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Oscillation properties of half-linear functional differential equations of the second order

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1. Introduction

This paper is devoted to the study of the oscillatory (and nonoscillatory) behavior of second order functional differential equations of the type

$$(|y'(t)|^{\alpha} \operatorname{sgn} y'(t))' + q(t)|y(g(t))|^{\alpha} \operatorname{sgn} y(g(t)) = 0 \tag{A}$$

for which the following conditions, collectively referred to as (H), are assumed to hold:

(a) α is a positive constant;
(b) $q(t)$ is a positive continuous function on $[a, \infty)$, $a \geqq 0$;
(c) $g(t)$ is a positive continuously differentiable function on $[a, \infty)$ such that $g'(t) > 0$ for $t \geqq a$ and $\lim_{t \to \infty} g(t) = \infty$.

By a solution of (A) we mean a function $y \in C^1(T_y, \infty)$, $T_y \geqq a$, which has the property $|y'|^{\alpha} \operatorname{sgn} y' \in C^1[T_y, \infty)$ and satisfies the equation for all sufficiently large t in $[T_y, \infty)$. Our attention will be restricted to those solutions which are nontrivial in the sense that $\sup\{|y(t)|: t \geqq T\} > 0$ for any $T > T_y$. Such a solution is said to be oscillatory if it has an infinite sequence of zeros clustering at ∞; otherwise it is said to be nonoscillatory. By definition, the equation (A) is oscillatory if all of its solutions are oscillatory and nonoscillatory otherwise.

The oscillation results for (A) to be proved in this paper are as follows.

THEOREM 1. *The equation* (A) *is oscillatory if*

$$\int_a^{\infty} q(t)dt = \infty. \tag{1.1}$$

THEOREM 2. *Suppose that*

$$\int_a^{\infty} q(t)dt < \infty \tag{1.2}$$

and that $g(t) \geqq t$ *for* $t \geqq a$.

(i) *The equation* (A) *is oscillatory if either*

$$\limsup_{t\to\infty} t^{\alpha} \int_t^{\infty} q(s)\,ds > 1 \tag{1.3}$$

or

$$\liminf_{t\to\infty} t^{\alpha} \int_t^{\infty} q(s)ds > \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}. \tag{1.4}$$

(ii) *The equation* (A) *is nonoscillatory if*

$$\limsup_{t\to\infty} (g(t))^{\alpha} \int_t^{\infty} q(s)ds < \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}. \tag{1.5}$$

THEOREM 3. *Suppose that* (1.2) *holds and that* $g(t) \leqq t$ *for* $t \geqq a$.

(i) *The equation* (A) *is oscillatory if either*

$$\limsup_{t\to\infty} (g(t))^{\alpha} \int_t^{\infty} q(s)ds > 1 \tag{1.6}$$

or

$$\liminf_{t\to\infty} (g(t))^{\alpha} \int_t^{\infty} q(s)ds > \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}. \tag{1.7}$$

(ii) *The equation* (A) *is nonoscillatory if*

$$\limsup_{t\to\infty} t^{\alpha} \int_t^{\infty} q(s)ds < \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}. \tag{1.8}$$

We also consider the parametrized equation

$$(|y'(t)|^{\alpha} \operatorname{sgn} y'(t))' + \lambda q(t)|y(g(t))|^{\alpha} \operatorname{sgn} y(t)) = 0, \qquad \lambda > 0, \tag{A_λ}$$

subject to the conditions (H), and examine its strong oscillation and strong nonoscillation. Here we say that (A_λ) is strongly oscillatory [respectively strongly nonoscillatory] if (A_λ) is oscillatory [respectively nonoscillatory] for all values of $\lambda > 0$ in the sense defined above. There is a class of equations of the form (A_λ) for which the situation of strong oscillation and that of strong nonoscillation can be completely characterized, as the following theorem shows.

