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Attractors for two-dimensional equations of thermal convection in the presence of the dissipation function

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Introduction

Thermal convection in an infinite fluid layer heated from below is extensively studied; it represents the simplest example of hydrodynamic instability and transition to turbulence ([3, 4, 13, 16]). Suppose that an infinite layer is occupied by a viscous incompressible fluid and that the lower boundary of the layer is warmer than the upper one. Then the conductive static state is a unique solution when the temperature difference, say $\lambda > 0$, is sufficinetly small. Beyond a certain value of λ, the conductive state loses its stability and the convection rolls appear. When λ is increased, the boundaries of rolls oscillate periodically in time. When λ is increased further, the boundaries of rolls oscillate in a less regular manner. Finally, for sufficiently large λ, the flow seems totally unstructured. This sequence of transitions leads to the concept of the strange attractor [25]. The complexity of the motion is due to the complicated structure of the attractor. Thus, an analysis of attractors is important for understanding the observed motions. In recent years, many authors have obtained the bounds for the dimension of the attractor for the Navier-Stokes equations in terms of the physical numbers ([1, 5, 6, 7, 10, 17, 18, 23, 24]). These works indicate that the dimension of the attractor for the Navier-Stokes equations may be identified with the number of degrees of freedom. For the study of attractors, the reader is referred to [2, 12, 17, 28] and references therein.

In this paper we consider the attractor for the two-dimensional Bénard convection problem in which the dissipative heating is taken into account, and give a bound for its Hausdorff dimension. The non-dimensional form of the governing equations for the velocity u, the pressure p and the fluctuation θ of the temperature from the static state is written as

$$\begin{aligned} &\frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u + \nabla p = e_2 f(\theta), \\ &\nabla \cdot u = 0, \end{aligned} \tag{0.1}$$

$$\frac{\partial \theta}{\partial t} - \kappa \Delta \theta + u \cdot \nabla \theta - e_2 \cdot u = \frac{\eta \nu}{2} D(u) : D(u). \tag{0.2}$$

Here, $e_2 = (0, 1)$ is the unit vector opposite to the direction of gravity; ν, κ and η are non-dimensional physical parameters to be defined in section 1; $D(u): D(u)$ is the dissipation function:

$$D(u): D(u) = \sum_{i,k=1}^{2} \left(\frac{\partial u^i}{\partial x_k} + \frac{\partial u^k}{\partial x_i} \right)^2 ;$$

and f is a smooth function on R satisfying

$$f(\theta) = \theta, \tag{0.3}$$

or

$$|f|_\infty \equiv \sup_{\theta \in R} |f(\theta)| < \infty, \ |f'|_\infty < \infty \quad \text{and} \quad |f''|_\infty < \infty. \tag{0.4}$$

Here f' and f'' denote the first and the second derivatives of f, respectively. Equations (0.1) and (0.2) are supplemented by the boundary condition:

$$u = 0; \ \theta = 0 \ \text{at} \ x_2 = 0, 1,$$

and the periodicity condition:

$$u|_{x_1=0} = u|_{x_1=\alpha}, \quad \theta|_{x_1=0} = \theta|_{x_1=\alpha};$$

$$\left.\frac{\partial u}{\partial x_1}\right|_{x_1=0} = \left.\frac{\partial u}{\partial x_1}\right|_{x_1=\alpha}, \quad \left.\frac{\partial \theta}{\partial x_1}\right|_{x_1=0} = \left.\frac{\partial \theta}{\partial x_1}\right|_{x_1=\alpha}.$$

In case $\eta = 0$ and $f(\theta) = \theta$, one obtains the usual Boussinesq equations in which the dissipation effect is ignored. For the Boussinesq equations, Foias, Manley and Temam [8] proved the existence of the associated global attractor to which all solutions converge as $t \to \infty$, and derived a bound for its Hausdorff dimension in terms of the significant physical parameters: the Rayleigh number Ra, the Prandtl number Pr and the Grashof number Gr. Their bound is

$$c|\Omega|(1 + Pr)(1 + Gr + Ra),$$

where $|\Omega|$ is Lebesgue measure of $\Omega = (0, \alpha) \times (0, 1)$ and c is a constant depending only on the flow geometry. Their bound is very similar to the bound for the number of degrees of freedom obtained by purely physical argument [20]. In this paper we prove the existence of the global attractor $\mathscr{A}$ for the convection problem in which the dissipative heating is not ignored and show that the Hausdorff dimension of $\mathscr{A}$ is bounded by

$$c|\Omega|(1 + Pr)(1 + Gr + Gr^{1/2} Ra + O(\eta)),$$

with a constant c depending only on the flow geometry.

The paper is organized as follows. In section 1 we deduce the non-dimensional form of the governing equations and set up the corresponding nonlinear evolutionary problem in a Hilbert space. The existence and uniqueness of the solutions and their continuous dependence on initial values, which are necessary for studying attractors, are discussed in section 2. We prove the existence of weak solutions for initial values in $\{u_0, \theta_0\} \in L^2 \times L^2$. When f satisfies (0.4), the solutions are unique if the initial values belong to $H_0^1 \times L^2$. In this case we can consider the corresponding dynamical system in $H_0^1 \times L^2$ and, consequently, discuss the associated attractor. Unfortunately, the uniqueness problem is unsettled when $f(\theta) = \theta$. However, we shall show in both cases that the weak solution converges, as $\eta \to 0$, to a (unique) weak solution of the Boussinesq equations. This shows the validity of the Boussinesq approximation for a viscous incompressible fluid with small η. We prove this convergence in section 3. From section 4 on we consider only the case where f satisfies (0.4) and discuss the global attractor associated with the dynamical system under consideration. To prove the existence of the global attractor in section 4, we first establish the existence of an absorbing set and then prove the uniform compactness for large t of the nonlinear semigroup defining the dynamical system. For the Boussinesq equations, i.e., when $\eta = 0$, Foias, Manley and Temam [8] proved the existence of the global attractor, applying the maximum principle to the governing equation of θ. In case $\eta > 0$, however, the maximum principle is not applicable because of the presence of the dissipation function. Besides, since the dissipation function contains the quadratic nonlinearity of ∇u, we need to estimate the higher order derivatives of the velocity in order to get the desired result. These estimates are obtained in sections 2 and 3 by applying the energy integral method for higher order derivatives of velocity and temperature, which are more complicated than those given in [8]. Section 5 is devoted to estimating the Hausdorff dimension of the global attractor. Foias, Manley and Temam derived their sharp estimate in [8], applying the Lieb-Thirring inequality [19]. We proceed basically in the same way as in [8] to obtain our bound for the Hausdorff dimension. However, since the maximum principle applied to the temperature does not provide useful estimates for solutions, our bound for the dimension of the attractor does not reduce to the bound of [8] as $\eta \to 0$.

In this paper we discuss only the two-dimensional problem, mainly applying the energy integral method. Even in this two-dimensional case, the presence of the dissipation function forces us to estimate higher order derivatives of unknown functions in order to get various results by passing to the limit in the equations. In the three-dimensional problem, the situation becomes much more complicated. In this case, the elementary energy estimate is obviously insufficient for ensuring the possibility of passage to the limit in the equations; and so we know nothing about the existence of a global weak solution as introduced in [14] for the case $\eta = 0$. As for the existence of strong solutions, we have been able to control the higher order norms of (approximate) solutions only locally in time for general initial data, and globally in time for small initial data. So we know only that the strong solutions exist loally in time for general initial data, and globally in time for small initial data, as proved in [15]. This situation does not seem to be improved even if we employ different approximation schemes. In fact, the problem is closely related to the problem of regularity of weak solutions of the three-dimensional Navier-Stokes equations.

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1. Preliminaries

We consider a two-dimensional infinite layer $\mathrm{R} \times (0, d)$ and assume that the layer is occupied by a viscous incompressible fluid. Suppose further that the temperature at the lower boundary $x_2 = 0$ equals θ_0, and the temperature at the upper boundary $x_2 = d$ equals θ_1. Here θ_0 and θ_1 are constants such that $\theta_0 > \theta_1$. Then the governing equations for the velocity $u = (u^1, u^2)$, the pressure p and the temperature θ are written as follows (see [3, 4]):

$$\rho_0 \frac{\partial u}{\partial t} - \rho_0 \nu_0 \Delta u + \rho_0 u \cdot \nabla u + \nabla p = \rho_0 g(1 - \gamma_0 f(\theta - \theta_0)) e_2,$$

$$\nabla \cdot u = 0,$$

$$\rho_0 C_v \frac{\partial \theta}{\partial t} - \rho_0 C_v \kappa_0 \Delta \theta + \rho_0 C_v u \cdot \nabla \theta = \frac{\nu_0}{2} D(u) \colon D(u).$$

Here, $e_2 = (0, 1)$ is the unit vector opposite to the direction of gravity; $-ge_2$ is the acceleration due to the gravity; ρ_0 is the constant mean density; γ_0 is the volume expansion coefficient; C_v is the specific heat at constant volume; ν_0 is the kinematic viscosity coefficient; κ_0 is the thermometric conductivity coefficient; and $D(u) \colon D(u)$ is the dissipation function

$$D(u) \colon D(u) = \sum_{i,k=1}^{2} \left(\frac{\partial u^i}{\partial x_k} + \frac{\partial u^k}{\partial x_i} \right)^2 .$$

At the boundaries $x_2 = 0, d$, the velocity u is prescribed by

$$u = 0 \quad \text{at} \quad x_2 = 0, d,$$

and the temperature θ is prescribed by

$$\theta = \theta_0 \quad \text{at} \quad x_2 = 0,$$
$$\theta = \theta_1 \quad \text{at} \quad x_2 = d.$$

We further require u, p and θ to be periodic in the x_1-direction with period α_0. To obtain the non-dimensional form of equations, we introduce the following non-dimensional variables:

$$\tilde{x} = \frac{x}{d}, \qquad \tilde{\theta} = \frac{\theta - \hat{\theta}}{\theta_0 - \theta_1},$$
$$\tilde{u} = \frac{u}{(\gamma_0 g(\theta_0 - \theta_1)d)^{1/2}}, \qquad \tilde{t} = \left(\frac{\gamma_0 g(\theta_0 - \theta_1)}{d}\right)^{1/2} t,$$
$$\tilde{p} = \frac{p - \hat{p}}{\rho_0 \gamma_0 g(\theta_0 - \theta_1)d}, \qquad \tilde{f}(\tilde{\theta}) = \frac{f(\theta - \theta_0) - f(\hat{\theta} - \theta_0)}{\theta_0 - \theta_1},$$

and non-dimensional parameters:

$$Ra = \frac{\gamma_0 g(\theta_0 - \theta_1)d^3}{\nu_0 \kappa_0}, \quad Pr = \frac{\nu_0}{\kappa_0}, \quad Gr = \frac{Ra}{Pr},$$

where

$$\hat{\theta} = \frac{\theta_1 - \theta_0}{d} x_2 + \theta_0;$$
$$\hat{p} = \rho_0 g(x_2 - \gamma_0 F(x_2)), \quad F(x_2) = \int_0^{x_2} f\left(\frac{(\theta_1 - \theta_0)}{2d}\tau\right) d\tau.$$

The non-dimensional numbers Ra, Pr and Gr are called the Rayleigh, the Prandtl and the Grashof numbers, respectively. Using these new variables, we obtain, after omitting tildes,

$$\begin{aligned} &\frac{\partial u}{\partial t} - \nu \Delta u + u \cdot \nabla u + \nabla p = f(\theta) e_2, \\ &\nabla \cdot u = 0, \end{aligned} \tag{1.1}$$

$$\frac{\partial \theta}{\partial t} - \kappa \Delta \theta + u \cdot \nabla \theta - e_2 \cdot u = \frac{\eta \nu}{2} D(u) : D(u), \tag{1.2}$$

where

$$\nu = \left(\frac{Pr}{Ra}\right)^{1/2}, \quad \kappa = \left(\frac{1}{PrRa}\right)^{1/2} \quad \text{and} \quad \eta = \frac{\gamma_0 g d}{C_v}.$$

The boundary conditions at $x_2 = 0, 1$ are

$$u = 0, \ \theta = 0 \quad \text{at} \quad x_2 = 0, 1,$$

and u, p and θ are required to be periodic in x_1 with period $\alpha = \alpha_0/d$. We consider equations (1.1) and (1.2) in the domain $\Omega = (0, \alpha) \times (0, 1)$, together with the above boundary conditions and initial conditions:

$$u|_{t=0} = u_0, \qquad \theta|_{t=0} = \theta_0.$$

We denote this initial boundary value problem by $(\mathrm{BE})_\eta$. In case $f(\theta) = \theta$, we formally obtain the Boussinesq equations from (1.1) and (1.2) by passing to the limit $\eta \to 0$.

We now introduce some notation. $L^p(\Omega)$ (resp. $H^m(\Omega)$) denotes the usual L^p-space (resp. the L^2-Sobolev space of order m) and its norm is denoted by $\|\cdot\|_p$ (resp. $\|\cdot\|_{m,2}$). We define the function spaces $C^2_{0,per}(\bar{\Omega})$, L^2_σ, $H^1_{0,per}$ and V by

$$C^2_{0,per}(\bar{\Omega}) = \{\psi|_\Omega;\ \psi \in C^2(\mathrm{R}^2),\ \psi|_{x_2=0,1} = 0,\ \psi(x_1 + \alpha, x_2) = \psi(x_1, x_2)\},$$

$$L^2_\sigma = \{u \in L^2(\Omega)^2;\ \nabla \cdot u = 0,\ u^2|_{x_2=0,1} = 0,\ u^1|_{x_1=0} = u^1|_{x_1=\alpha}\},$$

$$H^1_{0,per} = \{\theta \in H^{1,2}(\Omega);\ \theta|_{x_2=0,1} = 0,\ \theta|_{x_1=0} = \theta|_{x_1=\alpha}\},$$

$$V = \{u \in (H^1_{0,per})^2;\ \nabla \cdot u = 0\}.$$

Then, the following Helmholtz decomposition holds [27, 28]:

$$L^2(\Omega)^2 = L^2_\sigma \oplus (L^2_\sigma)^\perp \equiv L^2_\sigma \oplus G_{per},$$

where

$$G_{per} = \{\nabla q;\ q \in H^{1,2}(\Omega),\ q|_{x_1=0} = q|_{x_1=\alpha}\}.$$

In terms of the associated orthogonal projector P onto L^2_σ, we define the Stokes operator A by

$$Au = -P\Delta u, \quad u \in D(A) = \left\{u \in V \cap H^2(\Omega)^2;\ \left.\frac{\partial u}{\partial x_1}\right|_{x_1=0} = \left.\frac{\partial u}{\partial x_1}\right|_{x_1=\alpha}\right\}.$$

It is well known that A is positive definite and self-adjoint in the Hilbert space L^2_σ satisfying $\|A^{1/2}u\|_2 = \|\nabla u\|_2$, and there exists a constant $C > 0$ such that

$$\|u\|_{2,2} \le C\|Au\|_2 \qquad \text{for } u \in D(A). \tag{1.3}$$

We also define the operator B by

$$B\theta = -\Delta\theta, \quad \theta \in D(B) = \left\{\theta \in H^1_{0,per} \cap H^2(\Omega);\ \left.\frac{\partial \theta}{\partial x_1}\right|_{x_1=0} = \left.\frac{\partial \theta}{\partial x_1}\right|_{x_1=\alpha}\right\}.$$

The operator B is positive definite and self-adjoint in $L^2(\Omega)$ satisfying $\|B^{1/2}\theta\|_2 = \|\nabla\theta\|_2$, and there exists a constant $C > 0$ such that

$$\|\theta\|_{2,2} \le C\|B\theta\|_2 \qquad \text{for } \theta \in D(B). \tag{1.4}$$

We sometimes use the following Gagliardo-Nirenberg inequalities [9]:

$$\|u\|_p \le C\|u\|_{1,2}^{1-2/p}\|u\|_2^{2/p}, \qquad (2 \le p < \infty), \tag{1.5}$$

and

$$\|u\|_\infty \le C\|u\|_{2,2}^{1/2}\|u\|_2^{1/2}, \tag{1.6}$$

where C is a constant depending only on p and α. In particular, for $u \in H^1_{0,per}$ we have

$$\|u\|_p \le C\|\nabla u\|_2^{1-2/p}\|u\|_2^{2/p}, \qquad (2 \le p < \infty). \tag{1.7}$$

Throughout this paper, the letter C denotes constants which may vary from line to line.

2. Existence of solutions of $(\mathrm{BE})_\eta$

We discuss in this section the existence and uniqueness of solutions and their continuous dependence on the initial data. We begin with

DEFINITION. Given $\{u_0, \theta_0\} \in L^2_\sigma \times L^2(\Omega)$, a pair of functions $\{u(t), \theta(t)\}$ defined for $t \ge 0$ is called a *weak solution* of problem $(\mathrm{BE})_\eta$ if

$$u \in L^\infty(0, T; L^2_\sigma) \cap L^2(0, T; V), \qquad \theta \in L^{4/3}(0, T; L^2(\Omega)) \cap L^2(0, T; L^{4/3}(\Omega))$$

for all $T > 0$, and the identities

$$\begin{aligned} -\int_0^T (u, v')dt + \int_0^T [\nu(\nabla u, \nabla v) + (u \cdot \nabla u, v)]dt \\ = (u_0, v(0)) + \int_0^T (f(\theta)e_2, v)dt \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} -\int_0^T (\theta, \psi')dt - \int_0^T [\kappa(\theta, \Delta\psi) + (u \cdot \nabla\psi, \theta)]dt \\ = (\theta_0, \psi(0)) + \int_0^T \left[(e_2 \cdot u, \psi) + \frac{\eta\nu}{2}(D(u): D(u), \psi)\right]dt \end{aligned} \tag{2.2}$$

hold for all $v \in W^{1,2}(0, T; L^2_\sigma) \cap L^2(0, T; V)$ with $v(T) = 0$ and all $\psi \in C^1([0, T];$

$C^2_{0,per}(\bar{\Omega})$) with $\psi(T)=0$. Here and in what follows $(\cdot\,,\,\cdot)$ denotes the scalar product of L^2.

REMARKS. (i) It follows that if $\{u, \theta\}$ is a weak solution, then u is continuous from $[0, T]$ to L^2_σ and θ is continuous from $[0, T]$ to the dual space $D(B)^*$ of $D(B)$. Since $L^2(\Omega) \subset D(B)^*$, we find that the initial conditions make sense. The continuity of u can be proved by using the standard theory of the two-dimensional Navier-Stokes equations as given in [27]. The continuity of θ is proved as folows. Let $\psi \in D(B)$. By (1.4), (1.5) and (1.7) with $p=4$, we see that

$$\begin{aligned} |(u\cdot\nabla\psi, \theta)| &\le \|u\|_4 \|\nabla\psi\|_4 \|\theta\|_2 \le C\|u\|_2^{1/2}\|\nabla u\|_2^{1/2}\|\nabla\psi\|_2^{1/2}\|\psi\|_{2,2}^{1/2}\|\theta\|_2 \\ &\le C\|u\|_2^{1/2}\|\nabla u\|_2^{1/2}\|B\psi\|_2\|\theta\|_2. \end{aligned}$$

Furthermore, inequalities (1.4) and (1.6) imply that

$$|(D(u)\colon D(u), \psi)| \le \|\nabla u\|_2^2 \|\psi\|_\infty \le \|\nabla u\|_2^2 \|B\psi\|_2.$$

We also have

$$|(\theta, \Delta\psi)| \le \|\theta\|_2 \|B\psi\|_2$$

and

$$|(e_2\cdot u, \psi)| \le \|\theta\|_2 \|\psi\|_2 \le C\|\theta\|_2 \|B\psi\|_2.$$

It thus follows from (2.2) that

$$\frac{d\theta}{dt} \in L^1(0, T; D(B)^*),$$

which implies the desired continuity of θ.

(ii) We can also prove that $\{u, \theta\}$ satisfies

$$\begin{aligned} (u, v)(t) - (u, v)(s) = &\int_s^t [(u\otimes u, \nabla v) - (\nabla u, \nabla v)]\,d\tau \\ &+ \int_s^t [(e_2 f(\theta), v) + (u, v')]\,d\tau, \end{aligned} \tag{2.3}$$

$$\begin{aligned} (\theta, \psi)(t) - (\theta, \psi)(s) = &\int_s^t [(u\cdot\nabla\psi, \theta) + (\theta, \Delta\psi) + (e_2\cdot u, \psi)]\,d\tau \\ &+ \int_s^t \left[\eta\frac{\nu}{2}(D(u)\colon D(u), \psi) + (\theta, \psi')\right] d\tau \end{aligned} \tag{2.4}$$

for all $0 \le s \le t \le T$, $v \in L^2(0, T; V) \cap W^{1,2}(0, T; L^2_\sigma)$ and all $\psi \in L^4(0, T; D(B))$

$\cap W^{1,4}(0, T; L^2(\Omega))$. Indeed, define the function $\zeta_\varepsilon \in C[0, T]$ by

$$\zeta_\varepsilon(\tau) = \begin{cases} 0 & \text{for } 0 \le \tau \le s, \; t \le s \le T, \\ 1 & \text{for } s + \varepsilon \le \tau \le t - \varepsilon, \\ \varepsilon^{-1}(\tau - s) & \text{for } s \le \tau \le s + \varepsilon, \\ -\varepsilon^{-1}(\tau - t + \varepsilon) & \text{for } t - \varepsilon \le \tau \le t, \end{cases}$$

and substitute $v\zeta_\varepsilon$ into (2.1) as a testing function, to obtain

$$\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t} (u, v)d\tau - \frac{1}{\varepsilon}\int_{s}^{s+\varepsilon} (u, v)d\tau$$
$$= \int_s^t [(u \otimes u, \nabla v) - (\nabla u, \nabla v) + (e_2 f(\theta), v) + (u, v')]\zeta_\varepsilon d\tau.$$

Since $(u, v)(\tau)$ is continuous on $[0, T]$, letting $\varepsilon \to 0$ yields the desired identity (2.3). The identity (2.4) is proved similarly, so the details are omitted.

