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On maximal Riemann surfaces

Dedicated to Professor F. Maeda for his 60th birthday

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ABSTRACT. We obtain two sufficient conditions for a Riemann surface to be maximal. One is the condition $\Gamma_{h0} \cap \Gamma_{h0}^* \neq \{0\}$ and the other is the existence of a function which has the special behavior in the neighborhood of the ideal boundary.

1. Introduction

Let R be a Riemann surface. If there exists a conformal mapping ι of R into a Riemann surface $\tilde{R}$, then we call $\tilde{R}$, or more precisely the pair $(\tilde{R}, \iota)$, an extension of R. According to this definition R itself is an extension of R. An extension $(\tilde{R}, \iota)$ is called a proper extension if $\tilde{R} \backslash \iota(R) \neq \emptyset$. A Riemann surface is called maximal if it has no proper extensions. An extension $\tilde{R}$ of R is called a maximal extension if $\tilde{R}$ is a maximal Riemann surface. On the maximality of Riemann surfaces many papers have been written. Bochner [3] proved that every Riemann surface has a maximal extension. We say that a Riemann surface R has a unique maximal extension if all maximal extensions of R are conformally equivalent to one another (cf. [6]). Clearly every maximal Riemann surface has a unique maximal extension. A closed subset E of a Riemann surface R is said to be an N_D-set if every compact subset of $\varphi(U \cap E)$ is an N_D-set in the complex plane for every local chart (U, φ) on R; see [10, p. 255] for an N_D-set. Renggli [7] determined the class of Riemann surfaces which have a unique maximal extension.

THEOREM A [7, Theorem 2]. *A Riemann surface R has a unique maximal extension if and only if R is conformally equivalent to some $\tilde{R} \backslash E$, where $\tilde{R}$ is a maximal Riemann surface and E is a closed N_D-set in $\tilde{R}$.*

By a neighborhood of the ideal boundary of R we mean the exterior of

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a compact set of R. A connected component V of a neighborhood of the ideal boundary is called an end if it is not relatively compact. Let D be a simply connected regular subregion of R, and consider a conformal map from D onto the unit disc U in the complex plane. Then the relative boundary ∂D corresponds to a relatively open subset of ∂U. We denote by I the complement of the image set of ∂D with respect to ∂U. We call D a disc with crowded ideal boundary if I is totally disconnected and is not an N_D-set. Recently Sakai [8] has obtained a new characterization of non-maximal Riemann surfaces.

THEOREM B [8, Theorem 4.1]. *Let R be a Riemann surface. Then R is not maximal if and only if one of the following conditions holds for R.*

(a) *R has a planar end.*

(b) *R has a border.*

(c) *R has a disc with crowded ideal boundary.*

In [2, V.14F] it is pointed out that the condition $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R) = \{0\}$ seems to be indicative of a strong boundary, where $\Gamma_{h0}(R)$ is a closed subspace of the space of square integrable harmonic differentials (see the next section). In this paper we shall consider the case $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R) \neq \{0\}$. We know by Accola ([1, Lemma 3 on p. 158]) that if R is a bordered Riemann surface with boundary γ, not necessarily compact, then every $\omega \in \Gamma_{h0}(R)$ can be extended to be harmonic on $R \cup \gamma$ and the extended ω is zero along γ. If ω^* also belongs to $\Gamma_{h0}(R)$, then ω^* is also zero along γ. Hence $\omega \equiv 0$ and we have shown

PROPOSITION 1. *If $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R) \neq \{0\}$ holds for R, then R does not have a border.*

We may expect that the condition $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R) \neq \{0\}$ gives us some information about the ideal boundary of R. We shall show in Theorem 1 that R of finite positive genus belongs to the class O_{AD} if and only if $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R) \neq \{0\}$ holds. In the paper [4] we have shown that there exists a Riemann surface of infinite genus which satisfies the condition $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R) \neq \{0\}$ but does not belong to the class O_{AD}; see Lemma 3 and [4, Proposition 1]. Hence the information is not about the "scale" of the ideal boundary, but the "complexity" of the ideal boundary in case of infinite genus. We use Sakai's characterization of non-maximal Riemann surfaces to show in Theorem 2 that if R has no planar ends and satisfies the condition $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R) \neq \{0\}$, then R is maximal.

In Theorem 3 we shall obtain another sufficient condition for a Riemann surface to be maximal. It will be shown that Proposition 6.1 in [8] follows from Theorem 3.

2. Preliminaries

We recall some definitions of first order differentials on R. A differential $\omega = a(x, y)dx + b(x, y)dy$ is called real if all local coefficients $a(x, y)$ and $b(x, y)$ are real-valued functions and called of C^∞ class if $a(x, y)$ and $b(x, y)$ are so. We say that ω is square integrable if local coefficients are measurable and

$$\int_R (a^2 + b^2)dxdy = \int_R \omega \wedge \omega^*$$

is finite, where $\omega^* = -b(x, y)dx + a(x, y)dy$ is the conjugate differential of ω. The positive square root of this integral is denoted by $\|\omega\|_R$, and we call it the norm of ω. Let $\Gamma = \Gamma(R)$ be the space of all real square integrable differentials on R. We know that Γ is a Hilbert space with the inner product

$$(\omega_1, \omega_2) = (\omega_1, \omega_2)_R = \int_R \omega_1 \wedge \omega_2^*.$$

Set

$$\Gamma_e^\infty(R) = \{df;\ \|df\|_R < \infty, f \in C^\infty(R)\} \quad \text{and} \quad \Gamma_{e0}^\infty(R) = \{df; f \in C_0^\infty(R)\},$$

where $C^\infty(R)$ is the class of infinitely differentiable functions on R and $C_0^\infty(R)$ is the class of infinitely differentiable functions with compact support on R. Denote by $\Gamma_e(R)$ and $\Gamma_{e0}(R)$ the closures of $\Gamma_e^\infty(R)$ and $\Gamma_{e0}^\infty(R)$ in Γ, respectively. We denote by $\Gamma_h = \Gamma_h(R)$ the subspace of $\Gamma(R)$ which consists of harmonic differentials. We introduce important subspaces of Γ_h. Let $\Gamma_{he}(R)$ (resp. $\Gamma_{hse}(R)$) be the subspace of $\Gamma_h(R)$ whose elements ω are exact (resp. semiexact) on R, that is

$$\int_\gamma \omega = 0 \qquad \text{for every (resp. every dividing) 1-cycle } \gamma \text{ on } R.$$

We have the orthogonal decompositions

$$\Gamma = \Gamma_h + \Gamma_{e0} + \Gamma_{e0}^* \quad \text{and} \quad \Gamma_e = \Gamma_{he} + \Gamma_{e0}$$

(cf. [2, V.10A, 10B, and 11G]).

Let Γ_y be a closed subspace of Γ_h. The orthogonal complement of Γ_y in Γ_h is denoted by $\Gamma_y^\perp$. Set $\Gamma_y^* = \{\omega^*; \omega \in \Gamma_y\}$. Since $(\omega_1, \omega_2) = (\omega_1^*, \omega_2^*)$ holds, we have $(\Gamma_y^*)^\perp = (\Gamma_y^\perp)^*$. Then we shall write it simply $\Gamma_y^{*\perp}$. We need the subspace of harmonic measures Γ_{hm} and Γ_{h0}; see [2, V.15C, 10B, and 14C] for definition. By [2, V.15D and 10C] we have $\Gamma_{hm} = \Gamma_{hse}^{*\perp}$ and $\Gamma_{h0} = \Gamma_{he}^{*\perp}$. By definition it follows that $\Gamma_h \supset \Gamma_{hse} \supset \Gamma_{he}$ and $\Gamma_{he} \supset \Gamma_{hm}$. We have $\Gamma_{hse} \supset \Gamma_{h0} \supset \Gamma_{hm}$ because they are orthogonal complements of Γ_{hm}^*, Γ_{he}^*, and Γ_{hse}^*, respectively. See also [2, V.15E]. We summarize the inclusion relations here:

$$\begin{array}{ccc} \Gamma_h \supset & \Gamma_{hse} \supset & \Gamma_{he} \\ & \cup & \cup \\ & \Gamma_{h0} \supset & \Gamma_{hm}. \end{array}$$

For a given 1-cycle c on R and a closed subspace Γ_y of Γ_h there exists uniquely a period reproducing differential $\sigma_y(c)$ in Γ_y such that

$$\int_c \omega = (\omega, \sigma_y(c))_R \qquad \text{for every } \omega \in \Gamma_y.$$

We are interested in $\sigma_h(c)$, $\sigma_{hse}(c)$, and $\sigma_{h0}(c)$. We consider the set $\{\sigma_{hse}(A_j), \sigma_{hse}(B_j)\}$, where $\{A_j, B_j\}$ is the canonical homology basis for R modulo dividing cycles. We note that every $\omega \in \Gamma_{he}$ satisfies

$$(\omega, \sigma_{hse}(A_j))_R = 0 \quad \text{and} \quad (\omega, \sigma_{hse}(B_j))_R = 0.$$

Hence the set $\{\sigma_{hse}(A_j), \sigma_{hse}(B_j)\}$ is included in $\Gamma_{hse} \cap \Gamma_{h0}^*$. If $\sigma \in \Gamma_{hse} \cap \Gamma_{h0}^*$ satisfies relations $(\sigma, \sigma_{hse}(A_j)) = (\sigma, \sigma_{hse}(B_j)) = 0$ for every j, then σ belongs to Γ_{he}. Hence it is equal to zero. This shows that $\{\sigma_{hse}(A_j), \sigma_{hse}(B_j)\}$ spans $\Gamma_{hse} \cap \Gamma_{h0}^*$. Moreover $\sigma_h(c)$, $\sigma_{hse}(c)$, and $\sigma_{h0}(c)$ have the following property:

$$\int_\gamma \sigma_h(c)^* = \int_\gamma \sigma_{hse}(c)^* = \int_\gamma \sigma_{h0}(c)^*$$

$$= \gamma \times c \qquad \text{for 1-cycle } \gamma$$

where $\gamma \times c$ is the intersection number of γ and c (cf. [11, Theorem 4] and [2, V. Theorem 21G]).