THEOREM 4. *Suppose that* (1.2) *holds and that*

$$0 < \liminf_{t\to\infty} \frac{g(t)}{t}, \qquad \limsup_{t\to\infty} \frac{g(t)}{t} < \infty. \tag{1.9}$$

(i) *The equation* (A_λ) *is strongly oscillatory if and only if*

$$\limsup_{t\to\infty} t^\alpha \int_t^\infty q(s)\,ds = \infty. \tag{1.10}$$

(ii) *The equation* (A_λ) *is strongly nonoscillatory if and only if*

$$\lim_{t\to\infty} t^\alpha \int_t^\infty q(s)\,ds = 0. \tag{1.11}$$

The above-mentioned theorems form a natural generalization to (A), (A_λ) of the basic oscillation results recently developed in [4] for the half-linear ordinary differential equations

$$(|y'|^\alpha \operatorname{sgn} y')' + q(t)|y|^\alpha \operatorname{sgn} y = 0, \tag{B}$$

$$(|y'|^\alpha \operatorname{sgn} y')' + \lambda q(t)|y|^\alpha \operatorname{sgn} y = 0. \tag{B_λ}$$

In proving these theorems a crucial role is played by two types of comparison principles which relate the oscillation or nonoscillation of functional differential equations of the form (A) to that of suitably associated equations without functional arguments of the form (B). Such comparison principles are presented in Section 2, and the proofs of our main results are given in Section 3. The final Section 4 concerns the possibility of extending our oscillation theory to equations of the forms

$$p(t)|y'(t)|^\alpha \operatorname{sgn} y'(t))' + q(t)|y(g(t))|^\alpha \operatorname{sgn} y(g(t)) = 0, \tag{C}$$

$$(p(t)|y'(t)|^\alpha \operatorname{sgn} y'(t))' + \lambda q(t)|y(g(t))|^\alpha \operatorname{sgn} y(g(t)) = 0. \tag{C_λ}$$

2. Comparison principles

We begin with a preliminary result which is an extension of the second order version of a result of Onose [7].

LEMMA 1. *Let* $F(t, x)$ *be a continuous function on* $[a, \infty) \times \mathbb{R}$ *which is nondecreasing in* x *and satisfies* $\operatorname{sgn} F(t, x) = \operatorname{sgn} x$ *for each fixed* $t \geqq a$. *Let* α *and* $g(t)$ *be as in* (A). *If the differential inequality*

$$(|x'(t)|^\alpha \operatorname{sgn} x'(t))' + F(t, x(g(t))) \leqq 0 \tag{2.1}$$

has an eventually positive solution, then so does the differential equation

$$(|y'(t)|^\alpha \operatorname{sgn} y'(t))' + F(t, y(g(t))) = 0. \tag{2.2}$$

PROOF. Let $x(t)$ be an eventually positive function satisfying (2.1). Let $T > a$ be such that $x(t) > 0$ and $x(g(t)) > 0$ for $t \geqq T$. Since (2.1) implies that

$x'(t)$ is decreasing for $t \geqq T$, $x'(t)$ is eventually one-signed, that is, either $x'(t) > 0$ for $t \geqq T$ or there is $T' > T$ such that $x'(t) < 0$ for $t \geqq T'$. The latter case is impossible, for if $x'(t) < 0$ for $t \geqq T'$, then, integrating the inequality $x'(t) \leqq x'(T')$, $t \geqq T'$, from T' to t and letting $t \to \infty$, we see that $x'(t) \to -\infty$ as $t \to \infty$, which contradicts the assumed positivity of $x(t)$. It follows that $x'(t) > 0$ for $t \geqq T$. Integrating (2.1) over $[t, \infty)$, we have

$$(x'(t))^\alpha \geqq \omega + \int_t^\infty F(s, x(g(s)))\,ds, \qquad t \geqq T,$$

where $\omega = \lim_{t\to\infty} (x'(t))^\alpha \geqq 0$, that is,

$$x'(t) \geqq \left(\omega + \int_t^\infty F(s, x(g(s)))\,ds\right)^{\frac{1}{\alpha}}, \qquad t \geqq T.$$

An integration of the above inequality over $[T, t]$ yields

$$x(t) \geqq x(T) + \int_T^t \left(\omega + \int_s^\infty F(r, x(g(r)))\,dr\right)^{\frac{1}{\alpha}} ds, \qquad t \geqq T. \tag{2.3}$$

We now put $T_* = \min\{T, \inf_{t\geqq T} g(t)\}$ and define the set $Y \subset C[T_*, \infty)$ and the mapping $\mathscr{F}: Y \to C[T_*, \infty)$ by

$$Y = \{y \in C[T_*, \infty): 0 \leqq y(t) \leqq x(t),\ t \geqq T_*\}$$

and

$$(\mathscr{F}y)(t) = \begin{cases} x(T) + \int_T^t \left(\omega + \int_s^\infty F(r, y(g(r)))\,dr\right)^{\frac{1}{\alpha}} ds, & t \geqq T, \\ x(t), & T_* \leqq t \leqq T. \end{cases}$$