Before stating our existence result, we recall the assumptions on f: f is a smooth function on R satisfying

$$f(\theta) = \theta \tag{0.3}$$

or

$$|f|_\infty \equiv \sup_{\theta \in R} |f(\theta)| < \infty, \; |f'|_\infty < \infty \quad \text{and} \quad |f''|_\infty < \infty. \tag{0.4}$$

Our existence result is the following

THEOREM 2.1. (i) *Let f satisfy* (0.3) *or* (0.4). *Assume that $\eta < 1$ if f satisfies* (0.3). *Then, for each initial value $\{u_0, \theta_0\} \in L^2_\sigma \times L^2(\Omega)$, there exists a weak solution $\{u, \theta\}$ of problem* $(\mathrm{BE})_\eta$ *defined for all $t \ge 0$.*

(ii) *Let f satisfy* (0.4) *and let $\{u_0, \theta_0\} \in V \times L^2(\Omega)$. Then*

$$u \in C([0, T]; V) \cap L^2(0, T; D(A)), \quad \theta \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_{0,per})$$

for all $T > 0$, and the weak solution $\{u, \theta\}$ is unique in the class

$$(C([0, T]; V) \cap L^2(0, T; D(A))) \times (C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_{0,per})).$$

(iii) *Let f satisfy* (0.4) *and let $\{u_{0,i}, \theta_{0,i}\} \in V \times L^2(\Omega)$, $i = 1, 2$. For each $T > 0$ there exists a constant $C = C(T)$ such that if $\{u_i(t), \theta_i(t)\}$, $i = 1, 2$, denote, respectively, the weak solutions corresponding to initial values $\{u_{0,i}, \theta_{0,i}\}$, $i = 1, 2$, then*

$$\|\nabla u_2(t) - \nabla u_1(t)\|_2^2 + \|\theta_2(t) - \theta_1(t)\|_2^2 \le C(\|\nabla u_{0,2} - \nabla u_{0,1}\|_2^2 + \|\theta_{0,2} - \theta_{0,1}\|_2^2).$$

(iv) *Under the assumption of* (ii) *any weak solution $\{u, \theta\}$ satisfies*

$$\frac{du}{dt} \in L^2(0, T; L^2_\sigma) \cap L^\infty(\delta, T; L^2_\sigma) \cap L^2(\delta, T; V), \quad \frac{d^2u}{dt^2} \in L^2(\delta, T; V^*),$$

and

$$\theta \in L^\infty(\delta, T; H^1_{0,per}) \cap L^2(\delta, T; D(B)), \quad \frac{d\theta}{dt} \in L^2(\delta, T; L^2(\Omega)),$$

for all $\delta > 0$ *and* $T > 0$ *such that* $0 < \delta < T < \infty$, *where* V^* *is the dual space of* V.

We first prove (i) in the case where $f(\theta) = \theta$. The proof is divided into two parts. In the first part we construct approximate solutions by the semigroup method. In the second one we discuss the convergence of the approxomate solutions to a weak solution with the aid of some a priroi estimates. We next prove (i), (ii) and (iii) under assumption (0.4), modifying the arguments given in the case of assumption (0.3). Finally, we prove assertion (iv). For simplicity in notation we assume in this section that

$$\nu = \kappa = 1.$$

CONSTRUCTION OF APPROXIMATE SOLUTIONS. We construct approximate solutions, solving the integral equation:

$$\begin{pmatrix} u_k(t) \\ \theta_k(t) \end{pmatrix} = \begin{pmatrix} e^{-tA} u_{0,k} + S_1[u_k, \theta_k] \\ e^{-tB}\theta_{0,k} + S_2[u_k, \theta_k] \end{pmatrix}, \tag{2.5}$$

where

$$S_1[u_k, \theta_k] = \int_0^t e^{-(t-s)A} P(-u_k \cdot \nabla u_k + e_2 \theta_k)(s) ds,$$

$$S_2[u_k, \theta_k] = \int_0^t e^{-(t-s)B}\left(-u_k \cdot \nabla \theta_k + e_2 \cdot u_k + \frac{\eta}{2} D(\overline{u_k}) : D(\overline{u_k})\right)(s) ds,$$

$$u_{0,k} = (I + k^{-1}A)^{-m} u_0, \quad \overline{u_k} = (I + k^{-1}A)^{-m} u_k, \quad \theta_{0,k} = (I + k^{-1}B)^{-m}\theta_0.$$

The integer $m > 0$ is taken sufficiently large so that $\overline{u_k}$ is bounded in $\bar{\Omega}$.

We will solve (2.5) by applying the Banach fixed point theorem. To do so, we need

LEMMA 2.2. *Let* $1 \le q \le 2 \le p \le \infty$. *If* $v \in L^2_\sigma \cap L^q$ *and* $\psi \in L^q(\Omega)$, *then*

$$\| A^{j/2} e^{-tA} v \|_p \le C t^{-j/2 - (1/q - 1/p)} \| v \|_q,$$

$$\| B^{j/2} e^{-tB} \psi \|_p \le C t^{-j/2 - (1/q - 1/p)} \| \psi \|_q$$

for $j = 0, 1$, *with* C *independent of* v, ψ *and* t.

PROOF OF LEMMA 2.2. We prove only the assertions for the Stokes operator A; the case of the operator B is treated similarly. Suppose first $q = 2 \le p < \infty$. By (1.7) and the well-known estimate

$$\| A^{m/2} e^{-tA} v \|_2 \le C t^{-m/2} \| v \|_2, \qquad (m \ge 0), \tag{2.6}$$

we have

$$\begin{aligned} \| A^{j/2} e^{-tA} v \|_p &\le C \| A^{j/2} e^{-tA} v \|_2^{2/p} \| \nabla A^{j/2} e^{-tA} v \|_2^{1-2/p} \\ &= C \| A^{j/2} e^{-tA} v \|_2^{2/p} \| A^{(j+1)/2} e^{-tA} v \|_2^{1-2/p} \\ &\le C t^{-j/2-(1/2-1/p)} \| v \|_2. \end{aligned}$$

Suppose next that $p = \infty$, $q = 2$. In this case, using (1.3) and (1.6), we obtain

$$\begin{aligned} \| A^{j/2} e^{-tA} v \|_\infty &\le C \| A^{j/2} e^{-tA} v \|_2^{1/2} \| A^{j/2} e^{-tA} v \|_{2,2}^{1/2} \\ &\le C \| A^{j/2} e^{-tA} v \|_2^{2/p} \| A^{1+j/2} e^{-tA} v \|_2^{1-2/p} \\ &\le C t^{-j/2-1/2} \| v \|_2. \end{aligned}$$

In case $1 \le q \le 2 = p$, we set $r = q/(q-1)$. Since $r \ge 2$, the foregoing results yield

$$|(e^{-tA} v, w)| = |(v, e^{-tA} w)| \le \| v \|_q \| e^{-tA} w \|_r \le C t^{-(1/q-1/2)} \| v \|_q \| w \|_2$$

for $w \in L^2_\sigma$. Thus, by duality, we have

$$\| e^{-tA} v \|_2 \le C t^{-(1/q-1/2)} \| v \|_q.$$

This, together with (2.6), implies that

$$\| A^{1/2} e^{-tA} v \|_2 \le C t^{-1/2} \| e^{-tA/2} v \|_2 \le C t^{-1/2-(1/q-1/2)} \| v \|_q.$$

Suppose finally that $1 \le q \le 2 \le p \le \infty$. The foregoing results then yield

$$\| A^{1/2} e^{-tA} v \|_p \le C t^{-1/2-(1/2-1/p)} \| e^{-tA/2} v \|_2 \le C t^{-1/2-(1/q-1/p)} \| v \|_q.$$

This completes the proof of Lemma 2.2.

We now turn to the proof of Theorem 2.1 (i), assuming (0.3). For given $M > 0$ and $T > 0$, we define the closed set $X(M, T)$ of $C([0, T]; D(A^{1/2})) \times C([0, T]; D(B^{1/2}))$ by

$$\begin{aligned} X(M, T) = \{ U = \{u, \theta\} \in C([0, T]; D(A^{1/2})) \\ \times C([0, T]; D(B^{1/2})); \; \| U \|_X \le M \}, \end{aligned}$$

where

$$|||U|||_X = \sup_{[0,T]} (\|A^{1/2}u(t)\|_2 + \|B^{1/2}\theta(t)\|_2).$$

and consider on $X(M, T)$ the mapping

$$\Psi_k(u, \theta) = \begin{pmatrix} e^{-tA}u_{0,k} + S_1[u, \theta] \\ e^{-tB}\theta_{0,k} + S_2[u, \theta] \end{pmatrix}.$$

Due to Lemma 2.2, we have

$$\begin{aligned} \|A^{j/2}e^{-(t-s)A}P(u \cdot \nabla u)\|_2 &\le C(t-s)^{-j/2-1/6}\|u \cdot \nabla u\|_{3/2} \\ &\le C(t-s)^{-j/2-1/6}\|u\|_6\|\nabla u\|_2 \\ &\le C(t-s)^{-j/2-1/6}\|\nabla u\|_2^2, \\ \|B^{j/2}e^{-(t-s)B}(u \cdot \nabla\theta)\|_2 &\le C(t-s)^{-j/2-1/6}\|u \cdot \nabla\theta\|_{3/2} \\ &\le C(t-s)^{-j/2-1/6}\|u\|_6\|\nabla\theta\|_2 \\ &\le C(t-s)^{-j/2-1/6}\|\nabla u\|_2\|\nabla\theta\|_2 \end{aligned}$$

for $j = 0, 1$. Here we have used the Poincaré-Sobolev inequality:

$$\|u\|_p \le C\|\nabla u\|_2, \qquad (1 \le p < \infty).$$

Since $\bar{u} = (I + k^{-1}A)^{-m}u$, it follows that

$$\|\nabla\bar{u}\|_2 \le C_k\|u\|_2, \quad \|A\bar{u}\|_2 \le C_k\|u\|_2.$$

Therefore, applying (1.3), and (1.5) with $p = 4$, we have

$$\begin{aligned} \|B^{j/2}e^{-(t-s)B}D(\bar{u})\colon D(\bar{u})\|_2 &\le C(t-s)^{-j/2}\|\nabla\bar{u}\|_4^2 \\ &\le C(t-s)^{-j/2}\|\nabla\bar{u}\|_2\|A\bar{u}\|_2 \\ &\le C_k(t-s)^{-j/2}\|u\|_2^2. \end{aligned}$$

We thus obtain

$$\begin{aligned} |||\Psi_k(U_k)|||_X &\le |||\{u_{0,k}, \theta_{0,k}\}|||_X \\ &\quad + C_1(T^{1/3} + T^{1/2} + T^{5/6} + T)M^2 + C_2(T^{1/2} + T)M \end{aligned} \tag{2.7}$$

with C_1 and C_2 depending on k. Similarly, we can prove

$$|||\Psi_k(U_{1,k}) - \Psi_k(U_{2,k})|||_X \le C_3(M, T)\,|||U_{1,k} - U_{2,k}|||_X, \tag{2.8}$$

with

$$C_3(M, T) = C_4[(T^{1/3} + T^{1/2} + T^{5/6} + T)M + T^{1/2} + T],$$

where C_4 depends on k. Now, choose $M > 0$ with $|||\{u_{0,k}, \theta_{0,k}\}|||_X \le M/2$ and then fix $T > 0$ such that

$$C_1(T^{1/3} + T^{1/2} + T^{5/6} + T)M^2 + C_2(T^{1/2} + T)M \leq M/2;$$
$$C_3(M, T) \leq 1/2.$$

Then, it follows from (2.7) and (2.8) that Ψ_k is a contraction from $X(M, T)$ into itself; so there exists a unique $U_k = \{u_k, \theta_k\}$ in $X(M, T)$ which solves (2.5) on $[0, T]$.

We next show the global existence of U_k. For this purpose, it suffices to derive an a priori bound for $|||U_k|||_X$. By a standard result in the theory of parabolic evolution equations, the function U_k satisfies

$$\frac{du_k}{dt} + Au_k = P(-u_k \cdot \nabla u_k + e_2\theta_k), \qquad u_k(0) = u_{0,k}, \tag{2.9}$$

$$\frac{d\theta_k}{dt} + B\theta_k = -u_k \cdot \nabla\theta_k + e_2 \cdot u_k + \frac{\eta}{2} D(\overline{u_k}) : D(\overline{u_k}), \qquad \theta_k(0) = \theta_{0,k}. \tag{2.10}$$

Multiplying (2.9) by u_k, we obtain

$$\frac{d}{dt}\|u_k\|_2^2 + \|\nabla u_k\|_2^2 = (e_2\theta_k, u_k). \tag{2.11}$$

Let ϕ_k be a function satisfying

$$\frac{d\phi_k}{dt} + B\phi_k = -u_k \cdot \nabla\phi_k + e_2 \cdot u_k, \qquad \phi_k(0) = \theta_{0,k}.$$

Then the function $\psi_k - \phi_k$ satisfies

$$\frac{d\psi_k}{dt} + B\psi_k + u_k \cdot \nabla\psi_k = \frac{\eta}{2} D(\overline{u_k}) : D(\overline{u_k}) \geq 0, \qquad \psi_k(0) = 0. \tag{2.12}$$

The maximum principle gives

$$\psi_k \geq 0 \quad \text{in} \quad \Omega, \quad \text{and} \quad \frac{\partial\psi_k}{\partial x_2} \leq 0 \quad \text{at} \quad x_2 = 0, 1.$$

Multiplying (2.12) by $1 - x_2$ and then integrating in Ω, we have

$$\frac{d}{dt}(\psi_k, 1 - x_2) + (B\psi_k, 1 - x_2) = -(e_2\psi_k, u_k) + \frac{\eta}{2}(D(\overline{u_k}) : D(\overline{u_k}), 1 - x_2).$$

This, together with (2.11), then yields

$$\frac{d}{dt}\left[\frac{1}{2}\|u_k\|_2^2 + (\psi, 1 - x_2)\right] + \|\nabla u_k\|_2^2 + (B\psi_k, 1 - x_2)$$
$$= (e_2\phi_k, u_k) + \frac{\eta}{2}(D(\overline{u_k}) : D(\overline{u_k}), 1 - x_2).$$

Since

$$(B\psi_k, 1-x_2) = -\int_{x_2=0} \frac{\partial \psi_k}{\partial x_2} dS \geq 0$$

and

$$\int_\Omega D(\overline{u_k}): D(\overline{u_k})dx = 2\|\nabla \overline{u_k}\|_2^2 = 2\|A^{1/2}\overline{u_k}\|_2^2 \leq 2\|A^{1/2}u_k\|_2^2 = 2\|\nabla u_k\|_2^2,$$

we see that

$$\frac{d}{dt}\left[\frac{1}{2}\|u_k\|_2^2 + (\psi_k, 1-x_2)\right] + (1-\eta)\|\nabla u_k\|_2^2 \leq (e_2\phi_k, u_k).$$

But, as shown in [8, Lemma 2.2], we have

$$\|\phi_k(t)\|_2^2 \leq |\Omega| + 2\|\theta_{0,k}\|_2^2 e^{-2t}.$$

This gives

$$(e_2\phi_k, u_k) \leq \|\phi_k\|_2\|u_k\|_2 \leq \|u_k\|_2^2 + \|\phi_k\|_2^2 \leq \|u_k\|_2^2 + |\Omega| + 2\|\theta_{0,k}\|_2^2 e^{-2t}.$$

We thus obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|u_k\|_2^2 + (\psi_k, 1-x_2)\right] + (1-\eta)\|\nabla u_k\|_2^2 \leq |\Omega| + 2\|\theta_{0,k}\|_2^2 e^{-2t} + \|u_k\|_2^2.$$

Integrating this on $[0, t]$ and using the fact that $\psi_k \geq 0$ in Ω, we get

$$\|u_k(t)\|_2^2 + (1-\eta)\int_0^t \|\nabla u_k\|_2^2 ds \leq \|u_{0,k}\|_2^2 + |\Omega|t + \|\theta_{0,k}\|_2^2 + \int_0^t \|u_k\|_2^2 ds.$$

This implies that, for any $T > 0$,

$$\|u_k(t)\|_2^2 + \int_0^t \|\nabla u_k\|_2^2 ds \leq C, \qquad (0 \leq t \leq T), \tag{2.13}$$

with $C = C(\|u_0\|_2, \|\theta_0\|_2, T, \eta)$ independent of k.

We next estimate $\|\theta_k(t)\|_2$. Multiplying (2.5) by θ_k, we have

$$\frac{1}{2}\frac{d}{dt}\|\theta_k\|_2^2 + \|\nabla\theta_k\|_2^2 = (e_2\cdot u_k, \theta_k) + \frac{\eta}{2}(D(\overline{u_k}): D(\overline{u_k}), \theta_k).$$

By (1.3), (1.5) with $p = 4$ and the Poincaré inequality, the right-hand side is estimated as

$$|(e_2\cdot u_k, \theta_k)| \leq \|u_k\|_2\|\theta_k\|_2 \leq \|u_k\|_2^2 + \frac{1}{4}\|\theta_k\|_2^2 \leq \|u_k\|_2^2 + \frac{1}{4}\|\nabla\theta_k\|_2^2;$$

$$|(D(\overline{u_k}) : D(\overline{u_k}), \theta_k)| \leq C \| \nabla \overline{u_k} \|_4^2 \| \theta_k \|_2 \leq C \| \nabla \overline{u_k} \|_2 \| A \overline{u_k} \|_2 \| \nabla \theta_k \|_2$$

$$\leq C(k) \| u_k \|_2^2 \| \nabla \theta_k \|_2 \leq C(k) \| u_k \|_2^4 + \frac{1}{4} \| \nabla \theta_k \|_2^2.$$

It thus follows that

$$\frac{d}{dt} \| \theta_k \|_2^2 + \| \nabla \theta_k \|_2^2 \leq C(k)(\| u_k \|_2^4 + \| u_k \|_2^2).$$

This, together with (2.13), implies that, for any $T > 0$,

$$\| \theta_k \|_2^2 + \int_0^t \| \nabla \theta_k \|_2^2 ds \leq C(\| u_0 \|_2, \| \theta_0 \|_2, T, k).$$

We can now deduce an a priori bound for $\| \nabla u_k \|_2$. We take the scalar product of (2.9) with Au_k, to get

$$\frac{1}{2} \frac{d}{dt} \| \nabla u_k \|_2^2 + \| Au_k \|_2^2 \leq |(u_k \cdot \nabla u_k, Au_k)| + |(e_2 \theta_k, Au_k)|$$

$$\leq C \| u_k \|_4 \| \nabla u_k \|_4 \| Au_k \|_2 + \| \theta_k \|_2 \| Au_k \|_2$$

$$\leq C \| u_k \|_2^{1/2} \| \nabla u_k \|_2 \| Au_k \|_2^{3/2} + \| \theta_k \|_2 \| Au_k \|_2$$

$$\leq \frac{1}{2} \| Au_k \|_2^2 + C \| u_k \|_2^2 \| \nabla u_k \|_2^2 \| \nabla u_k \|_2^2 + 2 \| \theta_k \|_2^2,$$

whence,

$$\frac{d}{dt} \| \nabla u_k \|_2^2 + \| Au_k \|_2^2 \leq C(\| u_k \|_2^2 \| \nabla u_k \|_2^2 \| \nabla u_k \|_2^2 + \| \theta_k \|_2^2).$$

Applying the classical Gronwall lemma and (2.13), we have, for any $T > 0$,

$$\| \nabla u_k(t) \|_2^2 \, ds \leq C(T, k).$$

Similarly, we have, for any $T > 0$,

$$\| \nabla \theta_k(t) \|_2^2 + \int_0^t \| B\theta_k \|_2^2 \, ds \leq C(T, k).$$

We thus obtain the desired a priori bound for $||| U_k |||_X$, which ensures the global existence of $U_k = \{u_k, \theta_k\}$.

Convergence of approximate solutions. To show the convergence of $\{u_k, \theta_k\}$ to a weak solution, we first derive a priori bounds for $\{u_k, \theta_k\}$, which are uniformly valid in k. By (2.13), u_k is bounded in $L^\infty(0, T; L^2_\sigma) \cap L^2(0, T; V)$

for all $T>0$. To deduce a priori bounds for θ_k, we introduce, as in [11, 14], the Green function $P_k(x, t; y, s)$ of the problem:

$$\frac{\partial\theta}{\partial t}-\kappa\Delta\theta+u_k\cdot\nabla\theta=0, \qquad \theta|_{t=0}=\theta_{0,k},$$

$$\theta|_{x_2=0,1}=0, \quad \theta|_{x_1=0}=\theta|_{x_1=\alpha}, \quad \left.\frac{\partial\theta}{\partial x_1}\right|_{x_1=0}=\left.\frac{\partial\theta}{\partial x_1}\right|_{x_1=\alpha}.$$

As is well known, $P_k(x, t; y, s)$ satisfies

$$\int_\Omega P_k(x, t; y, s)dx\leq 1, \qquad \int_\Omega P_k(x, t; y, s)dy\leq 1. \tag{2.14}$$

Moreover, the estimate

$$0<P_k(x, t; y, s)\leq C(t-s)^{-1} \tag{2.15}$$

of Nash holds with C independent of k (see [21, 22]). In terms of the function $P_k(x, t; y, s)$, θ_k is represented as

$$\begin{aligned}\theta_k(x, t)&=\int_\Omega P_k(x, t; y, 0)\theta_{0,k}(y)dy\\&+\int_0^t\int_\Omega P_k(x, t; y, s)\left[e_2\cdot u_k+\frac{\eta}{2}D(\overline{u_k})\colon D(\overline{u_k})\right](y, s)dyds.\end{aligned}$$

Therefore, we see from (2.14) and (2.15) that

$$\begin{aligned}\|\theta_k(t)\|_p&\leq C\left(t^{-(1-1/p)}\|\theta_{0,k}\|_1+\int_0^t\|u_k\|_p ds+\eta\int_0^t(t-s)^{-(1-1/p)}\|\nabla\overline{u_k}\|_2^2ds\right)\\&\leq C\left(t^{-(1-1/p)}\|\theta_{0,k}\|_1+t^{1/2}\left(\int_0^t\|\nabla u_k\|_2^2ds\right)^{1/2}+\eta\int_0^t(t-s)^{-(1-1/p)}\|\nabla u_k\|_2^2ds\right),\end{aligned}$$

which shows that θ_k is bounded in $L^q(0, T; L^p(\Omega))$ for all $T>0$, with p and q satisfying $1/p+1/q>1$, $1<q<\infty$. In particular, θ_k is bounded in $L^{4/3}(0, T; L^2(\Omega))\cap L^2(0, T; L^{4/3}(\Omega))$ for all $T>0$.