REMARK. In [11, Theorem 4] the period reproducing differential $\sigma_\chi(c)^*$ in a closed subspace $\Gamma_\chi^{*\perp}$ is defined by

$$\int_c \omega = (\omega, \sigma_\chi(c)^*)_R \qquad \text{for every } \omega \in \Gamma_\chi^{*\perp}.$$

Then $\sigma_h(c)$, $\sigma_{hse}(c)$ and $\sigma_{h0}(c)$ in this paper are equal to $\sigma_{\{0\}}(c)^*$, $\sigma_{hm}(c)^*$ and $\sigma_{he}(c)^*$ in [11], respectively.

If the differential dh of a function h of the class C^1 is square integrable, then we call the integral $\int_R (h_x^2 + h_y^2) dx dy = \|dh\|_R^2$ the Dirichlet integral of h and say that h has a finite Dirichlet integral. Let $HD(R)$ be the class of real-valued harmonic functions on R with finite Dirichlet integral and $KD(R)$ be the subclass of $HD(R)$ whose elements u have the property

$$\int_\gamma du^* = 0 \qquad \text{for every dividing 1-cycle } \gamma \text{ on } R.$$

Let $AD(R)$ be the class of analytic functions on R with finite Dirichlet integral. We denote by $\Re AD(R)$ the class of real-valued harmonic functions u such that there is a single-valued conjugate harmonic function u^* of u and $u + iu^*$ belongs to $AD(R)$. By the Cauchy-Riemann equation we have

$$du^* = -u_y dx + u_x dy = (u^*)_x dx + (u^*)_y dy = d(u^*).$$

It is easily seen that $u \in \Re AD(R)$ if and only if $u \in HD(R)$ and

$$\int_\gamma du^* = 0 \qquad \text{for every 1-cycle } \gamma \text{ on } R.$$

The relations between subclasses of $HD(R)$ and subspaces of $\Gamma_h(R)$ are the following:

$$\{du; u \in HD(R)\} = \Gamma_{he}(R)$$
$$\{du; u \in KD(R)\} = \Gamma_{he}(R) \cap \Gamma^*_{hse}(R)$$
$$\{du; u \in \Re AD(R)\} = \Gamma_{he}(R) \cap \Gamma^*_{he}(R).$$

We say that a Riemann surface R belongs to the class O_{AD} (resp. O_{KD}) if and only if $AD(R)$ or equivalently $\Re AD(R)$ (resp. $KD(R)$) consists of only constant functions.

Let ω be a real differential defined in a neighborhood of the ideal boundary of R and Γ_χ be any closed subspace of Γ_{he}. Then ω is said to have Γ_χ-behavior if the following representation holds in some neighborhood of the ideal boundary of R:

$$\begin{cases} \omega = \omega_1 + df, \\ \omega^* = \omega_2 + dg, \end{cases}$$

where $\omega_1 \in \Gamma_\chi$, $\omega_2 \in \Gamma_\chi^{*\perp}$, f and g are C^∞-functions on R such that df and dg belong to Γ_{e0}. We say that a function u has Γ_χ-behavior if du does. We know by [11, Theorem 4] that $\sigma_{hse}(c)^*$ (resp. $\sigma_{h0}(c)^*$) has Γ_{hm}- (resp. Γ_{he}-) behavior.

REMARK. Suppose that ω is defined in a neighborhood V of the ideal boundary of R and has Γ_χ-behavior.

1) The above representation may not hold in V, but it holds in some neighborhood $V' \subset V$ of the ideal boundary. Since ω and ω^* are closed differentials in V', from Weyl's lemma it follows that ω is harmonic in V' (cf. [2, V.9A and 9B]).

2) Let V_0 be a neighborhood of the ideal boundary which is a subset of V. It is easily seen that $\omega|_{V_0}$ also has Γ_χ-behavior.

The following basic properties for special Γ_χ's will be used later.

LEMMA 1. *Let V be a neighborhood of the ideal boundary of R such that the relative boundary consists of a finite number of mutually disjoint analytic Jordan curves. Let u be a harmonic function on $\bar{V} = V \cup \partial V$. Denote by $HD(V)$ the set of harmonic functions on $\bar{V}$ with finite Dirichlet integral over V and set $KD(V) = \{v \in HD(V);\ dv^*$ is semi-exact in $V\}$.*

(1) *If u has Γ_{he}-behavior, then*

$$(du, dv)_V = \int_{\partial V} v du^* \qquad \text{for every } v \in HD(V). \tag{1-1}$$

If u has Γ_{hm}-behavior, then

$$(du, dv)_V = \int_{\partial V} u dv^* \qquad \text{for every } v \in KD(V). \tag{1-2}$$

(2) *For every $h \in H(\partial V)$, there exist u_0, $u_1 \in KD(V)$ such that $u_0 = u_1 = h$ on ∂V, u_0 has Γ_{he}-behavior and u_1 has Γ_{hm}-behavior, where $H(\partial V)$ is the class of harmonic functions defined in some neighborhood of ∂V.*

See [11, Propositions 1 and 2] for the assertions (1-1) and (1-2), and [11, Theorem 6 and p. 203] for (2).

Let $(\tilde{R}, \iota)$ be an extension of R. We denote by $\iota^\#(\tilde{\omega})$ the pull back of a differential $\tilde{\omega}$ on $\tilde{R}$ induced by ι. For a closed subspace $\Gamma_y(\tilde{R})$ of $\Gamma_h(\tilde{R})$ we set $\iota^\#(\Gamma_y(\tilde{R})) = \{\iota^\#(\tilde{\omega}); \tilde{\omega} \in \Gamma_y(\tilde{R})\}$. It is easily seen that $\iota^\#(\tilde{\omega}^*) = \iota^\#(\tilde{\omega})^*$ and $\iota^\#(d\tilde{u}) = d(\tilde{u} \circ \iota)$ hold for $\tilde{\omega} \in \Gamma_h(\tilde{R})$ and $\tilde{u} \in HD(\tilde{R})$.

3. Results

We know that if R is of finite genus and belongs to the class O_{AD} then R has a unique maximal extension. (See [6].) In this case a maximal extension of R is a compact Riemann surface $\tilde{R}$ of the same genus as that of R and we may assume that R is a subregion of $\tilde{R}$ such that $\tilde{R} \backslash R$ is an N_D-set; see [9, II.15A]. We say that a Riemann surface R has (W)-property if

$$\Gamma_{he}(R) \cap \Gamma_{hse}^*(R) \subset \Gamma_{he}^*(R)$$

holds; see [5]. We obtain the next theorem.

THEOREM 1. *Let R be a Riemann surface of finite positive genus. Then the following properties are equivalent:*

(a) $\Gamma_{h0}(R) \cap \Gamma_{h0}^{*}(R) \neq \{0\}$ *holds.*

(b) *R belongs to the class* O_{AD}.

(c) *R has (W)-property.*

Furthermore if $R \in O_{AD}$ *and* $\tilde{R}$ *is the unique maximal extension of* R, *then* $\Gamma_{h0}(R) \cap \Gamma_{h0}^{*}(R) = \Gamma_{h}(\tilde{R})|_{R}$.

Our results in case of infinite genus are Theorems 2 and 2′.