Then it is verified without difficulty that $\mathscr{F}$ is a countinuous mapping which sends Y into a relatively compact subst of Y. So, $\mathscr{F}$ has a fixed element $y \in Y$ by the Schauder-Tychonoff fixed point theorem. This fixed element $y = y(t)$ clearly satisfies the functional integral equation

$$y(t) = x(T) + \int_T^t \left(\omega + \int_s^\infty F(r, y(g(r)))\,dr\right)^{\frac{1}{\alpha}} ds, \qquad t \geqq T,$$

from which it readily follows that $y(t)$ is a positive solution of the differential equation (2.2) on $[T, \infty)$.

We now state and prove two comparison principles on which the proofs of our main results (Theorems 2–4) are based. The first one is a natural extension of the classical Sturmian comparison theorem for second order linear ordinary differential equations. See [1] and [3].

THEOREM 5. *Consider the equations* (2.2) *and*

$$(|y'(t)|^{\alpha} \operatorname{sgn} y'(t))' + G(t, y(h(t))) = 0, \tag{2.4}$$

where α, $g(t)$ *and* $F(t, x)$ *are as in Lemma* 1, *and* $h(t)$ *and* $G(t, y)$ *satisfy the following conditions*:

(a) $h(t)$ *is a positive continuously differentiable function on* $[a, \infty)$ *such that* $h'(t) > 0$ *for* $t \geqq a$ *and* $\lim_{t \to \infty} h(t) = \infty$;

(b) $G(t, y)$ *is a continuous function on* $[a, \infty) \times \mathbb{R}$ *which is nondecreasing in* y *and satisfies* $\operatorname{sgn} G(t, y) = \operatorname{sgn} y$ *for each fixed* $t \geqq a$.

Suppose moreover that

$$g(t) \geqq h(t), \qquad t \geqq a, \tag{2.5}$$

and

$$F(t, x) \operatorname{sgn} x \geqq G(t, x) \operatorname{sgn} x, \qquad (t, x) \in [a, \infty) \times \mathbb{R}. \tag{2.6}$$

If (2.2) *is nonoscillatory, then so is* (2.4); *or equivalently, if* (2.4) *is oscillatory, then so is* (2.2).

PROOF. Suppose that (2.2) is nonoscillatory. Then, (2.2) has a nonoscillatory solution $x(t)$ which may be assumed to be eventually positive. As in the proof of Lemma 1 we see that $x'(t)$ is eventually positive, so that in view of (2.5) and (2.6) there is $T > a$ such that

$$x(g(t)) \geqq x(h(t)) \text{ and } F(t, x(g(t))) \geqq G(t, x(h(t))) \qquad \text{for } t \geqq T.$$

It follows that $x(t)$ satisfies the differential inequality

$$(|x'(t)|^{\alpha} \operatorname{sgn} x'(t))' + G(t, x(h(t))) \leqq 0, \qquad t \geqq T.$$

Lemma 1 then shows that the equation (2.4) possesses a positive solution $y(t)$ for $t \geqq T$, which implies that (2.4) is nonoscillatory.

The second result in this section is a variant of a comparison theorem of Mahfoud [5]. See also [3].

THEOREM 6. *Let* (2.2) *be as in Lemma* 1 *and suppose that it is nonoscillatory. Then, for any continuously differentiable function* $k(t)$ *on* $[a, \infty)$

such that $k'(t) > 0$ *and* $k(t) \geqq g(t)$ *for* $t \geqq a$, *the equation*

$$(2.7) \qquad (|z'(t)|^{\alpha} \operatorname{sgn} z'(t))' + \frac{k'(t)}{g'(g^{-1}(k(t)))} F(g^{-1}(k(t)),\ z(k(t))) = 0$$

is nonoscillatory, where $g^{-1}(s)$ *denotes the inverse function of* $g(t)$.

