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Nonhomogeneity of Picard dimensions of rotation free hyperbolic densities

Dedicated to Professor Mitsuru Nakai on his 60th birthday

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Abstract. A real valued C^∞ function $P(z)$ on the punctured disk $0<|z|\le 1$ is constructed in such a way that there exists only one Martin minimal boundary point for the time independent Schrödinger equation $(-\Delta+P(z))u(z)=0$ over $z=0$ and, neverthless, there exist more than one Martin minimal boundary points for $(-\Delta+P(z)/4)u(z)=0$ over $z=0$.

We denote by Ω the punctured disc $0<|z|<1$ and consider a time independent Schrödinger equation

$$(1)\qquad (-\Delta+P(z))u(z)=0 \qquad \left(\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\ z=x+yi\right)$$

on Ω. The potential P is assumed to be a locally Hölder continuous function on $0<|z|\le 1$ and referred to as a *density* on Ω. Then a density P may take both positive and negative values. With a density P we associate the class $PP(\Omega;\Gamma)$ of nonnegative C^2 functions u on $\Omega\cup\Gamma$ satisfying the equation (1) in Ω and vanishing on the unit circle Γ: $|z|=1$. We also denote by $PP_1(\Omega;\Gamma)$ the subclass of $PP(\Omega;\Gamma)$ consisting of functions u with the normalization

$$-\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\partial}{\partial t}u(te^{i\theta})\right]_{t=1}d\theta=1.$$

The Choquet theorem (cf. e.g. [12]) yields that there exists a bijective correspondence $u\leftrightarrow\mu$ between the convex cone $PP(\Omega;\Gamma)$ and the set of Borel measures μ on the set ex.$PP_1(\Omega;\Gamma)$ of extremal points of the convex set $PP_1(\Omega;\Gamma)$ such that

$$u=\int_{\mathrm{ex}.PP_1(\Omega;\Gamma)} v\,d\mu(v).$$

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Thus the set $\text{ex}.PP_1(\Omega; \Gamma)$ is essential for the class $PP(\Omega; \Gamma)$, and the cardinal number $\#(\text{ex}.PP_1(\Omega; \Gamma))$ of $\text{ex}.PP_1(\Omega; \Gamma)$ is referred to as the *Picard dimension* of a density P at $z = 0$, $\dim P$ in notation, i.e.

$$\dim P = \#(\text{ex}.PP_1(\Omega; \Gamma)).$$

We say that a density P is *hyperbolic* on Ω if $\dim P \geq 1$ and there exists the Green's function on Ω with respect to the equation (1). Then nonnegative densities are hyperbolic on Ω ([6]).

A density P is said to be *rotation free* if P satisfies $P(z) = P(|z|)(z \in \Omega)$. Let P be a nonnegative rotation free density. Then $\dim P$ is equal to 1 or the cardinal number $\mathfrak{c}$ of the continuum ([7]) and satisfies

$$\dim P = \dim (cP) \qquad (c > 0)$$

([3]). We call this property the *homogeneity* of Picard dimensions of nonnegative rotation free densities.

Let P be a signed rotation free density. Then $\dim P$ is also 1 or $\mathfrak{c}$ if P is hyperbolic on Ω ([7], [11], [4]). Moreover if P is hyperbolic on Ω, the density cP $(0 < c \leq 1)$ is hyperbolic on Ω and satisfies

$$\dim P \leq \dim (cP) \qquad (0 < c \leq 1)$$

([11]). The purpose of this paper is to prove the following theorem which shows the nonhomogeneity of Picard dimensions of rotation free hyperbolic densities:

THEOREM. *There exists a rotation free hyperbolic density P on Ω such that*

$$\dim P = 1 \quad \text{and} \quad \dim\left(\frac{1}{4}P\right) = \mathfrak{c}.$$

It was shown in [11] that the above inequality $\dim P \leq \dim (cP)$ $(0 < c \leq 1)$ is also valid for every rotation free density P which is not hyperbolic on Ω. At the same time it was shown that the inequality sign in $\dim P \leq \dim (cP)$ can not be replaced by the equality sign ([2], [11]): There exists a rotation free density P on Ω such that $\dim P = 0$ and $\dim (cP) = 1$ $(0 < c < 1)$. Precisely speaking, the Picard dimension of P (cP, rsep.) considered on $0 < |z| < a$ is 0 (1, resp.) for every $a \in (0, 1]$ and $c \in (0, 1)$. Then it was asked a question in [11] whether there exists a rotation free density P such that $1 \leq \dim P < \dim (cP)$ $(0 < c < 1)$. The above theorem gives an answer to this question. We remark that the Picard dimension of a rotation free density P considered on Ω coincides with the one considered on $0 < |z| < a$ $(0 < a < 1)$ if P is hyperbolic on Ω ([8], [5], [11]).

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1. Subunit criterions

Hearafter *every* density P on Ω in consideration is assumed to be rotation free and is mainly viewed as a function $P(r)$ of r in the interval $(0, 1]$. For a density P we consider the differential equation

$$L_P u(r) \equiv -\frac{d^2}{dr^2}u(r) - \frac{1}{r}\frac{d}{dr}u(r) + P(r)u(r) = 0$$

for C^2 functions $u(r)$ in $(0, 1)$. The unique solution f_P of this equation with initial conditions

$$f_P(1) = 0 \quad \text{and} \quad f_P'(1) = -1$$

is referred to as the P-*subunit*. Then we have the following characterization of hyperbolicity for P in terms of f_P:

THEOREM A ([11], [4]). *A density P is hyperbolic on Ω if and only if*

$$f_P(r) > 0 \ (0 < r < 1) \quad \textit{and} \quad \int_0^{1/2} \frac{dr}{rf_P(r)^2} < \infty.$$

Moreover we have the following test of $\dim P = 1$ for hyperbolic densities P:

THEOREM B ([11], [4]). *A hyperbolic density P on Ω satisfies* $\dim P = 1$ *if and only if*

$$\int_0^a \frac{f_P(r)^2}{r} \int_0^r \frac{ds}{sf_P(s)^2}\, dr = \infty$$

for some a, and hence for any a, in $(0, 1]$.

2. P-subunits for discontinuous densities P

2.1. We take a positive numbers θ with $\theta \le \pi/2$ and sequences $\{\alpha_n\}_1^\infty$, $\{\beta_n\}_1^\infty$ of positive number α_n, β_n. With θ and $\{\beta_n\}$ we associate sequences $\{a_n\}_1^\infty$, $\{b_n\}_1^\infty$ of positive numbers $a_n = a_n(\theta, \{\beta_n\})$, $b_n = b_n(\theta, \{\beta_n\})$ defined by

$$\rho = 100,\ a_1 = 1,\ b_n = \rho^{-1}a_n,\ a_{n+1} = e^{-\theta/\beta_n}b_n \qquad (n = 1, 2, \cdots). \tag{2}$$

The sequences $\{a_n\}$ and $\{b_n\}$ satisfy

$$1 \ge a_n > b_n > a_{n+1} \ (n = 1, 2, \cdots),\ \lim_{n\to\infty} a_n = 0.$$

Now we consider a discontinuous function $P = P(\,\cdot\,; \theta, \{\alpha_n\}, \{\beta_n\})$ given by

$$P(r) = \begin{cases} \dfrac{\alpha_n^2}{r^2} & (b_n \leq r \leq a_n;\ n = 1, 2, \cdots) \\ -\dfrac{\beta_n^2}{r^2} & (a_{n+1} < r < b_n;\ n = 1, 2, \cdots) \end{cases} \tag{3}$$

and a C^1 function F_P on $(0, 1]$ satisfying

$$F_P(1) = 0,\ F_P'(1) = -1,\ L_P F_P = 0 \text{ on } \bigcup_{n=1}^{\infty} \{(a_{n+1}, b_n) \cup (b_n, a_n)\}. \tag{4}$$

The definition of a density on Ω can be generalized ([1], [4], [9], [10]). In this sense, the above function P is a discontinuous density on Ω. Moreover F_P is equal to the P-subunit f_P and both Theorems A and B are valid for P and $f_P = F_P$. However we do not use these facts in this paper.