THEOREM 2. *Let R be a Riemann surface of infinite genus having no planar ends. If R satisfies the condition* $\Gamma_{h0}(R) \cap \Gamma_{h0}^{*}(R) \neq \{0\}$, *then R is maximal.*

THEOREM 2′. *Let R be a Riemann surface of infinite genus. If R satisfies the condition* $\Gamma_{h0}(R) \cap \Gamma_{h0}^{*}(R) \neq \{0\}$, *then R has a unique maximal extension and the following relation holds for a maximal extension* $(\tilde{R}, \iota)$ *of R*:

$$\iota^{\#}(\Gamma_{h0}(\tilde{R}) \cap \Gamma_{h0}^{*}(\tilde{R})) = \Gamma_{h0}(R) \cap \Gamma_{h0}^{*}(R).$$

In particular Theorem 2′ is a generalization of the last part of Theorem 1. We obtain

COROLLARY 1. *If R satisfies* $\Gamma_{h0}(R) \cap \Gamma_{h0}^{*}(R) \neq \{0\}$, *then any extension* $(\tilde{R}, \iota)$ *of R satisfies*

$$\iota^{\#}(\Gamma_{h0}(\tilde{R}) \cap \Gamma_{h0}^{*}(\tilde{R})) = \Gamma_{h0}(R) \cap \Gamma_{h0}^{*}(R).$$

We have another sufficient condition for R to be maximal.

THEOREM 3. *Let R be a Riemann surface of infinite genus having no planar ends. Suppose that there exists a harmonic function u on a neighborhood V of the ideal boundary of R such that u is non-constant in each component of V and has* Γ_{he}- and Γ_{hm}*-behaviors simultaneously. Then R is maximal.*

The next is a generalization of Theorem 3.

THEOREM 3′. *Let R be a Riemann surface of infinite genus. Suppose that there exists a harmonic function u on a neighborhood V of the ideal boundary of R such that u is non-constant in each component of V and has* Γ_{he}*- and* Γ_{hm}*-behaviors simultaneously. Then R has a unique maximal extension and u is extended over a maximal extension* $\tilde{R}$ *of R so that the extended one has* $\Gamma_{he}(\tilde{R})$*- and* $\Gamma_{hm}(\tilde{R})$*-behaviors simultaneously.*

Let $P \in R$ and z be a local parameter in a neighborhood of P. There exists the principal function p_0 (resp. p_1) with respect to the singularity $\Re(1/z)$ at P and the operator L_0 (resp. $(Q)L_1$); see for example [9, Chapters I and II]. We say that R belongs to the class $\mathscr{S}_{KD}^{1}$ if $p_0 = p_1$ for some $P \in R$ and z.

This class $\mathscr{S}_{KD}^1$ includes O_{KD}. In fact by [10, II.2E] R belongs to the class O_{KD} if and only if $p_0 = p_1$ for every $P \in R$ and every local parameter about P. We know by [11, Theorem 6] that every principal function with respect to L_0 (resp. $(Q)L_1$) has Γ_{he} (resp. Γ_{hm})-behavior. If $R \in \mathscr{S}_{KD}^1$, then $p_0 (= p_1)$ satisfies the condition of Theorem 3. Now we have

Corollary 2 [8, Proposition 6.1]. *Let R be a Riemann surface having no planar ends. If R belongs to the class $\mathscr{S}_{KD}^1$, then R is maximal.*

By Lemma 1 (2) there exist many principal functions other than p_0 and p_1. For example $\sigma_{hse}(A_j)^*$ and $\sigma_{h0}(A_j)^*$ are exact in $R \backslash A_j$. Then harmonic functions $\int \sigma_{hse}(A_j)^*$ and $\int \sigma_{h0}(A_j)^*$ in $R \backslash A_j$ have Γ_{hm}- and Γ_{he}-behaviors, respectively. Hence Theorem 3 is a generalization of Sakai's result [8].

4. Proofs

If R has a planar end G whose relative boundary ∂G consists of one analytic Jordan curve, then G is mapped conformally into the unit disc $U = \{|z| < 1\}$ so that ∂G corresponds to ∂U. Denote by E the inner boundary of the image of G, which is considered as a realization of the ideal boundary of G (cf. [9, I.8E]). We show

Proposition 2. *Let R, G, and E be as above. If $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R) \neq \{0\}$ holds, then E is an N_D-set.*

Proof. We shall use the following lemma.

Lemma 2 [5, Lemma 4]. *Let R be a Riemann surface of finite genus. Suppose that there exists a non-constant harmonic function u in a neighborhood of the ideal boundary of R which satisfies the relations* (1-1) *and* (1-2) *in Lemma* 1. *Then R belongs to the class O_{AD}.*

It suffices to show the existence of u. By assumption we can take a non-zero $\omega \in \Gamma_{h0}(R) \cap \Gamma_{h0}^*(R)$. This ω is semi-exact on R, and hence exact on G. Thus we can choose $u \in HD(G)$ such that $du = \omega$. Every $v \in HD(G)$ can be extended over $R \backslash G$ as a C^∞-function such that $v = 0$ in the exterior of some neighborhood of ∂G, which will be still denoted by v. Since dv belongs to $\Gamma_e^\infty(R) \subset \Gamma_e(R)$ and $\Gamma_e(R) = \Gamma_{he}(R) + \Gamma_{e0}(R)$, it follows that $(\omega, dv)_R = 0$. Now we have

$$(du, dv)_G = (\omega, dv)_G = (\omega, dv)_R - (\omega, dv)_{R \backslash G} = 0 + \int_{\partial G} v\omega^* = \int_{\partial G} v du^*.$$

This shows that u satisfies the relation (1-1).

Moreover if $v \in KD(G)$, then the conjugate harmonic function v^* of v belongs to $HD(G)$. Since ω^* is also a non-zero element in $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R)$, by the above equality we have

$$(du, dv)_G = (du^*, dv^*)_G = -\int_{\partial G} (v^*)du.$$

From integration by parts it follows that

$$(du, dv)_G = \int_{\partial G} udv^*.$$

This shows that u satisfies also the relation (1-2). Since G is a neighborhood of the ideal boundary of a Riemann surface $S = \hat{\mathbb{C}} \backslash E$, S belongs to the class O_{AD} by Lemma 2 so that E is an N_D-set. □

We recall the following result.

THEOREM C [5]. *Let R be a Riemann surface and $\sigma_{hse}(c)$ (resp. $\sigma_{h0}(c)$) be the Γ_{hse} (resp. Γ_{h0}) period reproducing differential for a* 1-*cycle c. Then the following properties are equivalent*:

(a) *R has (W)-property.*

(b) $\|\sigma_{hse}(c)\| = \|\sigma_{h0}(c)\|$ (*equivalently* $\sigma_{hse}(c) = \sigma_{h0}(c)$) *for every* 1-*cycle c.*

Furthermore, if R is of finite positive genus, then the next properties are also equivalent to (a):

(c) *R belongs to the class O_{AD}.*

(d) $\|\sigma_{hse}(c)\| = \|\sigma_{h0}(c)\|$ (*equivalently* $\sigma_{hse}(c) = \sigma_{h0}(c)$) *for some non-dividing* 1-*cycle c.*

By this theorem if R has (W)-property, then the set $\{\sigma_{hse}(A_j), \sigma_{hse}(B_j)\}$ is included in Γ_{h0}. Hence $\Gamma_{hse} \cap \Gamma_{h0}^*$ which is spanned by $\{\sigma_{hse}(A_j), \sigma_{hse}(B_j)\}$ is included in Γ_{h0} and we have $\Gamma_{hse} \cap \Gamma_{h0}^* = \Gamma_{h0} \cap \Gamma_{h0}^*$. Conversely if $\Gamma_{hse} \cap \Gamma_{h0}^* = \Gamma_{h0} \cap \Gamma_{h0}^*$ holds, then every Γ_{hse} period reproducing differential $\sigma_{hse}(c)$, which is represented as a finite linear combination of $\{\sigma_{hse}(A_j), \sigma_{hse}(B_j)\}$, belongs to Γ_{h0}. This means that $\sigma_{hse}(c)$ is also Γ_{h0} period reproducing differential. By the uniqueness of the period reproducing differential $\sigma_{hse}(c) = \sigma_{h0}(c)$ holds. Therefore R has (W)-property. We have shown that R has (W)-property if and only if $\Gamma_{hse} \cap \Gamma_{h0}^* = \Gamma_{h0} \cap \Gamma_{h0}^*$.

Suppose that a non-planar surface R has (W)-property. Since we have $\int_{A_j} \sigma_{hse}(B_j)^* = A_j \times B_j = 1$, $\sigma_{hse}(B_j)$ is not zero. Then $\Gamma_{h0} \cap \Gamma_{h0}^*$ contains a non-zero element $\sigma_{hse}(B_j)$. Hence R satisfies the condition $\Gamma_{h0} \cap \Gamma_{h0}^* \neq \{0\}$ so that (W)-property implies the condition $\Gamma_{h0} \cap \Gamma_{h0}^* \neq \{0\}$.