PROOF. Let $y(t)$ be a nonoscillatory solution of (2.2). We may assume that $y(t) > 0$ for $t \geqq T > a$. Repeating the procedure which led to (2.3) in the proof of Lemma 1, we see that

$$(2.8) \qquad y(t) = y(T) + \int_T^t \left(\omega + \int_s^{\infty} F(r,\ y(g(r)))\, dr \right)^{\frac{1}{\alpha}} ds, \qquad t \geqq T,$$

where $\omega = \lim_{t \to \infty} (y'(t))^{\alpha} \geqq 0$. We use the change of variable $r = g^{-1}(k(\rho))$ in $\int_s^{\infty} F(r,\ y(g(r)))\, dr$. Noting that $k^{-1}(g(s)) \leqq s$, we find

$$\begin{aligned} \int_s^{\infty} F(r,\ y(g(r)))\, dr &= \int_{k^{-1}(g(s))}^{\infty} F(g^{-1}(k(\rho)),\ y(k(\rho))) \frac{k'(\rho)}{g'(g^{-1}(k(\rho)))} d\rho \\ &\geqq \int_s^{\infty} F(g^{-1}(k(\rho)),\ y(k(\rho))) \frac{k'(\rho)}{g'(g^{-1}(k(\rho)))} d\rho, \end{aligned}$$

and so from (2.8) we obtain

$$y(t) \geqq y(T) + \int_T^t \left(\omega + \int_s^{\infty} F(g^{-1}(k(\rho)),\ y(k(\rho))) \frac{k'(\rho)}{g'(g^{-1}(k(\rho)))} d\rho \right)^{\frac{1}{\alpha}} ds, \qquad t \geqq T.$$

Let $T_* = \min \{T,\ \inf_{t \geqq T} g(t)\}$. Define

$$Z = \{z \in C[T_*,\ \infty) : 0 \leqq z(t) \leqq y(t),\ t \geqq T_*\}$$

and let $\mathscr{G}$ denote the mapping from Z to $C[T_*,\ \infty)$ defined by

$$(\mathscr{G}z)(t) = \begin{cases} y(T) + \displaystyle\int_T^t \left(\omega + \int_s^{\infty} F(g^{-1}(k(\rho)),\ z(k(\rho))) \frac{k'(\rho)}{g'(g^{-1}(k(\rho)))} d\rho \right)^{\frac{1}{\alpha}} ds, & t \geqq T. \\ y(t), \qquad T_* \leqq t \leqq T. \end{cases}$$

Then, as is easily verified, all the conditions of the Schauder-Tychonoff fixed point theorem are satisfied for $\mathscr{G}$ and Z, and hence there exists a $z \in Z$ such that $z = \mathscr{G}z$, i.e.,

$$z(t) = y(T) + \int_T^t \left(\omega + \int_s^\infty F(g^{-1}(k(\rho)), z(k(\rho))) \frac{k'(\rho)}{g'(g^{-1}(k(\rho)))} d\rho \right)^{\frac{1}{\alpha}} ds, \qquad t \geqq T.$$

Differentiation of the above equation shows that $z(t)$ is a solution of (2.7) for $t \geqq T$. Since $z(t)$ is positive for $t \geqq T$, we conclude that (2.7) is indeed nonoscillatory.

Important special cases of Theorem 6 are contained in the following corollary.

COROLLARY. *Consider the functional differential equation* (2.2) *and the ordinary differential equation*

$$(|z'(t)|^\alpha \operatorname{sgn} z'(t))' + \frac{1}{g'(g^{-1}(t))} F(g^{-1}(t), z(t)) = 0. \tag{2.9}$$

(i) *Suppose that* $g(t) \leqq t$ *for* $t \geqq a$. *If* (2.2) *is nonoscillatory, then so is* (2.9).
(ii) *Suppose that* $g(t) \geqq t$ *for* $t \geqq a$. *If* (2.9) *is nonoscillatory, then so is* (2.2).

3. Proofs of oscillation theorems

PROOF OF THEOREM 1. Assume that (A) has an eventually positive solution $y(t)$. Since $y'(t)$ is also eventually positive, there exist positive constants c and T such that $y(g(t)) \geqq c$ for $t \geqq T$. Combining this inequality with

$$\int_T^\infty q(t)(y(g(t)))^\alpha dt < \infty,$$

which follows from integration of (A), we see that

$$c^\alpha \int_T^\infty q(t) dt < \infty.$$

This contradics the assumption (1.1). Similarly, (A) has no eventually negative solution, and so (A) is oscillatory.

In order to prove Theorems 2–4 we need the following oscillation and nonoscillation criteria for the half-linear ordinary differential equations (B) and (B_λ).

