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The products $\beta_s\beta_{tp/p}$ in the stable homotopy of L_2-localized spheres

Dedicated to Professor Seiya Sasao on his 60th birthday

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1. Introduction

The β-elements in the stable homotopy groups $\pi_*(S^0)$ of spheres at the prime > 3 are introduced by H. Toda ([20]) and generalized by L. Smith ([19]) and S. Oka ([4], [5], [6]). In [3], H. Miller, D. Ravenel and S. Wilson presented that the Adams-Novikov spectral sequence is powerful to study the stable homotopy groups of spheres, and gave the way to define the generalized Greek letter elements in its E_2-term including β-elements. S. Oka [7], [8] and H. Sadofsky [11] showed that some of those β-elements are permanent cycles.

S. Oka and the author has studied about the product of these β-elements in the homotopy groups $\pi_*(S^0)$ ([9], [12], [13], [14], [15]) and show whether or not the products of the form $\beta_s\beta_{tp/j}$ are trivial except for the case where

$$j = p,\ s = rp + 1,\ p \nmid t \quad \text{and} \quad p^{n+1} \mid r + t + p^n \qquad \text{for some } n \geq 0.$$

Here β_s for $s > 0$ and $\beta_{tp/j}$ for $j, t > 1$ are the β-elements given by L. Smith and S. Oka. In the recent work [18], A. Yabe and the author have determined the homotopy groups of L_2-local spheres, where L_2 stands for the Bousfield localization functor with respect to the Johnson-Wilson spectrum $E(2)$ with the coefficient ring $Z_{(p)}[v_1, v_2, v_2^{-1}]$ (cf. [1], [10]). In this paper we show the triviality of the product of β-elements in the homotopy groups $\pi_*(L_2S^0)$ for the above exceptional case (see Theorem 3.3). Consider the map $l_*: \pi_*(S^0) \to \pi_*(L_2S^0)$ induced from the localization map $l: S^0 \to L_2S^0$. We notice that if $l_*(x) = l_*(y)$ in the homotopy groups $\pi_*(L_2S^0)$, then $x \equiv y \bmod F_5$ in $\pi_*(S^0)$, where F_i denotes the Adams-Novikov filtration.

Together with known results, we obtain

THEOREM 1.1. *Let s and t be positive integers. Then in the homotopy groups $\pi_*(L_2S^0)$, $\beta_s\beta_{tp/p} = 0$ if and only if one of the following condition holds:*

1) *$p \mid st$ or*

2) $s = rp + 1$ *and* $p^{n+1} | r + t + p^n$ *for some integers* r *and* $n \geq 0$.

Note that $\beta_{p/p}$ is not a homotopy element of $\pi_*(S^0)$, but of $\pi_*(L_2 S^0)$. Using the relation $\beta_s \beta_{tp^2/p,2} = \beta_{s+t(p^2-p)} \beta_{tp/p}$ of [9, Prop. 6.1] in the E_2-term, we have

COROLLARY 1.2. *For positive integers* s *and* t, *in the homotopy groups* $\pi_*(L_2 S^0)$, $\beta_s \beta_{tp^2/p,2} = 0$ *if and only if one of the following condition holds*:

1) $p | st$ *or*
2) $s = rp + 1$ *and* $p^{n+1} | r + tp + p^n$ *for some* $n \geq 0$.

Theorem 1.1 must be a corollary of the result of [18], but it seems hard to tell which generator of $\pi_*(L_2 S^0)$ given there corresponds to our product. So we here prove the theorem directly.

2. β-elements

Let (A, Γ) denote the Hopf algebroid associated to the Johnson-Wilson spectrum $E(2)$, that is,

$$A = E(2)_* = \mathbf{Z}_{(p)}[v_1, v_2, v_2^{-1}] \text{ and}$$
$$\Gamma = E(2)_*(E(2)) = E(2)_*[t_1, t_2, \cdots]/(\eta_R(v_i): i > 2),$$

where $\eta_R: BP_* \to BP_*(BP) \to E(2)_*[t_1, t_2, \cdots]$ denotes the right unit map of the Hopf algebroid associated to the Brown-Peterson spectrum BP at the prime p. Here p denotes a prime number greater than 3. Then there is the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(L_2 S^0)$ of $E(2)_*$-local spheres S^0 with E_2-term $E_2^* = \mathrm{Ext}_\Gamma^*(A, A)$ (*cf.* [10], [1]). In order to compute the E_2-term, Miller, Ravenel and Wilson [3] introduced the chromatic spectral sequence associated to the short exact sequences

$$0 \longrightarrow N_0^i \hookrightarrow M_0^i \longrightarrow N_0^{i+1} \longrightarrow 0 \tag{2.1}$$

for $i \geq 0$, where $N_0^0 = A = E(2)_*$, $M_0^i = v_i^{-1} N_0^i$ and N_0^{i+1} is the cokernel of the inclusion $N_0^i \subset M_0^i$. Note that $M_0^2 = N_0^2$ and $M_0^i = 0$ if $i > 2$. The E_1-term is $\mathrm{Ext}^*(M_0^i)$ and the abutment is $\mathrm{Ext}^*(N_0^0)$ that is the E_2-term of the Adams-Novikov spectral sequence. Hereafter we use the abbreviation