2.2. By the condition (4), the function F_P has the following form on each interval:

$$F_P(r) = \begin{cases} x_n\left\{\left(\dfrac{a_n}{r}\right)^{\alpha_n} - \left(\dfrac{r}{a_n}\right)^{\alpha_n}\right\} + y_n\left\{\left(\dfrac{r}{b_n}\right)^{\alpha_n} - \left(\dfrac{b_n}{r}\right)^{\alpha_n}\right\} \\ \hfill (b_n \leq r \leq a_n) \\ -z_n \sin\left(\beta_n \log \dfrac{r}{b_n}\right) + w_n \sin\left(\beta_n \log \dfrac{r}{a_{n+1}}\right) \\ \hfill (a_{n+1} < r < b_n) \end{cases} \tag{5}$$

$$(n = 1, 2, \cdots).$$

The coefficients $x_n = x_n(P)$, $y_n = y_n(P)$, $z_n = z_n(P)$, $w_n = w_n(P)$ depend on P and hence θ, $\{\alpha_n\}$, $\{\beta_n\}$. In particular

$$x_1 = \frac{1}{2\alpha_1}, \quad y_1 = 0 \tag{6}$$

by (4). Since F_P is of class C^1, these coefficients satisfy conditions

$$\begin{gathered} w_n \sin\theta = (\rho^{\alpha_n} - \rho^{-\alpha_n}) x_n, \\ -\beta_n z_n + \beta_n w_n \cos\theta = -\alpha_n(\rho^{\alpha_n} + \rho^{-\alpha_n}) x_n + 2\alpha_n y_n, \\ (\rho^{\alpha_{n+1}} - \rho^{-\alpha_{n+1}}) y_{n+1} = z_n \sin\theta, \\ -2\alpha_{n+1} x_{n+1} + \alpha_{n+1}(\rho^{\alpha_{n+1}} + \rho^{-\alpha_{n+1}}) y_{n+1} = -\beta_n z_n \cos\theta + \beta_n w_n \end{gathered}$$

for every $n = 1, 2, \cdots$ which are equivalent to

$$\begin{pmatrix} 0 & \sin\theta \\ -\beta_n & \beta_n\cos\theta \end{pmatrix}\begin{pmatrix} z_n \\ w_n \end{pmatrix} = \begin{pmatrix} \rho^{\alpha_n}-\rho^{-\alpha_n} & 0 \\ -\alpha_n(\rho^{\alpha_n}+\rho^{-\alpha_n}) & 2\alpha_n \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix},$$

$$\begin{pmatrix} 0 & \rho^{\alpha_{n+1}}-\rho^{-\alpha_{n+1}} \\ -2\alpha_{n+1} & \alpha_{n+1}(\rho^{\alpha_{n+1}}+\rho^{-\alpha_{n+1}}) \end{pmatrix}\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \sin\theta & 0 \\ -\beta_n\cos\theta & \beta_n \end{pmatrix}\begin{pmatrix} z_n \\ w_n \end{pmatrix}.$$

These conditions are also equivalent to

$$\begin{pmatrix} z_n \\ w_n \end{pmatrix}\begin{pmatrix} \cot\theta & -\dfrac{1}{\beta_n} \\ \operatorname{cosec}\theta & 0 \end{pmatrix}\begin{pmatrix} \rho^{\alpha_n}-\rho^{-\alpha_n} & 0 \\ -\alpha_n(\rho^{\alpha_n}+\rho^{-\alpha_n}) & 2\alpha_n \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix},$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \dfrac{\rho^{\alpha_{n+1}}+\rho^{-\alpha_{n+1}}}{2(\rho^{\alpha_{n+1}}-\rho^{-\alpha_{n+1}})} & -\dfrac{1}{2\alpha_{n+1}} \\ \dfrac{1}{\rho^{\alpha_{n+1}}-\rho^{-\alpha_{n+1}}} & 0 \end{pmatrix}\begin{pmatrix} \sin\theta & 0 \\ -\beta_n\cos\theta & \beta_n \end{pmatrix}\begin{pmatrix} z_n \\ w_n \end{pmatrix}$$

so that we have

$$\begin{pmatrix} z_n \\ w_n \end{pmatrix} = \rho^{\alpha_n}\begin{pmatrix} (1-\rho^{-2\alpha_n})\cot\theta + \dfrac{\alpha_n}{\beta_n}(1+\rho^{-2\alpha_n}) & -\dfrac{2\alpha_n}{\beta_n}\rho^{-\alpha_n} \\ (1-\rho^{-2\alpha_n})\operatorname{cosec}\theta & 0 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix}, \tag{7}$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \frac{\rho^{\alpha_n}}{2(1-\rho^{-2\alpha_{n+1}})}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix} \tag{8}$$

$$(n = 1, 2, \cdots),$$

where

$$A_{11} = \left\{(1-\rho^{-2\alpha_n})(1+\rho^{-2\alpha_{n+1}}) + \frac{\alpha_n}{\alpha_{n+1}}(1+\rho^{-2\alpha_n})(1-\rho^{-2\alpha_{n+1}})\right\}\cos\theta$$
$$+ \left\{\frac{\alpha_n}{\beta_n}(1+\rho^{-2\alpha_n})(1+\rho^{-2\alpha_{n+1}}) - \frac{\beta_n}{\alpha_{n+1}}(1-\rho^{-2\alpha_n})(1-\rho^{-2\alpha_{n+1}})\right\}\sin\theta,$$

$$A_{12} = -2\rho^{-\alpha_n}\left\{\frac{\alpha_n}{\alpha_{n+1}}(1-\rho^{-2\alpha_{n+1}})\cos\theta + \frac{\alpha_n}{\beta_n}(1+\rho^{-2\alpha_{n+1}})\sin\theta\right\},$$

$$A_{21} = 2\rho^{-\alpha_{n+1}}\left\{(1-\rho^{-2\alpha_n})\cos\theta + \frac{\alpha_n}{\beta_n}(1+\rho^{-2\alpha_n})\sin\theta\right\},$$

$$A_{22} = -4\frac{\alpha_n}{\beta_n}\rho^{-\alpha_n}\rho^{-\alpha_{n+1}}\sin\theta.$$

2.3. Assume that $x_n > y_n \geq 0$ for some n. Then from (7) and (8) it follows that

$$\frac{z_n}{x_n} > \rho^{\alpha_n}\left\{(1-\rho^{-2\alpha_n})\cot\theta + \frac{\alpha_n}{\beta_n}(1-\rho^{-\alpha_n})^2\right\} > 0,$$

$$\frac{w_n}{x_n} = \rho^{\alpha_n}(1-\rho^{-2\alpha_n})\operatorname{cosec}\theta > 0,$$

$$y_{n+1} = \frac{z_n \sin\theta}{\rho^{\alpha_{n+1}} - \rho^{-\alpha_{n+1}}} > 0.$$

Therefore we obtain the following lemma:

LEMMA 1. *If* $x_n > y_n \geq 0$ *for some* n, *then* $z_n > 0$, $w_n > 0$, $y_{n+1} > 0$.

3. Calculations of integrals

3.1. In this section we assume that $x_n > 0$, $z_n > 0$, $w_n > 0$, $y_{n+1} > 0$ $(n = 1, 2, \cdots)$. This assumption is equivalent to $F_P(r) > 0$ $(0 < r < 1)$. In this no. we calculate integrals below.

LEMMA 2. *If* $n = 2, 3, \cdots$, *then*

(i)
$$\int_{b_n}^{a_n} \frac{dr}{rF_P(r)^2} = \frac{1}{2\alpha_n x_n y_n(\rho^{\alpha_n} - \rho^{-\alpha_n})},$$

(ii)
$$\int_{b_n}^{a_n} \frac{F_P(r)^2}{r}\int_{b_n}^{r}\frac{ds}{sF_P(s)^2}\,dr = \frac{1}{4\alpha_n^2(\rho^{\alpha_n} - \rho^{-\alpha_n})}$$
$$\times\left\{\frac{y_n}{x_n}(\rho^{2\alpha_n} - \rho^{-2\alpha_n} - 2\log\rho^{2\alpha_n})\right.$$
$$\left. + \rho^{-\alpha_n}((\rho^{2\alpha_n}+1)\log\rho^{2\alpha_n} - 2(\rho^{2\alpha_n}-1))\right\}.$$

PROOF. Consider the function

$$E(r) = F_P(r)\int_{b_n}^{r}\frac{ds}{sF_P(s)^2}$$

of r in $[b_n, a_n]$ which is a solution of $L_P u = 0$ on (b_n, a_n) along with F_P. Since $E(b_n) = 0$, $E(r)$ has the form

$$E(r) = c\left\{\left(\frac{r}{b_n}\right)^{\alpha_n} - \left(\frac{b_n}{r}\right)^{\alpha_n}\right\} \tag{9}$$

with a positive constant c. By setting $r = b_n$ in an equality

$$F_P'(r)\int_{b_n}^{r}\frac{ds}{sF_P(s)^2}+\frac{1}{rF_P(r)}=\frac{c\alpha_n}{r}\left\{\left(\frac{r}{b_n}\right)^{\alpha_n}+\left(\frac{b_n}{r}\right)^{\alpha_n}\right\}$$

and (5) we have

$$c=\frac{1}{2\alpha_n x_n(\rho^{\alpha_n}-\rho^{-\alpha_n})}.$$

Hence the equality (9) for $r=a_n$ is (i). The equality (ii) follows from calculations

$$\begin{aligned}
&2\alpha_n x_n(\rho^{\alpha_n}-\rho^{-\alpha_n})\int_{b_n}^{a_n}\frac{F_P(r)^2}{r}\int_{b_n}^{r}\frac{ds}{sF_P(s)^2}\,dr\\
&=\int_{b_n}^{a_n}\frac{1}{r}\left\{x_n\left(\left(\frac{a_n}{r}\right)^{\alpha_n}-\left(\frac{r}{a_n}\right)^{\alpha_n}\right)+y_n\left(\left(\frac{r}{b_n}\right)^{\alpha_n}-\left(\frac{b_n}{r}\right)^{\alpha_n}\right)\right\}\\
&\quad\times\left\{\left(\frac{r}{b_n}\right)^{\alpha_n}-\left(\frac{b_n}{r}\right)^{\alpha_n}\right\}dr\\
&=\int_{1}^{\rho}\frac{1}{t}\left\{x_n\left(\left(\frac{\rho}{t}\right)^{\alpha_n}-\left(\frac{t}{\rho}\right)^{\alpha_n}\right)+y_n\left(t^{\alpha_n}-\frac{1}{t^{\alpha_n}}\right)\right\}\left\{t^{\alpha_n}-\frac{1}{t^{\alpha_n}}\right\}dt\\
&=\frac{1}{2\alpha_n}\{(y_n-x_n\rho^{-\alpha_n})(\rho^{2\alpha_n}-1)+(y_n-x_n\rho^{\alpha_n})(1-\rho^{-2\alpha_n})\}\\
&\qquad+\{x_n(\rho^{\alpha_n}+\rho^{-\alpha_n})-2y_n\}\log\rho.
\end{aligned}$$