We next estimate the time-derivative du_k/dt. Applying (1.7), we have

$$|(P(u_k\cdot\nabla u_k), v)|=|(u_k\cdot\nabla v, u_k)|\leq C\|u_k\|_4^2\|\nabla v\|_2\leq C\|u_k\|_2\|\nabla u_k\|_2\|\nabla v\|_2$$

for all $v\in V$. This implies that $P(u_k\cdot\nabla u_k)$ is bounded in $L^2(0, T; V^*)$ for all $T>0$. (V^* denotes the dual space of V.) We also get the estimate

$$|(P(e_2\theta_k), v)|=|(e_2\theta_k, v)|\leq\|\theta_k\|_{4/3}\|v\|_4\leq C\|\theta_k\|_{4/3}\|\nabla v\|_2$$

for all $v\in V$, which implies that $P(g\theta_k)$ is bounded in $L^2(0, T; V^*)$ for all

$T>0$. So, we easily see that du_k/dt is bounded in $L^2(0, T; V^*)$ for all $T>0$.

We can now discuss the convergence of $\{u_k, \theta_k\}$. By the above argument, there exists a subsequence, denoted also by $\{u_k, \theta_k\}$, which converges to a $\{u, \theta\}$ in the sense that:

$$(2.16)\quad \begin{aligned} &u_k \longrightarrow u && *\text{-weakly in } L^\infty(0, T; L^2_\sigma) \text{ and weakly in } L^2(0, T; V),\\ &u_k \longrightarrow u && \text{strongly in } L^2(0, T; L^2_\sigma),\\ &\frac{du_k}{dt} \longrightarrow \frac{du}{dt} && \text{weakly in } L^2(0, T; V^*),\\ &\theta_k \longrightarrow \theta && \text{weakly in } L^{4/3}(0, T; L^2(\Omega)) \cap L^2(0, T; L^{4/3}(\Omega)) \end{aligned}$$

for all $T>0$.

We next show that $\{u, \theta\}$ is a weak solution of problem $(\mathrm{BE})_\eta$. Since (2.16) implies

$$u \in L^\infty(0, T; L^2_\sigma) \cap L^2(0, T; V), \quad \theta \in L^{4/3}(0, T; L^2(\Omega)) \cap L^2(0, T; L^{4/3}(\Omega))$$

for all $T>0$, we have only to show that $\{u, \theta\}$ satisfies identities (2.1) and (2.2). Observe that $\{u_k, \theta_k\}$, $k \geq 1$, satisfy

$$(2.17)\quad \begin{aligned} -\int_0^T (u_k, v')dt &+ \int_0^T [(\nabla u_k, \nabla v) + (u_k \cdot \nabla u_k, v)]\,dt \\ &= (u_{0,k}, v(0)) + \int_0^T (e_2 \theta_k, v)dt, \end{aligned}$$

and

(2.18)
$$\begin{aligned} -\int_0^T (\theta_k, \psi')dt &- \int_0^T [(\theta_k, \Delta\psi) + (u_k \cdot \nabla\psi, \theta_k)]\,dt \\ &= (\theta_{0,k}, \psi(0)) + \int_0^T \left[(e_2 \cdot u_k, \psi) + \frac{\eta}{2}(D(\overline{u_k}) : D(\overline{u_k}), \psi)\right]dt \end{aligned}$$

for all $v \in W^{1,2}(0, T; L^2_\sigma) \cap L^2(0, T; V)$ with $v(T)=0$ and all $\psi \in C^1([0, T]; C^2_{0,per}(\bar{\Omega}))$ with $\psi(T)=0$. Due to (2.16), passing to the limit $k \to \infty$ in (2.17) yields

$$\begin{aligned} -\int_0^T (u, v')dt &+ \int_0^T [(\nabla u, \nabla v) + (u \cdot \nabla u, v)]\,dt \\ &= (u_0, v(0)) + \int_0^T (e_2 \theta, v)dt. \end{aligned}$$

We next show, by letting $k \to \infty$ in (2.18), that $\{u, \theta\}$ satisfies

$$\begin{aligned}(2.19)\qquad & -\int_0^T (\theta, \psi')dt - \int_0^T [(\theta, \Delta\psi) + (u \cdot \nabla\psi, \theta)]dt \\ &= (\theta_0, \psi(0)) + \int_0^T \left[(e_2 \cdot u, \psi) + \frac{\eta}{2}(D(u): D(u), \psi) \right] dt\end{aligned}$$

for all $\psi \in C^1([0, T]; C^2_{0,per}(\bar{\Omega}))$ with $\psi(T) = 0$. To do so, we need only show that:

$$(2.20)\qquad u_k \longrightarrow u \text{ strongly in } L^2(0, T; V) \qquad \text{for all } T > 0,$$

which enables us to pass to the limit $k \to \infty$ in (2.18) in the term involving the dissipation function. Let $v \in W^{1,2}(0, T; L^2_\sigma) \cap L^2(0, T; V)$. Multiplying (2.9) by $v - u_k$ and then integrating over Ω, we have

$$\|\nabla u_k\|_2^2 = \langle u_k', v \rangle - \frac{1}{2}\frac{d}{dt}\|u_k\|_2^2 + (\nabla u_k, \nabla v) + (u_k \cdot \nabla u_k, v) - (e_2\theta_k, v - u_k),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between V^* and V. We integrate this on $(0, t)$ to get

$$\begin{aligned}\int_0^t \|\nabla u_k\|_2^2 ds &= \int_0^t \langle u_k', v \rangle ds - \frac{1}{2}\|u_k(t)\|_2^2 + \frac{1}{2}\|u_{0,k}\|_2^2 \\ &\quad + \int_0^t (\nabla u_k, \nabla v)ds + \int_0^t (u_k \cdot \nabla u_k, v)ds - \int_0^t (e_2\theta_k, v - u_k)ds.\end{aligned}$$

Taking the upper limit then yields

$$\begin{aligned}(2.21)\qquad \limsup_{k\to\infty} \int_0^t \|\nabla u_k\|_2^2 ds &= \int_0^t \langle u', v \rangle ds - \frac{1}{2}\|u(t)\|_2^2 + \frac{1}{2}\|u_0\|_2^2 \\ &\quad + \int_0^t (\nabla u, \nabla v)ds + \int_0^t (u \cdot \nabla u, v)ds - \int_0^t (e_2\theta, v - u)ds\end{aligned}$$

for a.e. t. Since we can take $v = u$ in (2.21), we obtain

$$(2.22)\qquad \limsup_{k\to\infty} \int_0^t \|\nabla u_k\|_2^2 ds = \int_0^t \|\nabla u\|_2^2 ds$$

for a.e. t. On the other hand, by the lower semicontinuity of $\|\nabla \cdot\|_2$ with respect to the weak convergence, we have

$$\liminf_{k\to\infty} \int_0^t \|\nabla u_k\|_2^2 ds \geq \int_0^t \|\nabla u\|_2^2 ds.$$

This, together with (2.22), implies that

$$\lim_{k\to\infty}\int_0^t \|\nabla u_k\|_2^2 ds = \int_0^t \|\nabla u\|_2^2 ds$$

for a.e. t. Since u_k converges to u weakly in $L^2(0, T; V)$ for all $T > 0$, we obtain (2.20). This completes the proof of Theorem 2.1 (i) in case $f(\theta) = \theta$.

Uniqueness and continuous dependence on inital data. We prove Theorem 2.1 (ii) and (iii), assuming that f satisfies (0.4). Let $\{u_0, \theta_0\} \in V \times L^2(\Omega)$. We construct approximate solutions by solving the integral equations (2.5) with $\theta_k e_2$ replaced by $f(\theta_k)e_2$. As in the proof of (i) for the case $f = \theta$, one can construct local solutions $U_k = \{u_k, \theta_k\}$. We will derive an a priori bound for $\|\nabla u_k(t)\|_2 + \|\nabla \theta_k(t)\|_2$ which ensures the global existence of each approximate solution. In this case, we shall also obtain stronger bounds which imply the convergence of approximate solutions to a weak solution in the uniqueness class

$$(C([0, T]; V)\cap L^2(0, T; D(A))) \times (C([0, T]; L^2(\Omega))\cap L^2(0, T; H^1_{0,per})).$$

The function U_k satisfies

$$\frac{du_k}{dt} + Au_k = P(-u_k\cdot\nabla u_k + e_2 f(\theta_k)), \qquad u_k(0) = u_{0,k}, \tag{2.23}$$

$$\frac{d\theta_k}{dt} + B\theta_k = -u_k\cdot\nabla\theta_k + e_2\cdot u_k + \frac{\eta}{2}D(\overline{u_k}) : D(\overline{u_k}), \qquad \theta_k(0) = \theta_{0,k}. \tag{2.24}$$

Taking the scalar product of (2.23) with u_k, we obtain

$$\frac{d}{dt}\|u_k\|_2^2 + \|\nabla u_k\|_2^2 = (e_2 f(\theta_k), u_k).$$

Since f is bounded, the right-hand side is estimated as

$$|e_2 f(\theta_k), u_k)| \le |\Omega|^{1/2}|f|_\infty \|u_k\|_2 \le \frac{1}{2}\|\nabla u_k\|_2^2 + |\Omega||f|_\infty^2.$$

We thus obtain

$$\frac{d}{dt}\|u_k\|_2^2 + \|\nabla u_k\|_2^2 \le |\Omega||f|_\infty^2,$$

which yields

$$\|u_k(t)\|_2^2 + \int_0^t \|\nabla u_k\|_2^2 ds \le \|u_{0,k}\|_2^2 + |\Omega||f|_\infty^2 t \le \|u_0\|_2^2 + |\Omega||f|_\infty^2 t \le C,$$

with $C = C(T)$ independent of k. Thus, u_k is bounded in $L^\infty(0, T; L^2_\sigma) \cap L^2(0, T; V)$. We next deduce an a priori bound for $\|\nabla u_k(t)\|_2$. Taking the scalar product of (2.23) with Au_k, we have, as in the case $f = \theta$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u_k\|_2^2 + \|Au_k\|_2^2 &\le |(u_k \cdot \nabla u_k, Au_k)| + |(e_2 f(\theta_k), Au_k)| \\ &\le C\|u_k\|_4\|\nabla u_k\|_4\|Au_k\|_2 + \|f(\theta_k)\|_2\|Au_k\|_2 \\ &\le C\|u_k\|_2^{1/2}\|\nabla u_k\|_2\|Au_k\|_2^{3/2} + \|f(\theta_k)\|_2\|Au_k\|_2 \\ &\le \frac{1}{2}\|Au_k\|_2^2 + C\|u_k\|_2^2\|\nabla u_k\|_2^2\|\nabla u_k\|_2^2 + 2\|f(\theta_k)\|_2^2.\end{aligned}$$

Since f is bounded, we obtain

$$\frac{d}{dt}\|\nabla u_k\|_2^2 + \|Au_k\|_2^2 \le C(\|u_k\|_2^2\|\nabla u_k\|_2^2\|\nabla u_k\|_2^2 + |\Omega||f|_\infty^2).$$

Applying the classical Gronwall lemma, we have, for any $T > 0$,

$$\|\nabla u_k(t)\|_2^2 + \int_0^t \|Au_k\|_2^2 ds \le C \tag{2.25}$$

with $C = C(\|\nabla u_0\|_2, |\Omega|, |f|_\infty, T)$ independent of k. Thus, u_k is bounded in $L^\infty(0, T; V) \cap L^2(0, T; D(A))$.

We next derive a priori bounds for θ_k. Taking the scalar product of (2.24) with θ_k gives

$$\frac{1}{2}\frac{d}{dt}\|\theta_k\|_2^2 + \|B^{1/2}\theta_k\|_2^2 = (e_2 \cdot u_k, \theta_k) + \frac{\eta}{2}(D(\bar{u}_k) : D(\bar{u}_k), \theta_k).$$

As in the case $f = \theta$, the right-hand side is estimated as:

$$|(e_2 \cdot u_k, \theta_k)| \le \|u_k\|_2\|\theta_k\|_2 \le \|u_k\|_2^2 + \|\theta_k\|_2^2;$$

$$\begin{aligned}|(D(\bar{u}_k) : D(\bar{u}_k), \theta_k)| &\le C\|\nabla \bar{u}_k\|_4^2\|\theta_k\|_2 \le C\|\nabla \bar{u}_k\|_2\|A\bar{u}_k\|_2\|\theta_k\|_2 \\ &\le C(\|\nabla u_k\|_2^2\|Au_k\|_2^2 + \|\theta_k\|_2^2).\end{aligned}$$

Hence we obtain

$$\frac{d}{dt}\|\theta_k\|_2^2 + \|\nabla\theta_k\|_2^2 \le C(\|u_k\|_2^2 + \|\nabla u_k\|_2^2\|Au_k\|_2^2 + \|\theta_k\|_2^2), \tag{2.26}$$

and so, for all $T > 0$,

$$\|\theta_k(t)\|_2^2 + \int_0^t \|\nabla\theta_k\|_2^2 ds \le C, \qquad (0 \le t \le T),$$

with $C = C(T)$ independent of k. Thus, θ_k is bounded in $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_{0,per})$. In the same way, we can deduce, for any $T > 0$,

$$\|\nabla \theta_k(t)\|_2^2 + \int_0^t \|B\theta_k\|_2^2 ds \leq C, \qquad (0 \leq t \leq T). \tag{2.27}$$

However, $C = C(T)$ here depends on k, because we assume only that $\theta_0 \in L^2(\Omega)$. It follows from (2.25) and (2.27) that U_k exists globally in time. On the other hand, we have already seen that u_k is bounded in $L^\infty(0, T; V) \cap L^2(0, T; D(A))$ and θ_k is bounded in $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_{0,per})$. Thus, we find that

$$\frac{du_k}{dt} \text{ is bounded in } \; L^2(0, T; L^2_\sigma),$$

$$\frac{d\theta_k}{dt} \text{ is bounded in } \; L^2(0, T; (H^1_{0,per})^*).$$

By a classical compactness theorem [27, Th. III. 3.1], we see that

$$\begin{array}{ll}
u_k \longrightarrow u & *\text{-weakly in } L^\infty(0, T; V) \text{ and weakly in } L^2(0, T; D(A)), \\
u_k \longrightarrow u & \text{strongly in } L^2(0, T; V), \\
\dfrac{du_k}{dt} \longrightarrow \dfrac{du}{dt} & \text{weakly in } L^2(0, T; L^2_\sigma), \\
\theta_k \longrightarrow \theta & *\text{-weakly in } L^\infty(0, T; L^2(\Omega)) \text{ and weakly in } L^2(0, T; H^1_{0,per}), \\
\theta_k \longrightarrow \theta & \text{strongly in } L^2(0, T; L^2(\Omega)), \\
\dfrac{d\theta_k}{dt} \longrightarrow \dfrac{d\theta}{dt} & \text{weakly in } L^2(0, T; (H^1_{0,per})^*).
\end{array}$$

It is easy to see that $\{u, \theta\}$ is a weak solution of problem $(\mathrm{BE})_\eta$. The fact that $\{u, \theta\} \in C([0, T]; V) \times C([0, T]; L^2(\Omega))$ follows from

$$u \in L^2(0, T; D(A)), \quad \frac{du}{dt} \in L^2(0, T; L^2_\sigma)$$

and

$$\theta \in L^2(0, T; H^1_{0,per}), \quad \frac{d\theta}{dt} \in L^2(0, T; (H^1_{0,per})^*).$$

We next show the uniqueness in the class as mentioned in Theorem 2.1 (ii), i.e., in the class

$$(C([0, T]; V) \cap L^2(0, T; D(A))) \times (C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_{0,per})).$$

Note that if a weak solution $\{u, \theta\}$ belongs to the above class, then

$$\frac{du}{dt} \in L^2(0, T; L^2_\sigma), \quad \frac{d\theta}{dt} \in L^2(0, T; (H^1_{0,per})^*).$$

To show the uniqueness, it suffices to prove Theorem 2.1 (iii). Let $\{u_{0,i}, \theta_{0,i}\} \in V \times L^2(\Omega)$, $i = 1, 2$. We wish to show that for any $T > 0$ there exists a $C = C(T)$ such that

$$\|\nabla u_2(t) - \nabla u_1(t)\|_2^2 + \|\theta_2(t) - \theta_1(t)\|_2^2 \le C(\|\nabla u_{0,2} - \nabla u_{0,1}\|_2^2 + \|\theta_{0,2} - \theta_{0,1}\|_2^2),$$

where $\{u_i, \theta_i\}$, $i = 1, 2$, are weak solutions with initial values $\{u_{0,i}, \theta_{0,i}\}$, $i = 1, 2$, respectively. Set $u = u_2 - u_1$, $\theta = \theta_2 - \theta_1$. Then

$$\frac{du}{dt} + Au + P(u \cdot \nabla u_2 + u_1 \cdot \nabla u) = P[(f(\theta_2) - f(\theta_1))e_2], \tag{2.28}$$

$$\begin{aligned} &\left\langle \frac{d\theta}{dt}, \psi \right\rangle + (\nabla \theta, \nabla \psi) + (u \cdot \nabla \theta_2 + u_1 \cdot \nabla \theta, \psi) \\ &\qquad = (e_2 \cdot u, \psi) + \frac{\eta}{2}(D(u) : D(u_1 + u_2), \psi) \end{aligned} \tag{2.29}$$

for all $\psi \in H^1_{0,per}$, and

$$u(0) = u_{0,2} - u_{0,1}; \theta(0) = \theta_{0,2} - \theta_{0,1}.$$

Here $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^1_{0,per}$ and $(H^1_{0,per})^*$. We take the scalar product of (2.28) with Au, to get

$$\frac{d}{dt}\|\nabla u\|_2^2 + \|Au\|_2^2 = -(u \cdot \nabla u_2 + u_1 \cdot \nabla u, Au) + ((f(\theta_2) - f(\theta_1))e_2, Au).$$

Using (1.5) and (1.7) with $p = 4$, we have

$$\begin{aligned} |(u \cdot \nabla u_2, Au)| &\le \|u\|_4 \|\nabla u_2\|_4 \|Au\|_2 \\ &\le C\|u\|_2^{1/2}\|\nabla u\|_2^{1/2}\|\nabla u_2\|_2^{1/2}\|Au_2\|_2^{1/2}\|Au\|_2 \\ &\le \frac{1}{8}\|Au\|_2^2 + \|\nabla u_2\|_2\|Au_2\|_2\|\nabla u\|_2^2. \end{aligned}$$

Here we have used the Poincaré inequality: $\|u\|_2 \le \|\nabla u\|_2$. Similarly,

$$|(u_1 \cdot \nabla u, Au)| \le \frac{1}{8}\|Au\|_2^2 + \|u_1\|_2^2\|\nabla u_1\|_2^2\|\nabla u\|_2^2.$$

Writing

$$f(\theta_2) - f(\theta_1) = \theta \int_0^1 f'(\theta_1 + \tau(\theta_2 - \theta_1))d\tau,$$

we obtain

$$|((f(\theta_2) - f(\theta_1))e_2, Au)| \le |f'|_\infty \|\theta\|_2 \|Au\|_2 \le \frac{1}{8} \|Au\|_2^2 + C|f'|_\infty^2 \|\theta\|_2^2.$$

Taking $\psi = \theta$ in (2.29), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_2^2 + \|\nabla\theta\|_2^2 = -(u\cdot\nabla\theta_2, \theta) + (e_2\cdot u, \theta) + \frac{\eta}{2}(D(u): D(u_1 + u_2), \theta).$$

In the same way as above, using (1.3), (1.5) and (1.7) with $p = 4$, we can estimate the right-hand side by

$$\frac{1}{8}\|Au\|_2^2 + \frac{1}{2}\|\theta\|_2^2$$
$$+ C[\|\theta_2\|_2^2 + \|\nabla(u_1 + u_2)\|_2^2 \|A(u_1 + u_2)\|_2^2 + 1]\|\nabla u\|_2^2 + C\|\theta\|_2^2.$$

We thus obtain

$$\begin{aligned}
&\frac{d}{dt}(\|\nabla u\|_2^2 + \|\theta\|_2^2) + \|Au\|_2^2 + \|\nabla\theta\|_2^2 \\
&\quad \le C[\|\nabla u_2\|_2 \|Au_2\|_2 + \|u_1\|_2^2 \|\nabla u_1\|_2^2]\|\nabla u\|_2^2 \\
&\qquad + C\|\nabla(u_1 + u_2)\|_2^2 \|A(u_1 + u_2)\|_2^2 \|\nabla u\|_2^2 \\
&\qquad + C(\|\theta_2\|_2^2 + 1)\|\nabla u\|_2^2 + C\|\theta\|_2^2.
\end{aligned}$$

Since

$u_i \in C([0, T]; V) \cap L^2(0, T; D(A))$ and $\theta_i \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_{0,per})$

for $i = 1, 2$, we see that

$$\frac{d}{dt}(\|\nabla u\|_2^2 + \|\theta\|_2^2) \le k(t)(\|\nabla u\|_2^2 + \|\theta\|_2^2)$$

for some $k(t) \in L^1(0, T)$. It thus follows that

$$\|\nabla u_2(t) - \nabla u_1(t)\|_2^2 + \|\theta_2(t) - \theta_1(t)\|_2^2 \le C(\|\nabla u_{0,2} - \nabla u_{0,1}\|_2^2 + \|\theta_{0,2} - \theta_{0,1}\|_2^2),$$

which proves Theorem 2.1 (iii).