$$\mathrm{Ext}^k(M) = \mathrm{Ext}_\Gamma^k(A, M)$$

for a Γ-comodule M. We deduce that $\mathrm{Ext}^i(M_0^2) = 0$ if $i > 4$ by the Bockstein spectral sequence from Morava's vanishing line theorem that says $\mathrm{Ext}^i(E(2)_*/(p, v_1)) = 0$ if $i > 4$. This implies $\mathrm{Ext}^i(N_0^0) = 0$ if $i > 6$ by the chromatic spectral sequence. By this, the Adams-Novikov spectral sequence collapses and arises no extension problem. Thus the E_2-term $\mathrm{Ext}^*(N_0^0)$ equals to the abutment $\pi_*(L_2S^0)$. And we identify these two algebras. Consider the connecting homomorphisms associated to the short exact sequences (2.1) for $i=0$ and 1:

$$\delta_0\colon \mathrm{Ext}^k(N_0^1) \longrightarrow \mathrm{Ext}^{k+1}(N_0^0) \text{ and}$$
$$\delta_1\colon \mathrm{Ext}^k(N_0^2) \longrightarrow \mathrm{Ext}^{k+1}(N_0^1).$$

Then the β-elements in the E_2-term of the Adams-Novikov spectral sequence are defined by:

$$\text{(2.2)} \qquad \begin{aligned} \beta_s &= \delta_0\delta_1(v_2^s/pv_1) \in \mathrm{Ext}^2(N_0^0) = \pi_*(L_2S^0) \text{ and} \\ \beta_{tp/p} &= \delta_0\delta_1(v_2^{tp}/pv_1^p) \in \mathrm{Ext}^2(N_0^0) = \pi_*(L_2S^0). \end{aligned}$$

Here we state the relation between these β-elements and the β-elements in $\pi_*(S^0)$. Combining the results of [20], [5], [2] and [3], we have

THEOREM 2.3. *β_s for $s>0$ and $\beta_{tp/p}$ for $t>1$ are pulled back to the homotopy groups $\pi_*(S^0)$ of spheres under the localization map $l_*\colon \pi_*(S^0)\to \pi_*(L_2S^0)$.*

As to the representative of $\beta_{tp/p}$ in the cobar complex, we recall [9, Lemma 4.4]

LEMMA 2.4. *In the cobar complex $\Omega_\Gamma^2 A = \Gamma\otimes_A\Gamma$,*

$$\beta_{tp/p} = -tv_2^{tp-1}g_0$$

for an integer t, where $g_0 = v_2^{-p}(t_1\otimes t_2^p + t_2\otimes t_1^{p^2})$.

3. Triviality of the products

In our proof of the triviality of the products, we construct cochains that bounds the products. For this sake, we recall [3, Prop. 5.4] the elements x_i such that

$$\text{(3.1)} \qquad d_0(x_i) \equiv v_1^{a_i}v_2^{(p-1)p^{i-1}}(2t_1 - v_1\zeta) \bmod (p, v_1^{2+a_i})$$

for $i>1$, where $d_0\colon A\to\Gamma=\Omega_\Gamma^1 A$ is the differential defined by $d_0=\eta_R-\eta_L$ for the right and the left units η_R and η_L, and $\zeta = v_2^{-1}t_2 + v_2^{-p}(t_2^p - t_1^{p^2+p})$.

LEMMA 3.2. *For any integers t and $n>0$, $v_2^{(tp-1)p^n}/pv_1\otimes g_0 = 0$ in $\mathrm{Ext}^2(M_0^2)$.*

PROOF. Consider the element $\xi = 1/tp^{n+2}v_1^{p^{n+1}+p^n}\otimes d_0(x_{n+1}^t)$ of the cobar complex $\Omega_{\Gamma}^{1}M_{0}^{2}=M_{0}^{2}\otimes_{A}\Gamma$. The differential $d_1\colon \Omega_{\Gamma}^{1}M_{0}^{2}\to\Omega_{\Gamma}^{2}M_{0}^{2}$ satisfies the relation $d_1(1/p^i v_1^j\otimes x)=1/p^i v_1^j\otimes d_1(x)+d_0(1/p^i v_1^j)\otimes x$ for $x\in\Omega_{\Gamma}^{1}A=\Gamma$. Furthermore, $d_1d_0=0$, $d_0(v_1)=pt_1\in\Gamma$, $d_0(v_2)\equiv v_1t_1^p \mod (p, v_1^p)\in\Gamma$ and $d_1(t_1)=0\in\Gamma\otimes_A\Gamma$. So we compute

$$\begin{aligned}d_1(\xi)&=-1/tpv_1^{p^{n+1}+p^n+1}\otimes t_1\otimes d_0(x_{n+1}^t)\\&=-1/pv_1^2\otimes t_1\otimes v_2^{(tp-1)p^n}(2t_1-v_1\zeta) \qquad \text{(by (3.1))}\\&=d_1(v_2^{(tp-1)p^n}/pv_1^2\otimes t_1^2)+v_2^{(tp-1)p^n}/pv_1\otimes t_1\otimes\zeta\end{aligned}$$

by noticing $n>0$. Thus $v_2^{(tp-1)p^n}/pv_1\otimes t_1\otimes\zeta$ is homologous to zero. On the other hand, in [17, Prop. 4.4], it is shown that $v_2^m/pv_1\otimes g_0$ is homologous to $v_2^m/pv_1\otimes t_1\otimes\zeta$ for any integer m. Note here that, in [17], we use a convention to denote $v_2^m g_0/pv_1$ for $v_2^m/pv_1\otimes g_0$. Hence we have the desired result. q.e.d.