□

LEMMA 3. *If* $n=2,3,\cdots$, *then*

$$\begin{aligned}
\int_{b_n}^{a_n}\frac{F_P(r)^2}{r}\,dr=&\frac{1}{2\alpha_n}\{(x_n^2+y_n^2)(\rho^{2\alpha_n}-\rho^{-2\alpha_n}-2\log\rho^{2\alpha_n})\\
&+2x_ny_n\rho^{-\alpha_n}((\rho^{2\alpha_n}+1)\log\rho^{2\alpha_n}-2(\rho^{2\alpha_n}-1))\}.
\end{aligned}$$

PROOF. Lemma follows from calculations

$$\begin{aligned}
\int_{b_n}^{a_n}\frac{F_P(r)^2}{r}\,dr&=\int_{b_n}^{a_n}\frac{1}{r}\left\{x_n\left(\left(\frac{a_n}{r}\right)^{\alpha_n}-\left(\frac{r}{a_n}\right)^{\alpha_n}\right)\right.\\
&\qquad\left.+y_n\left(\left(\frac{r}{b_n}\right)^{\alpha_n}-\left(\frac{b_n}{r}\right)^{\alpha_n}\right)\right\}^2dr\\
&=\int_{1}^{\rho}\frac{1}{t}\left\{x_n\left(\left(\frac{\rho}{t}\right)^{\alpha_n}-\left(\frac{t}{\rho}\right)^{\alpha_n}\right)+y_n\left(t^{\alpha_n}-\frac{1}{t^{\alpha_n}}\right)\right\}^2dt\\
&=\int_{1}^{\rho}\frac{1}{t}\left\{(x_n^2+y_n^2)\left(t^{\alpha_n}-\frac{1}{t^{\alpha_n}}\right)^2\right.
\end{aligned}$$

$$+ 2x_n y_n\left(\rho^{\alpha_n} + \rho^{-\alpha_n} - \frac{t^{2\alpha_n}}{\rho^{\alpha_n}} - \frac{\rho^{\alpha_n}}{t^{2\alpha_n}}\right)\Bigg\} dt$$

$$= (x_n^2 + y_n^2)\left(\frac{\rho^{2\alpha_n} - \rho^{-2\alpha_n}}{2\alpha_n} - 2\log\rho\right)$$

$$+ 2x_n y_n\left((\rho^{\alpha_n} + \rho^{-\alpha_n})\log\rho - \frac{\rho^{\alpha_n} - \rho^{-\alpha_n}}{\alpha_n}\right),$$

where we use

$$\int_1^\rho \frac{1}{t}\left\{\left(\frac{\rho}{t}\right)^{\alpha_n} - \left(\frac{t}{\rho}\right)^{\alpha_n}\right\}^2 dt = \int_1^\rho \frac{1}{t}\left\{\frac{1}{t^{\alpha_n}} - t^{\alpha_n}\right\}^2 dt. \qquad \square$$

LEMMA 4. *If* $n = 1, 2, \cdots$, *then*

(i)
$$\int_{a_{n+1}}^{b_n} \frac{dr}{rF_P(r)^2} = \frac{1}{\beta_n z_n w_n \sin\theta},$$

(ii)
$$\int_{a_{n+1}}^{b_n} \frac{F_P(r)^2}{r}\int_{a_{n+1}}^{r} \frac{ds}{sF_P(s)^2}\,dr = \frac{1}{2\beta_n^2\sin\theta}\left\{\frac{w_n}{z_n}(\theta - \sin\theta\cos\theta) + \sin\theta - \theta\cos\theta\right\}.$$

PROOF. Consider the function

$$E(r) = F_P(r)\int_{a_{n+1}}^{r} \frac{ds}{sF_P(s)^2}$$

of r in (a_{n+1}, b_n) which is a solution of $L_P u = 0$ on (a_{n+1}, b_n) along with F_P. Since $E(a_{n+1}) = 0$, $E(r)$ has the form

(10)
$$E(r) = c\sin\left(\beta_n \log\frac{r}{a_{n+1}}\right)$$

with a positive constant c. By making $r \downarrow a_{n+1}$ in the equality

$$F_P'(r)\int_{a_{n+1}}^{r} \frac{ds}{sF_P(s)^2} + \frac{1}{rF_P(r)} = \frac{c\beta_n}{r}\cos\left(\beta_n\log\frac{r}{a_{n+1}}\right)$$

and (5) we have

$$c = \frac{1}{\beta_n z_n \sin\theta}.$$

Hence the equality (10) for $r = b_n$ is (i). The equality (ii) follows from calculations

$$\beta_n z_n \sin\theta \int_{a_{n+1}}^{b_n} \frac{F_P(r)^2}{r} \int_{a_{n+1}}^{r} \frac{ds}{sF_P(s)^2}\, dr$$
$$= \int_{a_{n+1}}^{b_n} \frac{1}{r}\left\{ - z_n \sin\left(\beta_n \log\frac{r}{b_n}\right) + w_n \sin\left(\beta_n \log\frac{r}{a_{n+1}}\right)\right\}$$
$$\times \sin\left(\beta_n \log\frac{r}{a_{n+1}}\right) dr$$
$$= \int_0^\theta \frac{1}{\beta_n}\{- z_n \sin(t-\theta) + w_n \sin t\} \sin t dt$$
$$= \frac{1}{2\beta_n}\int_0^\theta \{z_n(\cos(2t-\theta) - \cos\theta) + w_n(1-\cos 2t)\}\, dt. \qquad \square$$

LEMMA 5. *If* $n = 1, 2, \cdots$, *then*

$$\int_{a_{n+1}}^{b_n} \frac{F_P(r)^2}{r} dr = \frac{1}{2\beta_n}\{(z_n^2 + w_n^2)(\theta - \sin\theta\cos\theta) + 2z_n w_n(\sin\theta - \theta\cos\theta)\}.$$

PROOF. Lemma follows from calculations

$$\int_{a_{n+1}}^{b_n} \frac{F_P(r)^2}{r} dr = \int_{a_{n+1}}^{b_n} \frac{1}{r}\left\{ - z_n \sin\left(\beta_n \log\frac{r}{b_n}\right) + w_n \sin\left(\beta_n \log\frac{r}{a_{n+1}}\right)\right\}^2 dr$$
$$= \int_0^\theta \frac{1}{\beta_n}\{- z_n \sin(t-\theta) + w_n \sin t\}^2 dt$$
$$= \frac{1}{\beta_n}\int_0^\theta \left\{\frac{z_n^2 + w_n^2}{2}(1 - \cos 2t) + z_n w_n(\cos(2t-\theta) - \cos\theta)\right\} dt.$$
$\square$

3.2. In the final section, a discontinuous function P will be aproximated by a density Q on Ω such that behaviour of the Q-subunit f_Q is similar to that of F_P. An estimation of f_Q will be given by using the following integral form of f_Q/F_P:

LEMMA 6. *If a density Q on Ω satisfy $Q(r) = P(r)$ $(b_1 \le r \le 1)$, then the Q-subunit f_Q satisfy*

$$\frac{f_Q(r)}{F_P(r)} = 1 + \int_r^{b_1} s\{Q(s) - P(s)\} f_Q(s) F_P(s) \int_r^s \frac{dt}{tF_P(t)^2}\, ds \qquad (0 < r < 1).$$

PROOF. Since f_Q and F_P are solutions of $L_Q u = 0$ and $L_P u = 0$ respectively, we have

$$\frac{d}{dr}\left\{ rF_P(r)^2 \frac{d}{dr}\frac{f_Q(r)}{F_P(r)}\right\} = r(Q(r) - P(r)) f_Q(r) F_P(r)$$

$$(r \in \bigcup_{n=1}^{\infty} ((a_{n+1}, b_n) \cup (b_n, a_n))).$$

Let b be a number in $(b_1, 1)$. The fact that F'_P is right and left differentiable yields

$$b\{f'_Q(b)F_P(b) - f_Q(b)F'_P(b)\} - rF_P(r)^2\left\{\frac{f_Q(r)}{F_P(r)}\right\}'$$
$$= \int_r^{b_1} s\{Q(s) - P(s)\}f_Q(s)F_P(s)ds \qquad (0 < r < 1).$$