LEMMA 2. *Suppose that* (1.2) *holds.*

(i) *The equation* (B) *is oscillatory if either*

$$(3.1)\qquad \limsup_{t\to\infty} t^{\alpha}\int_t^{\infty} q(s)\,ds > 1$$

or

$$(3.2)\qquad \liminf_{t\to\infty} t^{\alpha}\int_t^{\infty} q(s)\,ds > \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}.$$

(ii) *The equation* (B) *is nonoscillatory if*

$$(3.3)\qquad \limsup_{t\to\infty} t^{\alpha}\int_t^{\infty} q(s)ds < \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}.$$

LEMMA 3. *Suppose that* (1.2) *holds.*
(i) *The equation* (B_λ) *is strongly oscillatory if and only if*

$$(3.4)\qquad \limsup_{t\to\infty} t^{\alpha}\int_t^{\infty} q(s)\,ds = \infty.$$

(ii) *The equation* (B_λ) *is strongly nonoscillatory if and only if*

$$(3.5)\qquad \lim_{t\to\infty} t^{\alpha}\int_t^{\infty} q(s)\,ds = 0.$$

For the proofs of these lemmas we refer to Kusano, Naito and Ogata [4].

PROOF OF THEOREM 2. (i) Suppose that (1.3) or (1.4) holds. Since (1.3) and (1.4) are the same as (3.1) and (3.2), respectively, the equation (B) is oscillatory by (i) of Lemma 2. Since $g(t) \geqq t$ for $t \geqq a$, Theorem 5 applied to (A) and (B) shows that the equation (A) is oscillatory.

(ii) We apply to (A) the second statement of the corollary to Theorem 6, according to which the nonoscillation of (A) is implied by that of the equation

$$(3.6)\qquad (|z'(t)|^{\alpha}\operatorname{sgn} z'(t))' + \frac{q(g^{-1}(t))}{g'(g^{-1}(t))}|z(t)|^{\alpha}\operatorname{sng} z(t) = 0.$$

By (ii) of Lemma 2, a sufficient condition for nonoscillation of (3.6) is

$$\limsup_{t\to\infty} t^{\alpha}\int_t^{\infty}\frac{q(g^{-1}(s))}{g'(g^{-1}(s))}ds < \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}.$$

Since this is equivalent to the condition (1.5), the desired conclusion follows.

PROOF OF THEOREM 3. (i) From the first statememt of the corollary to Theorem 6 it follows that the oscillation of (A) is implied by that of the equation (3.6). By (i) of Lemma 2 the oscillation of (3.6) is in turn implied by either of the following conditions:

$$\limsup_{t\to\infty} t^{\alpha} \int_t^{\infty} \frac{q(g^{-1}(s))}{g'(g^{-1}(s))}\, ds > 1, \tag{3.7}$$

$$\limsup_{t\to\infty} t^{\alpha} \int_t^{\infty} \frac{q(g^{-1}(s))}{g'(g^{-1}(s))}\, ds > \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}. \tag{3.8}$$

To complete the proof it suffices to observe that (3.7) and (3.8) are equivalent to (1.6) and (1.7), respectively.

(ii) Suppose that (1.8) holds. Since (1.8) is identical with (3.3), (ii) of Lemma 2 then implies that the equation (B) is nonoscillatory. In view of Theorem 5 we see that the retarded differential equation (A) must be oscillatory.

Proof of Theorem 4. The proof is divided into two steps.

(The first step) We consider the two equations

$$(|z'(t)|^{\alpha} \operatorname{sgn} z'(t))' + \lambda q(t)|z(\gamma t)|^{\alpha} \operatorname{sgn} z(\gamma t) = 0, \tag{3.9}$$

$$(|z'(t)|^{\alpha} \operatorname{sgn} z'(t))' + \lambda q(t)|z(\gamma^{-1} t)|^{\alpha} \operatorname{sgn} z(\gamma^{-1} t) = 0, \tag{3.10}$$

where $\gamma > 1$ is a constant, and show that the strong oscillation or strong nonoscillation of (3.9) is equivalent to that of (3.10).

