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On the existence of tangential limits of monotone BLD functions

Dedicated to Professor Fumi-Yuki Maeda on the occasion of his sixtieth birthday

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Abstract. Our aim in this paper is to deal with the existence of tangential limits for monotone functions u in the upper half space R^n_+ of R^n satisfying

$$\int_D |\text{grad } u(x)|^p \omega(x) dx < \infty \qquad \text{for any bounded open set } D \subset R^n_+,$$

where $p > 1$ and ω is a non-negative measurable function on R^n_+. We are mainly concerned with the case when $\omega(x) = x_n^{p-n}$, $p > n-1$, and show that u has tangential limits at boundary points except those in a small set. For this purpose, we first give a fine limit result for BLD (or p-precise) functions on R^n_+, and then apply the estimate of the oscillations of monotone functions by the p-th means of partial derivatives over balls.

In case $\omega(x)$ is of the form $g(|x|)x_n^{p-n}$, we give a condition on g for u to have a tangential limit at the origin; in case $\omega(x) = g(x_n)x_n^{p-n}$, the same condition on g will assure that u has a usual boundary limit at any point of ∂R^n_+.

1 Introduction

Our aim in this paper is to study the existence of tangential boundary limits of monotone functions u in the half space $R^n_+ = \{x = (x_1, \dots, x_n) : x_n > 0\}$, $n \geqq 2$, which satisfy

$$(1) \qquad \int_D |\nabla u(x)|^p x_n^{p-n} dx < \infty \qquad \text{for any bounded open set } D \subset R^n_+,$$

where ∇ denotes the gradient; note that u is locally p-precise in R^n_+ in the sense of Ohtsuka [16]; see also Ziemer [21]. Here a continuous function u is said to be *monotone* (in the sense of Lebesgue) on an open set $G \subset R^n$ if

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$$\max_{\bar{D}} u(x) = \max_{\partial D} u(x) \qquad \text{and} \qquad \min_{\bar{D}} u(x) = \min_{\partial D} u(x)$$

hold for any relatively compact open set D such that $\bar{D} \subset G$ (see [4]).

The class of monotone functions is considerably wide. We give some examples of monotone functions.

EXAMPLE 1. Harmonic functions on an open set G are monotone in G. More generally, solutions of a wider class of partial differential equations are monotone (see Gilbarg-Trudinger [2]).

EXAMPLE 2. Weak solutions for variational problems may be monotone; in particular, weak solutions of the p-Laplacian are monotone. Moreover, if f is a quasi-regular mapping on G, then the coordinate functions of f are monotone in G. For these facts, see Heinonen-Kilpeläinen-Martio [3], Reshetnyak [17], Serrin [18] and Vuorinen [19], [20].

EXAMPLE 3. Let $f(r)$ be a non-increasing (or non-decreasing) continuous function on $(0, \infty)$. If we define $u(x) = f(|x - \xi|)$ for $x \in R^n_+$ and $\xi \in \partial R^n_+$, then u is monotone in R^n_+.

To evaluate the size of exceptional sets, we use the capacity

$$C_{\alpha,p,\omega}(E; G) = \inf \int f(y)^p \omega(|y_n|) dy,$$

where E is a subset of an open set G in R^n, ω is a non-negative measurable function on $(0, \infty)$ and the infimum is taken over all non-negative measurable functions f such that $f = 0$ outside G and

$$\int |x - y|^{\alpha - n} f(y) dy \geqq 1 \qquad \text{for every } x \in E;$$

see [6] and [14] for the basic properties of capacity. We write $C_{\alpha,p,\omega}(E) = 0$ if

$$C_{\alpha,p,\omega}(E \cap G; G) = 0 \qquad \text{for every bounded open set } G.$$

In case $\omega(r) = r^\beta$, we write $C_{\alpha,p,\beta}(E; G)$ for $C_{\alpha,p,\omega}(E; G)$; if $\beta = 0$, then we simply write $C_{\alpha,p}(E; G)$ for $C_{\alpha,p,\beta}(E; G)$.