We next prove the assertion (i) for the case f satisfies (0.4). Let $\{u_0, \theta_0\} \in L^2_\sigma \times L^2(\Omega)$. In view of the proof of the assertion (i) for the case $f(\theta) = \theta$, we can see that there exists a subsequence, denoted also by $\{u_k, \theta_k\}$, which convergesto a $\{u, \theta\}$ in the sense that:

$$\begin{aligned}
&u_k \longrightarrow u && \text{*-weakly in } L^\infty(0, T; L^2_\sigma) \text{ and weakly in } L^2(0, T; V),\\
&u_k \longrightarrow u && \text{strongly in } L^2(0, T; L^2_\sigma),\\
&\frac{du_k}{dt} \longrightarrow \frac{du}{dt} && \text{weakly in } L^2(0, T; V^*),\\
&\theta_k \longrightarrow \theta && \text{weakly in } L^{4/3}(0, T; L^2(\Omega)) \cap L^2(0, T; L^{4/3}(\Omega)).
\end{aligned}$$

In this case, however, the convergence of θ_k is too weak to pass to the limit in $f(\theta_k)$, since f is not linear. To overcome this obstacle, we will show that

$$\theta_k(x, t) \longrightarrow \theta(x, t) \quad \text{a.e.} \quad (x, t) \in \Omega \times (0, T). \tag{2.30}$$

If this is proved, then we can see, by the dominated convergence theorem, that

$$\int_0^T (f(\theta_k)e_2, v)dt \longrightarrow \int_0^T (f(\theta)e_2, v)dt$$

for all $v \in L^2(0, T; V)$ as $k \to \infty$, and so we can conclude that $\{u, \theta\}$ is a weak solution of $(\mathrm{BE})_\eta$. (2.30) is proved as follows. Taking the scalar product of (2.23) with tAu_k, we have

$$\frac{1}{2}\frac{d}{dt}(t\|\nabla u_k\|_2^2) + t\|Au_k\|_2^2 = \frac{1}{2}\|\nabla u_k\|_2^2 - t(u_k \cdot \nabla u_k, Au_k) + t(f(\theta_k), Au_k).$$

We majorize the right-hand side as before to obtain

$$\frac{d}{dt}(t\|\nabla u_k\|_2^2) + t\|Au_k\|_2^2 \leq C(t\|\nabla u_k\|_2^2\|u_k\|_2^2\|\nabla u_k\|_2^2 + \|\nabla u_k\|_2^2 + t\|f(\theta_k)\|_2^2).$$

By the classical Gronwall lemma, we have, for $0 \leq t \leq T$,

$$t\|\nabla u_k(t)\|_2^2 + \int_0^T \tau\|Au_k(\tau)\|_2^2 d\tau \leq C(T),$$

which implies that

$$u_k \quad \text{is bounded in} \quad L^\infty(\delta, T; V) \cap L^2(\delta, T; D(A))$$

for all $0 < \delta < T < \infty$. This, together with (2.26), yields that

$$\theta_k \quad \text{is bounded in} \quad L^\infty(\delta, T; L^2(\Omega)) \cap L^2(\delta, T; H^1_{0,per}).$$

It is easy to see from (2.24) that

$$\frac{d\theta_k}{dt} \quad \text{is bounded in} \quad L^2(\delta, T; (H^1_{0,per})^*).$$

Thus, by a classical compactness theorem [27, Th. III. 3.1], we see that

$$\theta_k \longrightarrow \theta \quad \text{strongly in} \quad L^2(\delta, T; L^2(\Omega)),$$

from which we can immediately obtain (2.30). This completes the proof of the assertion (i) for the case f satisfies (0.4).

REFULARITY IN TIME. In this paragraph we prove Theorem 2.1 (iv). The fact that $du/dt \in L^2(0, T; L^2_\sigma)$ is already shown in the proof of convergence of $\{u_k, \theta_k\}$ (see the discussion after (2.27)). Here we first show that

$$\theta \in L^\infty(\delta, T; H^1_{0,per}) \cap L^2(\delta, T; D(B)) \quad \text{and} \quad \frac{d\theta}{dt} \in L^2(\delta, T; L^2(\Omega)). \tag{2.31}$$

We multiply (2.24) by $tB\theta_k$ and integrate over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}(t\,\|\nabla\theta_k\|_2^2) + t\,\|B\theta_k\|_2^2 = \frac{1}{2}\|\nabla\theta_k\|_2^2 - t(u_k\cdot\nabla\theta_k, B\theta_k)$$

$$+ t\frac{\eta}{2}(D(\overline{u_k}) : D(\overline{u_k}), B\theta_k) + t(e_2\cdot u_k, B\theta_k).$$

Estimating the right-hand side as in the proof of (i), we have

$$\frac{d}{dt}(t\,\|\nabla\theta_k\|_2^2) + t\,\|B\theta_k\|_2^2 \le C(\|\nabla\theta_k\|_2^2 + \|Au_k\|_2^2 + \|u_k\|_2^2),$$

with $C = C(T)$ independent of k. Upon integration, this gives

$$t\,\|\nabla\theta_k\|_2^2 + \int_0^t s\,\|B\theta_k\|_2^2 ds \le C\int_0^t (\|\nabla\theta_k\|_2^2 + \|Au_k\|_2^2 + \|u_k\|_2^2)ds \le C,$$

where $C = C(T)$ is independent of k. Hence,

$$\theta_k \ \text{is bounded in} \ L^\infty(\delta, T; H^1_{0,per}) \cap L^2(\delta, T; D(B)), \tag{2.32}$$

and so the first assertion of (2.31) is obtained by letting $k \to \infty$. Using (2.32) and the relation

$$\frac{d\theta_k}{dt} = -B\theta_k - u_k\cdot\nabla\theta_k + e_2\cdot u_k + \frac{\eta}{2}D(\overline{u_k}) : D(\overline{u_k})$$

as well as the estimates

$$\|u_k\cdot\nabla\theta_k\|_2 \le \|u_k\|_4\,\|\nabla\theta_k\|_4 \le C\,\|u_k\|_2^{1/2}\,\|\nabla u_k\|_2^{1/2}\,\|\nabla\theta_k\|_2^{1/2}\,\|B\theta_k\|_2^{1/2};$$

$$\|D(\overline{u_k}) : D(\overline{u_k})\|_2^2 \le C\,\|\nabla\overline{u_k}\|_4^4 \le C\,\|\nabla u_k\|_2\,\|Au_k\|_2,$$

we see that

$$\frac{d\theta_k}{dt} \text{ is bounded in } L^2(\delta, T; L^2(\Omega)),$$

and so the second assertion of (2.31) is obtained by passing to the limit.

We next show that

$$\frac{du}{dt} \in L^\infty(\delta, T; L^2_\sigma) \cap L^2(\delta, T; V) \text{ and } \frac{d^2u}{dt^2} \in L^2(\delta, T; V^*). \tag{2.33}$$

Consider the difference quotients

$$(u_k)_\tau(t) = \tau^{-1}(u_k(t+\tau) - u_k(t)), \quad (\theta_k)_\tau(t) = \tau^{-1}(\theta_k(t+\tau) - \theta_k(t)).$$

From (2.23) we have

$$\frac{d(u_k)_\tau}{dt} + A(u_k)_\tau = -P((u_k)_\tau \cdot \nabla u_k + u_k \cdot \nabla(u_k)_\tau - e_2(\theta_k)_\tau H(\theta_k)),$$

where

$$H(\theta_k)(t) = \int_0^1 f'(\theta_k(t) + \zeta(\theta_k(t+\tau) - \theta_k(t)))d\zeta.$$

The standard method gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(u_k)_\tau\|_2^2 + \|\nabla(u_k)_\tau\|_2^2 &\le |((u_k)_\tau \cdot \nabla u_k, (u_k)_\tau)| + |(e_2(\theta_k)_\tau H(\theta_k), (u_k)_\tau)| \\ &\le |((u_k)_\tau \cdot \nabla(u_k)_\tau, u_k)| + |f'|_\infty \|(\theta_k)_\tau\|_2 \|(u_k)_\tau\|_2.\end{aligned}$$

By (1.3), (1.5) and (1.7) with $p = 4$, the first term on the right-hand side is estimated as

$$\begin{aligned}|((u_k)_\tau \cdot \nabla(u_k)_\tau, u_k)| &\le \|(u_k)_\tau\|_4 \|\nabla(u_k)_\tau\|_2 \|u_k\|_4 \\ &\le C\|(u_k)_\tau\|_2^{1/2} \|\nabla(u_k)_\tau\|_2^{3/2} \|u_k\|_2^{1/2} \|\nabla u_k\|_2^{1/2} \\ &\le \frac{1}{2}\|\nabla(u_k)_\tau\|_2^2 + C\|(u_k)_\tau\|_2^2,\end{aligned}$$

since u_k is bounded in $C([0, T]; V)$. Hence we get

$$\frac{d}{dt}\|(u_k)_\tau\|_2^2 + \|\nabla(u_k)_\tau\|_2^2 \le C(\|(u_k)_\tau\|_2^2 + \|(\theta_k)_\tau\|_2^2),$$

and so, denoting $u_k' = du_k/dt$ and $\theta_k' = d\theta_k/dt$,

$$\|(u_k)_\tau(t)\|_2^2 + \int_s^t \|\nabla(u_k)_\tau\|_2^2 d\sigma \le \|(u_k)_\tau(s)\|_2^2 + C\int_s^t (\|(u_k)_\tau\|_2^2 + \|(\theta_k)_\tau\|_2^2)d\sigma$$

$$\leq \|(u_k)_\tau(s)\|_2^2 + C\int_0^T (\|u_k'\|_2^2 + \|\theta_k'\|_2^2)d\sigma$$

$$\leq \|(u_k)_\tau(s)\|_2^2 + C,$$

where $C = C(T)$ is independent of k and τ. Here we have used the boundedness of u_k' and θ_k' in $L^2(0, T; L_\sigma^2)$ and $L^2(0, T; L^2(\Omega))$, respectively. Integrating this with respect to s gives

$$t\|(u_k)_\tau(t)\|_2^2 + \int_0^t s\|\nabla(u_k)_\tau\|_2^2 ds \leq C\int_0^t \|(u_k)_\tau\|_2^2 ds + C$$

$$\leq C\int_0^T \|u_k'\|_2^2 ds + C \leq C$$

with $C = C(T)$ independent of k and τ. Hence,

$$(u_k)_\tau \quad \text{is bounded in} \quad L^\infty(\delta, T; L_\sigma^2) \cap L^2(\delta, T; V)$$

uniformly for k and τ; and the first assertion of (2.33) follows by passing to the limit. To show the second assertion, observe first that $(u_k)_\tau$ satisfies

$$-\int_0^T ((u_k)_\tau, v)h'dt + \int_0^T [(\nabla(u_k)_\tau, \nabla v) - ((u_k)_\tau \cdot \nabla v, u_k)]hdt$$

$$-\int_0^t (u_k \cdot \nabla v, (u_k)_\tau)hdt = \int_0^T (e_2(\theta_k)_\tau H(\theta_k), v)hdt$$

for any fixed $v \in V$ and $h \in C_0^1(0, T)$ provided that $|\tau|$ is sufficiently small. Letting $\tau \to 0$ and then $k \to \infty$ yields

$$-\int_0^T (u', v)h'dt + \int_0^T [(\nabla u', \nabla v) - (u' \cdot \nabla v, u)]hdt$$

$$-\int_0^T (u \cdot \nabla v, u')hdt = \int_0^T (e_2 f'(\theta)\theta', v)hdt.$$

Since $|(u' \cdot \nabla v, u)|$ and $|(u \cdot \nabla v, u')|$ are both majorized by

$$C\|u'\|_2^{1/2}\|\nabla u'\|_2^{1/2}\|u\|_2^{1/2}\|\nabla u\|_2^{1/2}\|\nabla v\|_2,$$

we see by duality that

$$\frac{d^2u}{dt^2} \in L^2(\delta, T; V^*).$$

This completes the proof of Theorem 2.1.

3. Passage to the limit $\eta \to 0$

This section establishes the following

THEOREM 3.1. (i) *Let f satisfy* (0.3) *and let $\{u_\eta, \theta_\eta\}$ be the family of weak solutions of* $(\mathrm{BE})_\eta$ *given in Theorem* 2.1 *corresponding to any fixed initial value $\{u_0, \theta_0\} \in L^2_\sigma \times L^2(\Omega)$. Let $\{u, \theta\}$ be a (unique) weak solution of the Boussinesq equations with the same initial value. Then, for all $T > 0$,*

$$u_\eta \longrightarrow u \quad \textit{strongly in} \quad C([0, T]; L^2_\sigma) \cap L^2(0, T; V),$$

$$\theta_\eta \longrightarrow \theta \quad \textit{strongly in} \quad L^{4/3}(0, T; L^2(\Omega)).$$

(ii) *Let f satisfy* (0.4) *and let $\{u_0, \theta_0\} \in V \times L^2(\Omega)$. Then, in the situation stated in* (i), *we have*

$$u_\eta \longrightarrow u \quad \textit{strongly in} \quad C([0, T]; V) \cap L^2(0, T; D(A)),$$

$$\theta_\eta \longrightarrow \theta \quad \textit{strongly in} \quad C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^1_{0,per}).$$

for all $T > 0$.

REMARK. By the same argument as in [14], we can show that the Boussinesq equations possess a unique weak solution in

$$(L^\infty(0, T; L^2_\sigma) \cap L^2(0, T; V)) \times L^{4/3}(0, T; L^2(\Omega))$$

in case f satisfies (0.3). Furthermore, our proof of Theorem 2.1 (ii) and (iii) automatically implies that if f satisfies (0.4), then the Boussinesq equations possess a unique solution in

$$(L^\infty(0, T; V) \cap L^2(0, T; D(A))) \times (L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_{0,per}))$$

provided $\{u_0, \theta_0\} \in V \times L^2(\Omega)$.

PROOF. Also in this section, we assume that

$$\nu = \kappa = 1$$

for simplicity in notation.

(i) We prove that

$$u_\eta \longrightarrow u \quad \text{strongly in} \quad C([0, T]; L^2_\sigma) \cap L^2(0, T; V), \tag{3.1}$$

and

$$\theta_\eta \longrightarrow \theta \quad \text{strongly in} \quad L^{4/3}(0, T; L^2(\Omega)). \tag{3.2}$$

In the same way as in section 2 we can show that

(3.1)' $$u_\eta \longrightarrow u \quad \text{strongly in } L^2(0, T; V).$$

But, since $u'_\eta = du_\eta/dt$ is bounded in $L^2(0, T; V^*)$, the relation

$$\| u_\eta(t) - u(t) \|_2^2 = 2 \int_0^t \langle u'_\eta - u', u_\eta - u \rangle ds,$$

together with (3.1)', implies that

$$\begin{aligned} \| u_\eta(t) - u(t) \|_2^2 &\le 2 \int_0^t \| \nabla (u_\eta - u) \|_2 \, \| u'_\eta - u' \|_{V^*} ds \\ &\le C \left(\int_0^T \| \nabla (u_\eta - u) \|_2^2 ds \right)^{1/2} \longrightarrow 0 \end{aligned}$$

as $\eta \to 0$. This completes the proof of (3.1). To show (3.2), we introduce the notation:

$$\| \theta \|_{p,r,J} = \left(\int_J \| \theta(\tau) \|_r^p d\tau \right)^{1/p},$$

where J is a measurable set in $(0, T)$. Let $\phi \in C^2_{0,per}(\bar{\Omega})$. Taking $\psi(\tau) = e^{-(t-\tau)B}\phi$ in (2.4), we have

$$\begin{aligned} (\theta_\eta(t), \phi) - (\theta_\eta(s), e^{-(t-s)B}\phi) &= \int_s^t (u \cdot \nabla e^{-(t-\tau)B}\phi, \theta_\eta) d\tau + \int_s^t (e_2 \cdot u_\eta, e^{-(t-\tau)B}\phi) d\tau \\ &\quad + \int_s^t \frac{\eta}{2} (D(u_\eta) : D(u_\eta), e^{-(t-\tau)B}\phi) d\tau. \end{aligned}$$

Let η_m be any sequence such that $\eta_m \to 0$ as $m \to \infty$. For simplicity, we write $u_m = u_{\eta_m}$ and $\theta_m = \theta_{\eta_m}$. Then we have

$$\begin{aligned} (\theta_m(t) - \theta_n(t), \phi) &= (\theta_m(s) - \theta_n(s), e^{-(t-s)B}\phi) + \int_s^t I_1(t, \tau) d\tau + \int_s^t I_2(t, \tau) d\tau \\ &\quad + \int_s^t I_3(t, \tau) d\tau + \int_s^t I_4(t, \tau) d\tau, \end{aligned}$$

where

$$\begin{aligned} I_1(t, \tau) &= -((u_m - u_n) \cdot \nabla e^{-(t-\tau)B}\phi, \theta_m) \\ I_2(t, \tau) &= -(u_n \cdot \nabla e^{-(t-\tau)B}\phi, \theta_m - \theta_n) \\ I_3(t, \tau) &= (e_2 \cdot (u_m - u_n), e^{-(t-\tau)B}\phi) \\ I_4(t, \tau) &= \frac{\eta_m}{2} (D(u_m) : D(u_m), e^{-(t-\tau)B}\phi) + \frac{\eta_n}{2} (D(u_n) : D(u_n), e^{-(t-\tau)B}\phi). \end{aligned}$$

Using (1.7) we have

$$\begin{aligned}|I_1(t,\tau)| &\le \|u_m - u_n\|_p \|\theta_m\|_2 \|\nabla e^{-(t-\tau)B}\phi\|_r \\ &\le \|u_m - u_n\|_2^{2/p} \|\nabla(u_m - u_n)\|_2^{1-2/p} \|\theta_m\|_2 \|\nabla e^{-(t-r)B}\phi\|_r\end{aligned}$$

where $1/p + 1/r = 1/2$, $p, r \ge 2$. Since

$$\begin{aligned}\|\nabla e^{-tB}\phi\|_r \le C\|\nabla e^{-tB}\phi\|_2^{2/r}\|Be^{-tB}\phi\|_2^{1-2/r} &= C\|B^{1/2}e^{-tB}\phi\|_2^{2/r}\|Be^{-tB}\phi\|_2^{1-2/r} \\ &\le Ct^{-1+1/r}\|\phi\|_2,\end{aligned}$$

it follows that

$$|I_1(t,\tau)| \le C(t-\tau)^{-1+1/r}\|u_m - u_n\|_2^{2/p}\|\nabla(u_m - u_n)\|_2^{1-2/p}\|\theta_m\|_2\|\phi\|_2.$$

Since $\{u_m\}$ is bounded in $L^\infty(0, T; L^2_\sigma)$, this gives

$$|I_1(t,\tau)| \le C(t-\tau)^{-1+1/r}\|\nabla(u_m - u_n)\|_2^{1-2/p}\|\theta_m\|_2\|\phi\|_2.$$

Similarly, we obtain

$$|I_2(t,\tau)| \le C(t-\tau)^{-1+1/r}\|\nabla u_n\|_2^{1-2/p}\|\theta_m - \theta_n\|_2\|\phi\|_2.$$

I_3 is estimated as

$$|I_3(t,\tau)| \le C\|u_m - u_n\|_2\|e^{-(t-\tau)B}\phi\|_2 \le C\|u_m - u_n\|_2\|\phi\|_2.$$

Applying Lemma 2.2, we have

$$\begin{aligned}|I_4(t,\tau)| &\le [\eta_m\|D(u_m): D(u_m)\|_1 + \eta_n\|D(u_n): D(u_n)\|_1]\|e^{-(t-\tau)B}\phi\|_\infty \\ &\le C(t-\tau)^{-1/2}(\eta_m\|\nabla u_m\|_2^2 + \eta_n\|\nabla u_n\|_2^2)\|\phi\|_2.\end{aligned}$$

Thus, by duality, we obtain

$$\|\theta_m(t) - \theta_n(t)\|_2 \le \|\theta_m(s) - \theta_n(s)\|_2 + S_1(t,s) + S_2(t,s) + S_3(t,s) + S_4(t,s),$$

where

$$S_1(t,s) = C\int_s^t (t-\tau)^{-1+1/r}\|\nabla(u_m - u_n)\|_2^{1-2/p}\|\theta_m\|_2 d\tau,$$

$$S_2(t,s) = C\int_s^t (t-\tau)^{-1+1/r}\|\nabla u_n\|_2^{1-2/p}\|\theta_m - \theta_n\|_2 d\tau,$$

$$S_3(t,s) = C\int_s^t \|u_m - u_n\|_2 d\tau,$$

$$S_4(t,s) = C\int_s^t (t-\tau)^{-1/2}(\eta_m\|\nabla u_m\|_2^2 + \eta_n\|\nabla u_n\|_2^2)d\tau.$$

Choose $p, r > 2$ so that $1/p + 1/r = 1/2$, $p < 4$. Then, since $(p-2)/2 + 3/4 < 1$ and $1 - 1/r + (p-2)/2 + 3/4 = 1 + 3/4$, we apply the Hardy-Littlewood-Sobolev inequality [26] to obtain

$$\left(\int_s^t |S_1(\sigma, s)|^{4/3} d\sigma\right)^{3/4} \le C \|\nabla(u_m - u_n)\|_{2,2,J}^{(p-2)/p} \|\theta_m\|_{4/3,2,J}$$

for $J = (s, t)$. Similarly,

$$\left(\int_s^t |S_2(\sigma, s)|^{4/3} d\sigma\right)^{3/4} \le \|\nabla u_m\|_{2,2,J}^{(p-2)/p} \|\theta_m - \theta_n\|_{4/3,2,J}$$

for $J = (s, t)$. Since $\{u_m\}$ is bounded in $L^\infty(0, T; L^2_\sigma)$, we have

$$\left(\int_s^t |S_4(\sigma, s)|^{4/3} d\sigma\right)^{3/4} \le CT^{1/4}(\eta_m + \eta_n) \le C(T)(\eta_m + \eta_n).$$

Since $\{\theta_m\}$ is bounded in $L^{4/3}(0, T; L^2(\Omega))$, we now deduce

$$\begin{aligned} \|\theta_m - \theta_n\|_{4/3,2,J} \le\ & C(T)(\|\theta_m(s) - \theta_n(s)\|_2 + \|\nabla(u_m - u_n)\|_{2,2,J}^{(p-2)/p}) \\ & + C(T)(\|\nabla u_n\|_{2,2,J}^{(p-2)/p} \|\theta_m - \theta_n\|_{4/3,2,J} + \|u_m - u_n\|_{2,2,J}) \\ & + C(T)(\eta_m + \eta_n), \end{aligned} \tag{3.4}$$

where $J = (s, t)$. Since $\nabla u_m \to \nabla u$ in $L^2(0, T; L^2)$ by (3.1)′, there exists a $\delta > 0$ such that if $0 < t - s \le \delta$ then

$$\|\nabla u_m\|_{2,2,J} < \left(\frac{1}{2C(T)}\right)^{p/(p-2)} \qquad \text{for all } m$$

with $J = (s, t)$. We take $s = 0$ and $t = \delta$ in (3.4). Since $\theta_m(0) - \theta_n(0) = 0$ by assumption, we have

$$\|\theta_m - \theta_n\|_{4/3,2,J_0} \le 2C(T)(\|\nabla(u_m - u_n)\|_{2,2,J_0}^{(p-2)/p} + \|u_m - u_n\|_{2,2,J_0} + \eta_m + \eta_n)$$

where $J_0 = (0, \delta)$. This implies that $\{\theta_m\}$ is a Cauchy sequence in $L^{4/3}(0, \delta; L^2(\Omega))$. Since θ_m converges to θ weakly in $L^{4/3}(0, T; L^2(\Omega))$, we see that θ_m converges to θ strongly in $L^{4/3}(0, \delta; L^2(\Omega))$ and, in particular, that there exist a subsequence, also denoted $\{\theta_m\}$, and $t_1 \in (\delta/2, \delta)$ such that $\theta_m(t_1)$ converges to $\theta(t_1)$ strongly in $L^2(\Omega)$. Now take $s = t_1$ and $t = 3\delta/2$ in (3.4). Then we have, for $J_1 = (t_1, 3\delta/2)$,

$$\begin{aligned} \|\theta_m - \theta_n\|_{4/3,2,J_1} \le\ & 2C(T)(\|\theta_m(t_1) - \theta_n(t_1)\|_2 + \|\nabla(u_m - u_n)\|_{2,2,J_1}^{(p-2)/p}) \\ & + 2C(T)(\|u_m - u_n\|_{2,2,J_1} + \eta_m + \eta_n). \end{aligned}$$

In the same way as above, we see that θ_m converges to θ strongly in

$L^{4/3}(t_1, 3\delta/2; L^2(\Omega))$ and, in particular, that there exist a subsequence, also denoted by $\{\theta_m\}$, and $t_2 \in (\delta, 3\delta/2)$ such that $\theta_m(t_2)$ converges to $\theta(t_2)$ strongly in $L^2(\Omega)$. Taking $s = t_2$ and $t = 2\delta$ in (3.4), we have, for $J_2 = (t_2, 2\delta)$,

$$\|\theta_m - \theta_n\|_{4/3,2,J_2} \leq 2C(T)(\|\theta_m(t_2) - \theta_n(t_2)\|_2 + \|\nabla(u_m - u_n)\|_{2,2,J_2}^{(p-2)/p}) + 2C(T)(\|u_m - u_n\|_{2,2,J_2} + \eta_m + \eta_n).$$

Thus, θ_m converges to θ strongly in $L^{4/3}(t_2, 2\delta; L^2(\Omega))$ and, in particular, there exist a subsequence, denoted again by $\{\theta_m\}$, and $t_3 \in (3\delta/2, 2\delta)$ such that $\theta_m(t_3)$ converges to $\theta(t_3)$ strongly in $L^2(\Omega)$.