If $b \uparrow 1$, then the first term of the above equality goes to 0. This implies

$$\frac{f_Q(r)}{F_P(r)} - \frac{f_Q(b)}{F_P(b)} = \int_r^{b_1} \frac{1}{tF_P(t)^2}\int_t^{b_1} s\{Q(s) - P(s)\}f_Q(s)F_P(s)dsdt.$$

By $b \uparrow 1$ again, we obtain the lemma. □

4. F_R for a special R

4.1. We fix values of θ, α_n, and β_n:

$$\theta = \frac{\pi}{2},\ \alpha_n = n^2,$$
$$\beta_n = \frac{(n+1)^2}{2}\left\{\sqrt{\rho^{-2n^2+2n} + \frac{4n^2}{(n+1)^2}} - \rho^{-n^2+n}\right\} \qquad (n = 1, 2, \cdots). \tag{11}$$

Hereafter R denotes a special discontinuous function $P = P(\cdot\,; \theta, \{\alpha_n\}, \{\beta_n\})$ given by (3) with these θ, α_n, and β_n, where a_n and b_n are special numbers defined by (2) with these θ and β_n. Then F_R means a special C^1 function satisfying (4) with these a_n, b_n, and $P = R$ so that the coefficients x_n, y_n, z_n, and w_n in (5) are also special numbers. They are fixed by the initial values

$$x_1 = \frac{1}{2\alpha_1} = \frac{1}{2},\ y_1 = 0 \tag{12}$$

and the recursion formulas (7), (8) with (11):

$$\begin{pmatrix} z_n \\ w_n \end{pmatrix} = \rho^{\alpha_n}\begin{pmatrix} \frac{\alpha_n}{\beta_n}(1 + \rho^{-2\alpha_n}) & -\frac{2\alpha_n}{\beta_n}\rho^{-\alpha_n} \\ 1 - \rho^{-2\alpha_n} & 0 \end{pmatrix}\begin{pmatrix} x_n \\ y_n \end{pmatrix}, \tag{13}$$

$$\text{(14)} \qquad \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \frac{\rho^{\alpha_n}}{2(1-\rho^{-2\alpha_{n+1}})} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$
$$(n = 1, 2, \cdots),$$

where

$$A_{11} = \frac{\alpha_n}{\beta_n}(1+\rho^{-2\alpha_n})(1+\rho^{-2\alpha_{n+1}}) - \frac{\beta_n}{\alpha_{n+1}}(1-\rho^{-2\alpha_n})(1-\rho^{-2\alpha_{n+1}}),$$

$$A_{12} = -\frac{2\alpha_n}{\beta_n}\rho^{-\alpha_n}(1+\rho^{-2\alpha_{n+1}}),$$

$$A_{21} = \frac{2\alpha_n}{\beta_n}\rho^{-\alpha_{n+1}}(1+\rho^{-2\alpha_n}),\ A_{22} = -\frac{4\alpha_n}{\beta_n}\rho^{-\alpha_n}\rho^{-\alpha_{n+1}}.$$

In the proof of the lemma below we use properties of β_n

$$\text{(15)} \qquad \frac{\alpha_n}{\beta_n} - \frac{\beta_n}{\alpha_{n+1}} = \rho^{-n^2+n},\ \frac{1}{2} \le \frac{\alpha_n}{\beta_n} \le 2 \qquad (n = 1, 2, \cdots)$$

which are derived from

$$\left(\frac{2\beta_n}{\alpha_{n+1}} + \rho^{-n^2+n}\right)^2 = \rho^{-2n^2+2n} + \frac{4\alpha_n}{\alpha_{n+1}},$$

$$\frac{\beta_n}{\alpha_n} = \frac{2}{\sqrt{\rho^{-2n^2+2n} + 4\alpha_n/\alpha_{n+1}} + \rho^{-n^2+n}}.$$

The following lemma shows that F_R is positive on $(0, 1)$:

LEMMA 7. *The numbers* x_n, y_n, z_n, *and* w_n *satisfy*

$$y_1 = 0,\ x_n > y_n,\ z_n > 0,\ w_n > 0,\ y_{n+1} > 0 \qquad (n = 1, 2, \cdots).$$

PROOF. In view of Lemma 1 and (12) we only need to prove that $x_n > y_n \ge 0$ implies $x_{n+1} > y_{n+1}$. Suppose $x_n > y_n \ge 0$. Then by (14) we have

$$\frac{x_{n+1} - y_{n+1}}{x_n} \frac{2(1-\rho^{-2\alpha_{n+1}})}{\rho^{\alpha_n}} = \frac{\alpha_n}{\beta_n}(1+\rho^{-2\alpha_n})(1-\rho^{-\alpha_{n+1}})^2$$
$$- \frac{\beta_n}{\alpha_{n+1}}(1-\rho^{-2\alpha_n})(1-\rho^{-2\alpha_{n+1}}) - \frac{2\alpha_n}{\beta_n}\frac{y_n}{x_n}\rho^{-\alpha_n}(1-\rho^{-\alpha_{n+1}})^2$$
$$\ge \frac{\alpha_n}{\beta_n}(1-\rho^{-\alpha_n})^2(1-\rho^{-\alpha_{n+1}})^2 - \frac{\beta_n}{\alpha_{n+1}}(1-\rho^{-2\alpha_n})(1-\rho^{-2\alpha_{n+1}}).$$

Hence by (15) we have

$$\frac{x_{n+1}-y_{n+1}}{x_n}\frac{2(1+\rho^{-\alpha_{n+1}})}{\rho^{\alpha_n}(1-\rho^{-\alpha_n})}\geq\left(\frac{\alpha_n}{\beta_n}-\frac{\beta_n}{\alpha_{n+1}}\right)(1+\rho^{-\alpha_n-\alpha_{n+1}})$$

$$-\left(\frac{\alpha_n}{\beta_n}+\frac{\beta_n}{\alpha_{n+1}}\right)(\rho^{-\alpha_n}+\rho^{-\alpha_{n+1}})$$

$$>\rho^{-n^2+n}-8\rho^{-n^2}.$$

This implies $x_{n+1}-y_{n+1}>0$ since $\rho=100$. □

4.2. Behaviour of the function F_R is determined by the coefficients x_n, y_n, z_n, and w_n. We estimate growth of these numbers as $n\to\infty$.

LEMMA 8. *There exists a positive constant C_1 with $C_1>1$ such that*

(i) $$x_n\geq C_1^{-n}\rho^{n^2/2}\qquad(n=1,2,\cdots),$$

(ii) $$\frac{y_n}{x_n}\geq C_1^{-1}\rho^{-3n}\qquad(n=2,3,\cdots),$$

(iii) $$\frac{z_n}{x_n}\geq C_1^{-1}\rho^{n^2},\ \frac{w_n}{x_n}\geq C_1^{-1}\rho^{n^2}\qquad(n=1,2,\cdots).$$

PROOF. The proof is based upon Lemma 7 and formulas (12)–(15). The letters m_i $(i=1,\cdots,7)$ used below denote positive constans satisfying $m_i>1$. Since

$$\frac{x_{n+1}}{x_n}\geq\frac{\rho^{\alpha_n}}{2(1-\rho^{-2\alpha_{n+1}})}\left\{\frac{\alpha_n}{\beta_n}(1-\rho^{-\alpha_n})^2(1+\rho^{-2\alpha_{n+1}})\right.$$

$$\left.-\frac{\beta_n}{\alpha_{n+1}}(1-\rho^{-2\alpha_n})(1-\rho^{-2\alpha_{n+1}})\right\}$$

$$=\frac{\rho^{\alpha_n}(1-\rho^{-\alpha_n})}{2(1-\rho^{-2\alpha_{n+1}})}\left\{\left(\frac{\alpha_n}{\beta_n}-\frac{\beta_n}{\alpha_{n+1}}\right)(1-\rho^{-\alpha_n-2\alpha_{n+1}})\right.$$

$$\left.-\left(\frac{\alpha_n}{\beta_n}+\frac{\beta_n}{\alpha_{n+1}}\right)(\rho^{-\alpha_n}-\rho^{-2\alpha_{n+1}})\right\}$$

$$\geq\frac{\rho^{n^2}(1-\rho^{-1})}{2}\{\rho^{-n^2+n}(1-\rho^{-9})-4\rho^{-n^2}\}$$

$$\geq\frac{\rho^n}{2}(1-100^{-1})(1-100^{-1}-4\cdot100^{-1})\geq m_1^{-1}\rho^n,$$

the inequality (i) holds:

$$x_n \geq m_1^{-n+1}\rho^{(n^2-n)/2}x_1 \geq m_2^{-n}\rho^{n^2/2}.$$

For the proof of (ii) we need an upper estimate of x_{n+1}/x_n:

$$\begin{aligned}\frac{x_{n+1}}{x_n} &\leq \frac{\rho^{\alpha_n}}{2(1-\rho^{-2\alpha_{n+1}})}\left\{\frac{\alpha_n}{\beta_n}(1+\rho^{-2\alpha_n})(1+\rho^{-2\alpha_{n+1}})\right.\\ &\quad \left. - \frac{\beta_n}{\alpha_{n+1}}(1-\rho^{-2\alpha_n})(1-\rho^{-2\alpha_{n+1}})\right\}\\ &= \frac{\rho^{\alpha_n}}{2(1-\rho^{-2\alpha_{n+1}})}\left\{\left(\frac{\alpha_n}{\beta_n}-\frac{\beta_n}{\alpha_{n+1}}\right)(1+\rho^{-2\alpha_n-2\alpha_{n+1}})\right.\\ &\quad \left. + \left(\frac{\alpha_n}{\beta_n}+\frac{\beta_n}{\alpha_{n+1}}\right)(\rho^{-2\alpha_n}+\rho^{-2\alpha_{n+1}})\right\}\\ &\leq \frac{\rho^{n^2}}{2(1-\rho^{-8})}\{\rho^{-n^2+n}(1+\rho^{-10})+4\rho^{-n^2+n}(\rho^{-2}+\rho^{-8})\} \leq m_3\rho^n.\end{aligned}$$