For $\gamma > 1$ and $\xi \in \partial R^n_+$, consider the set

$$T_\gamma(\xi) = \{x = (x_1, \ldots, x_n) \in R^n_+ : |x - \xi|^\gamma < x_n\},$$

which is tangential at ξ. If $\lim_{x \to \xi, x \in T_\gamma(\xi)} u(x) = \ell$ for every $\gamma > 1$, then u is said to have a T_∞-limit ℓ at ξ (cf. [11]).

Our main aim in this paper is to establish the following result.

THEOREM 1. *If u is a monotone function on R^n_+ satisfying* (1) *for $p > n-1$, then u has a finite T_∞-limit at every boundary point except that in a set $E \subset \partial R^n_+$ such that $C_{n/p,p}(E) = 0$.*

The case $p = n$ was treated in [15, Theorem 1]. For the non-tangential case, we refer to the result by Manfredi and Villamor [5]. For harmonic functions, see [1], [9], [12]; for weak solutions of the p-Laplacian, see [10].

For a proof of Theorem 1, we need the fact that if u is monotone on $B(x, 2r)$, then

$$|u(x) - u(y)|^p \leqq Mr^{p-n} \int_{B(x,2r)} |\nabla u(z)|^p dz \qquad \text{whenever } y \in B(x, r),$$

where $B(x, r)$ denotes the open ball centered at x with radius r. This estimate is obtained by an application of Sobolev's inequality over the spherical surface. For this purpose, we need the restriction $p > n-1$; see Manfredi-Villamor [5, Remark after Lemma 4.1], which is an extension of [20, Corollary 16.7, Chap. IV]).

Condition (1) may not assure the existence of T_∞-limit at any given point, which may be assumed to be the origin. In studying the existence of T_∞-limit at the origin, we consider a positive non-increasing function g on the half interval $(0, \infty)$ satisfying the doubling condition

$$M^{-1}g(t) \leqq g(2t) \leqq Mg(t) \qquad \text{for } t > 0$$

with a positive constant M and the condition

$$\int_0^1 g(t)^{-1/(p-1)} t^{-1} dt < \infty. \tag{2}$$

For $\xi \in \partial R^n_+$ and $r > 0$, set

$$B_+(\xi, r) = R^n_+ \cap B(\xi, r) \qquad \text{and} \qquad B_-(\xi, r) = B(\xi, r) \setminus \overline{R^n_+}.$$

THEOREM 2. *Let g be as above. If u is a monotone function on $B_+(0, 1)$ satisfying*

$$\int_{B_+(0,1)} |\nabla u(x)|^p g(|x|) x_n^{p-n} dx < \infty \tag{3}$$

for $p > n-1$, then u has a finite T_∞-limit at the origin.

We shall also show by an example that condition (2) is necessary for u to have a finite T_∞-limit at 0 (see Remark 3 given later).

THEOREM 3. *Let g be as above and $p > n-1$. If u is a monotone function on R^n_+ satisfying*

(4) $\displaystyle\int_D |\nabla u(x)|^p g(x_n) x_n^{p-n} dx < \infty$ *for every bounded open set* $D \subset R^n_+$,

then u has a finite limit at every boundary point.

2 Preliminary lemmas

Throughout this paper, let M denote various constants independent of the variables in question.

For a function u on $B_+(0, N)$, $N > 0$, define

$$\bar{u}(x', x_n) = \begin{cases} u(x', x_n), & x \in B_+(0, N), \\ u(x', -x_n), & x \in B_-(0, N). \end{cases}$$

If u is p-precise in $B_+(0, N)$, then $\bar{u}$ is extended to a p-precise function on $B(0, N)$; see Ziemer [21] for the definition of p-precise functions.