Repeating these processes finitely many times, we conclude that there exists a subsequence of $\{\theta_m\}$ which converges to θ strongly in $L^{4/3}(0, T; L^2(\Omega))$. Since $\{\eta_m\}$ is arbitrary, $\{\theta_\eta\}$ converges to θ strongly in $L^{4/3}(0, T; L^2(\Omega))$. This completes the proof of (i).

(ii) We may assume $\eta \leq 1$. First we show that

$$u_\eta \longrightarrow u \quad \text{strongly in} \quad L^2(0, T; V). \tag{3.5}$$

Observe that letting $k \to \infty$ in (2.25) implies that

$$u_\eta \quad \text{is bounded in} \quad L^\infty(0, T; V) \cap L^2(0, T; D(A)).$$

Also, as in section 2, we can show that

$$\frac{du_\eta}{dt} \quad \text{is bounded in} \quad L^2(0, T; L^2_\sigma).$$

Applying [27, Th. III. 3.1], we can take a subsequence $u_m = u_{\eta_m}$ of u_η which converges to u in $L^2(0, T; V)$. Since u is unique, we have proved (3.5).

We next prove

$$\theta_\eta \longrightarrow \theta \quad \text{strongly in} \quad L^2(0, T; L^2(\Omega)). \tag{3.6}$$

As in section 2, we estimate approximate solutions $\{u_k, \theta_k\} = \{u_{k,\eta}, \theta_{k,\eta}\}$ and then pass to the limit $k \to \infty$ to obtain

$$\|\theta_\eta(t)\|_2^2 + \int_0^t \|\nabla\theta_\eta\|_2^2 ds \leq \|\theta_0\|_2^2 + C\left(\int_0^t \|\theta_\eta\|_2^2 ds + 1\right),$$

for a.e. $t \in (0, T)$, with $C = C(T)$ independent of $\eta \leq 1$. Applying the classical Gronwall lemma yields

$$\|\theta_\eta(t)\|_2^2 + \int_0^t \|\nabla\theta_\eta\|_2^2 ds \leq C$$

for a.e. $t \in (0, T)$, with $C = C(T)$ independent of $\eta \leq 1$. This shows that

$$\theta_\eta \quad \text{is bounded in} \quad L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_{0,per}).$$

One can also deduce that

$$\frac{d\theta_\eta}{dt} \quad \text{is bounded in} \quad L^2(0, T; (H^1_{0,per})^*).$$

Thus, we can apply [27, Th. III. 3.1] to extract a subsequence θ_m such that

$$\theta_m \longrightarrow \theta \quad \text{strongly in} \quad L^2(0, T; L^2(\Omega)),$$

which shows (3.6) because the limit function θ is unique.

Using (3.5) and (3.6), we now show that

$$u_\eta \longrightarrow u \quad \text{strongly in} \quad C([0, T]; V) \cap L^2(0, T; D(A)), \tag{3.7}$$

as in the proof of Theorem 2.1 (ii). The functions $v = u_\eta - u_{\eta'}$ and $\psi = \theta_\eta - \theta_{\eta'}$ satisfy

$$\frac{dv}{dt} + Av + P(v \cdot \nabla u_\eta + u_{\eta'} \cdot \nabla v) = P[(f(\theta_\eta) - f(\theta_{\eta'}))e_2], \tag{3.8}$$

and

(3.9)

$$\left\langle \frac{d\psi}{dt}, \phi \right\rangle + (\nabla \psi, \nabla \phi) + (v \cdot \nabla \theta_\eta + u_{\eta'} \cdot \nabla \psi, \phi)$$
$$= (e_2 \cdot v, \phi) + \frac{\eta}{2}(D(u_\eta) : D(u_\eta), \phi) - \frac{\eta'}{2}(D(u_{\eta'}) : D(u_{\eta'}), \phi)$$

for $\phi \in H^1_{0,per}$. From (3.8) we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla v\|_2^2 + \|Av\|_2^2 = -(v \cdot \nabla u_\eta + u_{\eta'} \cdot \nabla v, Av)$$
$$+ (f(\theta_\eta) - f(\theta_{\eta'}), Av)$$

and, as in the proof of Theorem 2.1 (ii), the right-hand side is majorized by

$$\frac{1}{2}\|Av\|_2^2 + C\|\nabla v\|_2^2[\|\nabla u_\eta\|_2 \|Au_\eta\|_2 + \|\nabla u_{\eta'}\|_2 \|Au_{\eta'}\|_2] + C|f'|_\infty^2 \|\psi\|_2^2.$$

We thus obtain

$$\frac{d}{dt}\|\nabla v\|_2^2 + \|Av\|_2^2 \le C(1 + \|Au_\eta\|_2 + \|Au_{\eta'}\|_2)\|\nabla v\|_2^2 + C\|\psi\|_2^2, \tag{3.10}$$

so that, by the classical Gronwall lemma,

$$\|\nabla v(t)\|_2^2 \le C \int_0^t \|\psi\|_2^2 ds \tag{3.11}$$

with $C = C(T)$ independent of η and η'. This shows the convergence of u_η in $C([0, T]; V)$. From (3.10) and (3.11), we also get

$$\begin{aligned}\int_0^t \|Av\|_2^2 ds &\le C \int_0^t (1 + \|Au_\eta\|_2 + \|Au_{\eta'}\|_2) \|\nabla v\|_2^2 ds + C \int_0^t \|\psi\|_2^2 ds \\ &\le C \sup_{[0,T]} \|\nabla v\|_2^2 + C \int_0^T \|\psi\|_2^2 ds\end{aligned}$$

which shows the convergence of u_η in $L^2(0, T; D(A))$. The proof of (3.7) is complete.

We finally prove

$$\theta_\eta \longrightarrow \theta \quad \text{strongly in} \quad C([0, T]; L^2(\Omega)) \cap L^2(0, T; H^2_{0,per}). \tag{3.12}$$

We insert $\phi = \psi(t)$ into (3.9) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\psi\|_2^2 + \|\nabla\psi\|_2^2 = &-(v \cdot \nabla\theta_\eta, \psi) + (e_2 \cdot v, \psi) \\ &+ \frac{\eta}{2}(D(u_\eta): D(u_\eta), \psi) - \frac{\eta'}{2}(D(u_{\eta'}): D(u_{\eta'}), \psi).\end{aligned}$$

The right-hand side is estimated as in the proof of Theorem 2.1 (ii), and we are led to

$$\begin{aligned}\frac{d}{dt}\|\psi\|_2^2 + \|\nabla\psi\|_2^2 \le &\, C\|\nabla v\|_2^2 (1 + \|\nabla\theta_\eta\|_2^2) \\ &+ C[\eta \|\nabla u_\eta\|_2^2 \|Au_\eta\|_2^2 + \eta' \|\nabla u_{\eta'}\|_2^2 \|Au_{\eta'}\|_2^2].\end{aligned}$$

Upon integration and application of the foregoing bounds, this gives

$$\|\psi(t)\|_2^2 + \int_0^t \|\nabla\psi\|_2^2 ds \le C(\sup_{[0,T]} \|\nabla v\|_2^2 + \eta + \eta')$$

with $C = C(T)$ independent of η and η', and this shows the desired convergence (3.12). This completes the proof of Theorem 3.1.

4. The global attractor for problem $(\mathrm{BE})_\eta$

From this section on we assume that f satisfies (0.4). Our convection problem is reformulated as

$$\text{(4.1)} \qquad \frac{du}{dt} + \nu Au + P(u \cdot \nabla u) = Pf(\theta)e_2,$$

$$\text{(4.2)} \qquad \frac{d\theta}{dt} + \kappa Bu + u \cdot \nabla\theta = \frac{\eta\nu}{2} D(u) : D(u) - e_2 \cdot u,$$

$$u(0) = u_0, \quad \theta(0) = \theta_0.$$

Let

$$H = V \times L^2(\Omega).$$

Then, by Theorem 2.1 (ii) and (iii), we can define the semigroup:

$$S(t) : H \ni \{u(0), \theta(0)\} \longmapsto \{u(t), \theta(t)\} \in H.$$

We will prove in this section the existence of the global attractor for $\{S(t)\}_{t \geq 0}$, to which all solutions converge as $t \to \infty$.

Following [28], we introrduce a notion of attractor.

Definition. A set $\mathscr{A}$ of a metric space H is called an *attractor* associated with $\{S(t)\}_{t \geq 0}$ if

(i) $\mathscr{A}$ is invariant, i.e., $S(t)\mathscr{A} = \mathscr{A}$ for all $t \geq 0$.

(ii) There exists an open neighborhood $\mathscr{U}$ of $\mathscr{A}$ such that for every $u_0 \in \mathscr{U}$, $S(t)u_0$ converges to $\mathscr{A}$ as $t \to \infty$ in the sense that

$$\lim_{t \to \infty} d(S(t)u_0, \mathscr{A}) = 0,$$

where $d(\cdot, \cdot)$ is the distance function of H.

We say that $\mathscr{A}$ attracts a set $\mathscr{B} \subset \mathscr{U}$ if

$$\lim_{t \to \infty} d(S(t)\mathscr{B}, \mathscr{A}) = 0,$$

where

$$d(\mathscr{B}_0, \mathscr{B}_1) = \sup_{x \in \mathscr{B}_0} \inf_{y \in \mathscr{B}_1} d(x, y).$$

An attractor $\mathscr{A} \subset H$ is called a *global attractor* if it is compact and attracts bounded sets of H.

We can now state our main result in this section.

Theorem 4.1. *Suppose that f satisfies* (0.4). *Then, the semigroup $\{S(t)\}_{t \geq 0}$ associated with problem* (4.1)–(4.2) *has a unique global attractor $\mathscr{A}$ which is* *bounded in* $D(A) \times H^1_{0,per}(\Omega)$, *compact and connected in* H. *Furthermore,* $\mathscr{A}$ *attracts bounded sets of* H.

To prove Theorem 4.1, we need two notions: the first is the notion of absorbing set, and the second is the uniform compactness of the semigroup $\{S(t)\}_{t\geq 0}$ for t large.

DEFINITION. A subset $\mathscr{B}$ of H is called *absorbing in H* if for each bounded set $\mathscr{B}_0$ of H there exists $t_0 = t_0(\mathscr{B}_0)$ such that $S(t)\mathscr{B}_0 \subset \mathscr{B}$ for all $t \geq t_0$.

DEFINITION. Let $\{S(t)\}_{t\geq 0}$ be a family of operators on H. Then $\{S(t)\}_{t\geq 0}$ is said to be *uniformly compact for t large* if for each bounded set $\mathscr{B}$, there exists $t_0 = t_0(\mathscr{B})$ such that

$$\bigcup_{t\geq t_0} S(t)\mathscr{B}$$

is relatively compact in H.

To verify the existence of a global attractor, we appeal to the following abstract result:

THEOREM 4.2 ([28, Th. I. 1.1]). *Let H be a metric space and let* $\{S(t)\}_{t\geq 0}$ *be a nonlinear semigroup of continuous transformations on H. If* $\{S(t)\}_{t\geq 0}$ *are uniformly compact for t large and if there exists an open set* $\mathscr{U}$ *and a bounded subset* $\mathscr{B}$ *of* $\mathscr{U}$ *such that* $\mathscr{B}$ *is absorbing in* $\mathscr{U}$, *then the* ω*-limit set* $\mathscr{A}$ *of* $\mathscr{B}$ *is a compact attractor which attracts the bounded sets of* $\mathscr{U}$. *Furthermore,* $\mathscr{A}$ *is the maximal bounded attractor in* $\mathscr{U}$. *If H is a Banach space and* $\mathscr{U}$ *is convex and connected, then* $\mathscr{A}$ *is connected.*

We prove Theorem 4.1, applying Theorem 4.2 to our semigroup. In the subsequent argument the following result will be frequently applied.

LEMMA 4.3. *Let g, h and y be three positive locally integrable functions on* (t_0, ∞). *Assume that* y' *is locally integrable on* (t_0, ∞) *and*

$$\frac{dy}{dt} \leq gy + h \tag{4.3}$$

holds for $t \geq t_0$. *If there exist positive functions* $a_i(t)$, $i = 1, 2, 3$, *such that*

$$\int_t^{t+r} g(s)ds \leq a_1(t), \quad \int_t^{t+r} h(s)ds \leq a_2(t), \quad \int_t^{t+r} y(s)ds \leq a_3(t) \qquad \text{for } t \geq t_0,$$

where r is a positive constant, then

$$y(t+r) \leq \left(\frac{a_3(t)}{r} + a_2(t)\right)\exp a_1(t) \qquad \text{for } t \geq t_0.$$

REMARK. When a_i are constant functions, Lemma 4.3 is called *the uniform Gronwall Lemma* [28, Lemma III. 1.1].

PROOF. Let $t_0 \leq t \leq s \leq t + r$. Multiplying (4.3) by

$$\exp\left(-\int_t^s g(\tau)d\tau\right)$$

we have

$$\frac{d}{ds}\left(y(s)\exp\left(-\int_t^s g(\tau)d\tau\right)\right) \leq h(s)\exp\left(-\int_t^s g(\tau)d\tau\right).$$

Integrating this on $[s, t+r]$ gives

$$y(t+r) \leq y(s)\exp\left(\int_t^{t+r} g(\tau)d\tau\right) + \int_s^{t+r} h(\tau)\exp\left(\int_\tau^{t+r} g(\sigma)d\sigma\right)d\tau$$
$$\leq (y(s) + a_2(t))\exp a_1(t).$$

Integrating this in s on $[t, t+r]$ yields the desired result. This proves Lemma 4.3.

We begin by establishing the existence of an absorbing set.

PROPOSITION 4.4. *There exisits an absorbing set for* $\{S(t)\}_{t\geq 0}$ *in* H.

PROOF. We first show the uniform boundedness of u in V for large t. From (4.1) we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \nu\|\nabla u\|_2^2 = (f(\theta), u) \leq |\Omega|^{1/2}|f|_\infty\|u\|_2$$
$$\leq |\Omega|^{1/2}|f|_\infty\|\nabla u\|_2 \leq \frac{\nu}{2}\|\nabla u\|_2^2 + \frac{|\Omega||f|_\infty^2}{2\nu},$$

and so

$$\frac{d}{dt}\|u\|_2^2 + \nu\|\nabla u\|_2^2 \leq \frac{|\Omega||f|_\infty^2}{\nu}. \tag{4.4}$$

We also obtain, by the Poincaré inequality,

$$\frac{d}{dt}\|u\|_2^2 + \nu\|u\|_2^2 \leq \frac{|\Omega||f|_\infty^2}{\nu}. \tag{4.5}$$

Integrating (4.5) gives

$$\|u(t)\|_2^2 \leq \|u_0\|_2^2 e^{-\nu t} + \frac{|\Omega||f|_\infty^2}{\nu^2}(1 - e^{-\nu t}). \tag{4.6}$$

Let $\mathscr{B}$ be an arbitrary bounded set of H. From now on we assume that $\{u_0, \theta_0\} \in \mathscr{B}$. There exists $R > 0$ such that $B_H(0, R) \supset \mathscr{B}$, where $B_H(0, R)$ denotes the ball of H with center 0 and radius R, and, in particular, $\|\nabla u_0\|_2^2 + \|\theta_0\|_2^2 \leq R^2$. Set

$$\rho_0 = \frac{|\Omega|^{1/2}|f|_\infty}{\nu}$$

and define

$$t_0 = \frac{1}{\nu}\log\frac{R^2}{\rho_0'^2 - \rho_0^2}$$

for $\rho_0' > \rho_0$. It follows from (4.6) that

$$\|u(t)\|_2^2 \leq \rho_0'^2$$

for $t \geq t_0$, since $\{u_0, \theta_0\} \in \mathscr{B}$. Note that ρ_0 and ρ_0' depend only on physical data and are independent of initial data. Here and in what follows we use letters ρ_j, ρ_j' and a_j to denote constants which depend only on physical data and are independent of initial data and time t, while the letters K_j denote constants which may depend on initial data or time t, etc. Now (4.4) gives

$$\nu\int_t^{t+1}\|\nabla u\|_2^2 ds \leq \frac{|\Omega||f|_\infty^2}{\nu} + \|u(t)\|_2^2,$$

which implies that

$$\int_t^{t+1}\|\nabla u\|_2^2 ds \leq \frac{|\Omega||f|_\infty^2}{\nu^2} + \frac{\rho_0'^2}{\nu}$$

for $t \geq t_0$, since $\{u_0, \theta_0\} \in \mathscr{B}$. Multiplying (4.1) by Au, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 + \nu\|Au\|_2^2 = -(u\cdot\nabla u, Au) + (f(\theta), Au).$$

We estimate each term on the right-hand side. The second term is estimated as

$$|(f(\theta), Au)| \leq |\Omega|^{1/2}|f|_\infty\|Au\|_2 \leq \frac{\nu}{4}\|Au\|_2^2 + \frac{|\Omega||f|_\infty^2}{\nu}.$$

By (1.3), (1.5) and (1.7) with $p = 4$, the first term is estimated as

$$\begin{aligned}|(u\cdot\nabla u, Au)| &\leq \|u\|_4\|\nabla u\|_4\|Au\|_2 \leq C\|u\|_2^{1/2}\|\nabla u\|_2\|Au\|_2^{3/2} \\ &\leq \frac{\nu}{4}\|Au\|_2^2 + \frac{C}{\nu^3}\|u\|_2^2\|\nabla u\|_2^4.\end{aligned}$$

It thus follows that

$$\frac{d}{dt}\|\nabla u\|_2^2 + \nu\|Au\|_2^2 \le \frac{2|\Omega||f|_\infty^2}{\nu} + \frac{C}{\nu^3}\|u\|_2^2\|\nabla u\|_2^4. \tag{4.7}$$

Since

$$\|\nabla u\|_2 \le \lambda_1^{-1/2}\|Au\|_2$$

for $u \in D(A)$, where λ_1 is the principal eigenvalue of the Stokes operator, we also have

$$\frac{d}{dt}\|\nabla u\|_2^2 + \nu\lambda_1\|\nabla u\|_2^2 \le \frac{2|\Omega||f|_\infty^2}{\nu} + \frac{C}{\nu^3}\|u\|_2^2\|\nabla u\|_2^4. \tag{4.8}$$

Applying Lemma 4.3 to (4.7), we obtain

$$\|\nabla u(t)\|_2^2 \le (a_2 + a_3)\exp a_1 \equiv \rho_1^2$$

for $t \ge t_0 + 1$, where

$$a_1 = \frac{C\rho_0'^2 a_3}{\nu^3}, \quad a_2 = \frac{2|\Omega||f|_\infty^2}{\nu}, \quad a_3 = \frac{|\Omega||f|_\infty^2}{\nu^2\lambda_1} + \frac{\rho_0'^2}{\nu}.$$

This shows that $\{u(t)\}_{t \ge t_0+1}$ is bounded in V uniformly for $\{u_0, \theta_0\} \in \mathscr{B}$. Integrating (4.7) on $[t, t+1]$, we also obtain

$$\begin{aligned}\int_t^{t+1}\|Au\|_2^2 ds &\le \frac{2|\Omega||f|_\infty^2}{\nu^2} + \frac{C}{\nu^4}\int_t^{t+1}\|u\|_2^2\|\nabla u\|_2^4 ds + \frac{\|\nabla u(t)\|_2^2}{\nu}\\ &\le \frac{2|\Omega||f|_\infty^2}{\nu^2} + \frac{C\rho_0'^2\rho_1^4}{\nu_4} + \frac{\rho_1^2}{\nu} \equiv a_4\end{aligned}$$

for $t \ge t_0 + 1$. We next deduce the uniform estimate of $\|\theta(t)\|_2$ for large t. For this purpose, we first estimate $\|\nabla u(t)\|_2$ for $0 \le t \le t_0 + 1$. From (4.4), we see that

$$\|u(t)\|_2^2 + \nu\int_0^t\|\nabla u\|_2^2 ds \le \|u_0\|_2^2 + \frac{|\Omega||f|_\infty^2}{\nu}t.$$