We have hence shown

LEMMA 3. *If a non planar Riemann surface R has (W)-property, then $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R) \neq \{0\}$ holds.*

REMARK. In general the converse is not true. We recall an example of a Riemann surface R on which there exist non-dividing 1-cycles c_1 and c_2 with the property

$$\|\sigma_{hse}(c_1)\|_R = \|\sigma_{h0}(c_1)\|_R \neq 0, \ \|\sigma_{hse}(c_2)\|_R \neq \|\sigma_{h0}(c_2)\|_R$$

(see [4, Example 2]). The Riemann surface R satisfies the condition $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R) \neq \{0\}$ because a non-zero element $\sigma_{hse}(c_1) = \sigma_{h0}(c_1)$ belongs to $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R)$. But R does not have (W)-property.

We now give

PROOF OF THEOREM 1. The equivalence (b)$\Leftrightarrow$(c) follows from Theorem C and by Lemma 3 (c) implies (a). The assertion (a)$\Rightarrow$(b) follows from Proposition 2.

To prove the last part of Theorem 1 suppose that R belongs to the class O_{AD}. Then there exists a compact Riemann surface $\tilde{R}$ of the same genus as that of R such that $E = \tilde{R} \backslash R$ is an N_D-set. By [6] $\tilde{R}$ is a unique maximal extension of R. Let R_0 be a regular subregion of R defined in [9, I.8E]. For every $\omega \in \Gamma_{h0}(R) \cap \Gamma_{h0}^*(R)$, ω and ω^* are exact on $R \backslash R_0 \backslash E$. Hence an analytic function $\int(\omega + i\omega^*)$ with finite Dirichlet integral can be extended to be analytic on $\tilde{R} \backslash R_0$. Therefore ω can be extended to be harmonic on $\tilde{R} \backslash R_0$ and the extended ω belongs to $\Gamma_h(\tilde{R})$. We obtain $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R) \subset \Gamma_h(\tilde{R})|_R$. Now we prove the inverse inclusion relation. Since R is of finite genus and $R \in O_{AD}$, R belongs to the class O_{KD}; see for example [9, II.15A]. Note that $KD(R)$ consists of only constant functions if and only if $\Gamma_{he}(R) \cap \Gamma_{hse}^*(R) = \{0\}$. On the other hand by the orthogonal decomposition $\Gamma_{hse} = \Gamma_{h0} + (\Gamma_{hse}^* \cap \Gamma_{he})^*$ we know that $\Gamma_{he}(R) \cap \Gamma_{hse}^*(R) = \{0\}$ if and only if $\Gamma_{hse}(R) = \Gamma_{h0}(R)$. Thus R belongs to the class O_{KD} if and only if $\Gamma_{hse}(R) = \Gamma_{h0}(R)$ holds. Let $\tilde{\omega}$ be an arbitrary differential in $\Gamma_h(\tilde{R})$. Since $\tilde{\omega}|_R$ is semiexact in R, it belongs to $\Gamma_{hse}(R) = \Gamma_{h0}(R)$. If we consider the conjugate differential $\tilde{\omega}^*$, then we see that $\tilde{\omega}^*|_R$ is an element of $\Gamma_{h0}(R)$, too. Hence we have $\tilde{\omega}|_R \in \Gamma_{h0}(R) \cap \Gamma_{h0}^*(R)$ and conclude $\Gamma_h(\tilde{R})|_R \subset \Gamma_{h0}(R) \cap \Gamma_{h0}^*(R)$. This completes the proof. $\square$

Let $(\tilde{R}, \iota)$ be an extension of R and $H_1(\tilde{R})$ (resp. $H_1(R)$) be the homology group of $\tilde{R}$ (resp. R). Then the mapping ι maps a closed curve on R to a closed curve on $\tilde{R}$. Thus ι induces a natural homomorphism ι^* of $H_1(R)$ into $H_1(\tilde{R})$. Since ι is injective, the image set $\{\iota^*(A_j), \iota^*(B_j)\}$ of the canonical homology basis $\{A_j, B_j\}$ for R modulo dividing cycles is linearly independent.

To prove Theorems 2 and 3 we need the following lemma.

LEMMA 4. *Let R be a Riemann surface of infinite genus having to planar ends. If R is not maximal, then there is an extension $(\tilde{R}, \iota)$ of R having the following properties*:

(1) *$\tilde{R} \backslash \iota(R)$ is a set of two dimensional Lebesgue measure zero.*

(2) *$\tilde{R}$ has a border.*

(3) *Every dividing cycle on R is mapped to a dividing cycle on $\tilde{R}$ by ι^* and $\{\iota^*(A_j), \iota^*(B_j)\}$ is the canonical homology basis for $\tilde{R}$ modulo dividing cycles.*

PROOF. By Theorem B we know that R has a border or a disc with crowded ideal boundary. If R has a border, then R itself is an extension of R which satisfies (1), (2), and (3).

Suppose that R has a disc D with crowded ideal boundary. Let ϕ be a one-to-one conformal mapping of D onto the unit disc U and I be the complement of the image set of ∂D with respect to ∂U. Let ψ be a Möbius transformation which maps U onto the upper half plane H with $\psi(I) = \tilde{I} \subset \partial H$. Let $\mathscr{V}$ be the family of univalent functions F on $\mathbb{C} \backslash \tilde{I}$ with the following expansion around ∞:

$$F(z) = z + \frac{a[F]}{z} + \cdots.$$

By [10, VI. Theorems 2B and 2C] there exists a unique function $P_1(z) = P_1(z; \infty)$ (resp. $P_0(z) = P_0(z; \infty)$) which minimizes (resp. maximizes) $\Re a[F]$ in $\mathscr{V}$ and P_1 (resp. P_0) maps $\mathbb{C} \backslash \tilde{I}$ onto a vertical (resp. horizontal) slit plane. Since $\overline{P_1(\bar{z})}$ has the expansion $z + \overline{a[P_1]}/z + \cdots$ around ∞, we conclude $P_1(z) = \overline{P_1(\bar{z})}$ by the uniqueness of P_1. Let $z_0 \in H$. If $|z_0|$ is sufficiently large, then from the expansion of P_1 it follows that $P_1(z_0) \in H$. For any $z \in H$ we join z with z_0 by a segment $l_{z_0 z}$ in H. By the univalency of P_1 and the property $P_1(z) = \overline{P_1(\bar{z})}$ the image set $P_1(l_{z_0 z})$ does not intersect the real axis. Hence $P_1(l_{z_0 z})$ is included in either the upper half plane or the lower half plane. Since $P_1(z_0) \in H$, $P_1(l_{z_0 z})$ is included in the upper half plane H and we conclude $P_1(z) \in H$. Therefore we obtain $P_1(H) = P_1(\mathbb{C} \backslash \tilde{I}) \cap H$. For the same reason we have $P_0(H) = P_0(\mathbb{C} \backslash \tilde{I}) \cap H$ and $\mathbb{C} \backslash P_0(H) = (\mathbb{C} \backslash P_0(\mathbb{C} \backslash \tilde{I})) \cup (\mathbb{C} \backslash H)$.

We shall show that $P_0(H) \equiv z$. Suppose that $\mathbb{C} \backslash P_0(\mathbb{C} \backslash \tilde{I})$ contains a point $z_1 \in H$. The connected component C_1 of $\mathbb{C} \backslash P_0(\mathbb{C} \backslash \tilde{I})$ which contains z_1 is a horizontal slit or a point. Then C_1 is included in H. Set $E_n = \{z; \operatorname{dist}(z, \mathbb{C} \backslash P_0(\mathbb{C} \backslash \tilde{I})) \leq 1/n\}$ for $n \in \mathbb{N}$. Let E_n^1 be the connected component of E_n which includes C_1. Since $\bigcap_{n=1}^{\infty} E_n^1 = C_1$, there is a number $n_0 \in \mathbb{N}$ such that $E_{n_0}^1 \subset H$. Then Int $E_{n_0}^1$ (= the interior of $E_{n_0}^1$) and $\mathbb{C} \backslash E_{n_0}^1$ are mutually disjoint open sets such that $(\text{Int } E_{n_0}^1) \cup (\mathbb{C} \backslash E_{n_0}^1) \supset \mathbb{C} \backslash P_0(H) = (\mathbb{C} \backslash P_0(\mathbb{C} \backslash \tilde{I})) \cup (\mathbb{C} \backslash H)$ and that $(\mathbb{C} \backslash P_0(H)) \cap (\text{Int } E_{n_0}^1)$, $(\mathbb{C} \backslash P_0(H)) \cap (\mathbb{C} \backslash E_{n_0}^1)$ are not empty. This shows that $\mathbb{C} \backslash P_0(H)$ is not connected, which contradicts the simply connectivity of

$P_0(H)$. Therefore $P_0(H)$ must coincide with H. Hence $P_0(z)$ is a Möbius transformation which preserves H. Since P_0 fixes ∞ and its expansion around ∞ has a vanishing constant term, we conclude $P_0(z) \equiv z$. Since $\tilde{I}$ is not of class N_D, by [10, VI. Theorem 2D] we have $P_1(z) \neq P_0(z) = z$. Therefore we see that $P_1(H) \neq H$. By the same arguement as above we can show that every connected component of the complement of $P_1(\mathbb{C}\backslash\tilde{I})$ is either a point on the real axis or a vertical slit which intersects the real axis. Let l be one of the vertical slits not degenerating to a point. We construct a Riemann surface $\tilde{R}$ as the union of R and $H\backslash l$ by identifying $p \in D$ with $(P_1 \circ \psi \circ \phi)(p) \in H\backslash l$. There is a natural inclusion mapping ι of R into $\tilde{R}$. Then $(\tilde{R}, \iota)$ is an extension of R. It is easily seen that $\tilde{R}$ satisfies properties (1) and (2).