LEMMA 1 (cf. [13, Lemma 3]). *Let $p > 1$ and u be a continuous p-precise function on $B_+(0, N)$. Then there exist a constant c depending only on n and a harmonic function v on $B(0, N)$ such that*

$$u(x) = c \sum_{j=1}^{n} \int_{B(0,N)} \frac{x_j - y_j}{|x - y|^n} \frac{\partial \bar{u}}{\partial y_j}(y) dy + v(x) \tag{5}$$

for almost every $x \in B_+(0, N)$; in fact, $c = \omega_n^{-1}$ with ω_n denoting the surface measure of $\partial B(0, 1)$.

PROOF. We first note that the extension $\bar{u}$ is p-precise in $B(0, N)$ as was remarked above. Consider

$$U(x) = \sum_{j=1}^{n} \int_{B(0,N)} \frac{x_j - y_j}{|x - y|^n} \frac{\partial \bar{u}}{\partial y_j}(y) dy.$$

We see that U is locally integrable on R^n. If $\varphi \in C_0^\infty(B(0, N))$, then

$$\begin{aligned} \int U \Delta\varphi dx &= \sum_{j=1}^{n} \int_{B(0,N)} \left(\int \frac{x_j - y_j}{|x - y|^n} \Delta\varphi(x) dx \right) \frac{\partial \bar{u}}{\partial y_j}(y) dy \\ &= -c^{-1} \sum_{j=1}^{n} \int_{B(0,N)} \frac{\partial \varphi}{\partial y_j}(y) \frac{\partial \bar{u}}{\partial y_j}(y) dy \\ &= c^{-1} \int_{B(0,N)} \Delta\varphi(y) \bar{u}(y) dy \end{aligned}$$

for a constant $c \neq 0$ depending only on n. With the aid of Weyl's lemma, we can find a harmonic function v on $B(0, N)$ such that $v = \bar{u} - cU$ a.e. on $B(0, N)$.

COROLLARY 1. *Let u be a continuous locally p-precise function on $B_+(0, N)$ satisfying*

$$\int_{B_+(0,N)} |\nabla u(y)|^p |y_n|^\alpha dy < \infty \tag{6}$$

for $p > 1$ and $\alpha < p - 1$. Then there exists a harmonic function v on $B(0, N)$ such that

$$u(x) = \sum_{j=1}^{n} \int_{B(0,N)} (x_j - y_j)|x - y|^{-n} u_j(y) dy + v(x)$$

for $x \in B_+(0, N) \backslash E'$ with a set E' such that $C_{1,p}(E') = 0$, where $(u_1, \dots, u_n) = c\nabla \bar{u}$.

PROOF. By Hölder's inequality we have

$$\int_{B_+(0,N)} |\nabla u(y)|^q dy < \infty$$

when $1 < q < p$ and $q(1 + \alpha) < p$. Hence $\bar{u}$ is q-precise in $B(0, N)$. By Lemma 1, we can find a harmonic function v on $B(0, N)$ such that (5) holds for almost every $x \in B_+(0, N)$. Since $\int_{B(a,2r)} (x_j - y_j)|x - y|^{-n} u_j(y) dy$ is p-precise in R^n whenever $\overline{B(a, 2r)} \subset B_+(0, N)$ (cf. [8, Lemma 3.3]), we see that

$$\int_{B(0,N)} (x_j - y_j)|x - y|^{-n} u_j(y) dy = \int_{B(a,2r)} (x_j - y_j)|x - y|^{-n} u_j(y) dy + \int_{B(0,N)\backslash B(a,2r)} (x_j - y_j)|x - y|^{-n} u_j(y) dy$$

is p-precise in $B(a, r)$; note here that the second term on the right hand side is infinitely differentiable on $B(a, r)$. Since u is continuous on $B_+(0, N)$, (5) holds for every $x \in B_+(0, N) \backslash E'$ with a set E' such that $C_{1,p}(E') = 0$ (cf. [8, Lemma 2.3]).