Now we have

$$\frac{x_{n+1}}{x_n} \geq \frac{\alpha_n\rho^{\alpha_n-\alpha_{n+1}}(1-\rho^{-\alpha_n})^2}{\beta_n(1-\rho^{-2\alpha_{n+1}})} \geq \frac{1}{2}\rho^{-2n-1}(1-\rho^{-1})^2 \geq m_4^{-1}\rho^{-2n}$$

and hence

$$\frac{y_{n+1}}{x_{n+1}} = \frac{y_{n+1}}{x_n}\frac{x_n}{x_{n+1}} \geq m_4^{-1}\rho^{-2n}m_3^{-1}\rho^{-n} \geq m_5^{-1}\rho^{-3(n+1)}.$$

The estimate (iii) follows from

$$\frac{z_n}{x_n} \geq \frac{\alpha_n}{\beta_n}\rho^{\alpha_n}(1-\rho^{-\alpha_n})^2 \geq \frac{(1-\rho^{-1})^2}{2}\rho^{n^2} \geq m_6^{-1}\rho^{n^2},$$

$$\frac{w_n}{x_n} = \rho^{\alpha_n}(1-\rho^{-2\alpha_n}) \geq m_7^{-1}\rho^{n^2}.$$ □

4.3. We are ready to show that the function F_R succeeds in the integral test in Theorems A and B.

LEMMA 9. *The function F_R satisfies*

$$\int_0^{b_1}\frac{dr}{rF_R(r)^2} < \infty.$$

PROOF. In view of Lemmas 2 and 8 we have

$$\int_{b_n}^{a_n}\frac{dr}{rF_R(r)^2} \leq \frac{C_1\rho^{3n}}{2\alpha_n x_n^2(\rho^{\alpha_n}-\rho^{-\alpha_n})} \leq \frac{C_1^{2n+1}\rho^{3n}}{2n^2\rho^{2n^2}(1-\rho^{-2})} \qquad (n=1, 2,\cdots).$$

We also have

$$\int_{a_{n+1}}^{b_n} \frac{dr}{rF_R(r)^2} \leq \frac{2C_1^2}{\alpha_n x_n^2 \rho^{2n^2}} \leq \frac{2C_1^{2n+2}}{n^2 \rho^{3n^2}} \qquad (n = 1, 2, \cdots)$$

by Lemma 4, 8 and (15). These inequalities prove the lemma. □

LEMMA 10. *The function F_R satisfies*

$$\int_0^{b_1} \frac{F_R(r)^2}{r} \int_0^r \frac{ds}{sF_R(s)^2} dr = \infty.$$

PROOF. Apply inequalities

$$\rho^{2\alpha_n} - \rho^{-2\alpha_n} - 2\log\rho^{2\alpha_n} \geq \rho^{2\alpha_n}(1 - \rho^{-4} - 2\rho^{-2}\log\rho^2) > 0 \qquad (n = 1, 2, \cdots),$$

$$(\rho^{2\alpha_n} + 1)\log\rho^{\alpha_n} - (\rho^{2\alpha_n} - 1) > 0 \qquad (n = 1, 2, \cdots)$$

and Lemma 8 to (ii) in Lemma 2. Then

$$\begin{aligned}\int_{b_n}^{a_n} \frac{F_R(r)^2}{r} \int_{b_n}^r \frac{ds}{sF_R(s)^2} dr &\geq \frac{1}{4\alpha_n^2(\rho^{\alpha_n} - \rho^{-\alpha_n})} \frac{y_n}{x_n} \rho^{2\alpha_n}(1 - \rho^{-4} - 2\rho^{-2}\log\rho^2) \\ &\geq \frac{1 - \rho^{-4} - 4\rho^{-2}\log\rho}{4C_1(1 - \rho^{-2})} \frac{\rho^{n^2 - 3n}}{n^2} \qquad (n = 2, 3, \cdots).\end{aligned}$$

This proves the lemma. □

5. F_S for $S = R/4$

5.1. We consider a discontinuous function $S = P(\cdot\,; \theta/2, \{\alpha_n/2\}, \{\beta_n/2\})$ on $(0, 1]$, where θ, α_n, and β_n are the numbers given by (11). Recall the definition of symbol $P = P(\cdot\,;\,,\,,\,)$ in No. 2.1. Then S has an expression $S = R/4$ with the discontinuous function R considered in No. 4.1 since the sequences $\{a_n(\theta/2, \{\beta_n/2\})\}$ and $\{b_n(\theta/2, \{\beta_n/2\})\}$ are equal to the sequences $\{a_n\}$ and $\{b_n\}$ defined in No. 4.1, respectively. We also associate F_S with S that is the C^1 function on $(0, 1]$ satisfying

$$F_S(1) = 0,\ F_S' = -1,\ L_S F_S = 0 \text{ on } \bigcup_{n=1}^{\infty} \{(a_{n+1}, b_n) \cup (b_n, a_n)\}.$$

Then by (5) F_S has the following form:

$$F_S(r) = \begin{cases} X_n\left\{\left(\dfrac{a_n}{r}\right)^{\alpha_n/2} - \left(\dfrac{r}{a_n}\right)^{\alpha_n/2}\right\} + Y_n\left\{\left(\dfrac{r}{b_n}\right)^{\alpha_n/2} - \left(\dfrac{b_n}{r}\right)^{\alpha_n/2}\right\} \\ \hfill (b_n \leq r \leq a_n) \\ -Z_n \sin\left(\dfrac{\beta_n}{2}\log\dfrac{r}{b_n}\right) + W_n \sin\left(\dfrac{\beta_n}{2}\log\dfrac{r}{a_{n+1}}\right) \\ \hfill (a_{n+1} < r < b_n) \end{cases}$$

$$(n = 1, 2, \cdots).$$

In view of (6)–(8) the coefficients X_n, Y_n, Z_n, and W_n are given by initial values

$$X_1 = \frac{1}{2(\alpha_1/2)} = 1, \quad Y_1 = 0 \tag{16}$$

and recursion formulas

$$\begin{pmatrix} Z_n \\ W_n \end{pmatrix} = \rho^{\alpha_n/2} \begin{pmatrix} 1 - \rho^{-\alpha_n} + \dfrac{\alpha_n}{\beta_n}(1 + \rho^{-\alpha_n}) & -\dfrac{2\alpha_n}{\beta_n}\rho^{-\alpha_n/2} \\ \sqrt{2}(1 - \rho^{-\alpha_n}) & 0 \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix}, \tag{17}$$

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \frac{\sqrt{2}\,\rho^{\alpha_n/2}}{4(1 - \rho^{-\alpha_{n+1}})} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} \tag{18}$$

$$(n = 1, 2, \cdots),$$

where

$$\begin{aligned} B_{11} &= (1 - \rho^{-\alpha_n})(1 + \rho^{-\alpha_{n+1}}) + \frac{\alpha_n}{\alpha_{n+1}}(1 + \rho^{-\alpha_n})(1 - \rho^{-\alpha_{n+1}}) \\ &\quad + \frac{\alpha_n}{\beta_n}(1 + \rho^{-\alpha_n})(1 + \rho^{-\alpha_{n+1}}) - \frac{\beta_n}{\alpha_{n+1}}(1 - \rho^{-\alpha_n})(1 - \rho^{-\alpha_{n+1}}), \\ B_{12} &= -2\rho^{-\alpha_n/2}\left\{\frac{\alpha_n}{\beta_n}(1 + \rho^{-\alpha_{n+1}}) + \frac{\alpha_n}{\alpha_{n+1}}(1 - \rho^{-\alpha_{n+1}})\right\}, \\ B_{21} &= 2\rho^{-\alpha_{n+1}/2}\left\{1 - \rho^{-\alpha_n} + \frac{\alpha_n}{\beta_n}(1 + \rho^{-\alpha_n})\right\}, \\ B_{22} &= -4\frac{\alpha_n}{\beta_n}\rho^{-\alpha_n/2}\rho^{-\alpha_{n+1}/2}. \end{aligned}$$