That the strong nonoscillation of (3.9) implies that of (3.10) is an immediate consequence of Theorem 5. To prove the converse we proceed as follows. Let (3.10) be strongly nonoscillatory. By (i) of the corollary to Theorem 6, the ordinary differential equation

$$(|u'(t)|^{\alpha} \operatorname{sgn} u'(t))' + \lambda \gamma q(\gamma t)|u(t)|^{\alpha} \operatorname{sgn} u(t) = 0 \tag{3.11}$$

is strongly nonoscillatory. The "only if" part of Lemma 3-(ii) then implies

$$\lim_{t\to\infty} t^{\alpha} \int_t^{\infty} \gamma q(\gamma s)\, ds = 0. \tag{3.12}$$

Note that (3.12) is equivalent to

$$\lim_{t\to\infty} t^{\alpha} \int_t^{\infty} \gamma^{-1} q(\gamma^{-1} s)\, ds = 0. \tag{3.13}$$

From the "if" part of Lemma 3-(ii) it then follows that the equation

$$(|u'(t)|^{\alpha} \operatorname{sgn} u'(t))' + \lambda \gamma^{-1} q(\gamma^{-1} t)|u(t)|^{\alpha} \operatorname{sgn} u(t) = 0 \tag{3.14}$$

is strongly nonoscillatory. Comparing (3.14) with (3.9) via (ii) of the corollary to Theorem 6 shows that (3.9) is strongly nonoscillatory as desired. Observe that (3.12) and (3.13) are equivalent to (1.11).

If (3.10) is strongly oscillatory, then so is (3.9) by Theorem 5. Conversely, if (3.9) is strongly oscillatory, then so is the equation (3.14) by (ii) of the corollary to Theorem 6. The strong oscillation of (3.14) implies that

$$\limsup_{t\to\infty} t^{\alpha} \int_t^{\infty} \gamma^{-1} q(\gamma^{-1} s)\, ds = \infty. \tag{3.15}$$

(cf. Lemma 3-(i).) Since (3.15) is equivalent to

$$\limsup_{t\to\infty} t^{\alpha} \int_t^{\infty} \gamma q(\gamma s)\, ds = \infty, \tag{3.16}$$

(i) of Lemma 3 ensures that (3.11) is strongly oscillatory. The desired strong oscillation of (3.10) now follows from (i) of the corollary to Theorem 6. Observe that (3.15) and (3.16) are equivalent to (1.10).

From the above observations we see that the equations (3.9) and (3.10) are strongly oscillatory [respectively strongly nonoscillatory] if and only if (1.10) [respectively (1.11)] is satisfied.

(The second step) Because of the assumption (1.9) on $g(t)$ there exist constants $\gamma > 1$ and $T > a$ such that

$$\gamma^{-1} t \leqq g(t) \leqq \gamma t, \qquad t \geqq T. \tag{3.17}$$

For this choice of γ consider the equations (3.9) and (3.10).

By virtue of Theorem 5 the strong oscillation of (A_λ) implies that of (3.9) and the strong oscillation of (3.10) implies that of (A_λ). This fact combined with the equivalence of (3.9) and (3.10) observed above shows that (A_λ) is strongly oscillatory if and only if so is (3.9) or (3.10). It follows that (A_λ) is strongly oscillatory if and only if (1.10) holds.

On the other hand, again by Theorem 5, the strong nonoscillation of (A_λ) implies that of (3.10) and the strong nonoscillation of (3.9) implies that of (A_λ). In view of the equivalence of (3.9) and (3.10) it turns out that (A_λ) is strongly nonoscillatory if and only if so is (3.9) or (3.10), and hence that the strong nonoscillation of (A_λ) implies and is implied by the condition (1.11). This completes the proof.

REMARK. When specialized to the case $\alpha = 1$, Theorem 4 covers the second order versions of the main results of Kusano [2] and Naito [6].

EXAMPLE 1. Consider the equation

$$(|y'(t)|^{\alpha} \operatorname{sgn} y'(t))' + \lambda t^{-\beta} |y(\gamma t + \delta)|^{\alpha} \operatorname{sgn} y(\gamma t + \delta) = 0, \qquad t \geqq a, \tag{3.18}$$

where $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and δ are constants. This is a special case of (A_λ)

in which $g(t) = \gamma t + \delta$ and $q(t) = t^{-\beta}$. Suppose that $a > 0$ is large enough so that $\gamma t + \delta > 0$ for $t \geqq a$.

Let $\beta \leqq 1$. Then, since $\int_a^\infty q(t)\,dt = \infty$, Theorem 1 implies that (3.18) is strongly oscillatory.