LEMMA 2. *Let $E' \subset R^n_+$. If $C_{1,p}(E') = 0$, then $C_{1,p,\omega}(E') = 0$ for any measurable function ω such that $\inf_{r \in [a,b]} \omega(r) > 0$ whenever $0 < a \leqq b < \infty$.*

PROOF. We show that $C_{1,p,\omega}(E' \cap B(a, r); B(a, 2r)) = 0$ whenever $\overline{B(a, 2r)} \subset R^n_+$. In fact, for $A \subset B(a, r)$, we can show that $C_{1,p,\omega}(A; B(a, 2r)) = 0$ if and only if $C_{1,p,\omega}(A) = 0$. By our assumption,

$$C_{1,p}(E' \cap B(a, r); B(a, 2r)) = 0, \tag{7}$$

so that we can find a non-negative function $f \in L^p(B(a, 2r))$ such that $\int |x - y|^{1-n} f(y) dy = \infty$ for every $x \in E' \cap B(a, r)$ (cf. [8, Theorem 3.2]). Since $\inf_{y \in B(a,2r)} \omega(y_n) > 0$,

$$\int |x-y|^{1-n} f(y)\omega(y_n)dy = \infty$$

for every $x \in E' \cap B(a, r)$, which implies

$$C_{1,p,\omega}(E' \cap B(a, r); B(a, 2r)) = 0.$$

Now the required conclusion follows.

For a positive measurable function ω on the interval $(0, \infty)$, define

$$h_\omega(r) = \left(\int_r^1 [t^{n-p}\omega(t)]^{-1/(p-1)} t^{-1} dt\right)^{1-p}$$

for $0 \leqq r \leqq 2^{-1}$; set $h_\omega(r) = h_\omega(2^{-1})$ for $r > 2^{-1}$.

LEMMA 3. *Let $\omega(r) = g(r)r^{p-n}$ for a non-increasing function g on $(0, \infty)$ such that*

$$1 \leqq g(r) \leqq Mg(2r) \qquad \textit{for all } r > 0.$$

If $x \in B_+(\xi, 1)$, then

$$\left(\int_{B(\xi, 2|x-\xi|)\setminus B(x, x_n/2)} |x-y|^{p'(1-n)} \omega(|y_n|)^{-p'/p} dy\right)^{1/p'} \leqq M[h_\omega(x_n)]^{-1/p},$$

where $1/p + 1/p' = 1$.

PROOF. For $x = (x_1, \ldots, x_{n-1}, x_n)$, write $x' = (x_1, \ldots, x_{n-1}, 0)$. Then we have

$$\int_{B(\xi, 2|x-\xi|)\setminus B(x, x_n/2)} |x-y|^{p'(1-n)} \omega(|y_n|)^{-p'/p} dy$$

$$\leqq \int_{B(x', 3|x-\xi|)\setminus(B(x, x_n/2)\cup B(x', x_n/2))} |x-y|^{p'(1-n)} \omega(|y_n|)^{-p'/p} dy$$

$$+ (x_n/2)^{p'(1-n)} \int_{B(x', x_n/2)} \omega(|y_n|)^{-p'/p} dy = I + J.$$

Note that $|x'-y| \leqq x_n + |x-y| \leqq 3|x-y|$ for $y \in B(x', 3|x-\xi|)\setminus B(x, x_n/2)$. Since $g(|y_n|) \geqq g(|x'-y|)$ and $-p'(p-n)/p > -1$, we have for $x \in B_+(\xi, 1)$

$$I \leqq M \int_{B(x', 3|x-\xi|)\setminus B(x', x_n/2)} |x'-y|^{p'(1-n)} \omega(|y_n|)^{-p'/p} dy$$

$$\leqq M \int_{x_n/2}^{3|x-\xi|} g(t)^{-1/(p-1)} t^{-1} dt \leqq M h_\omega(x_n)^{1/(1-p)}.$$

On the other hand, since $g(r)^{-1}$ is non-decreasing, we have

$$J \leqq (x_n/2)^{p'(1-n)} g(1)^{-p'/p} \int_{B(x', x_n/2)} |y_n|^{-p'(p-n)/p} dy \leqq M.$$

Thus Lemma 3 is established.