The following lemma shows that F_S is positive on $(0, 1)$:

LEMMA 11. *The numbers X_n, Y_n, Z_n, and W_n satisfy*

$$Y_1 = 0,\ X_n > Y_n,\ Z_n > 0,\ W_n > 0,\ Y_{n+1} > 0 \qquad (n = 1, 2, \cdots).$$

PROOF. In view of Lemma 1 and (16) we only need to prove that $X_n > Y_n \geq 0$ implies $X_{n+1} > Y_{n+1}$. Suppose $X_n > Y_n \geq 0$. Then by (18) we have

$$\begin{aligned}
&\frac{X_{n+1} - Y_{n+1}}{X_n} \frac{4(1 - \rho^{-\alpha_{n+1}})}{\sqrt{2}\,\rho^{\alpha_n/2}} \\
&= (1 - \rho^{-\alpha_n})(1 - \rho^{-\alpha_{n+1}/2})^2 + \frac{\alpha_n}{\alpha_{n+1}}(1 + \rho^{-\alpha_n})(1 - \rho^{-\alpha_{n+1}}) \\
&\quad + \frac{\alpha_n}{\beta_n}(1 + \rho^{-\alpha_n})(1 - \rho^{-\alpha_{n+1}/2})^2 - \frac{\beta_n}{\alpha_{n+1}}(1 - \rho^{-\alpha_n})(1 - \rho^{-\alpha_{n+1}}) \\
&\quad - 2\frac{Y_n}{X_n}\rho^{-\alpha_n/2}\left\{\frac{\alpha_n}{\beta_n}(1 - \rho^{-\alpha_{n+1}/2})^2 + \frac{\alpha_n}{\alpha_{n+1}}(1 - \rho^{-\alpha_{n+1}})\right\} \\
&> (1 - \rho^{-\alpha_n})(1 - \rho^{-\alpha_{n+1}/2})^2 + \frac{\alpha_n}{\alpha_{n+1}}(1 - \rho^{-\alpha_n/2})^2(1 - \rho^{-\alpha_{n+1}}) \\
&\quad + \frac{\alpha_n}{\beta_n}(1 - \rho^{-\alpha_n/2})^2(1 - \rho^{-\alpha_{n+1}/2})^2 - \frac{\beta_n}{\alpha_{n+1}}(1 - \rho^{-\alpha_n})(1 - \rho^{-\alpha_{n+1}}).
\end{aligned}$$

Hence by (15) we have

$$\begin{aligned}
&\frac{X_{n+1} - Y_{n+1}}{X_n} \frac{4(1 + \rho^{-\alpha_{n+1}/2})}{\sqrt{2}\,\rho^{\alpha_n/2}(1 - \rho^{-\alpha_n/2})} \\
&> 1 - \rho^{-\alpha_{n+1}/2} + \frac{\alpha_n}{\alpha_{n+1}}(1 - \rho^{-\alpha_n/2}) + \frac{\alpha_n}{\beta_n}(1 - \rho^{-\alpha_n/2} - \rho^{-\alpha_{n+1}/2}) \\
&\quad - \frac{\beta_n}{\alpha_{n+1}}(1 + \rho^{-\alpha_n/2} + \rho^{-\alpha_{n+1}/2} + \rho^{-(\alpha_n+\alpha_{n+1})/2}) \\
&= \left(1 + \frac{\alpha_n}{\alpha_{n+1}} + \frac{\alpha_n}{\beta_n} - \frac{\beta_n}{\alpha_{n+1}}\right) - \left(\frac{\alpha_n}{\alpha_{n+1}} + \frac{\alpha_n}{\beta_n} + \frac{\beta_n}{\alpha_{n+1}}\right)\rho^{-\alpha_n/2} \\
&\quad - \left(1 + \frac{\alpha_n}{\beta_n} + \frac{\beta_n}{\alpha_{n+1}}\right)\rho^{-\alpha_{n+1}/2} - \frac{\beta_n}{\alpha_{n+1}}\rho^{-(\alpha_n+\alpha_{n+1})/2} \\
&\geq \frac{5}{4} - 5\rho^{-1/2} - 5\rho^{-2} - 2\rho^{-5/2}.
\end{aligned}$$

This implies $X_{n+1} - Y_{n+1} > 0$ since $\rho = 100$. □

5.2. Behaviour of the function F_S is determined by the coefficients X_n, Y_n, Z_n, and W_n. We estimate growth of these numbers as $n \to \infty$.

Lemma 12. *There exists a positive constant C_2 with $C_2 > 1$ such that*

(i) $$C_2^{-1}\rho^{n^2/2} \leq \frac{X_{n+1}}{X_n} \leq C_2\rho^{n^2/2} \qquad (n = 1, 2, \cdots),$$

(ii) $$C_2^{-1}\rho^{-n^2/2} \leq \frac{Y_n}{X_n} \leq C_2\rho^{-n^2/2} \qquad (n = 2, 3, \cdots),$$

(iii) $$C_2^{-1}\rho^{n^2/2} \leq \frac{Z_n}{X_n} \leq C_2\rho^{n^2/2} \qquad (n = 1, 2, \cdots),$$

(iv) $$C_2^{-1}\rho^{n^2/2} \leq \frac{W_n}{X_n} \leq C_2\rho^{n^2/2} \qquad (n = 1, 2, \cdots).$$

Proof. The proof is based upon Lemma 11 and formulas (15)–(18). The letters $m_i (i = 1, \cdots, 8)$ used below denote positive constants satisfying $m_i > 1$. The inequality (i) follows from

$$\begin{aligned}
\frac{X_{n+1}}{X_n} &\geq \frac{\sqrt{2}\rho^{\alpha_n/2}(1-\rho^{-\alpha_n/2})}{4(1-\rho^{-\alpha_{n+1}})}\Big\{(1+\rho^{-\alpha_n/2})(1+\rho^{-\alpha_{n+1}}) \\
&\quad + \frac{\alpha_n}{\alpha_{n+1}}(1-\rho^{-\alpha_n/2})(1-\rho^{-\alpha_{n+1}}) + \frac{\alpha_n}{\beta_n}(1-\rho^{-\alpha_n/2})(1+\rho^{-\alpha_{n+1}}) \\
&\quad - \frac{\beta_n}{\alpha_{n+1}}(1+\rho^{-\alpha_n/2})(1-\rho^{-\alpha_{n+1}})\Big\} \\
&\geq \frac{\sqrt{2}(1-\rho^{-1/2})}{4}\rho^{\alpha_n/2}\Big\{1 + \frac{\alpha_n}{\alpha_{n+1}}(1-\rho^{-\alpha_n/2}-\rho^{-\alpha_{n+1}}) \\
&\quad + \frac{\alpha_n}{\beta_n}(1-\rho^{-\alpha_n/2}) - \frac{\beta_n}{\alpha_{n+1}}(1+\rho^{-\alpha_n/2})\Big\} \\
&= \frac{\sqrt{2}(1-\rho^{-1/2})}{4}\rho^{\alpha_n/2}\Big\{1 + \frac{\alpha_n}{\alpha_{n+1}} + \frac{\alpha_n}{\beta_n} - \frac{\beta_n}{\alpha_{n+1}} \\
&\quad - \left(\frac{\alpha_n}{\alpha_{n+1}} + \frac{\alpha_n}{\beta_n} + \frac{\beta_n}{\alpha_{n+1}}\right)\rho^{-\alpha_n/2} - \frac{\alpha_n}{\alpha_{n+1}}\rho^{-\alpha_{n+1}}\Big\} \\
&\geq \frac{\sqrt{2}(1-\rho^{-1/2})}{4}\left(\frac{5}{4} - 5\rho^{-1/2} - \rho^{-4}\right)\rho^{\alpha_n/2} \\
&= \frac{\sqrt{2}}{4}(1-10^{-1})\left(\frac{5}{4} - \frac{1}{2} - 100^{-4}\right)\rho^{n^2/2} \geq m_1^{-1}\rho^{n^2/2}
\end{aligned}$$

and

$$\frac{X_{n+1}}{X_n} \leq \frac{\sqrt{2}\,\rho^{\alpha_n/2}}{4(1-\rho^{-\alpha_{n+1}})}\left\{1+\rho^{-\alpha_{n+1}}+\frac{\alpha^n}{\alpha_{n+1}}(1+\rho^{-\alpha_n})\right.$$

$$\left.+\frac{\alpha_n}{\beta_n}(1+\rho^{-\alpha_n})(1+\rho^{-\alpha_{n+1}})\right\}$$

$$\leq \frac{\sqrt{2}}{4(1-\rho^{-4})}\{1+\rho^{-4}+1+\rho^{-1}+2(1+\rho^{-1})(1+\rho^{-4})\}\rho^{\alpha_n/2} \leq m_2\rho^{n^2/2}.$$