This, together with (4.6), implies that

$$\begin{aligned}\frac{C}{\nu^3}\int_0^t\|u\|_2^2\|\nabla u\|_2^2 ds &\le \frac{C}{\nu^3}\left(\|u_0\|_2^2 + \frac{|\Omega||f|_\infty^2}{\nu^2}\right)\int_0^t\|\nabla u\|_2^2 ds\\ &\le \frac{C}{\nu^4}\left(R^2 + \frac{|\Omega||f|_\infty^2}{\nu^2}\right)\left(R^2 + \frac{|\Omega||f|_\infty^2(t_0+1)}{\nu}\right) \qquad (4.9)\\ &\equiv K_1\end{aligned}$$

for $0 \le t \le t_0 + 1$. Applying the classical Gronwall lemma to (4.7), we obtain

$$
\begin{aligned}
\|\nabla u(t)\|_2^2 &\le \|\nabla u_0\|_2^2 + \int_0^t \frac{2|\Omega||f|_\infty^2}{\nu} \exp\left(\int_0^s \frac{C}{\nu^3}\|u\|_2^2\|\nabla u\|_2^2 d\tau\right) ds \\
&\le R^2 + \frac{2|\Omega||f|_\infty^2(t_0+1)}{\nu} \exp K_1 \equiv K_2
\end{aligned}
\tag{4.10}
$$

for $0 \le t \le t_0 + 1$. Integrating (4.7) on $[0, t]$, we also obtain

$$
\begin{aligned}
\int_0^t \|Au\|_2^2 ds &\le \|\nabla u(t)\|_2^2 + \frac{2|\Omega||f|_\infty^2}{\nu^2} t + \int_0^t \frac{C}{\nu^4}\|u\|_2^2\|\nabla u\|_2^4 ds \\
&\le R^2 + \frac{2|\Omega||f|_\infty^2}{\nu^2}(t_0+1) + \frac{1}{\nu} K_1 K_2 \equiv K_3
\end{aligned}
\tag{4.11}
$$

for $0 \le t \le t_0 + 1$. Taking the scalar product of (4.2) with θ, we have

$$
\frac{1}{2}\frac{d}{dt}\|\theta\|_2^2 + \kappa\|\nabla\theta\|_2^2 = \eta\frac{\nu}{2}(D(u): D(u), \theta) - (e_2 \cdot u, \theta).
$$

We majorize the right-hand side as follows: By (1.5) with $p = 4$,

$$
\begin{aligned}
\left|\frac{\eta\nu}{2}(D(u): D(u), \theta)\right| &\le \eta\nu\|\nabla u\|_4^2\|\theta\|_2 \le C\eta\nu\|\nabla u\|_2\|Au\|_2\|\nabla\theta\|_2 \\
&\le \frac{\kappa}{4}\|\theta\|_2^2 + \eta^2\frac{C\nu^2}{\kappa}\|\nabla u\|_2^2\|Au\|_2^2.
\end{aligned}
$$

A simple estimate gives

$$
|(e_2 \cdot u, \theta)| \le \|u\|_2\|\theta\|_2 \le \|\nabla u\|_2\|\nabla\theta\|_2 \le \frac{\kappa}{4}\|\nabla\theta\|_2^2 + \frac{1}{\kappa}\|\nabla u\|_2^2.
$$

We thus obtain

$$
\frac{d}{dt}\|\theta\|_2^2 + \kappa\|\nabla\theta\|_2^2 \le \frac{C\eta^2\nu^2}{\kappa}\|\nabla u\|_2^2\|Au\|_2^2 + \frac{2}{\kappa}\|\nabla u\|_2^2,
\tag{4.12}
$$

which implies

$$
\|\theta(t)\|_2^2 \le \|\theta_0\|_2^2 + \frac{C\eta^2\nu^2}{\kappa}\int_0^t \|\nabla u\|_2^2\|Au\|_2^2 ds + \frac{2}{\kappa}\int_0^t \|\nabla u\|_2^2 ds.
$$

It then follows from (4.10) and (4.11) that

$$
\|\theta(t_0+1)\|_2^2 \le R^2 + \frac{C\nu^2}{\kappa}K_2K_3 + \frac{C}{\kappa\lambda_1}K_3 \equiv K_4.
$$

Using the Poincaré inequality, we also obtain, from (4.12),

$$\frac{d}{dt}\|\theta\|_2^2 + \kappa\|\theta\|_2^2 \le \frac{Cv^2}{\kappa}\|\nabla u\|_2^2\|Au\|_2^2 + \frac{2}{\kappa}\|\nabla u\|_2^2,$$

and therefore,

$$\begin{aligned}\|\theta(t)\|_2^2 &\le e^{-\kappa(t-t_0-1)}\|\theta(t_0+1)\|_2^2 \\ &\quad + \int_{t_0+1}^t e^{-\kappa(t-s)}\left(\frac{Cv^2}{\kappa}\|\nabla u\|_2^2 + \frac{C}{\kappa\lambda_1}\right)\|Au\|_2^2 ds \\ &\le K_4 e^{-\kappa(t-t_0-1)} + \int_{t_0+1}^t e^{-\kappa(t-s)}\left(\frac{C\rho_1^2 v^2}{\kappa} + \frac{C}{\kappa\lambda_1}\right)\|Au\|_2^2 ds,\end{aligned} \tag{4.13}$$

for $t \ge t_0 + 1$. To estimate the last term, we multiply (4.7) by $e^{-\kappa(t-s)}$ to get

$$\begin{aligned}&\frac{d}{ds}(e^{-\kappa(t-s)}\|\nabla u\|_2^2) + ve^{-\kappa(t-s)}\|Au\|_2^2 \\ &\qquad \le e^{-\kappa(t-s)}\left(\kappa\|\nabla u\|_2^2 + \frac{2|\Omega||f|_\infty^2}{v} + \frac{C}{v^3}\|u\|_2^2\|\nabla u\|_2^4\right).\end{aligned}$$

Integrating this with respect to $s\in[t_0+1, t]$ then yields

$$\begin{aligned}v\int_{t_0+1}^t e^{-\kappa(t-s)}\|Au\|^2 ds &\le e^{-\kappa(t-t_0-1)}\|\nabla u(t_0+1)\|_2^2 \\ &\quad + \int_{t_0+1}^t e^{-\kappa(t-s)}\left(\kappa\|\nabla u\|_2^2 + \frac{2|\Omega||f|_\infty^2}{v} + \frac{C}{v^3}\|u\|_2^2\|\nabla u\|_2^4\right) \\ &\le \rho_1^2 + \left(\kappa\rho_1^2 + \frac{2|\Omega||f|_\infty^2}{v} + \frac{C\rho_0'^2\rho_1^4}{v^3}\right)\int_{t_0+1}^t e^{-\kappa(t-s)}ds \\ &\le \rho_1^2 + \frac{1}{\kappa}\left(\kappa\rho_1^2 + \frac{2|\Omega||f|_\infty^2}{v} + \frac{C\rho_0'^2\rho_1^4}{v^3}\right) \equiv a_5.\end{aligned}$$

We thus deduce from (4.13) that

$$\|\theta(t)\|_2^2 \le K_4 e^{-\kappa(t-t_0-1)} + \left(\frac{C\rho_1^2 v}{\kappa} + \frac{C}{\kappa\lambda_1 v}\right)a_5 \tag{4.14}$$

for $t \ge t_0 + 1$. Now set $\rho_2^2 = \left(\dfrac{C\rho_1^2 v}{\kappa} + \dfrac{C}{\kappa\lambda_1 v}\right)a_5$, and define

$$t_1 = t_0 + 1 + \frac{1}{\kappa}\log\frac{K_4}{\rho_2'^2 - \rho_2^2} \qquad \text{for } \rho_2' > \rho_2.$$

Then, $\|\theta(t)\|_2 \leq \rho_2' \leq \rho_2'$ for $t \geq t_1$, which implies that $\{\theta(t)\}_{t\geq t_1}$ is bounded in $L^2(\Omega)$ uniformly in $(u_0, \theta_0) \in \mathscr{B}$. Since we have already seen that $\|\nabla u(t)\|_2 \leq \rho_1$ for $t \geq t_0 + 1$, it follows that $(u(t), \theta(t)) \in \mathscr{B}(0, \rho_3)$ for $t \geq t_1$, where $\rho_3^2 = \rho_1^2 + \rho_2'^2$. This shows that $\mathscr{B}(0, \rho_3)$ is an absorbing set in H and the proof is complete.

We next prove the uniform compactness of $\{S(t)\}_{t\geq 0}$ for t large.

PROPOSITION 4.5. *$\{S(t)\}_{t\geq 0}$ is uniformly compact for t large.*

PROOF. Let $\mathscr{B}$ be a bounded set of H. It suffices to show that there exists a $t_2 = t_2(\mathscr{B})$ such that $\bigcup_{t\geq t_2} S(t)\mathscr{B}$ is bounded in $D(A) \times H^1_{0,per}$, since the embedding $D(A) \times H^1_{0,per} \subset H$ is compact. Let $(u_0, \theta_0) \in \mathscr{B}$. Recall that $\|\nabla u_0\|_2^2 + \|\theta_0\|_2^2 \leq R^2$ for some $R > 0$, since $\mathscr{B}$ is bounded in H. Integrating (4.12) on $[t, t+1]$ gives

$$\begin{aligned}\int_t^{t+1} \|\nabla\theta\|_2^2 ds &\leq \frac{1}{\kappa}\|\theta(t)\|_2^2 + \frac{C\nu^2}{\kappa}\int_t^{t+1} \|\nabla u\|_2^2 \|Au\|_2^2 ds + \frac{C}{\kappa}\int_t^{t+1} \|\nabla u\|_2^2 ds \\ &\leq \frac{\rho_2'^2}{\kappa} + \frac{C\nu^2\rho_1^2 a_4}{\kappa^2} + \frac{C\rho_1^2}{\kappa^2} \equiv a_6.\end{aligned}$$

Multiplying (4.2) by $B\theta$, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\theta\|_2^2 + \kappa\|B\theta\|_2^2 = -(u\cdot\nabla\theta, B\theta) + \eta\frac{\nu}{2}(D(u): D(u), B\theta) - (e_2\cdot u, B\theta).$$

By (1.3), (1.4) and (1.5) with $p = 4$, the right-hand sides are estimated as follows:

$$\begin{aligned}\frac{\eta\nu}{2}|(D(u): D(u), B\theta)| &\leq \eta\nu\|\nabla u\|_4^2\|B\theta\|_2 \leq C\eta\nu\|\nabla u\|_2\|Au\|_2\|B\theta\|_2 \\ &\leq \frac{\kappa}{6}\|B\theta\|_2^2 + \frac{C\eta^2\nu^2}{\kappa}\|\nabla u\|_2^2\|Au\|_2^2, \\ |(u\cdot\nabla\theta, B\theta)| &\leq \|u\|_4\|\nabla\theta\|_4\|B\theta\|_2 \leq C\|u\|_2^{1/2}\|\nabla u\|_2^{1/2}\|\nabla\theta\|_2^{1/2}\|B\theta\|_2^{3/2} \\ &\leq \frac{\kappa}{6}\|B\theta\|_2^2 + \frac{C}{\kappa^3}\|u\|_2^2\|\nabla u\|_2^2\|\nabla\theta\|_2^2, \\ |(e_2\cdot u, B\theta)| &\leq \|u\|_2\|B\theta\|_2 \leq \|\nabla u\|_2\|B\theta\|_2 \\ &\leq \frac{\kappa}{6}\|B\theta\|_2^2 + \frac{3}{2\kappa}\|\nabla u\|_2^2.\end{aligned}$$

We thus obtain

$$\frac{d}{dt}\|\nabla\theta\|_2^2+\kappa\|B\theta\|_2^2\le\frac{C\eta^2\nu^2}{\kappa}\|\nabla u\|_2^2\|Au\|_2^2+\frac{C}{\kappa^3}\|u\|_2^2\|\nabla u\|_2^2\|\nabla\theta\|_2^2+\frac{3}{\kappa}\|\nabla u\|_2^2.$$

On the other hand, if $t\ge t_1$, then

$$\frac{C}{\kappa^3}\int_t^{t+1}\|u\|_2^2\|\nabla u\|_2^2ds\le\frac{C}{\kappa^3}\rho_0'^2\rho_1^2\equiv a_7;$$

$$\frac{C\nu^2}{\kappa}\int_t^{t+1}\|\nabla u\|_2^2\|Au\|_2^2ds+\frac{3}{\kappa}\int_t^{t+1}\|\nabla u\|_2^2ds\le\frac{C\nu^2}{\kappa}\rho_1^2K_1+\frac{3}{\kappa}\rho_1^2\equiv a_8;$$

$$\int_t^{t+1}\|\nabla\theta\|_2^2ds\le a_6.$$

Applying Lemma 4.3, we have

$$\|\nabla\theta(t)\|_2^2\le(a_6+a_8)\exp(a_7)\equiv\rho_4^2 \tag{4.15}$$

for $t\ge t_1$. This shows that $\{\theta(t)\}_{t\ge t_1}$ is bounded in $H^1_{0,per}$ uniformly in $\{u_0,\theta_0\}\in\mathscr{B}$.

We next estimate $\|Au(t)\|_2$ for large t. To do so, we first estimate $\int_t^{t+1}\|\theta_t\|_2^2ds$, where θ_t is the time-derivative of θ. Note that this integral exists by Theorem 2.1 (iv). We multiply (4.2) by θ_t to get

$$\|\theta_t\|_2^2+\frac{\kappa}{2}\frac{d}{dt}\|\nabla\theta\|_2^2=-(u\cdot\nabla\theta,\theta_t)+\eta\frac{\nu}{2}(D(u):D(u),\theta_t)-(e_2\cdot u,\theta_t).$$

The right-hand side is estimated as follows: First, by (1.3) and (1.6),

$$\begin{aligned}|(u\cdot\nabla\theta,\theta_t)|&\le\|u\|_\infty\|\nabla\theta\|_2\|\theta_t\|_2\le C\|Au\|_2\|\nabla\theta\|_2\|\theta_t\|_2\\&\le\frac{1}{6}\|\theta_t\|_2^2+C\|Au\|_2^2\|\nabla\theta\|_2^2.\end{aligned}$$

Secondly, by (1.5) with $p=4$,

$$\begin{aligned}\frac{\eta\nu}{2}|(D(u):D(u),\theta_t)|&\le\nu\|\nabla u\|_4^2\|\theta_t\|_2\le C\nu\|\nabla u\|_2\|Au\|_2\|\theta_t\|_2\\&\le\frac{1}{6}\|\theta_t\|_2^2+C\nu^2\|\nabla u\|_2^2\|Au\|_2^2.\end{aligned}$$

Third, by an elementary calculation,

$$|(e_2\cdot u,\theta_t)|\le\|u\|_2\|\theta_t\|_2\le\|\nabla u\|_2\|\theta_t\|_2\le\frac{1}{6}\|\theta_t\|_2^2+3\|\nabla u\|_2^2.$$

We thus have

$$\|\theta_t\|_2^2 + \kappa \frac{d}{dt}\|\nabla\theta\|_2^2 \le 2C(\|\nabla\theta\|_2^2 + \nu^2\|\nabla u\|_2^2)\|Au\|_2^2 + 6\|\nabla u\|_2^2.$$

Integrating this on $[t, t+1]$, for $t \ge t_1 + 1$, then yields

$$\begin{aligned}\int_t^{t+1}\|\theta_t\|_2^2 ds &\le \kappa\|\nabla\theta(t)\|_2^2 \\ &\quad + 2\int_t^{t+1}[C(\|\nabla\theta\|_2^2 + \nu^2\|\nabla u\|_2^2)\|Au\|_2^2 + 3\|\nabla u\|_2^2]ds \\ &\le \kappa\rho_4^2 + 2C(\rho_4^2 + \nu^2\rho_1^2)a_4 + 6\rho_1^2 \equiv a_9.\end{aligned} \tag{4.16}$$

We next estimate $\int_t^{t+1}\|u_t\|_2^2 ds$. Multiplying (4.1) by u_t, we have

$$\|u_t\|_2^2 + \frac{\nu}{2}\frac{d}{dt}\|\nabla u\|_2^2 = -(u\cdot\nabla u, u_t) + (f(\theta), u_t).$$

In the same way as above, we obtain

$$\begin{aligned}|(u\cdot\nabla u, u_t)| &\le \|u\|_4\|\nabla u\|_4\|u_t\|_2 \le C\|u\|_2^{1/2}\|\nabla u\|_2\|Au\|_2^{1/2}\|u_t\|_2 \\ &\le \frac{1}{4}\|u_t\|_2^2 + C\|u\|_2\|\nabla u\|_2^2\|Au\|_2,\end{aligned}$$

$$|(f(\theta), u_t)| \le \|f(\theta)\|_2\|u_t\|_2 \le |\Omega|^{1/2}|f|_\infty\|u_t\|_2 \le \frac{1}{4}\|u_t\|_2^2 + |\Omega||f|_\infty^2.$$

It thus follows that

$$\begin{aligned}&\int_t^{t+1}\|u_t\|_2^2 ds \\ &\le \nu\|\nabla u(t)\|_2^2 + 2|\Omega||f|_\infty^2 + C\int_t^{t+1}\|u\|_2\|\nabla u\|_2^2\|Au\|_2 ds \\ &\le \nu\|\nabla u(t)\|_2^2 + 2|\Omega||f|_\infty^2 + C\left(\int_t^{t+1}\|u\|_2^2\|\nabla u\|_2^4 ds\right)^{1/2}\left(\int_t^{t+1}\|Au\|_2^2 ds\right)^{1/2},\end{aligned}$$

so that

$$\int_t^{t+1}\|u_t\|_2^2 ds \le \nu\rho_1^2 + 2|\Omega||f|_\infty^2 + C\rho_0'\rho_1^2 a_4^{1/2} \equiv a_{10}$$

for $t \ge t_0 + 1$, since $\{u_0, \theta_0\} \in \mathscr{B}$.

We now deduce a differential inequality for $\|u_t\|_2^2$, applying the foregoing estimates. Differentiating (4.1) with respect to t, we have

$$u_{tt} + Au_t + P(u \cdot \nabla u_t + u_t \cdot \nabla u) = Pf'(\theta)\theta_t e_2.$$

Multiplying this by u_t then yields

$$\frac{1}{2}\frac{d}{dt}\|u_t\|_2^2 + \nu\|\nabla u_t\|_2^2 = -(u_t \cdot \nabla u, u_t) + (f'(\theta)\theta_t, e_2 \cdot u_t).$$

We apply (1.5) and (1.7) with $p = 4$ to obtain

$$\begin{aligned}|(u_t \cdot \nabla u, u_t)| &\leq \|u_t\|_4 \|\nabla u\|_4 \|u_t\|_2 \leq C\|u_t\|_2^{3/2}\|\nabla u_t\|_2^{1/2}\|\nabla u\|_2^{1/2}\|Au\|_2^{1/2} \\ &\leq \frac{\nu}{4}\|\nabla u_t\|_2^2 + \frac{C}{\nu^{1/3}}\|u_t\|_2^2\|\nabla u\|_2^{2/3}\|Au\|_2^{2/3},\end{aligned}$$

$$\begin{aligned}|(f'(\theta)\theta_t, e_2 \cdot u_t)| &\leq |f'|_\infty \|\theta_t\|_2\|u_t\|_2 \leq |f'|_\infty\|\theta_t\|_2\|\nabla u_t\|_2 \\ &\leq \frac{\nu}{4}\|\nabla u_t\|_2^2 + \frac{|f'|_\infty^2}{\nu}\|\theta_t\|_2^2,\end{aligned}$$

so that

$$\frac{d}{dt}\|u_t\|_2^2 + \nu\|\nabla u_t\|_2^2 \leq \frac{2|f'|_\infty^2}{\nu}\|\theta_t\|_2^2 + \frac{C}{\nu^{1/3}}\|u_t\|_2^2\|\nabla u\|_2^{2/3}\|Au\|_2^{2/3}. \tag{4.17}$$

Since, by (4.11),

$$\begin{aligned}\frac{C}{\nu^{1/3}}\int_t^{t+1}\|\nabla u\|_2^{2/3}\|Au\|_2^{2/3}ds &\leq \frac{C}{\nu^{1/3}}\left(\int_t^{t+1}\|\nabla u\|_2 ds\right)^{2/3}\left(\int_t^{t+1}\|Au\|_2^2 ds\right)^{1/3} \\ &\leq \frac{C\rho_1^{2/3}a_4^{1/3}}{\nu^{1/3}} \equiv a_{11}\end{aligned}$$

for $t \geq t_1 + 1$, applying Lemma 4.3 then yields

$$\|u_t(t)\|_2^2 \leq (a_9 + a_{10})\exp(a_{11}) \equiv \rho_5^2$$

for $t \geq t_1 + 2$.

We can now deduce a uniform estimate of $\|Au(t)\|_2$ for large t. From (4.1) we see that

$$Au(t) = \frac{1}{\nu}(-u_t(t) - P(u \cdot \nabla u)(t) + Pf(\theta(t))e_2),$$

which implies

$$\|Au(t)\|_2 \leq \frac{1}{\nu}(\|u_t(t)\|_2 + \|u \cdot \nabla u(t)\|_2 + \|f(\theta(t))\|_2).$$

Since

$$\|u\cdot\nabla u\|_2\le\|u\|_4\|\nabla u\|_4\le C\|u\|_2^{1/2}\|\nabla u\|_2\|Au\|_2^{1/2}$$
$$\le\frac{\nu}{2}\|Au\|_2+\frac{C}{\nu}\|u\|_2\|\nabla u\|_2^2,$$
$$\|f(\theta)\|_2\le|\Omega|^{1/2}|f|_\infty,$$

we have

$$\|Au(t)\|_2\le\frac{2\rho_5}{\nu}+\frac{2c_3^2\rho_0\rho_1^2}{\nu^2}+\frac{2|\Omega|^{1/2}|f|_\infty}{\nu}$$

for $t\ge t_1+2$, which implies that $\{u(t)\}_{t\ge t_1+2}$ is bounded in $D(A)$ uniformly for $\{u_0,\theta_0\}\in\mathscr{B}$. This, together with (4.15), implies that $\bigcup_{t\ge t_1+2}S(t)\mathscr{B}$ is contained in some bounded set of $D(A)\times H^1_{0,per}(\Omega)$. Thus, $\{S(t)\}_{t\ge 0}$ is uniformly compact for t large. This completes the proof of Proposition 4.5, and so Theorem 4.1 is proved.