Let h be a non-decreasing positive function on the interval $(0, \infty)$ satisfying the doubling condition. We use H_h to denote the Hausdorff measure with the measure function h. For a measurable function f, set

$$A_f = \left\{ \xi \in \partial R^n_+ : \int_{B(\xi, 1)} |\xi - y|^{1-n} |f(y)| dy = \infty \right\}$$

and

$$A_{h,f} = \left\{ \xi \in \partial R^n_+ : \limsup_{r \to 0} [h(r)]^{-1} \int_{B_+(\xi, r)} |f(y)|^p \omega(y_n) dy > 0 \right\}.$$

The following is easily shown:

Lemma 4. *Let f be a non-negative function on R^n_+ satisfying*

(8) $$\int_G f(y)^p \omega(|y_n|) dy < \infty \qquad \text{for any bounded open set } G \subset R^n.$$

Then

$$C_{1,p,\omega}(A_f) = 0 \qquad \text{and} \qquad H_h(A_{h,f}) = 0.$$

In view of [14, Lemma 12.4], we can show the following (see also [6], [7]).

Corollary 2. *If f satisfies* (8) *with $\omega(r) = r^{p-n}$ for $p > n - 1$, then*

$$C_{n/p,p}(A_f \cup A_{h,f}) = 0,$$

where $h(r) = h_\omega(r)$ $(= [\log(1/r)]^{1-p}$ for $0 < r < 2^{-1})$.

Lemma 5 (cf. [11, Theorem 2′ and Remark 1]). *Let $\omega(r) = g(r) r^{p-n}$ be as in Lemma 3. For a positive non-decreasing function h on $(0, \infty)$ satisfying the doubling condition and $a > 0$, define*

$$T_h(\xi, a) = \{x \in R^n_+ : h(|x - \xi|) < a h_\omega(x_n)\}.$$

Let f be a non-negative measurable function on R^n_+ satisfying (8) *and vanishing outside a bounded set. For each positive integer j, $1 \leqq j \leqq n$, set*

$$U(x) = \int \frac{x_j - y_j}{|x - y|^n} f(y) dy.$$

If $\xi \in \partial R_+^n \setminus (A_f \cup A_{h,f})$, *then there exists a set* $E(\xi) \subset R_+^n$ *such that*

(i) $$\lim_{x \to \xi, x \in T_h(\xi, a) \setminus E(\xi)} U(x) \text{ exists and is finite for any } a > 1;$$

(ii) $$\lim_{k \to \infty} [h(2^{-k})]^{-1} C_{1,p,\omega}(E_k(\xi); B_k(\xi)) = 0,$$

where $E_k(\xi) = \{x \in E(\xi); 2^{-k} \leqq |\xi - x| < 2^{-k+1}\}$ *and* $B_k(\xi) = \{x \in R^n; 2^{-k-1} < |\xi - x| < 2^{-k+2}\}$.

PROOF. For $x \in R_+^n$, write

$$U_1(x) = \int_{R^n \setminus B(\xi, 2|x-\xi|)} (x_j - y_j)|x - y|^{-n} f(y) dy,$$

$$U_2(x) = \int_{B(\xi, 2|x-\xi|) \setminus B(x, x_n/2)} (x_j - y_j)|x - y|^{-n} f(y) dy,$$

$$U_3(x) = \int_{B(x, x_n/2)} (x_j - y_j)|x - y|^{-n} f(y) dy.$$

If $y \in R^n \setminus B(\xi, 2|x - \xi|)$, then we have $|x - y| \geqq 2^{-1}|y - \xi|$, so that

$$|(x_j - y_j)|x - y|^{-n} f(y)| \leqq |x - y|^{1-n} f(y) \leqq 2^{n-1}|y - \xi|^{1-n} f(y).$$