Since we have inequalities

$$\frac{Y_{n+1}}{X_n} \geq \frac{\rho^{(\alpha_n-\alpha_{n+1})/2}}{\sqrt{2}(1-\rho^{-\alpha_{n+1}})}\left\{1-\rho^{-\alpha_n}+\frac{\alpha_n}{\beta_n}(1-\rho^{-\alpha_n/2})^2\right\}$$

$$\geq \frac{1-\rho^{-\alpha_n/2}}{\sqrt{2}}\rho^{(\alpha_n-\alpha_{n+1})/2}\left\{1+\rho^{-\alpha_n/2}+\frac{\alpha_n}{\beta_n}(1-\rho^{-\alpha_n/2})\right\}$$

$$\geq \frac{1}{\sqrt{2}}(1-\rho^{-1/2})\left\{1+\frac{1}{2}(1-\rho^{-1/2})\right\}\rho^{-n-1/2} \geq m_3^{-1}\rho^{-n}$$

and

$$\frac{Y_{n+1}}{X_n} \leq \frac{\rho^{(\alpha_n-\alpha_{n+1})/2}}{\sqrt{2}(1-\rho^{-\alpha_{n+1}})}\left\{1-\rho^{-\alpha_n}+\frac{\alpha_n}{\beta_n}(1+\rho^{-\alpha_n})\right\}$$

$$\leq \frac{\rho^{-n-1/2}}{\sqrt{2}(1-\rho^{-4})}\{1+2(1+\rho^{-1})\} \leq m_4\rho^{-n},$$

we obtain (ii):

$$\frac{Y_{n+1}}{X_{n+1}} = \frac{Y_{n+1}}{X_n}\frac{X_n}{X_{n+1}} \geq m_3^{-1}\rho^{-n}m_2^{-1}\rho^{-n^2/2} \geq m_5^{-1}\rho^{-(n+1)^2/2},$$

$$\frac{Y_{n+1}}{X_{n+1}} \leq m_4\rho^{-n}m_1\rho^{-n^2/2} \leq m_6\rho^{-(n+1)^2/2}.$$

The estimates (iii) and (iv) follow from

$$\frac{Z_n}{X_n} \geq \rho^{\alpha_n/2}\left\{1-\rho^{-\alpha_n}+\frac{\alpha_n}{\beta_n}(1-\rho^{-\alpha_n/2})^2\right\} \geq \rho^{\alpha_n/2}(1-\rho^{-1}) \geq m_7^{-1}\rho^{n^2/2},$$

$$\frac{Z_n}{X_n} \leq \rho^{\alpha_n/2}\left\{1+\frac{\alpha_n}{\beta_n}(1+\rho^{-\alpha_n})\right\} \leq \rho^{\alpha_n/2}\{1+2(1+\rho^{-1})\} \leq m_8\rho^{n^2/2}$$

and

$$\frac{W_n}{X_n} \geq \sqrt{2}(1-\rho^{-1})\rho^{n^2/2},\ \frac{W_n}{X_n} \leq \sqrt{2}\,\rho^{n^2/2},$$

respectively. □

5.3. We are ready to show that the function F_S fails in the integral test in Theorem B although it succeeds in the integral test in Theorem A. For the purpose we consider integrals

$$I_{1,n}=\int_{b_n}^{a_n}\frac{dr}{rF_S(r)^2},\quad I_{2,n}=\int_{a_{n+1}}^{b_n}\frac{dr}{rF_S(r)^2},$$

$$J_{1,n}=\int_{b_n}^{a_n}\frac{F_S(r)^2}{r}dr,\quad J_{2,n}=\int_{a_{n+1}}^{b_n}\frac{F_S(r)^2}{r}dr,$$

$$K_{1,n}=\int_{b_n}^{a_n}\frac{F_S(r)^2}{r}\int_{b_n}^{r}\frac{ds}{sF_S(s)^2}dr,\quad K_{2,n}=\int_{a_{n+1}}^{b_n}\frac{F_S(r)^2}{r}\int_{a_{n+1}}^{r}\frac{ds}{F_S(s)^2}dr.$$

These integrals satisfy the following inequalities:

LEMMA 13. *There exists a positive constant C_3 such that*

(i) $$\sum_{k=n}^{\infty}I_{1,k}\le\frac{C_3}{n^2X_n^2}\qquad(n=2,3,\cdots),$$

(ii) $$\sum_{k=n}^{\infty}I_{2,k}\le\frac{C_3}{n^2X_n^2\rho^{n^2}}\qquad(n=1,2,\cdots),$$

(iii) $$J_{1,n}\le\frac{C_3X_n^2\rho^{n^2}}{n^2},\quad K_{1,n}\le\frac{C_3}{n^2}\qquad(n=2,3,\cdots),$$

(iv) $$J_{2,n}\le\frac{C_3X_n^2\rho^{n^2}}{n^2},\quad K_{2,n}\le\frac{C_3}{n^4}\qquad(n=1,2,\cdots).$$

PROOF. The proof is based upon Lemmas 2–5, 11–12, and the formula (15). The letters m_i $(i=1,\cdots,7)$ used below denote positive constants. Since for every $n=2,3,\cdots$ and $k=1,2,\cdots$ we have

$$\begin{aligned}\frac{I_{1,n+k}}{I_{1,n}}&=\frac{n^2X_nY_n(\rho^{n^2/2}-\rho^{-n^2/2})}{(n+k)^2X_{n+k}Y_{n+k}(\rho^{(n+k)^2/2}-\rho^{-(n+k)^2/2})}\\&\le\frac{C_2^2X_n^2}{X_{n+k}^2}\,\frac{1-\rho^{-n^2}}{1-\rho^{-(n+k)^2}}\\&\le C_2^{2k+2}\rho^{-\{n^2+\cdots+(n+k-1)^2\}}\\&\le C_2^{2k+2}\rho^{-\{2^2+\cdots+(k+1)^2\}}\end{aligned}$$

and $I_{1,n}\le m_1n^{-2}X_n^{-2}$, we obtain (i):

$$\sum_{k=n}^{\infty}I_{1,k}=I_{1,n}\sum_{k=0}^{\infty}\frac{I_{1,n+k}}{I_{1,n}}\le\frac{m_2}{n^2X_n^2}.$$

We have also for evry $n = 1, 2, \cdots$ and $k = 1, 2, \cdots$

$$\frac{I_{2,n+k}}{I_{2,n}} = \frac{\beta_n Z_n W_n}{\beta_{n+k} Z_{n+k} W_{n+k}} \leq \frac{4C_2^4 \alpha_n X_n^2 \rho^{n^2}}{\alpha_{n+k} X_{n+k}^2 \rho^{(n+k)^2}} \leq \frac{4C_2^4 X_n^2}{X_{n+k}^2}.$$

Therefore $I_{2,n} \leq m_3 n^{-2} X_n^{-2} \rho^{-n^2}$ yields (ii). The inequalities (iii) and (iv) hold since

$$J_{1,n} \leq \frac{1}{n^2} \{X_n^2(1 + C_2^2 \rho^{-n^2})\rho^{n^2} + 2C_2 X_n^2 \rho^{-n^2}(\rho^{n^2} + 1) \log \rho^{n^2}\} \leq \frac{m_4 X_n^2 \rho^{n^2}}{n^2},$$

$$K_{1,n} \leq \frac{1}{n^4 \rho^{n^2/2}(1 - \rho^{-n^2})} \{C_2 \rho^{n^2/2} + \rho^{-n^2/2}(\rho^{n^2} + 1) \log \rho^{n^2}\} \leq \frac{m_5}{n^2},$$

$$J_{2,n} \leq \frac{2}{n^2} \left\{2C_2^2 X_n^2 \rho^{n^2}\left(\frac{\pi}{4} - \frac{1}{2}\right) + 2C_2^2 X_n^2 \rho^{n^2}\left(\frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}\right)\right\} \leq \frac{m_6 X_n^2 \rho^{n^2}}{n^2},$$

$$K_{2,n} \leq \frac{2\sqrt{2}}{\beta_n^2} \left\{C_2^2\left(\frac{\pi}{4} - \frac{1}{2}\right) + \frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}\right\} \leq \frac{m_7}{n^4}. \qquad \square$$