5. Estimate of the Hausdorff dimension of the attractor $\mathscr{A}$

In this section we give an upper bound for the Hausdorff dimension of the global attractor $\mathscr{A}$ obtained in the previous section. The method is basically the same as that given in [8]. We begin with

PROPOSITION 5.1. *The mapping $S(t)$ is "uniformly differentiable on $\mathscr{A}$" with respect to the L^2-norm, i.e., for any $\varphi_0=\{u_0,\theta_0\}$, there exists a continuous linear operator $L(t;\varphi_0)$ on $L^2_\sigma\times L^2(\Omega)$ such that*

$$\lim_{\varepsilon\to 0}\sup_{\substack{\varphi_0,\varphi_1\in\mathscr{A}\\ 0<\|\varphi_1-\varphi_0\|_{L^2}\le\varepsilon}}\frac{\|S(t)\varphi_1-S(t)\varphi_0-L(t;\varphi_0)(\varphi_1-\varphi_0)\|_{L^2}}{\|\varphi_1-\varphi_0\|_{L^2}}=0,$$

where $\|\cdot\|_{L^2}$ is the norm of $L^2_\sigma\times L^2(\Omega)$:

$$\|\varphi\|_{L^2}=(\|u\|_2^2+\|\theta\|_2^2)^{1/2},\quad \varphi=\{u,\theta\}\in L^2_\sigma\times L^2(\Omega).$$

The proof of Proposition 5.1 is long and technical; and so will be given at the end of this section. One will see in the proof of Proposition 5.1 below that $L(t,\varphi_0)$ is the linear mapping

$$L^2_\sigma\times L^2(\Omega)\ni\xi=\{U_0,\Theta_0\}\longmapsto\Phi(t)=\{U(t),\Theta(t)\}\in L^2_\sigma\times L^2(\Omega),$$

where $\Phi(t)=\{U(t),\Theta(t)\}$ is the solution of the linear problem

$$\frac{dU}{dt}+\nu AU+P(u\cdot\nabla U+U\cdot\nabla u)=f'(\theta)\Theta e_2$$

$$\text{(LP)}\qquad \frac{d\Theta}{dt} + \kappa B\Theta + (u\cdot\nabla\Theta + U\cdot\nabla\theta) = e_2\cdot U + \eta\nu D(u)\colon D(U),$$

$$U(0) = U_0,\quad \Theta(0) = \Theta_0.$$

We write this linear problem as

$$\frac{d\Phi}{dt} = \mathscr{F}(\varphi)\Phi,\quad \Phi(0) = \xi,$$

where

$$\mathscr{F}(\varphi)\Phi = \begin{pmatrix} -\nu AU - P(u\cdot\nabla U + U\cdot\nabla u) + f'(\theta)\Theta e_2 \\ -\kappa B\Theta - (u\cdot\nabla\Theta + U\cdot\nabla\theta) + e_2\cdot U + \eta\nu D(u)\colon D(U) \end{pmatrix}.$$

REMARK. Since $\mathscr{A}$ is bounded in $D(A)\times H^1_{0,per}(\Omega)$, one can easily show that for each $\xi = \{U_0, \Theta_0\}\in L^2_\sigma\times L^2(\Omega)$, there exists a unique solution $\{U, \Theta\}$ of (LP) such that

$$U\in C([0,T]; L^2_\sigma)\cap L^2(0,T; V),\quad \Theta\in C([0,T]; L^2(\Omega))\cap L^2(0,T; H^1_{0,per}(\Omega))$$

for all $T > 0$.

Our goal in this section is to show the following

THEOREM 5.2. *The Hausdorff dimension of $\mathscr{A}$ is bounded above by*

$$c|\Omega|(1 + Pr)(1 + Gr + Gr^{1/2}Ra + O(\eta)),$$

with a constant c depending only on α.

REMARK. Foias, Manley and Temam [8] obtained a bound of the form

$$c|\Omega|(1 + Pr)(1 + Gr + Ra)$$

for the attractor of the Boussinesq equations, i.e., for the case $f(\theta) = \theta$ and $\eta = 0$. The same result can be derived also in the case that f satisfies (0.4) and $\eta = 0$.

PROOF. We will prove Theorem 5.2, following the procedure in [28, Chap. V]. Let $\Phi_1,\cdots,\Phi_l$ be the solutions of (LP) with initial values $\xi_1,\cdots,\xi_l$, respectively. In the same way as in [8], we can show

$$\|\Phi_1\wedge\cdots\wedge\Phi_l\|_{\wedge L^2} = \|\xi_1\wedge\cdots\wedge\xi_l\|_{\wedge L_2}\exp\left(\int_0^t \operatorname{tr}\mathscr{F}(\varphi(s))\circ Q_l(s)ds\right),$$

where ΛL^2 denotes l-exterior product of $L^2_\sigma\times L^2(\Omega)$, with norm $\|\cdot\|_{\wedge L^2}$; $Q_l(s) = Q_l(s, \varphi_0; \xi_1, \xi_l)$ is the orthogonal projector in $L^2_\sigma\times L^2(\Omega)$ onto the space spanned by $\Phi_1,\cdots,\Phi_l$. Using these notations, we define

$$q_l = \limsup_{t\to\infty} \sup_{\varphi\in\mathscr{A}} \sup_{\substack{\xi_j\in L^2_\sigma\times L^2(\Omega)\\ \|\xi_j\|_{L^2}\le 1,\, j=1,\dots,l}} \left(\frac{1}{t}\int_0^t \operatorname{tr}\mathscr{F}(\varphi(s))\circ Q_l(s)ds\right).$$

According to [8, 28], the Hausdorff dimension of $\mathscr{A}$ is less than or equal to the integer l for which $q_l < 0$. Thus, we need to estimate q_l and find an l such that $q_l < 0$.

At a given time s, let $\{\varphi_j(s)\}_{j=1}^{\infty} = \{v_j(s), \psi_j(s)\}_{j=1}^{\infty}$ be an orthonormal basis of $L^2_\sigma \times L^2(\Omega)$ such that $\{\varphi_j(s)\}_{j=1}^{l}$ spans $Q_l(s)(L^2_\sigma \times L^2(\Omega))$. Then we have

$$\begin{aligned}\operatorname{tr}\mathscr{F}(\varphi(s))\circ Q_l(s) &= \sum_{j=1}^{\infty} (\mathscr{F}(\varphi(s))\circ Q_l(s)\varphi_j(s), \varphi_j(s))\\ &= \sum_{j=1}^{l} (\mathscr{F}(\varphi(s))\varphi_j(s), \varphi_j(s)).\end{aligned}$$

Since

$$\begin{aligned}(\mathscr{F}(\varphi(s))Q_l(s)\varphi_j(s), \varphi_j(s)) = &-\nu\|\nabla v_j(s)\|_2^2 - (v_j\cdot\nabla u(s), v_j(s)) + (f'(\theta(s)), e_2\cdot v_j(s))\\ &-\kappa\|\nabla\psi_j(s)\|_2^2 - (v_j\cdot\nabla\theta(s), \psi_j(s)) + (e_2\cdot v_j(s), \psi_j(s))\\ &+\eta\nu(D(u): D(v_j)(s), \psi_j(s)),\end{aligned}$$

we have

$$\begin{aligned}&\operatorname{tr}\mathscr{F}(\varphi(s))\circ Q_l(s)\\ &\quad = -\nu\sum_{j=1}^{l}\|\nabla v_j(s)\|_2^2 - \sum_{j=1}^{l}(v_j\cdot\nabla u(s), v_j(s)) + \sum_{j=1}^{l}(f'(\theta(s)), e_2\cdot v_j(s))\\ &\quad\quad -\kappa\sum_{j=1}^{l}\|\nabla\psi_j(s)\|_2^2 - \sum_{j=1}^{l}(v_j\cdot\nabla\theta(s), \psi_j(s))\\ &\quad\quad +\sum_{j=1}^{l}(e_2\cdot v_j(s), \psi_j(s)) + \eta\nu\sum_{j=1}^{l}(D(u): D(v_j)(s), \psi_j(s))\\ &\quad \equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7.\end{aligned}$$

In the same way as in [8], we can majorize I_2 as follows. By the Schwarz inequality, we have

$$|(v_j\cdot\nabla u)v_j(x))| \le |\nabla u(x)||v_j(x)|^2,$$

hence

$$|I_2| \le \int_\Omega |\nabla u(x, s)||v_j(x, s)|^2 dx \le \|\nabla u(s)\|_2\,\|\rho(s)\|_2,$$

where $\rho(x, s) = \sum_{j=1}^{l} (|v_j(x, s)|^2 + |\psi_j(x, s)|^2)$. Here we apply the Lieb-Thirring inequality [19, 28]:

$$\|\rho(s)\|_2^2 \le c_1' \sum_{j=1}^{l} (\|\nabla v_j(s)\|_2^2 + \|\nabla \psi_j(s)\|_2^2), \tag{5.1}$$

with c_1' depending only on α, to get

$$\begin{aligned}
|I_2| &\le \|\nabla u(s)\|_2 \{c_1' \sum_{j=1}^{l} (\|\nabla v_j(s)\|_2^2 + \|\nabla \psi_j(s)\|_2^2)\}^{1/2} \\
&\le c_1'^{1/2} \|\nabla u(s)\|_2 \{(\sum_{j=1}^{l} \|\nabla v_j(s)\|_2^2)^{1/2} + (\sum_{j=1}^{l} \|\nabla \psi_j(s)\|_2^2)^{1/2}\} \\
&\le \frac{1}{6} \sum_{j=1}^{l} (\nu \|\nabla v_j(s)\|_2^2 + \kappa \|\nabla \psi_j(s)\|_2^2) + c_2'\left(\frac{1}{\nu} + \frac{1}{\kappa}\right) \|\nabla u(s)\|_2^2,
\end{aligned}$$

where $c_2' = 2c_1'$. Similarly, we have

$$\begin{aligned}
|I_5| &\le \sum_{j=1}^{l} \int_\Omega |\nabla \theta(x, s)| |v_j(x, s)| |\psi_j(x, s)| dx \\
&\le \|\nabla \theta(s)\|_2 \|\rho(s)\|_2 \\
&\le \frac{1}{6} \sum_{j=1}^{l} (\nu \|\nabla v_j(s)\|_2^2 + \kappa \|\nabla \psi_j(s)\|_2^2) + c_2'\left(\frac{1}{\nu} + \frac{1}{\kappa}\right) \|\nabla \theta(s)\|_2^2.
\end{aligned}$$

Since $\{\varphi_j(s)\}$ is orthonormal in $L_\sigma^2 \times L^2(\Omega)$, we have $\|v_j(s)\|_2^2 + \|\psi_j(s)\|_2^2 = 1$. Therefore, $I_3 + I_6$ is estimated as

$$|I_3 + I_6| \le \sum_{j=1}^{l} (1 + |f'|_\infty) \|v_j(s)\|_2 \|\psi_j(s)\|_2 \le (1 + |f'|_\infty) l.$$

We next estimate I_7:

$$\begin{aligned}
|I_7| &= |\eta\nu \sum_{j=1}^{l} (D(u): D(v_j)(s), \psi_j(s))| \\
&= \left| \nu \int_\Omega \sum_{j=1}^{l} \sum_{i,k=1}^{2} \left(\frac{\partial u^i}{\partial x_k} + \frac{\partial u^k}{\partial x_i}\right)(x, s) \left(\frac{\partial v_j^i}{\partial x_k} + \frac{\partial v_j^k}{\partial x_i}\right)(x, s) \psi_j(x, s) dx \right|.
\end{aligned}$$

By the Schwarz inequality, we have

$$\left| \sum_{j=1}^{l} \sum_{i,k=1}^{2} \left(\frac{\partial u^i}{\partial x_k} + \frac{\partial u^k}{\partial x_i}\right)(x, s) \left(\frac{\partial v_j^i}{\partial x_k} + \frac{\partial v_j^k}{\partial x_i}\right)(x, s) \psi_j(x, s) \right|$$

$$\leq \sum_{i,k=1}^{2}\left|\left(\frac{\partial u^i}{\partial x_k}+\frac{\partial u^k}{\partial x_i}\right)(x, s)\right|\left\{\sum_{j=1}^{l}\left(\frac{\partial v_j^i}{\partial x_k}+\frac{\partial v_j^k}{\partial x_i}\right)^2(x, s)\right\}^{1/2}\{\sum_{j=1}^{l}|\psi_j(x, s)|^2\}^{1/2}$$

$$\leq 4|\nabla u(x, s)|\{\sum_{j=1}^{l}|\nabla v_j(x, s)|^2\}^{1/2}\{\sum_{j=1}^{l}|\psi_j(x, s)|^2\}^{1/2},$$

whence,

$$|\eta\nu\sum_{j=1}^{l}(D(u): D(v_j)(s), \psi_j(s))|$$

$$\leq 4\eta\nu\int_{\Omega}|\nabla u(x, s)|\{\sum_{j=1}^{l}|\nabla v_j(x, s)|^2\}^{1/2}\{\sum_{j=1}^{l}|\psi_j(x, s)|^2\}^{1/2}dx$$

$\leq$ (by the Hölder inequality)

$$\leq 4\eta\nu\|\nabla u(s)\|_4\left\{\sum_{j=1}^{l}\int_{\Omega}|\nabla v_j(x, s)|^2dx\right\}^{1/2}\|\rho(s)\|_2^{1/2}$$

$\leq$ (by (5.1))

$$\leq C\eta\nu\|\nabla u(s)\|_2^{1/2}\|Au(s)\|_2^{1/2}\left\{\sum_{j=1}^{l}\int_{\Omega}|\nabla v_j(x, s)|^2dx\right\}^{1/2}\|\rho(s)\|_2^{1/2}$$

$\leq$ (by Cauchy's inequality)

$$\leq \frac{\nu}{12}\sum_{j=1}^{l}\|\nabla v_j(s)\|_2^2 + C\eta^2\nu\|\nabla u(s)\|_2\|Au(s)\|_2\|\rho(s)\|_2$$

$\leq$ (by (5.1) and the Schwarz inequality)

$$\leq \frac{1}{6}\sum_{j=1}^{l}(\nu\|\nabla v_j(s)\|_2^2+\kappa\|\nabla\psi_j(s)\|_2^2) + C\eta^4\nu\left(1+\frac{\nu}{\kappa}\right)\|\nabla u(s)\|_2^2\|Au(s)\|_2^2.$$

Collecting these inequalites, we obtain

$$\operatorname{tr}\mathscr{F}(\varphi(s))\circ Q_l(s)$$

$$\leq -\frac{1}{2}\sum_{j=1}^{l}(\nu\|\nabla v_j(s)\|_2^2+\kappa\|\nabla\psi_j(s)\|_2^2) + 2c_2'\left(\frac{1}{\nu}+\frac{1}{\kappa}\right)(\|\nabla u(s)\|_2^2+\|\nabla\theta(s)\|_2^2)$$

$$+ (1+|f'|_\infty)l + C\eta^4\nu\left(1\nu+\frac{\nu}{\kappa}\right)\|\nabla u(s)\|_2^2\|Au(s)\|_2^2.$$

On the other hand, as shown in [8], there exists a positive constant c_3' depending only on α such that

$$\sum_{j=1}^{l}(\|\nabla v_j(s)\|_2^2+\|\nabla\psi_j(s)\|_2^2) \geq 2c_3'\frac{l^2}{|\Omega|}$$

for a.e. $s > 0$. It follows that

$$\sum_{j=1}^{l} (\nu \| \nabla v_j(s) \|_2^2 + \kappa \| \nabla \psi_j(s) \|_2^2)$$
$$\geq \frac{\nu\kappa}{\nu+\kappa} \sum_{j=1}^{l} (\| \nabla v_j(s) \|_2^2 + \| \nabla \psi_j(s) \|_2^2) \geq 2c_3' \frac{\nu\kappa}{\nu+\kappa} \frac{l^2}{|\Omega|}$$

for a.e. $s > 0$. We thus obtain

$$\operatorname{tr} \mathscr{F}(\varphi(s)) \circ Q_l(s) \leq -c_3' \frac{\nu\kappa}{\nu+\kappa} \frac{l^2}{|\Omega|} + 2c_2'\left(\frac{1}{\nu} + \frac{1}{\kappa}\right)(\| \nabla u(s) \|_2^2 + \| \nabla \theta(s) \|_2^2)$$
$$+ (1 + |f'|_\infty) l + C\eta^4 \nu \left(1 + \frac{\nu}{\kappa}\right) \| \nabla u(s) \|_2^2 \, \| Au(s) \|_2^2.$$

We now estimate

$$q_l = \limsup_{t \to \infty} \sup_{\varphi \in \mathscr{A}} \sup_{\substack{\xi_j \in L^2_\sigma \times L^2(\Omega) \\ \| \xi_j \|_{L^2} \leq 1, j=1,\dots,l}} \left(\frac{1}{t} \int_0^t \operatorname{tr} \mathscr{F}(\varphi(s)) \circ Q_l(s) ds \right)$$

by applying

LEMMA 5.3. *There hold*

(i) $$\limsup_{t \to \infty} \sup_{\varphi_0 \in \mathscr{A}} \frac{1}{t} \int_0^t \| \nabla u \|_2^2 ds \leq \frac{|\Omega| |f|_\infty^2}{\nu^2};$$

(ii) $$\limsup_{t \to \infty} \sup_{\varphi_0 \in \mathscr{A}} \frac{1}{t} \int_0^t \| \nabla u \|_2^2 \, \| Au \|_2^2 ds \leq L_1;$$

and

(iii) $$\limsup_{t \to \infty} \sup_{\varphi_0 \in \mathscr{A}} \frac{1}{t} \int_0^t \| \nabla \theta \|_2^2 ds \leq L_3,$$

where

$$L_1 = L_2 \frac{|\Omega| |f|_\infty^2}{\nu} \left(2 + \frac{1}{\nu}\right) \exp\left(\frac{C |\Omega|^2 |f|_\infty^4}{\nu^7}\right);$$
$$L_2 = \frac{C |\Omega|^3 |f|_\infty^6}{\nu^9} \left(2 + \frac{1}{\nu}\right) \exp\left(\frac{C |\Omega|^2 |f|_\infty^4}{\nu^7}\right) + \frac{2 |\Omega| |f|_\infty^2}{\nu^2};$$

and

$$L_3 = \frac{C \nu^2 L_1}{\kappa^2} \eta^2 + \frac{|\Omega| |f|_\infty^2}{\nu^2 \kappa^2}.$$

The proof will be given later. Lemma 5.3 now implies

$$q_l \leq -c_3' \frac{\nu\kappa}{\nu+\kappa}\frac{l^2}{|\Omega|} + 2c_2'\left(\frac{1}{\nu}+\frac{1}{\kappa}\right)\left(\frac{|\Omega||f|_\infty^2}{\nu^2}+L_3\right)$$
$$+ (1+|f'|_\infty)l + \eta^4\nu\left(1+\frac{\nu}{\kappa}\right)\eta L_1.$$

Therefore, by the Schwarz inequality, we obtain

$$q_l \leq -c_4' \frac{\nu\kappa}{\nu+\kappa}\frac{l^2}{|\Omega|} + L_4,$$

where

$$c_4' = \frac{1}{2}c_3'$$

and

$$L_4 = \frac{\nu+\kappa}{\nu\kappa}\frac{|\Omega|}{c_4'}\left[(1+|f'|_\infty)^2 + c_4'c_2'\left(\frac{|f|_\infty^2}{\nu^2}+\frac{|f|_\infty^2}{\nu^2\kappa^2}+\frac{c_1^4\nu^2L_1}{\kappa^2|\Omega|}\eta^2\right)\right]$$
$$+ C\nu\left(1+\frac{\nu}{\kappa}\right)\eta^4 L_1.$$

We now easily obtain the desired result of Theorem 5.2 from the inequality

$$L_4 - c_4' \frac{\nu\kappa}{\nu+\kappa}\frac{l^2}{|\Omega|} < 0.$$

It remains to prove Lemma 5.3. Integrating (4.4) on $[0, t]$, we obtain

$$\nu\int_0^t \|\nabla u\|_2^2 ds \leq \|u_0\|_2^2 + \frac{|\Omega||f|_\infty^2}{\nu^2},$$

which yields (i) since $\mathscr{A}$ is bounded in H. We next prove part (ii). From (4.4) and (4.6), we see that

$$\int_t^{t+1} \|\nabla u\|_2^2 ds \leq \frac{1}{\nu}\|u(t)\|_2^2 + \frac{|\Omega||f|_\infty^2}{\nu^2}$$
$$\leq \frac{e^{-\nu t}}{\nu}\|u_0\|_2^2 + \frac{|\Omega||f|_\infty^2}{\nu^3}e^{-\nu t} + \frac{|\Omega||f|_\infty^2}{\nu^2}.$$

This, together with (4.5), implies that

$$\int_t^{t+1} \|u\|_2^2 \|\nabla u\|_2^2 ds$$

$$\leq \int_t^{t+1} \left(e^{-\nu s}\|u_0\|_2^2 + \frac{|\Omega||f|_\infty^2}{\nu^2}\right)\|\nabla u\|_2^2 ds$$

$$\leq \left(e^{-\nu t}\|u_0\|_2^2 + \frac{|\Omega||f|_\infty^2}{\nu^2}\right)\left(\frac{e^{-\nu t}}{\nu}\|u_0\|_2^2 + \frac{|\Omega||f|_\infty^2}{\nu^3}e^{-\nu} + \frac{|\Omega||f|_\infty^2}{\nu^2}\right)$$

$$\equiv h_1(t)h_2(t).$$

Taking (4.7) into account we have by Lemma 4.3,

$$\|\nabla u(t+1)\|_2^2 \leq \left(h_2(t) + \frac{2|\Omega||f|_\infty^2}{\nu}\right)\exp\left(\frac{C}{\nu^3}h_1(t)h_2(t)\right). \tag{5.2}$$

We also get from (4.7)

$$\frac{1}{t}\int_0^t \|Au\|_2^2 ds \leq \frac{1}{\nu t}\|\nabla u_0\|_2^2 + \frac{C}{\nu^4}\frac{1}{t}\int_0^t \|u\|_2^2\|\nabla u\|_2^4 ds + \frac{2|\Omega||f|_\infty^2}{\nu^2}. \tag{5.3}$$