THEOREM 3.3. *In the homotopy groups* $\pi_*(L_2S^0)$,

$$\beta_{rp+1}\beta_{tp/p}=0$$

if $p^n\mid r+t+p^{n-1}$ *for some integer* $n>0$.

PROOF. Since the connecting homomorphisms are maps of $\mathrm{Ext}^*(A)$-modules, we have

$$\begin{aligned}\beta_{rp+1}\beta_{tp/p}&=\delta_0\delta_1(v_2^{rp+1}/pv_1)\beta_{tp/p}\\&=\delta_0\delta_1(v_2^{rp+1}/pv_1\otimes\beta_{tp/p})\end{aligned}$$

by (2.2). Now substitute $-tv_2^{tp-1}g_0$ for $\beta_{tp/p}$ by Lemma 2.4, and we see the triviality $v_2^{(r+t)p}/pv_1\otimes g_0=0$ by Lemma 3.2 if $r+t=(up-1)p^{n-1}$ for some u and $n>0$. q.e.d.

References

[1] A. K. Bousfield, The localization of spectra with respect to homology, Topology, **18** (1979), 257–281.

[2] D. C. Johnson, H. R. Miller, W. S. Wilson, and R. S. Zahler, Boundary homomorphisms in the generalised Adams spectral sequence and the non-triviality of infinitely many γ_t in stable homotopy, Reunion sobre teorie de homotopia, Northwestern Univ. 1974, Soc. Mat. Mexicana, (1975), 47–59.

[3] H. R. Miller, D. C. Ravenel, and W. S. Wilson, Periodic phenomena in the Adams-Novikov spectral sequence, Ann. of Math., **106** (1977), 469–516.

[4] S. Oka, A new family in the stable homotopy groups of spheres, Hiroshima Math. J., **5** (1975), 87–114.

[5] S. Oka, A new family in the stable homotopy groups of spheres II, Hiroshima Math. J., **6** (1976), 331–342.

[6] S. Oka, Realizing some cyclic BP_* modules and applications to stable homotopy of spheres, Hiroshima Math. J., **7** (1977), 427–447.

[7] S. Oka, Small ring spectra and p-rank of the stable homotopy of spheres, Proceedings of the 1982 Northwestern Conference in Homotopy Theory, Contemp. Math. **19**, Amer. Math. Soc., (1983), 267–308.

[8] S. Oka, Multiplicative structure of finite ring spectra and stable homotopy of spheres, Proceedings of the Aahus Algebraic Topology Conference 1982, Lecture Notes in Math., **1051** (1984), 418–441.

[9] S. Oka and K. Shimomura, On products of the β-elements in the stable homotopy groups of spheres, Hiroshima Math. J., **12** (1982), 611–626.

[10] D. C. Ravenel, Localization with respect to certain periodic homology theories, Amer. J. Math., **106** (1984), 351–414.

[11] H. Sadofsky, Higher p-torsion in the β-family, Proc. of A.M.S., **108** (1990), 1063–1071.

[12] K. Shimomura, On the Adams-Novikov spectral sequence and products of β-elements, Hiroshima Math. J., **16** (1986), 209–224.

[13] K. Shimomura, Non-triviality of some products of β-elements in the stable homotopy of spheres, Hiroshima Math. J., **17** (1987), 349–353.

[14] K. Shimomura, Triviality of products of β-elements in the stable homotopy group of spheres, J. Math. Kyoto Univ., **29** (1989), 57–67.

[15] K. Shimomura, On the products $\beta_s\beta_t$ in the stable homotopy groups of spheres, Hiroshima Math. J., **19** (1989), 347–354.

[16] K. Shimomura and H. Tamura, Non-triviality of some compositions of β-elements in the stable homotopy of the Moore spaces, Hiroshima Math. J., **16** (1986), 121–133.

[17] K. Shimomura and A. Yabe, On the chromatic E_1-term $H^*M_0^2$, Cont. Math. Series of the A.M.S., Topology and Representation Theory, **158** (1994), 217–228.

[18] K. Shimomura and A. Yabe, The homotopy group $\pi_*(L_2S^0)$, Topology, **34** (1995), 261–289.

[19] L. Smith, On realizing complex bordism modules, IV, Applications to the stable homotopy groups of spheres, Amer. J. Math., **99** (1971), 418–436.

[20] H. Toda, Algebra of stable homotopy of Z_p-spaces and applications, J. Math. Kyoto Univ., **11** (1971), 197–251.

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