Let γ be a piecewise analytic dividing curve on R. If $\gamma \cap D = \emptyset$, then it is clear that $\iota(\gamma)$ is dividing also on $\tilde{R}$. We consider the case when γ intersects D. Since ∂D and γ are piecewise analytic, $\gamma \cap D$ consists of a finite number of components and $H\backslash l\backslash\iota(\gamma)$ consists of a finite number of components which are simply connected regions with piecewise analytic boundary. If $\tilde{R}\backslash\iota(\gamma)$ is connected, then there exist points z_1, $z_2 \in R\backslash\gamma$ which are not in the same component of $R\backslash\gamma$ and a piecewise analytic arc $\tilde{\gamma}$ which joins $\tilde{z}_1 = \iota(z_1)$ with $\tilde{z}_2 = \iota(z_2)$ in $\tilde{R}\backslash\iota(\gamma)$. It is easily seen that $\tilde{\gamma} \cap (H\backslash l)$ is not empty and consists of a finite number of components, $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_n$. Since each endpoint of $\tilde{\gamma}_j$ is $\tilde{z}_1$, $\tilde{z}_2$, or some point on $\partial H\backslash\tilde{I}$, we can deform $\tilde{\gamma}_j$ to a piecewise analytic arc γ_j on $P_1(H)$ whose endpoints are equal to those of $\tilde{\gamma}_j$ continuously in $H\backslash l\backslash\iota(\gamma)$. If we replace $\tilde{\gamma}_j$ by γ_j, then we obtain a new piecewise analytic arc γ^* on $R\backslash\gamma$ which joins z_1 with z_2. But this is a contradiction. Therefore γ is a dividing curve on $\tilde{R}$. Since every dividing cycle is homologous to a finite linear combination of piecewise analytic dividing curves, we deduce that ι^* maps dividing cycles on R to those on $\tilde{R}$.

Let $\tilde{c}$ be a piecewise analytic Jordan curve on $\tilde{R}$. By the same deformation as above we can show that $\tilde{c}$ is homologous to some piecewise analytic Jordan curve c on R. Then $\tilde{c}$ is homologous to a finite linear combination of $\{\iota^*(A_j), \iota^*(B_j)\}$ modulo dividing cycles. Hence $\{\iota^*(A_j), \iota^*(B_j)\}$ is the canonical homology basis for $\tilde{R}$ modulo dividing cycles. This completes the proof.

□

PROOF OF THEOREM 2. Suppose that R is not maximal. By Lemma 4 there exists $(\tilde{R}, \iota)$, an extension of R, which satisfies conditions (1), (2), and (3). Let ω be a non-zero element in $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R)$. Then $L_\omega(\tilde{\sigma}) = (\iota^{\#}(\tilde{\sigma}), \omega)_R$ is a bounded linear functional of $\tilde{\sigma} \in \Gamma_{hse}(\tilde{R})$. Hence there exists a unique $\tilde{\omega} \in \Gamma_{hse}(\tilde{R})$ such that $L_\omega(\tilde{\sigma}) = (\tilde{\sigma}, \tilde{\omega})_{\tilde{R}}$ for every $\tilde{\sigma} \in \Gamma_{hse}(\tilde{R})$. We shall show that $\tilde{\omega}$ belongs to $\Gamma_{h0}(\tilde{R}) \cap \Gamma_{h0}^*(\tilde{R})$. Since every $\tilde{u} \in KD(\tilde{R})$ satisfies $(d\tilde{u}^*, \tilde{\omega})_{\tilde{R}} = (d(\tilde{u} \circ \iota)^*, \omega)_R = 0$, we deduce that $\tilde{\omega}$ belongs to $\Gamma_{h0}(\tilde{R})$ from the orthogonal decomposition $\Gamma_{hse}(\tilde{R}) = \Gamma_{h0}(\tilde{R}) + \Gamma_{hse}(\tilde{R}) \cap \Gamma_{he}^*(\tilde{R})$. Similarly we obtain $L_\omega(d\tilde{u}) = (d(\tilde{u} \circ \imath), \omega)_R = 0$ for every $\tilde{u} \in HD(\tilde{R})$ and so $(d\tilde{u}, \tilde{\omega})_{\tilde{R}} = 0$. Hence $\tilde{\omega}$ belongs also to $\Gamma_{h0}^*(\tilde{R})$. It is shown that $\tilde{\omega} \in \Gamma_{h0}(\tilde{R}) \cap \Gamma_{h0}^*(\tilde{R})$.

If we show that $\tilde{\omega} \neq 0$, then $\Gamma_{h0}(\tilde{R}) \cap \Gamma_{h0}^*(\tilde{R}) \neq \{0\}$. By Proposition 1 $\tilde{R}$ does not have a border. This contradicts the property (2) of $\tilde{R}$ in Lemma 4.

Now we show that $\tilde{\omega}$ is not 0. By Lemma 4 (1) and (3) we see that $\imath^{\#}(\Gamma_{hse}(\tilde{R}) \cap \Gamma_{hse}^*(\tilde{R}))$ is a (closed) subspace of $\Gamma_{hse}(R) \cap \Gamma_{hse}^*(R)$. Let π be the orthogonal projection of $\Gamma_{hse}(R) \cap \Gamma_{hse}^*(R)$ onto $\imath^{\#}(\Gamma_{hse}(\tilde{R}) \cap \Gamma_{hse}^*(\tilde{R}))$. We remark that $\pi(\sigma^*) = \pi(\sigma)^*$ holds for every $\sigma \in \Gamma_{hse}(R) \cap \Gamma_{hse}^*(R)$, because $\pi(\sigma^*) - \pi(\sigma)^*$ satisfies the equation

$$\begin{aligned}(\pi(\sigma^*) - \pi(\sigma)^*, \tau)_R &= (\pi(\sigma^*), \tau)_R - (\pi(\sigma)^*, \tau)_R \\ &= (\sigma^*, \tau)_R - (-\pi(\sigma), \tau^*)_R \\ &= (-\sigma, \tau^*)_R - (-\sigma, \tau^*)_R \\ &= 0\end{aligned}$$

for every $\tau \in \imath^{\#}(\Gamma_{hse}(\tilde{R}) \cap \Gamma_{hse}^*(\tilde{R}))$.

If $\tilde{\sigma} \in \Gamma_{hse}(\tilde{R}) \cap \Gamma_{hse}^*(\tilde{R})$ and c is a non-dividing 1-cycle of R, then we have

$$\begin{aligned}\int_{\tilde{c}} \tilde{\sigma} = \int_c \imath^{\#}(\tilde{\sigma}) &= (\imath^{\#}(\tilde{\sigma}), \sigma_{hse}(c))_R \\ &= (\imath^{\#}(\tilde{\sigma}), \pi(\sigma_{hse}(c)))_R,\end{aligned}$$

where $\tilde{c} = \imath^*(c)$. On the other hand

$$\int_{\tilde{c}} \tilde{\sigma} = (\tilde{\sigma}, \tilde{\sigma}_{hse}(\tilde{c}))_{\tilde{R}} = (\imath^{\#}(\tilde{\sigma}), \imath^{\#}(\tilde{\sigma}_{hse}(\tilde{c})))_R$$

holds, where $\tilde{\sigma}_{hse}(\tilde{c}) \in \Gamma_{hse}(\tilde{R}) \cap \Gamma_{h0}^*(\tilde{R})$ is $\Gamma_{hse}(\tilde{R})$ period reproducing differential for a 1-cycle $\tilde{c}$. By the uniqueness of the period reproducing differential in $\imath^{\#}(\Gamma_{hse}(\tilde{R}) \cap \Gamma_{hse}^*(\tilde{R}))$ we obtain $\pi(\sigma_{hse}(c)) = \imath^{\#}(\tilde{\sigma}_{hse}(\tilde{c}))$. If we restrict the linear functional L_ω to $\Gamma_{hse}(\tilde{R}) \cap \Gamma_{hse}^*(\tilde{R})$, then