We need to estimate the second term on the right-hand side. By (4.9) and(4.10) we see that, for all $T > 0$,

$$\|\nabla u(t)\|_2^2 \leq C(T), \qquad 0 \leq t \leq T, \tag{5.4}$$

with $C(T)$ independent of $\varphi_0 = \{u_0, \theta_0\}$. Let $\varepsilon > 0$; by (4.5) there exists $T_1 = T_1(\varepsilon, \mathscr{A})$ such that

$$\|u(t)\|_2^2 \leq \varepsilon + \frac{|\Omega||f|_\infty^2}{\nu^2} \equiv g_1(\varepsilon)$$

for all $t \geq T_1$ and all $\varphi_0 = \{u_0, \theta_0\} \in \mathscr{A}$. We can also deduce from (5.2) that there exists $T_2 = T_2(\varepsilon, \mathscr{A})$ such that

$$\begin{aligned}\|\nabla u(t)\|_2^2 &\leq \left(\frac{|\Omega||f|_\infty^2}{\nu^2} + \frac{2|\Omega||f|_\infty^2}{\nu} + \varepsilon\right)\exp\frac{C}{\nu^3}\left(\frac{|\Omega||f|_\infty^2}{\nu^2} + \varepsilon\right)^2 \\ &\equiv g_2(\varepsilon)\end{aligned} \tag{5.5}$$

for all $t \geq T_2 + 1$ and all $\varphi_0 = \{u_0, \theta_0\} \in \mathscr{A}$. So, if we set $T_0 = \max(T_1, T_2 + 1)$, then

$$\|u(t)\|_2^2 \|\nabla u(t)\|_2^2 \leq g_1(\varepsilon)g_2(\varepsilon)$$

for all $t \geq T_0$ and all $\varphi_0 = \{u_0, \theta_0\} \in \mathscr{A}$. We thus obtain

$$\frac{1}{t}\int_0^t \|u\|_2^2\|\nabla u\|_2^4 ds \leq \frac{1}{t}\int_0^{T_0} \|u\|_2^2\|\nabla u\|_2^4 ds + \frac{1}{t}\int_{T_0}^t \|u\|_2^2\|\nabla u\|_2^4 ds$$

$$\leq \frac{1}{t} C(T_0) + g_1(\varepsilon) g_2(\varepsilon) \frac{1}{t} \int_0^t \|\nabla u\|_2^2 ds,$$

with $C(T_0)$ independent of $\varphi_0 = \{u_0, \theta_0\} \in \mathscr{A}$. This, together with (i), implies that

$$\limsup_{t\to\infty} \sup_{\varphi_0 \in \mathscr{A}} \frac{1}{t} \int_0^t \|u\|_2^2 \|\nabla u\|_2^4 ds \leq g_1(\varepsilon) g_2(\varepsilon) \frac{|\Omega| |f|_\infty^2}{\nu^2}.$$

Since $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0$ yields

$$\limsup_{t\to\infty} \sup_{\varphi_0 \in \mathscr{A}} \frac{1}{t} \int_0^t \|u\|_2^2 \|\nabla u\|_2^4 ds \leq \frac{|\Omega|^3 |f|_\infty^6}{\nu^5} \left(2 + \frac{1}{\nu}\right) \exp\left(\frac{C |\Omega|^2 |f|_\infty^4}{\nu^7}\right).$$

Combining this with (5.3), we have

$$\begin{aligned} &\limsup_{t\to\infty} \sup_{\varphi_0 \in \mathscr{A}} \frac{1}{t} \int_0^t \|Au\|_2^2 ds \\ &\leq \frac{C |\Omega|^3 |f|_\infty^6}{\nu^9} \left(2 + \frac{1}{\nu}\right) \exp\left(\frac{C |\Omega|^2 |f|_\infty^4}{\nu^7}\right) + \frac{2 |\Omega| |f|_\infty^2}{\nu^2} \qquad (5.6) \\ &= L_2. \end{aligned}$$

We now estimate $\frac{1}{t} \int_0^t \|\nabla u\|_2^2 \|Au\|_2^2 ds$. We see from (4.6), (4.7) and (5.4) that for all $T > 0$,

$$\begin{aligned} \nu \int_0^t \|Au\|_2^2 ds &\leq \|\nabla u_0\|_2^2 + \frac{C}{\nu^3} \int_0^t \|u\|_2^2 \|\nabla u\|_2^4 ds + \frac{2 |\Omega| |f|_\infty^2}{\nu} t \\ &\leq C(T), \qquad (0 \leq t \leq T), \end{aligned}$$

with $C(T)$ independent of $\varphi_0 = \{u_0, \theta_0\}$, since $\mathscr{A}$ is bounded in H. It then follows from (5.4) and (5.5) that

$$\begin{aligned} \frac{1}{t} \int_0^t \|\nabla u\|_2^2 \|Au\|_2^2 ds &\leq \frac{1}{t} \int_0^{T_0} \|\nabla u\|_2^2 \|Au\|_2^2 ds + \frac{1}{t} \int_{T_0}^t \|\nabla u\|_2^2 \|Au\|_2^2 ds \\ &\leq \frac{1}{t} C(T_0) + g_2(\varepsilon) \frac{1}{t} \int_0^t \|Au\|_2^2 ds \end{aligned}$$

for $t \geq T_0$, with $C(T_0)$ independent of $\varphi_0 = \{u_0, \theta_0\}$. This, together with (5.6), implies that

$$\limsup_{t\to\infty} \sup_{\varphi_0 \in \mathscr{A}} \frac{1}{t} \int_0^t \|\nabla u\|_2^2 \|Au\|_2^2 ds \leq g_2(\varepsilon) L_2.$$

Since $\varepsilon > 0$ is arbitrary, letting $\varepsilon \to 0$ yields

$$\limsup_{t\to\infty} \sup_{\varphi_0 \in \mathscr{A}} \frac{1}{t} \int_0^t \|\nabla u\|_2^2 \|Au\|_2^2 ds \le L_2 \frac{|\Omega||f|_\infty^2}{\nu}\left(2 + \frac{1}{\nu}\right)\exp\left(\frac{C|\Omega|^2|f|_\infty^4}{\nu^7}\right) = L_1,$$

and (ii) is proved. Part (iii) now immediately follows: From (4.12), we see that

$$\frac{1}{t}\int_0^t \|\nabla\theta\|_2^2 ds \le \frac{1}{\kappa t}\|\theta_0\|_2^2 + \frac{C\nu^2\eta^2}{\kappa^2}\frac{1}{t}\int_0^t \|\nabla u\|_2^2\|Au\|_2^2 ds + \frac{2}{\kappa^2}\frac{1}{t}\int_0^t \|\nabla u\|_2^2 ds.$$

Combining this with (i) and (ii) gives

$$\limsup_{t\to\infty}\sup_{\varphi_0\in\mathscr{A}} \int_0^t \|\nabla\theta\|_2^2 ds \le \frac{C\nu^2\eta^2}{\kappa} L_1 + \frac{|\Omega|}{\nu^2\kappa^2} = L_3.$$

This completes the proof.

Proof of Proposition 5.1. Throughout the proof we fix an arbitrary $T > 0$. Let $\{u_i, \theta_i\}$, $i = 1, 2$, be the solutions of (4.1) and (4.2) with initial values $\{u_{0,i}, \theta_{0,i}\} \in \mathscr{A}$, $i = 1, 2$, respectively, and set $u = u_2 - u_1$, $\theta = \theta_2 - \theta_1$. Let $\{U_1, \Theta_1\}$ be the solution of (LP) with initial value $\{u_{0,2} - u_{0,1}, \theta_{0,2} - \theta_{0,1}\}$ and set $v = u_2 - u_1 - U_1$, $\psi = \theta_2 - \theta_1 - \Theta_1$. Then

$$\frac{du}{dt} + \nu Au + P(u\cdot\nabla u_2 + u_1\cdot\nabla u) = P[(f(\theta_2) - f(\theta_1))e_2], \tag{5.7}$$

$$\frac{d\theta}{dt} + \kappa Bu + u\cdot\nabla\theta_2 + u_1\cdot\nabla\theta = \frac{\eta\nu}{2}D(u_1+u_2): D(u) - e_2\cdot u, \tag{5.8}$$

$$u(0) = u_{0,2} - u_{0,1}, \qquad \theta(0) = \theta_{0,2} - \theta_{0,1}.$$

and

(5.9)

$$\frac{dv}{dt} + \nu Av + P(u_1\cdot\nabla v + v\cdot\nabla u_1 + u\cdot\nabla u) = P[(f(\theta_2) - f(\theta_1) - f'(\theta_1)\Theta_1)e_2],$$

$$\begin{aligned}&\frac{d\psi}{dt} + \kappa B\psi + u_1\cdot\nabla\psi + v\cdot\nabla\theta_1 + u\cdot\nabla\theta\\ &= 2\eta\nu D(u_1): D(u) + \eta\nu D(u): D(u) - e_2\cdot v,\\ &v(0) = 0, \qquad \psi(0) = 0.\end{aligned} \tag{5.10}$$

We shall show that there exists a C, possibly depending on T, such that

$$\|v(t)\|_2^2 + \|\psi(t)\|_2^2 \le C(\|u_{0,2} - u_{0,1}\|_2^2 + \|\theta_{0,2} - \theta_{0,1}\|_2^2)^\beta \tag{5.11}$$

for some $\beta > 1$. The desired result then follows immediately. To deduce (5.11), we first show the Lipschitz-continuity of the operator $S(t)$. Note that there exists a $C > 0$ such that

$$\|Au_i(t)\|_2 + \|\nabla\theta_i(t)\|_2 \le C \qquad \text{for all } t,\ i = 1, 2, \tag{5.12}$$

with C is independent of u_i, θ_i, and $i = 1, 2$, since $\mathscr{A}$ is invariant and bounded in $D(A) \times H_0^1(\Omega)$. Multiplying (5.7) by u, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_2^2 + \nu\|\nabla u\|_2^2 \le |(u\cdot\nabla u_2, u)| + |(f(\theta_2) - f(\theta_1), e_2\cdot u)|.$$

With the aid of (1.7) with $p = 4$, the first term on the right-hand side is estimated as

$$\begin{aligned}|(u\cdot\nabla u_2, u)| &\le \|u\|_4^2\|\nabla u_2\|_2 \le C\|u\|_2\|\nabla u\|_2\|\nabla u_2\|_2\\ &\le \frac{\nu}{8}\|\nabla u\|_2^2 + C\|\nabla u\|_2^2\|u\|_2^2.\end{aligned}$$

As for the second term, writing

$$\begin{aligned}f(\theta_2) - f(\theta_1) &= \theta\int_0^1 f'(\theta_1 + \zeta(\theta_2 - \theta_1))d\zeta\\ &\equiv h(\theta_1, \theta_2),\end{aligned}$$

we have

$$|(f(\theta_2) - f(\theta_1), e_2\cdot u)| \le |f'|_\infty\|\theta\|_2\|u\|_2 \le \frac{|f'|}{2}(\|u\|_2^2 + \|\theta\|_2^2).$$

We next multiply (5.8) by θ to get

$$\frac{1}{2}\frac{d}{dt}\|\theta\|_2^2 + \kappa\|\nabla\theta\|_2^2 \le |(u\cdot\nabla\theta_2, \theta)| + \eta\frac{\nu}{2}|(D(u_1 + u_2){:}\, D(u), \theta)| + |(e_2\cdot u, \theta)|.$$

Using (1.7) with $p = 4$, we estimate the first term on the right-hand side as

$$\begin{aligned}|(u\cdot\nabla\theta_2, \theta)| &\le \|u\|_4\|\theta\|_4\|\nabla\theta_2\|_2\\ &\le C\|u\|_2^{1/2}\|\nabla u\|_2^{1/2}\|\theta\|_2^{1/2}\|\nabla\theta\|_2^{1/2}\|\nabla\theta_2\|_2\\ &\le \frac{\nu}{8}\|\nabla u\|_2^2 + \frac{\kappa}{4}\|\nabla\theta\|_2^2 + C\|\nabla\theta_2\|_2^2(\|u\|_2^2 + \|\theta\|_2^2).\end{aligned}$$

We also have, by (1.3), (1.5) and (1.7) with $p = 4$,

$$\frac{\eta\nu}{2}|(D(u_1+u_2)\colon D(u),\theta)|$$
$$\leq C\|\nabla(u_1+u_2)\|_4\|\nabla(u_1+u_2)\|_4\|\nabla u\|_2\|\theta\|_4$$
$$\leq C\|\nabla(u_1+u_2)\|_2^{1/2}\|A(u_1+u_2)\|_2^{1/2}\|\nabla u\|_2\|\theta\|_2^{1/2}\|\nabla\theta\|_2^{1/2}.$$

We thus deduce from (5.12) that

$$\frac{\eta\nu}{2}|(D(u_1+u_2)\colon D(u),\theta)|\leq\frac{\nu}{8}\|\nabla u\|_2^2+\frac{\kappa}{4}\|\nabla\theta\|_2^2+C\|\theta\|_2^2.$$

Since

$$|(e_2\cdot u,\theta)|\leq\frac{1}{2}(\|u\|_2^2+\|\theta\|_2^2).$$

we conclude that if $\{u_{0,i},\theta_{0,i}\}\in\mathscr{A}$, then

$$\frac{d}{dt}(\|u\|_2^2+\|\theta\|_2^2)+\nu\int_0^t\|\nabla u\|_2^2ds+\kappa\int_0^t\|\nabla\theta\|_2^2ds\leq k(t)(\|u\|_2^2+\|\theta\|_2^2)$$

for some $k(t)\in L^1(0,T)$. This implies that

$$\|u(t)\|_2^2+\|\theta(t)\|_2^2\leq C(\|u_{0,2}-u_{0,1}\|_2^2+\|\theta_{0,2}-\theta_{0,1}\|_2^2)\tag{5.13}$$

and

$$\int_0^t\|\nabla u\|_2^2ds+\int_0^t\|\nabla\theta\|_2^2ds\leq C(\|u_{0,2}-u_{0,1}\|_2^2+\|\theta_{0,2}-\theta_{0,1}\|_2^2)\tag{5.14}$$

for $0\leq t\leq T$ if $\{u_{0,i},\theta_{0,i}\}\in\mathscr{A}$, $i=1,2$. We have thus proved the Lipschitz-continuity.

We can now prove the uniform differentiability of $S(t)$ with respect to the L^2-norm. Multiplying (5.9) by v, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|_2^2+\nu\|\nabla v\|_2^2\leq|(v\cdot\nabla u_1,v)|+|(u\cdot\nabla u,v)|\\+|(f(\theta_2)-f(\theta_1)-f'(\theta_1)\Theta_1,v)|.\end{aligned}\tag{5.15}$$

Using (1.7) with $p=4$, we see from (5.12) that

$$|(v\cdot\nabla u_1,v)|\leq\|v\|_4^2\|\nabla u_1\|_2\leq C\|v\|_2\|\nabla v_2\|\nabla u_1\|_2$$
$$\leq\frac{\nu}{8}\|\nabla v\|_2^2+C\|v\|_2^2,$$
$$|(u\cdot\nabla u,v)|=|(u\cdot\nabla v,u)|\leq\|u\|_4^2\|\nabla v\|_2\leq C\|u\|_2\|\nabla u\|_2\|\nabla v\|_2$$
$$\leq\frac{\nu}{8}\|\nabla v\|_2^2+C\|u\|_2^2\|\nabla u\|_2^2.$$

To estimate the last term on the right-hand side of (5.15), we write

$$\begin{aligned} f(\theta_2) - f(\theta_1) - f'(\theta_1)\Theta_1 &= h(\theta_1, \theta_2)\theta - f'(\theta_1)\Theta_1 \\ &= (h(\theta_1, \theta_2) - f'(\theta_1))\theta + f'(\theta_1)\psi \\ &= g(\theta_1, \theta_2)\theta^2 + f'(\theta_1)\psi, \end{aligned}$$

with

$$g(\theta_1, \theta_2) = \int_0^1 \int_0^1 \tau f''(\theta_1 + \tau\zeta(\theta_2 - \theta_1))d\tau d\zeta,$$

to get

$$\begin{aligned} |(f(\theta_2) - f(\theta_1) - f'(\theta_1)\Theta_1, v)| &\le |(g(\theta_1, \theta_2)\theta^2 + f'(\theta_1)\psi, v)| \\ &\le C(|f'|_\infty, |f''|_\infty)(\|\theta\|_4^2 + \|\psi\|_2)\|v\|_2 \\ &\le C(\|\theta\|_2 \|\nabla\theta\|_2 + \|\psi\|_2)\|v\|_2 \\ &\le C(\|v\|_2^2 + \|\psi\|_2^2 + \|\theta\|_2^2 \|\nabla\theta\|_2^2). \end{aligned}$$

To estimate ψ, we multiply (5.10) by θ to get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\psi\|_2^2 + \kappa\|\nabla\theta\|_2^2 \le |(v \cdot \nabla\theta_1, \psi)| + |(u \cdot \nabla\theta, \psi)| + 2\eta\nu|(D(u_1)\colon D(u), \psi)| \\ + \eta\nu|(D(u)\colon D(u), \psi)| + |(e_2 \cdot v, \theta)|. \end{aligned}$$

The first and second terms on the right-hand side are estimated as

$$\begin{aligned} |(v \cdot \nabla\theta_1, \psi)| \le \|v\|_4 \|\psi\|_4 \|\nabla\theta_1\|_2 &\le C\|v\|_2^{1/2}\|\nabla v\|_2^{1/2}\|\psi\|_2^{1/2}\|\nabla\psi\|_2^{1/2}\|\nabla\theta_1\|_2 \\ &\le \frac{\nu}{8}\|\nabla v\|_2^2 + \frac{\kappa}{8}\|\nabla\psi\|_2^2 + C\|\nabla\theta_1\|_2^2(\|v\|_2^2 + \|\psi\|_2^2) \\ &\le \frac{\nu}{8}\|\nabla v\|_2^2 + \frac{\kappa}{8}\|\nabla\psi\|_2^2 + C(\|v\|_2^2 + \|\psi\|_2^2), \end{aligned}$$

$$\begin{aligned} |(u \cdot \nabla\theta, \psi)| = |(u \cdot \nabla\psi, \theta)| &\le \|u\|_4 \|\nabla\psi\|_2 \|\theta\|_4 \\ &\le C\|u\|_2^{1/2}\|\nabla u\|_2^{1/2}\|\nabla\psi\|_2\|\theta\|_2^{1/2}\|\nabla\theta\|_2^{1/2} \\ &\le \frac{\kappa}{8}\|\nabla\psi\|_2^2 + C(\|u\|_2^2\|\nabla u\|_2^2 + \|\theta\|_2^2\|\nabla\theta\|_2^2). \end{aligned}$$

By (5.12), we have

$$\begin{aligned} 2\eta\nu|(D(u_1)\colon D(u), \psi)| &\le C\|\nabla u_1\|_4 \|\nabla v\|_2 \|\psi\|_4 \\ &\le C\|\nabla u_1\|_2^{1/2}\|Au_1\|_2^{1/2}\|\nabla v\|_2\|\psi\|_2^{1/2}\|\nabla\psi\|_2^{1/2} \end{aligned}$$

$$\leq \frac{\nu}{8}\|\nabla v\|_2^2 + \frac{\kappa}{8}\|\nabla\psi\|_2^2 + C\|\nabla u_1\|_2^2\|Au_1\|_2^2\|\psi\|_2^2$$

$$\leq \frac{\nu}{8}\|\nabla v\|_2^2 + \frac{\kappa}{8}\|\nabla\psi\|_2^2 + C\|\psi\|_2^2.$$

To estimate $\eta\nu|D(u)\colon D(u), \psi)|$, we use the inequalities

$$\|\nabla u\|_2 \leq C\|u\|_2^{1/2}\|Au\|_2^{1/2}, \qquad u\in D(A)$$

and

$$\|\nabla u\|_3 \leq C\|\nabla u\|_2^{2/3}\|Au\|_2^{1/3}, \qquad u\in D(A).$$

The first is obtained by integrating by parts, while the second follows from (1.3) and (1.5) with $p = 3$. We then obtain

$$\begin{aligned}\eta\nu|(D(u)\colon D(u), \psi)| &\leq C\|\nabla u\|_3\|\nabla u\|_2\|\psi\|_6\\ &\leq C\|\nabla u\|_2\|Au\|_2^{1/3}\|u\|_2^{1/3}\|Au\|_2^{1/3}\|\psi\|_6\\ &\leq \frac{\kappa}{8}\|\nabla\psi\|_2^2 + C\|\nabla u\|_2^2\|Au\|_2^{4/3}\|u\|_2^{2/3}\\ &\leq \frac{\kappa}{8}\|\nabla\psi\|_2^2 + C\|\nabla u\|_2^2\|u\|_2^{2/3}.\end{aligned}$$

Here we have used the Poincaré-Sobolev inequality $\|\psi\|_6 \leq C\|\nabla\psi\|_2$. Also, we have

$$|(e_2\cdot v, \psi)| \leq \|v\|_2\|\psi\|_2 \leq \|v\|_2^2 + \|\psi\|_2^2.$$

In conclusion, we obtain

$$\frac{d}{dt}(\|v\|_2^2 + \|\psi\|_2^2) \leq C[\|v\|_2^2 + \|\psi\|_2^2 + (\|u\|_2^{2/3} + \|u\|_2^2)\|\nabla u\|_2^2 + \|\theta\|_2^2\|\nabla\theta\|_2^2].$$

Combining this with (5.13) and (5.14), we find

$$\begin{aligned}&\|v(t)\|_2^2 + \|\psi(t)\|_2^2\\ &\quad\leq C\int_0^t[(\|u\|_2^{2/3} + \|u\|_2^2)\|\nabla u\|_2^2 + \|\theta\|_2^2\|\nabla\theta\|_2^2]ds\\ &\quad\leq C(\|u_{0,2} - u_{0,1}\|_2^2 + \|\theta_{0,2} - \theta_{0,1}\|_2^2)^{1/3}\int_0^t(\|\nabla u\|_2^2 + \|\theta\|_2^2)ds\\ &\quad\leq C(\|u_{0,2} - u_{0,1}\|_2^2 + \|\theta_{0,2} - h_{0,1}\|_2^2)^{4/3}.\end{aligned}$$

This shows (5.11) with $\beta = 4/3$. The proof is complete.

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