Since $\xi \notin A_f$, Lebesgue's dominated convergence theorem implies that

$$\lim_{x \to \xi, x \in R_+^n} U_1(x) = U(\xi).$$

Next, if $x \in B_+(\xi, 1/4)$, then we have Hölder's inequality and Lemma 3

$$\begin{aligned} |U_2(x)| &\leqq \int_{B(\xi, 2|x-\xi|) \setminus B(x, x_n/2)} |x - y|^{1-n} f(y) dy \\ &\leqq \left(\int_{B(\xi, 2|x-\xi|)} f(y)^p \omega(|y_n|) dy \right)^{1/p} \\ &\quad \times \left(\int_{B(\xi, 2|x-\xi|) \setminus B(x, x_n/2)} |x - y|^{p'(1-n)} \omega(|y_n|)^{-p'/p} dy \right)^{1/p'} \\ &\leqq M \left([h_\omega(x_n)]^{-1} \int_{B(\xi, 2|x-\xi|)} f(y)^p \omega(|y_n|) dy \right)^{1/p}. \end{aligned}$$

Since $\xi \notin A_{h,f}$ and $[h_\omega(x_n)]^{-1} \leqq a[h(|x - \xi|)]^{-1}$ for $x \in T_h(\xi, a)$, we see that

$$\lim_{x \to \xi, x \in T_h(\xi, a)} U_2(x) = 0.$$

Finally, let $\{b_k\}$ be a sequence of positive numbers such that $b_k \to \infty$. We set $E_k = \{x; 2^{-k} \leqq |x - \xi| < 2^{-k+1}, |U_3(x)| \geqq b_k^{-1}\}$ and $E(\xi) = \bigcup_k E_k$. For $x \in$

E_k, we have

$$b_k^{-1} \leqq |U_3(x)| \leqq \int_{B(x, x_n/2)} |x-y|^{1-n} f(y) dy \leqq \int_{B_k} |x-y|^{1-n} f(y) dy,$$

where we set $B_k = B_k(\xi)$ for simplicity. By the definition of $C_{1,p,\omega}$-capacity, we have

$$C_{1,p,\omega}(E_k; B_k) \leqq b_k^p \int_{B_k} f(y)^p \omega(|y_n|) dy.$$

Here, since $\xi \notin A_{h,f}$, we can choose the sequence $\{b_k\}$ in such a way that

$$[h(2^{-k})]^{-1} b_k^p \int_{B_k} f(y)^p \omega(|y_n|) dy \to 0 \qquad \text{as} \qquad k \to \infty.$$

With this choice of $\{b_k\}$, condition (ii) of the lemma is satisfied, while

$$\limsup_{x \to \xi, x \in R_+^n \setminus E(\xi)} |U_3(x)| \leqq \limsup_{k \to \infty} b_k^{-1} = 0.$$

Thus Lemma 5 is established.

LEMMA 6. *There exists a positive constant M such that*

$$\int_{B(x,r)} |z-y|^{1-n} dy \leqq M r^n (r + |x-z|)^{1-n}$$

for any x, $z \in R^n$ and $r > 0$.

PROOF. First note that

$$\int_{B(x,r)} |z-y|^{1-n} dy \leqq \int_{B(x,r)} |x-y|^{1-n} dy = Mr$$

for all z. Hence the required inequality holds if $|x-z| \leqq 2r$. If $|x-z| > 2r$, then $|z-y| \geqq |z-x| - |x-y| \geqq 2^{-1}|z-x|$ for $y \in B(x,r)$, so that

$$\int_{B(x,r)} |z-y|^{1-n} dy \leqq 2^{n-1} |x-z|^{1-n} \int_{B(x,r)} dy \leqq M |x-z|^{1-n} r^n.$$

Since $r + |x-z| \leqq (3/2)|x-z|$, we obtain the required inequality in case $|x-z| > 2r$.

LEMMA 7 (cf. [14, Lemma 7.3]). *Let $\omega(r) = g(r) r^{p-n}$ be as in Lemma 3. Then there exists $M > 0$ such that*

$$C_{1,p,\omega}(B(x, x_n/4); B(\xi, 1)) \geqq M h_\omega(x_n)$$

whenever $x \in B_+(\xi, 2^{-1})$.