$$\begin{aligned}L_\omega(\tilde{\sigma}) &= (\imath^{\#}(\tilde{\sigma}), \omega)_R = (\imath^{\#}(\tilde{\sigma}), \pi(\omega))_R \\ &= (\tilde{\sigma}, \tilde{\omega})_{\tilde{R}} = (\imath^{\#}(\tilde{\sigma}), \imath^{\#}(\tilde{\omega}))_R\end{aligned}$$

holds for every $\tilde{\sigma} \in \Gamma_{hse}(\tilde{R}) \cap \Gamma_{hse}^*(\tilde{R})$. Since $\pi(\omega)$ and $\imath^{\#}(\tilde{\omega})$ belong to $\imath^{\#}(\Gamma_{hse}(\tilde{R}) \cap \Gamma_{hse}^*(\tilde{R}))$, we conclude $\pi(\omega) = \imath^{\#}(\tilde{\omega})$. Since $\omega \in \Gamma_{h0}(R) \cap \Gamma_{h0}^*(R)$ and $\Gamma_{he}^*(R) = \Gamma_{h0}^{\perp}(R)$, ω does not belong to $\Gamma_{he}^*(R)$. Hence there exists a non-dividing 1-cycle γ such that

$$(\omega^*, \sigma_{hse}(\gamma))_R = \int_\gamma \omega^* = \alpha \neq 0,$$

where $\sigma_{hse}(\gamma)$ is $\Gamma_{hse}(R)$-period reproducing differential for γ. We construct a complete orthonormal system $\{\phi_n\}$ of $\Gamma_{hse}^*(R)\cap\Gamma_{h0}(R)$ from $\{\sigma_{hse}(A_j)^*, \sigma_{hse}(B_j)^*\}$ by Schmidt's orthogonarization. Expand $\omega^* = \sum_{n=1}^{\infty} a_n\phi_n$, where $a_n = (\omega^*, \phi_n)_R$. The orthogonal projection of ω^* is equal to $\sum_{n=1}^{\infty} a_n\pi(\phi_n)$, so that

$$\iota^{\#}(\tilde{\omega}^*) = \pi(\omega)^* = \sum_{n=1}^{\infty} a_n\pi(\phi_n).$$

Set $\tilde{A}_j = \iota^*(A_j)$, $\tilde{B}_j = \iota^*(B_j)$, and $\tilde{\gamma} = \iota^*(\gamma)$. Let us recall the relations

$$(\sigma_{hse}(A_j)^*, \sigma_{hse}(\gamma))_R = \int_\gamma \sigma_{hse}(A_j)^* = \gamma \times A_j$$

and

$$(\iota^{\#}(\tilde{\sigma}_{hse}(\tilde{A}_j)^*), \sigma_{hse}(\gamma))_R = (\tilde{\sigma}_{hse}(\tilde{A}_j)^*, \tilde{\sigma}_{hse}(\tilde{\gamma}))_{\tilde{R}} = \tilde{\gamma} \times \tilde{A}_j = \gamma \times A_j$$

given in Section 2. The same is true for B_j. Since ϕ_n is a finite linear combination of $\{\sigma_{hse}(A_j)^*, \sigma_{hse}(B_j)^*\}$ which is denoted by $\sum_{j=1}^{\nu_n}\{\alpha_j^{(n)}\sigma_{hse}(A_j)^* + \beta_j^{(n)}\sigma_{hse}(B_j)^*\}$, we have

$$\begin{aligned}
(\pi(\phi_n), \sigma_{hse}(\gamma))_R &= \sum_{j=1}^{\nu_n} \{\alpha_j^{(n)}(\pi(\sigma_{hse}(A_j)^*), \sigma_{hse}(\gamma))_R + \beta_j^{(n)}(\pi(\sigma_{hse}(B_j)^*), \sigma_{hse}(\gamma))_R\} \\
&= \sum_{j=1}^{\nu_n} \{\alpha_j^{(n)}(\iota^{\#}(\tilde{\sigma}_{hse}(\tilde{A}_j)^*), \sigma_{hse}(\gamma))_R + \beta_j^{(n)}(\iota^{\#}(\tilde{\sigma}_{hse}(\tilde{B}_j)^*), \sigma_{hse}(\gamma))_R\} \\
&= \sum_{j=1}^{\nu_n} \{\alpha_j^{(n)}\cdot\gamma \times A_j + \beta_j^{(n)}\cdot\gamma \times B_j\} \\
&= \sum_{j=1}^{\nu_n} \{\alpha_j^{(n)}(\sigma_{hse}(A_j)^*, \sigma_{hse}(\gamma))_R + \beta_j^{(n)}(\sigma_{hse}(B_j)^*, \sigma_{hse}(\gamma))_R\} \\
&= (\phi_n, \sigma_{hse}(\gamma))_R.
\end{aligned}$$

We conclude that

$$\begin{aligned}
\int_{\tilde{\gamma}} \tilde{\omega}^* = \int_\gamma \iota^{\#}(\tilde{\omega}^*) &= \left(\sum_{n=1}^{\infty} a_n\pi(\phi_n), \sigma_{hse}(\gamma)\right)_R \\
&= \left(\sum_{n=1}^{\infty} a_n\phi_n, \sigma_{hse}(\gamma)\right)_R \\
&= \int_\gamma \omega^* = \alpha \neq 0.
\end{aligned}$$

Now it is shown that $\tilde{\omega} \neq 0$ and it deduces that R is maximal. □

In order to prove Theorem 3, we need the following.

LEMMA 5. *Let R be a bordered Riemann surface with boundary γ and (V, φ) be a parametric half disc about a point on γ (cf. [2, II. 7C]). Suppose that a C^∞ function f on R whose differential df belongs to $\Gamma_{e0}(R)$ is harmonic in $V \cap R$. Then f is extended to be constant on $\gamma \cap V$.*

PROOF. Set $U = \{z; |z| < 1\}$, $U^+ = \{z; |z| < 1, \Im z > 0\}$, $U^- = \{z; |z| < 1, \Im z < 0\}$ and $I = \{z; |z| < 1, \Im z = 0\}$. A parametric half disc (V, φ) satisfies $\varphi(V) = \{z; |z| < 1, \Im z \geq 0\}$ and $\varphi(\gamma \cap V) = I$. The local representation of f in (V, φ) is also denoted by f. Set $g(z) = -f(\bar{z})$ for z in U^- and

$$\omega = \begin{cases} df & \text{in } U^+ \\ dg & \text{in } U^-. \end{cases}$$

If we can prove that ω is a harmonic differential in U, then there exists a harmonic function u in U such that $du = \omega$. We have

$$u(z) = \begin{cases} f(z) + C & \text{in } U^+ \\ g(z) + C' & \text{in } U^- \end{cases}$$

with constants C and C'. Consider a harmonic function $v(z) = u(z) + u(\bar{z})$ in U. Then we have $v(z) = f(z) + C + g(\bar{z}) + C' = C + C'$ in U^+ and $v \equiv C + C'$ in U. We conclude

$$\lim_{U^+ \ni z \to I} u(z) = \frac{C + C'}{2}$$

so that

$$\lim_{U^+ \ni z \to I} f(z) = \frac{C' - C}{2}.$$

Thus f can be extended continuously over I and the extended f is a constant function on $I = \varphi(\gamma \cap V)$.

Now we shall show that ω is harmonic. Since df is an element of $\Gamma_{e0}(R)$, there is a sequence $\{h_n\}$ in $C_0^\infty(R)$ such that $\lim_{n\to\infty} \|df - dh_n\|_R = 0$. The local representation of h_n in (V, φ) is also denoted by h_n. Set

$$\tilde{h}_n(z) = \begin{cases} h_n(z) & \text{in } U^+ \\ -h_n(\bar{z}) & \text{in } U^-. \end{cases}$$

Since every h_n vanishes in some neighborhood of I, $\{\tilde{h}_n\}$ is a sequence in $C^\infty(U)$ which satisfies $\lim_{n\to\infty} \|\omega - d\tilde{h}_n\|_U = 0$. Let $\phi(z)$ be an arbitrary function in $C_0^\infty(U)$. We have $(\omega, d\phi^*)_U = \lim_{n\to\infty} (d\tilde{h}_n, d\phi^*)_U = 0$. On the other hand

$$(\omega, d\phi)_U = (df, d\phi)_{U^+} + (dg, d\phi)_{U^-} = (df, d\tilde{\phi})_{U^+}$$

holds, where $\tilde{\phi}(z) = \phi(z) - \phi(\bar{z})$ in U. Set $U_\delta^+ = \{z; |z| < 1, \Im z > \delta\}$ for $\delta > 0$. From here we use the same argument as in [2, V.13B]. We note

$$\begin{aligned}(df, d\tilde{\phi})_{U^+} = \lim_{\delta\to 0}(df, d\tilde{\phi})_{U_\delta^+} &= \lim_{\delta\to 0}\iint_{U_\delta^+} d\tilde{\phi}\wedge df^* \\ &= \lim_{\delta\to 0}\left\{-\iint_{U_\delta^+}\tilde{\phi}\Delta f dxdy + \int_{\partial U_\delta^+}\tilde{\phi} df^*\right\}.\end{aligned}$$