Let $\beta > 1$. Then, $\int_a^\infty q(t)\,dt < \infty$ and

$$\lim_{t\to\infty} t^\alpha \int_t^\infty q(s)\,ds = \begin{cases} \infty & (1 < \beta < \alpha + 1), \\ \dfrac{1}{\alpha} & (\beta = \alpha + 1), \\ 0 & (\beta > \alpha + 1). \end{cases}$$

Therefore, by Theorem 4, (3.18) is strongly oscillatory if $1 < \beta < \alpha + 1$ and strongly nonoscillatory if $\beta > \alpha + 1$.

Finally let $\beta = \alpha + 1$. Then, applying Theorems 2 and 3, we conclude that (3.18) is oscillatory if either

$$\gamma > 1 \quad \text{and} \quad \lambda > \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$$

or

$$\gamma \leqq 1 \quad \text{and} \quad \lambda > \gamma^{-\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1},$$

and that (3.18) is nonoscillatory if either

$$\gamma \leqq 1 \quad \text{and} \quad \lambda < \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}$$

or

$$\gamma > 1 \quad \text{and} \quad \lambda < \gamma^{-\alpha}\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}.$$

4. Extensions

Our purpose here is to show that, by a simple transformation, the results regarding (A), (A_λ) can easily be carried over to the more general equations (C), (C_λ):

(C) $$(p(t)|y'(t)|^\alpha \operatorname{sgn} y'(t))' + q(t)|y(g(t))|^\alpha \operatorname{sgn} y(g(t)) = 0,$$

(C_λ) $$(p(t)|y'(t)|^\alpha \operatorname{sgn} y'(t))' + \lambda q(t)|y(g(t))|^\alpha \operatorname{sgn} y(g(t)) = 0,$$

provided that $p(t)$ is a positive continuous function on $[a, \infty)$ such that

$$\int_a^\infty \frac{dt}{(p(t))^{1/\alpha}} = \infty. \tag{4.1}$$

Define the function $P(t)$ by

$$P(t) = \int_a^t \frac{ds}{(p(s))^{1/\alpha}}, \qquad t \geqq a, \tag{4.2}$$

and introduce the transformation $(t, y) \to (\tau, Y)$ given by

$$\tau = P(t), \ Y(\tau) = y(t). \tag{4.3}$$

This transformation reduces (C) and (C_λ) to

(D) $$(|\dot{Y}(\tau)|^\alpha \operatorname{sgn} \dot{Y}(\tau))^{\boldsymbol{\cdot}} + Q(\tau)|Y(G(\tau))|^\alpha \operatorname{sgn} Y(G(\tau)) = 0$$

and

(D_λ) $$(|\dot{Y}(\tau)|^\alpha \operatorname{sgn} \dot{Y}(\tau))^{\boldsymbol{\cdot}} + \lambda Q(\tau)|Y(G(\tau))|^\alpha \operatorname{sgn} Y(G(\tau)) = 0,$$

respectively, where a dot denotes differentiation with respect to τ, and

$$Q(\tau) = (p(t))^{1/\alpha} q(t), \ G(\tau) = P \circ g \circ P^{-1}(\tau). \tag{4.4}$$

Since (D) and (D_λ) are in the form of (A) and (A_λ), oscillation and nonoscillation criteria for (D) and (D_λ) are derived directly from Theorems 1–4. Taking into account the fact that $G(\tau) \leqq \tau$ [or $G(\tau) \geqq \tau$] if $g(t) \leqq t$ [or $g(t) \geqq t$] and using the relations

$$\int_0^\infty Q(\tau) = \int_a^\infty q(t)\,dt,$$

$$\tau^\alpha \int_\tau^\infty Q(\sigma)\,d\sigma = (P \circ P^{-1}(\tau))^\alpha \int_{P^{-1}(\tau)}^\infty q(s)\,ds,$$

$$(G(\tau))^\alpha \int_\tau^\infty Q(\sigma)\,d\sigma = (P \circ g \circ P^{-1}(\tau))^\alpha \int_{P^{-1}(\tau)}^\infty q(s)\,ds,$$

one can translate the results for (D) and (D_λ) into the following oscillation and nonoscillation theorems for (C) and (C_λ).

THEOREM 1′. *The equation* (C) *is oscillatory if*

$$\int_a^\infty q(t)\,dt = \infty. \tag{4.5}$$

THEOREM 2′. *Suppose that*

$$\int_a^\infty q(t)\,dt < \infty. \tag{4.6}$$

and that $g(t) \geqq t$ *for* $t \geqq a$.

