n-1)}(t) < 0$ for $t \geq t_3 \geq t_1$. Since n is odd, $x'(t) > 0$ for all $t \geq t_4 \geq t_3$ and so $k \equiv x(t_4) \leq x(t)$ for all $t \geq t_4$. Now we integrate (*) from t_4 to t, $t \geq t_4$, to get

$$\begin{aligned} x^{(n-1)}(t) + p(t)x^{(n-2)}(t) &= x^{(n-1)}(t_4) + p(t_4)x^{(n-2)}(t_4) \\ &\quad + \int_{t_4}^{t} [p'(s) - q(s)]x^{(n-2)}(s)ds - \int_{t_4}^{t} H(s, x(s))ds \\ &= M + f(t) - \int_{t_4}^{t} H(s, x(s))ds\,, \end{aligned}$$

where M is a constant and $f(t)$ is the first integral in the above expression. If $z(t) = x^{(n-2)}(t)$, then z satisfies a first-order linear differential equation and thus can be written as

$$\begin{aligned} z(t) = \exp\left[-\int_{t_4}^{t} p(s)ds\right]\Bigg\{z(t_4) + \int_{t_4}^{t}\left[\exp\int_{t_4}^{s} p(r)ds\right] \\ \cdot\left[M + f(s) - \int_{t_4}^{s} H(r, x(r))dr\right]ds\Bigg\}. \end{aligned}$$

Since $f(t) \geq 0$ and $x(t) \geq k > 0$, the above equality can be written as

$$z(t) \geq \int_{t_4}^{t}\left[\exp\left(-\int_{s}^{t} p(r)dr\right)\right]\left[M - \int_{t_4}^{s} H(r, k)dr\right]ds\,.$$

Due to the integral assumption on H, there exists $s_0 \in \mathfrak{R}^+$ such that

$$\int_{t_4}^{s_0} H(t, k)dt \leq M - 1\,.$$

Therefore,

$$\begin{aligned} z(t) &\geq \int_{t_4}^{t}\left[\exp\left(-\int_{s}^{t} p(r)dr\right)\right][M - (M - 1)]ds \\ &= \int_{t_4}^{t} \exp\left(-\int_{s}^{t} p(r)dr\right)ds \\ &\geq \int_{t_4}^{t} 1ds = t - t_4 \to +\infty \text{ as } t \to +\infty\,. \end{aligned}$$

Thus, $\lim_{t\to\infty} x^{(n-2)}(t) = +\infty$ which contradicts the fact that $x(t)$ is bounded. Hence, $x^{(n-2)}(t) \leq 0$ eventually.

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