Since f is harmonic, we have

$$(df, d\tilde{\phi})_{U^+} = -\lim_{\delta\to 0}\int_{-\sqrt{1-\delta^2}}^{\sqrt{1-\delta^2}}\tilde{\phi}(x+i\delta)f_y(x+i\delta)dx.$$

Hence for any $\varepsilon > 0$ there is a positive number δ_0 such that if $0 < y < \delta_0$ then

$$(df, d\tilde{\phi})_{U^+} - \varepsilon < -\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\tilde{\phi}(x+iy)f_y(x+iy)dx < (df, d\tilde{\phi})_{U^+} + \varepsilon$$

holds. Integrate by y from 0 to δ, $\delta < \delta_0$. We have

$$\delta\{(df, d\tilde{\phi})_{U^+} - \varepsilon\} < -\int_0^\delta dy\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\tilde{\phi}f_y dx < \delta\{(df, d\tilde{\phi})_{U^+} + \varepsilon\}$$

and

$$(df, d\tilde{\phi})_{U^+} - \varepsilon < -\frac{1}{\delta}\int_0^\delta dy\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\tilde{\phi}f_y dx < (df, d\tilde{\phi})_{U^+} + \varepsilon.$$

Therefore we have

$$(df, d\tilde{\phi})_{U^+} = -\lim_{\delta\to 0}\frac{1}{\delta}\int_0^\delta dy\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\tilde{\phi}f_y dx.$$

Since $\tilde{\phi}(x) = 0$ on the real axis and $\tilde{\phi}\in C_0^\infty(U)$, we have an estimate $|\tilde{\phi}(x+iy)| \le My$ with constant M. We obtain

$$\begin{aligned}\left|\frac{1}{\delta}\int_0^\delta dy\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\tilde{\phi}f_y dx\right| &\le \frac{1}{\delta}\int_0^\delta dy\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} My|f_y|dx \le \int_0^\delta\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} M|f_y|dxdy \\ &\le M\left(\int_0^\delta\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dxdy\right)^{1/2}\left(\int_0^\delta\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} |f_y|^2 dxdy\right)^{1/2} \\ &\le M(2\delta)^{1/2}\|df\|_R.\end{aligned}$$

This shows that $(\omega, d\phi)_U = 0$ for all $\phi\in C_0^\infty(U)$. By Weyl's lemma we conclude that ω is a harmonic differential. □

Now we show

PROOF OF THEOREM 3. We may assume that ∂V consists of a finite number of mutually disjoint analytic Jordan curves and u is harmonic on ∂V too.

Assume that R is not maximal. Let $(\tilde{R}, \iota)$ be an extension of R satisfying properties (1), (2), and (3) in Lemma 4. Set $\tilde{V} = \tilde{R} \backslash \iota(R \backslash V)$. By Lemma 1 (2), there exist $\tilde{u}_0$ and $\tilde{u}_1 \in KD(\tilde{V})$ such that $\tilde{u}_0$ and $\tilde{u}_1$ are equal to $u \circ \iota^{-1}$ on $\partial \tilde{V}$ and $\tilde{u}_0$ (resp. $\tilde{u}_1$) has $\Gamma_{he}(\tilde{R})$ (resp. $\Gamma_{hm}(\tilde{R})$)-behavior.

Since $\tilde{R}$ has the property (3) in Lemma 4, $\tilde{u}_0 \circ \iota$ and $\tilde{u}_1 \circ \iota$ also belong to $KD(V)$. Since u has $\Gamma_{he}(\tilde{R})$- and $\Gamma_{hm}(\tilde{R})$-behaviors, from Lemma 1 (1-1) and (1-2) it follows that

$$(du, d(\tilde{u}_i \circ \iota))_V = \int_{\partial V} u d(\tilde{u}_i \circ \iota)^* = \int_{\partial V} (\tilde{u}_i \circ \iota) du^* = \|du\|_V^2.$$

We have by Lemma 4 (1) $\|d\tilde{u}_i\|_{\tilde{V}}^2 = \|d(\tilde{u}_i \circ \iota)\|_V^2$ and by Lemma 1 (1-1)

$$\|d\tilde{u}_i\|_{\tilde{V}}^2 = \int_{\partial \tilde{V}} \tilde{u}_i d\tilde{u}_i^* = \int_{\partial V} u d(\tilde{u}_i \circ \iota)^* = \|du\|_V^2.$$

Therefore

$$\begin{aligned} \|d(\tilde{u}_i \circ \iota) - du\|_V^2 &= \|d(\tilde{u}_i \circ \iota)\|_V^2 + \|du\|_V^2 - 2(d(\tilde{u}_i \circ \iota), du)_V \\ &= 0 \qquad \text{for } i = 0, 1 \end{aligned}$$

and we have $\tilde{u}_0 \circ \iota = \tilde{u}_1 \circ \iota = u$ on V. So $\tilde{u}_0 (= \tilde{u}_1)$ has $\Gamma_{he}(\tilde{R})$- and $\Gamma_{hm}(\tilde{R})$-behaviors. Now represent $d\tilde{u}_0$ as follows

$$\begin{cases} d\tilde{u}_0 = \omega_1 + df, \, \omega_1 \in \Gamma_{hm}(\tilde{R}), \, df \in \Gamma_{e0}(\tilde{R}) \\ d\tilde{u}_0^* = \omega_2 + dg, \, \omega_2 \in \Gamma_{h0}(\tilde{R}), \, dg \in \Gamma_{e0}(\tilde{R}) \end{cases}$$

By [1, Lemma 3 on p. 158] ω_1 and ω_2 are extended to be harmonic on a border γ of $\tilde{R}$ and zero along γ. By Lemma 5 df and dg are zero along γ. Then $d\tilde{u}_0$ and $d\tilde{u}_0^*$ are also extended to be harmonic on a border γ of $\tilde{R}$ and zero along γ. Hence $d\tilde{u}_0$ must be 0 on some component of $\tilde{V}$. But this contradicts that u is non-constant on each component of V. Therefore R is maximal. □

To prove Theorem 2′ and Theorem 3′ we construct an extension $\tilde{R}$ of R such that $\tilde{R}$ has no planar ends as follows: Let $\{R_n\}_{n=0,1,2,\dots}$ be a canonical exhaustion of R. Let $G_n^{(1)}, \dots, G_n^{(k_n)}$ be the planar components of $R \backslash \overline{R_n}$ each of which is not included in any planar component of $R \backslash \overline{R_{n-1}}$. Map $G_n^{(j)}$ conformally into the unit disc $U_n^{(j)} = \{|z| < 1\}$ so that $\partial G_n^{(j)}$ corresponds to $\partial U_n^{(j)}$, and denote by $E_n^{(j)}$ the inner boundary of the image of $G_n^{(j)}$. We obtain a new Riemann surface as the union of R and $\bigcup_{n \geq 1, j=1,\dots k_n} U_n^{(j)}$ by identifying

$G_n^{(j)}$ with $U_n^{(j)} \backslash E_n^{(j)}$. There is a natural inclusion mapping $\imath$ of R into $\tilde{R}$. In this way we obtain an extension $(\tilde{R}, \imath)$ of R which has no planar ends. We use the extension $(\tilde{R}, \imath)$ in the proofs of Theorem 2′ and 3′.

Proof of Theorem 2′. By Proposition 2 $E_n^{(j)}$ is an N_D-set. Let σ be an arbitrary differential in $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R)$. Since $\Gamma_{h0} \subset \Gamma_{hse}$ holds, σ and σ^* are exact in $G_n^{(j)}$. Since $\int(\sigma + i\sigma^*)$ belongs to $AD(G_n^{(j)})$, it can be extended to be analytic on $U_n^{(j)}$. Then σ is extended to be harmonic on $U_n^{(j)}$. Therefore σ, more precisely the differential $(\imath^{-1})^\#(\sigma)$ on $\imath(R)$, is extended to be harmonic over $\tilde{R}$. We denote the extended one by $\tilde{\sigma}$. Since $\tilde{R} \backslash \imath(R)$ is a set of two dimensional Lebesgue measure 0 (cf. [9, I. Theorem 8C]), $(d\tilde{u}, \tilde{\sigma}^*)_{\tilde{R}} = (d(\tilde{u} \circ \imath), \sigma^*)_R = 0$ and $(d\tilde{u}, \tilde{\sigma})_{\tilde{R}} = (d(\tilde{u} \circ \imath), \sigma)_R = 0$ hold for any $\tilde{u} \in HD(\tilde{R})$. Hence we have $\tilde{\sigma} \in \Gamma_{h0}(\tilde{R}) \cap \Gamma_{h0}^*(\tilde{R})$. Therefore $\imath^\#(\Gamma_{h0}(\tilde{R}) \cap \Gamma_{h0}^*(\tilde{R})) \supset \Gamma_{h0}(R) \cap \Gamma_{h0}^*(R) \neq \{0\}$. By Theorem 2 we conclude that $\tilde{R}$ is a maximal Riemann surface. Hence R has a maximal extension $\tilde{R}$ such that $\tilde{R} \backslash \imath(R)$ is a closed N_D-set. By Theorem A, R has a unique maximal extension.