(i) *The equation* (C) *is oscillatory if either*

$$\limsup_{t\to\infty} (P(t))^\alpha \int_t^\infty q(s)\,ds > 1 \tag{4.7}$$

or

$$\liminf_{t\to\infty} (P(t))^\alpha \int_t^\infty q(s)\,ds > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}. \tag{4.8}$$

(ii) *The equation* (C) *is nonoscillatory if*

$$\limsup_{t\to\infty} (P(g(t)))^\alpha \int_t^\infty q(s)\,ds < \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}. \tag{4.9}$$

THEOREM 3′. *Suppose that* (4.6) *holds and that* $g(t) \leqq t$ *for* $t \geqq a$.

(i) *The equation* (C) *is oscillatory if either*

$$\limsup_{t\to\infty} (P(g(t)))^\alpha \int_t^\infty q(s)\,ds > 1 \tag{4.10}$$

or

$$\liminf_{t\to\infty} (P(g(t)))^\alpha \int_t^\infty q(s)\,ds > \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}. \tag{4.11}$$

(ii) *The equation* (C) *is nonoscillatory if*

$$\limsup_{t\to\infty} (P(t))^\alpha \int_t^\infty q(s)\,ds < \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}. \tag{4.12}$$

THEOREM 4′. *Suppose that* (4.6) *holds and that*

$$0 < \liminf_{t\to\infty} \frac{P(g(t))}{P(t)}, \qquad \limsup_{t\to\infty} \frac{P(g(t))}{P(t)} < \infty. \tag{4.13}$$

(i) *The equation* (C_λ) *is strongly oscillatory if and only if*

$$\limsup_{t\to\infty} (P(t))^\alpha \int_t^\infty q(s)\,ds = \infty. \tag{4.14}$$

(ii) *The equation* (C_λ) *is strongly nonoscillatory if and only if*

(4.15) $$\lim_{t\to\infty} (P(t))^{\alpha} \int_t^{\infty} q(s)\,ds = 0.$$

These theorems are illustrated by the following example.

EXAMPLE 2. Consider the equation

(4.16) $$(t^{\alpha}|y'(t)|^{\alpha} \operatorname{sgn} y'(t))' + \lambda t^{-1}(\log t)^{-\beta}|y(t^{\gamma})|^{\alpha} \operatorname{sgn} y(t^{\gamma}) = 0, \qquad t \geqq a,$$

where α, β and γ are positive constants and $a > 1$. This equation is a special case of (D_{λ}) in which

$$p(t) = t^{\alpha},\ g(t) = t^{\gamma} \quad \text{and} \quad q(t) = t^{-1}(\log t)^{-\beta}.$$

The function $p(t)$ clearly satisfies (4.1) and the function $P(t)$ defined by (4.2) can be taken to be $P(t) = \log t$.

If $\beta \leqq 1$, then (4.16) is strong oscillatory by Theorem 1′.

If $\beta > 1$, then

$$\lim_{t\to\infty} (P(t))^{\alpha} \int_t^{\infty} q(s)\,ds = \begin{cases} \infty & (1 < \beta < \alpha + 1), \\ \dfrac{1}{\alpha} & (\beta = \alpha + 1), \\ 0 & (\beta > \alpha + 1). \end{cases}$$

From Theorem 4′ it follows that (4.16) is strongly oscillatory if $1 < \beta < \alpha + 1$ and strongly nonoscillatory if $\beta > \alpha + 1$.

Suppose that $\beta = \alpha + 1$. Then, applying Theorems 2′ and 3′, we conclude that:

(i) in case $\gamma < 1$, (4.16) is oscillatory if

$$\lambda > \gamma^{-\alpha}\left(\frac{\alpha}{\alpha + 1}\right)^{\alpha + 1}$$

and nonoscillatory if

$$\lambda < \left(\frac{\alpha}{\alpha + 1}\right)^{\alpha + 1};$$

(ii) in case $\gamma > 1$, (4.16) is oscillatory if

$$\lambda > \left(\frac{\alpha}{\alpha + 1}\right)^{\alpha + 1}$$

and nonoscillatory if

$$\lambda < \gamma^{-\alpha}\left(\frac{\alpha}{\alpha + 1}\right)^{\alpha + 1}.$$

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