From this lemma it follows that

$$\int_0^{b_1} \frac{dr}{rF_S(r)^2} = \sum_{n=1}^{\infty} (I_{1,n+1} + I_{2,n}) \leq \frac{C_3}{4X_2^2} + \frac{C_3}{X_1^2 \rho} < \infty$$

and

$$\begin{aligned}
&\int_0^{b_1} \frac{F_S(r)^2}{r} \int_0^r \frac{ds}{sF_S(s)^2} dr \\
&\quad = \sum_{n=1}^{\infty} \{K_{2,n} + J_{2,n} \sum_{k=n+1}^{\infty} (I_{1,k} + I_{2,k})\} \\
&\qquad + \sum_{n=2}^{\infty} \{K_{1,n} + J_{1,n} \sum_{k=n}^{\infty} (I_{2,k} + I_{1,k+1})\} \\
&\quad \leq \sum_{n=1}^{\infty} \left\{\frac{C_3}{n^4} + \frac{C_3 X_n^2 \rho^{n^2}}{n^2}\left(\frac{C_3}{(n+1)^2 X_{n+1}^2} + \frac{C_3}{(n+1)^2 X_{n+1}^2 \rho^{(n+1)^2}}\right)\right\} \\
&\qquad + \sum_{n=2}^{\infty} \left\{\frac{C_3}{n^2} + \frac{C_3 X_n^2 \rho^{n^2}}{n^2}\left(\frac{C_3}{n^2 X_n^2 \rho^{n^2}} + \frac{C_3}{(n+1)^2 X_{n+1}^2}\right)\right\} \\
&\quad \leq \sum_{n=1}^{\infty} \left\{\frac{C_3}{n^4} + \frac{C_2^2 C_3^2}{n^2(n+1)^2}(1 + \rho^{-(n+1)^2})\right\} \\
&\qquad + \sum_{n=2}^{\infty} \left\{\frac{C_3}{n^2} + \frac{C_3^2}{n^4}\left(1 + \frac{C_2^2 n^2}{(n+1)^2}\right)\right\}.
\end{aligned}$$

Thus we proved the following lemma:

LEMMA 14. *The function F_S satisfies*

(i)
$$\int_0^{b_1} \frac{dr}{rF_S(r)^2} < \infty,$$

(ii)
$$\int_0^{b_1} \frac{F_S(r)^2}{r} \int_0^r \frac{ds}{sF_S(s)^2} dr < \infty.$$

6. Proof of Theorem

Recall the discontinuous function R ($S = R/4$, resp.) and the C^1 function F_R (F_S, resp.) considered in Section 4 (5, resp.). In the definition of these functions, we used the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{a_n\}$, $\{b_n\}$ given by (11) and (2), respectively. In the proof of Theorem, these letters denote the same.

Let $\delta = \{\delta_n\}_1^\infty$ be a sequence of numbers δ_n satisfying

(19)
$$0 < \delta_n \leq \frac{b_n - a_{n+1}}{2} \qquad (n = 1, 2, \cdots).$$

With δ and R we associate a density R_δ on Ω defined by

$$R_\delta = \begin{cases} R(r) = \dfrac{\alpha_n^2}{r^2} & (b_n \leq r \leq a_n) \\[2ex] \dfrac{1}{\delta_n}\left\{\dfrac{\alpha_n^2}{b_n^2} + \dfrac{\beta_n^2}{(b_n - \delta_n)^2}\right\}(r - b_n) + \dfrac{\alpha_n^2}{b_n^2} & (b_n - \delta_n < r < b_n) \\[2ex] R(r) = -\dfrac{\beta_n^2}{r^2} & (a_{n+1} + \delta_n \leq r \leq b_n - \delta_n) \\[2ex] -\dfrac{1}{\delta_n}\left\{\dfrac{\beta_n^2}{(a_{n+1} + \delta_n)^2} + \dfrac{\alpha_{n+1}^2}{a_{n+1}^2}\right\}(r - a_{n+1}) + \dfrac{\alpha_{n+1}^2}{a_{n+1}^2} & (a_{n+1} < r < a_{n+1} + \delta_n) \\[2ex] (n = 1, 2, \cdots). \end{cases}$$

Apply Lemma 6 to $P = R$ and $Q = R_\delta$. Then $R_\delta \geq R$ implies $f_{R_\delta} \geq F_R$, where f_{R_δ} is the R_δ-subunit. In particular we denote by R_0 the density R_δ with $\delta_n = (b_n - a_{n+1})/2 (n = 1, 2, \cdots)$ and f_{R_0} the R_0-subunit. For a general δ satisfying (19), $R_\delta \leq R_0$ implies $f_{R_\delta} \leq f_{R_0}$ ([8]). Moreover in view of this and Lemma 6 and 9, following inequalities hold for positive constant C_4:

(20)
$$1 \leq \frac{f_{R_\delta}(r)}{F_R(r)} \leq 1 + C_4 \int_0^{b_1} s\{R_\delta(s) - R(s)\} f_{R_0}(s) F_R(s) ds \qquad (0 < r < 1).$$

We also consider densities $R_\delta/4$ and $R_0/4$. Then the $R_\delta/4$-subunit $f_{R_\delta/4}$ is dominated by the $R_0/4$-subunit $f_{R_0/4}$ so that following inequalities hold for positive constant C_5:

$$(21)\qquad 1 \le \frac{f_{R_\delta/4}(r)}{F_S(r)} \le 1 + C_5 \int_0^{b_1} s\left\{\frac{R_\delta(s)}{4} - \frac{R(s)}{4}\right\} f_{R_0/4}(s)F_S(s)ds \qquad (0 < r < 1)$$

by Lemmas 6 and 14.

Now we set

$$\gamma_n = \frac{b_n(\alpha_{n+1}^2 + \beta_n^2)}{a_{n+1}^2},\ U_n = (a_{n+1}, a_{n+1} + \delta_n) \cup (b_n - \delta_n, b_n) \qquad (n = 1, 2, \cdots).$$

We can choose and fix $\delta = \{\delta_n\}$ satisfying (19) and the following condition:

$$(22)\qquad \sum_{n=1}^{\infty} \gamma_n \int_{U_n} \{f_{R_0}(s)F_R(s) + f_{R_0/4}(s)F_S(s)\}\, ds < \infty.$$

Let P be the density R_δ with this δ. By (20)–(22) the P-subunit f_P and the function F_R (the $P/4$-subunit $f_{P/4}$ and the function F_S, resp.) are comparable since $s\{R_\delta(s) - R(s)\}$ is dominated by γ_n on U_n $(n = 1, 2, \cdots)$ and vanishies otherwise: there exists a positive constant C_6 such that

$$1 \le \frac{f_P(r)}{F_R(r)} \le C_6 \left(1 \le \frac{f_{P/4}(r)}{F_S(r)} \le C_6, \text{ resp.}\right) \qquad (0 < r < 1).$$

Hence Lemma 9 and (i) of Lemma 14 yield

$$\int_0^{b_1} \frac{dr}{rf_P(r)^2} < \infty, \quad \int_0^{b_1} \frac{dr}{rf_{P/4}(r)^2} < \infty$$

so that P and $P/4$ are both hyperbolic by Theorem A. Moreover Lemma 10 and (ii) of Lemma 14 yield

$$\int_0^{b_1} \frac{f_P(r)^2}{r} \int_0^r \frac{ds}{sf_P(s)^2}\, dr = \infty, \quad \int_0^{b_1} \frac{f_{P/4}(r)^2}{r} \int_0^r \frac{ds}{sf_{P/4}(s)^2}\, dr < \infty.$$

Thus we conclude $\dim P = 1$ and $\dim (P/4) = \mathfrak{c}$ by Theorem B. □

In the above proof, the density P can be replaced by a C^∞ density. In fact we can construct a C^∞ density Q on Ω such that $0 \le Q(s) - R(s) \le \gamma_n/b_n$ $(s \in U_n;\ n = 1, 2, \cdots)$ and $Q(s) - R(s)$ vanishes otherwise. By the same reason as that of P, the functions Q and R ($Q/4$ and S, resp.) are also comparable. Hence Q is hyperbolic and satisfy $\dim Q = 1$, $\dim (Q/4) = \mathfrak{c}$.

References

[1] M. Aizenman and B. Simon, Brownian motion and Harnack inequality for Schrödinger operators, Comm. Pure Appl. Math., **35** (1982), 209–273.
[2] H. Imai, On Picard dimensions of nonpositive densities in Schrödinger equations, Complex Variables, (to appear).
[3] M. Kawamura and M. Nakai, A test of Picard principle for rotation free densities, II, J. Math. Soc. Japan, **28** (1976), 323–342.
[4] M. Murata, Structure of positive solutions to $(-\Delta + V)u = 0$ in $\mathbb{R}^n$, Duke Math. J., **53** (1986), 869–943.
[5] M. Murata, Isolated singularites and positive solutions of elliptic equations in $\mathbb{R}^n$, Aarhus Univ. Preprint Series, **14** (1986/87), 1–39.
[6] L. Myrberg, Über die Existenz der Greenschen Funktion der Gleichung $\Delta u = c(P)u$ auf Riemannschen Flächen, Ann. Acad. Sci. Fenn., **170** (1954), 1–8.
[7] M. Nakai, Martin boundary over an isolated singularity of rotation free density, J. Math. Soc. Japan, **26** (1974), 483–507.
[8] M. Nakai, Picard principle and Riemann theorem, Tohoku Math. J., **28** (1976), 277–292.
[9] M. Nakai, Comparison of Martin boundaries for Schrödinger operators, Hokkaido Math. J., **18** (1989), 245–261.
[10] M. Nakai, Continuity of solutions of Schrödinger equations, Math. Proc. Camb. Phil. Soc., **110** (1991), 581–597.
[11] M. Nakai and T. Tada, Monotoneity and homogeneity of Picard dimensions for signed radial densities, NIT Sem. Rep., **99** (1993), 1–51.
[12] R. Phelps, Lectures on Choquet's Theorem, Van Nostrand Math. Studies #7, 1965.

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