Let $\tilde{\omega}$ be an arbitrary element in $\Gamma_{h0}(\tilde{R}) \cap \Gamma_{h0}^*(\tilde{R})$. Since each boundary component of ∂R_n is a dividing curve on $\tilde{R}$ or homologous to 0 on $\tilde{R}$, $\imath^\#(\tilde{\omega})$ belongs to $\Gamma_{hse}(R) \cap \Gamma_{hse}^*(R)$. In $G_n^{(j)}$ any $u \in KD(R)$ has a single-valued conjugate harmonic function u^*. Since $u + iu^*$ belongs to $AD(G_n^{(j)})$, it can be extended to be analytic on $U_n^{(j)}$. Then u is extended to be harmonic on $U_n^{(j)}$. Therefore u, more precisely $u \circ \imath^{-1}$, is extended to be harmonic over $\tilde{R}$. We denote the extended one by $\tilde{u}$. It is easily seen that $\tilde{u}$ belongs to $HD(\tilde{R})$. We have

$$(\imath^\#(\tilde{\omega}), du^*)_R = (\tilde{\omega}, d\tilde{u}^*)_{\tilde{R}} = 0 \quad \text{and} \quad (\imath^\#(\tilde{\omega}^*), du^*)_R = (\tilde{\omega}^*, d\tilde{u}^*)_{\tilde{R}} = 0.$$

By the orthogonal decomposition $\Gamma_{hse} = \Gamma_{h0} + \Gamma_{hse} \cap \Gamma_{he}^*$ we conclude that $\imath^\#(\tilde{\omega})$ belongs to $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R)$. We have shown the relation $\imath^\#(\Gamma_{h0}(\tilde{R}) \cap \Gamma_{h0}^*(\tilde{R})) = \Gamma_{h0}(R) \cap \Gamma_{h0}^*(R)$ for the special maximal extension $(\tilde{R}, \imath)$. Let $(\tilde{R}', \imath')$ be another maximal extension of R. By [7, Lemma 4] the conformal mapping $\imath' \circ \imath^{-1}$ is extended to that of $\tilde{R}$ onto $\tilde{R}'$. We denote the extended one by F. Then the pull back $F^\#$ induced by F is a bijection of $\Gamma_h(\tilde{R}')$ onto $\Gamma_h(\tilde{R})$. It is easily seen that $\tilde{\omega}' \in \Gamma_{h0}(\tilde{R}') \cap \Gamma_{h0}^*(\tilde{R}')$ if and only if the pull back $F^\#(\tilde{\omega}') \in \Gamma_{h0}(\tilde{R}) \cap \Gamma_{h0}^*(\tilde{R})$. Since $\imath'^\#(\tilde{\omega}') = \imath^\#(F^\#(\tilde{\omega}'))$ holds, we conclude that $\imath'^\#(\Gamma_{h0}(\tilde{R}') \cap \Gamma_{h0}^*(\tilde{R}')) = \Gamma_{h0}(R) \cap \Gamma_{h0}^*(R)$. This completes the proof. □

Proof of Theorem 3′. We may assume $\partial V \subset R_0$. Since u has Γ_{he}- and Γ_{hm}-behaviors, by Lemma 1,

$$(du, dv)_{G_n^{(j)}} = \int_{\partial G_n^{(j)}} v du^* \qquad \text{for every } v \in HD(G_n^{(j)})$$

and

$$(du, dv)_{G_n^{(j)}} = \int_{\partial G_n^{(j)}} u dv^* \qquad \text{for every } v \in KD(G_n^{(j)}).$$

By Lemma 2 we conclude that $E_n^{(j)}$ is an N_D-set. Since du and du^* are exact in $G_n^{(j)}$, u can be extended to be harmonic over $E_n^{(j)}$. By the same argument as in the proof of Theorem 3 we can show that the extended u has $\Gamma_{he}(\tilde{R})$- and $\Gamma_{hm}(\tilde{R})$-behaviors. By Theorem 3 $\tilde{R}$ is maximal. By Theorem A, R has a unique maximal extension. Denote the extended u over $\tilde{R}$ by $\tilde{u}$. Let $(\tilde{R}', \imath')$ be another maximal extension of R. Then $\imath' \circ \imath^{-1}$ is extended to the conformal mapping F of $\tilde{R}$ onto $\tilde{R}'$. It is easily seen that $\tilde{u} \circ F^{-1}$ has $\Gamma_{he}(\tilde{R}')$- and $\Gamma_{hm}(\tilde{R}')$-behaviors simultaneously. Since $u \circ \imath'^{-1} = \tilde{u} \circ F^{-1}$ holds on $\imath'(R)$, $u \circ \imath'^{-1}$ is extended to $\tilde{u} \circ F^{-1}$. This completes the proof. □

The last one is

Proof of Corollary 1. Let $(\tilde{R}, \imath)$ be an extension of R, not necessarily maximal. There is a maximal extension $(\hat{R}, \imath')$ of $\tilde{R}$. Then $(\hat{R}, \imath' \circ \imath)$ is a maximal extension of R. By Theorem 2′ $\hat{R} \backslash (\imath' \circ \imath)(R)$ is an N_D-set and there exists a non-zero element $\hat{\omega}$ in $\Gamma_{h0}(\hat{R}) \cap \Gamma_{h0}^*(\hat{R})$. Since $\tilde{R} \backslash \imath(R)$ is also an N_D-set, we have for any $\tilde{u} \in HD(\tilde{R})$

$$(\imath'^{\#}(\hat{\omega}), d\tilde{u})_{\tilde{R}} = (\imath^{\#}(\imath'^{\#}(\hat{\omega})), d(\tilde{u} \circ \imath))_R = 0$$

and

$$(\imath'^{\#}(\hat{\omega}^*), d\tilde{u})_{\tilde{R}} = (\imath^{\#}(\imath'^{\#}(\hat{\omega}^*)), d(\tilde{u} \circ \imath))_R = 0$$

because $\imath^{\#}(\imath'^{\#}(\hat{\omega})) = (\imath' \circ \imath)^{\#}(\hat{\omega})$ and $\imath^{\#}(\imath'^{\#}(\hat{\omega}^*)) = (\imath' \circ \imath)^{\#}(\hat{\omega}^*)$ belong to $\Gamma_{h0}(R) \cap \Gamma_{h0}^*(R)$. Therefore a non-zero element $\imath'^{\#}(\hat{\omega})$ exists in $\Gamma_{h0}(\tilde{R}) \cap \Gamma_{h0}^*(\tilde{R})$. Again by Theorem 2′

$$\imath'^{\#}(\Gamma_{h0}(\hat{R}) \cap \Gamma_{h0}^*(\hat{R})) = \Gamma_{h0}(\tilde{R}) \cap \Gamma_{h0}^*(\tilde{R})$$

holds. Since $(\imath' \circ \imath)^{\#} = \imath^{\#} \circ \imath'^{\#}$ holds, we have a conclusion. □

References

[1] R. Accola, The bilinear relation on open Riemann surfaces, T. A. M. S. **96** (1960), 143–161.
[2] L. Ahlfors and L. Sario, Riemann surface, Princeton University Press, Princeton, 1960.
[3] S. Bochner, Fortsetzung Riemannscher Flächen, Math. Ann. **98** (1928), 406–421.
[4] N. Jin, Some remarks on the class of the Riemann surfaces with (W)-property, Proc. Japan Acad. Ser. A. Math. Sci. **69** (1993), 322–326.

[5] N. Jin and Y. Kusunoki, On a class of Riemann surfaces characterized by period reproducing differentials, Pitman Research Notes in Math., vol. 212, 1989, pp. 13–20.

[6] K. Oikawa, On the uniqueness of the prolongation of an open Riemann surface of finite genus, P. A. M. S. **11** (1960), 785–787.

[7] H. Renggli, Structural instability and extensions of Riemann surfaces, Duke Math. J. **42** (1975), 211–224.

[8] M. Sakai, Continuations of Riemann surfaces, Canad. J. Math. **44** (1992), 357–367.

[9] L. Sario and M. Nakai, Classification theory of Riemann surfaces, Springer, Berlin-Heidelberg-New York, 1970.

[10] L. Sario and K. Oikawa, Capacity functions, Springer, Berlin-Heidelberg-New York, 1969.

[11] M. Yoshida, The method of orthogonal decomposition for differentials on open Riemann surfaces, J. Sci. Hiroshima Univ. Sci. A-I **32** (1968), 181–210.

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