PROOF. Let f be a non-negative measurable function such that $f=0$ outside $B(\xi, 1)$ and

$$\int |z-y|^{1-n} f(y)dy \geqq 1 \qquad \text{for every } z \in B(x, x_n/4).$$

For $x=(x_1, \ldots, x_{n-1}, x_n)$, where $x'=(x_1, \ldots, x_{n-1}, 0)$. Since $x_n+|x'-y| \leqq 2(x_n+|x-y|)$, we have by Fubini's theorem, Lemma 6 and Hölder's inequality

$$\begin{aligned}
\int_{B(x,x_n/4)} dz &\leqq \int_{B(x,x_n/4)} \left(\int |z-y|^{1-n} f(y)dy \right) dz \\
&= \int f(y)dy \int_{B(x,x_n/4)} |z-y|^{1-n} dz \\
&\leqq M x_n^n \int_{B(\xi,1)} f(y)\{x_n+|x'-y|\}^{1-n} dy \\
&\leqq M x_n^n \left(\int_{B(\xi,1)} f(y)^p \omega(|y_n|) dy \right)^{1/p} \\
&\quad \times \left(\int_{B(\xi,1)} [(x_n+|x'-y|)^{1-n} \omega(|y_n|)^{-1/p}]^{p'} dy \right)^{1/p'}.
\end{aligned}$$

Here note that

$$\begin{aligned}
&\int_{B(\xi,1)} [(x_n+|x'-y'|)^{1-n} \omega(|y_n|)^{-1/p}]^{p'} dy \\
&\qquad \leqq \int_{B(x',2)} [(x_n+|x'-y|)^{1-n} \omega(|y_n|)^{-1/p}]^{p'} dy \\
&\qquad \leqq M \int_0^2 (x_n+r)^{p'(1-n)} g(r)^{-p'/p} r^{-p'(p-n)/p} r^{n-1} dr \\
&\qquad \leqq M \left(\int_{x_n}^2 r^{p'(1-n)} g(r)^{-1/(p-1)} r^{p'(n-1)-1} dr \right. \\
&\qquad\qquad \left. + x_n^{p'(1-n)} g(1)^{-1/(p-1)} \int_0^{x_n} r^{p'(n-1)-1} dr \right) \\
&\leqq M \left(\int_{x_n}^1 g(r)^{-1/(p-1)} r^{-1} dr + 1 \right) \leqq M h_\omega(x_n)^{1/(1-p)}.
\end{aligned}$$

Thus we have

$$Mh_\omega(x_n) \leqq \int_{B(\xi,1)} f(y)^p \omega(|y_n|)dy,$$

which yields the required inequality.

3 Proof of Theorem 1

Let $\omega(r) = r^{p-n}$. Then $h_\omega(r) = [\log(1/r)]^{1-p}$ for $0 < r < 2^{-1}$. In view of (1), we see that $f = |\nabla \bar{u}|$ satisfies (8). Consider A_f and $A_{h,f}$ with $h(r) = h_\omega(r)$. In what follows we show that u has a finite T_∞-limit at every $\xi \in \partial R^n_+ \setminus (A_f \cup A_{h,f})$.

For $N > 0$, in view of Corollary 1, u is of the form

$$u(x) = \sum_{j=1}^{n} \int_{B(0,N)} (x_j - y_j)|x - y|^{-n} u_j(y)dy + v_N(x)$$

for $x \in B_+(0, N)\setminus E'$, where $C_{1,p}(E') = 0$ and v_N is a harmonic function on $B(0, N)$. Note that

$$T_\gamma(\xi) \subset T_h(\xi, \gamma^{p-1}) \qquad \text{whenever } \gamma > 1.$$

Further Lemma 2 implies that $C_{1,p,\omega}(E') = 0$. By Lemma 5, for $\xi \in (B(0, N) \cap \partial R^n_+)\setminus(A_f \cup A_{h,f})$, there exists a set $E(\xi) \subset R^n_+$ such that

$$\lim_{x \to \xi, x \in T_h(\xi,a)\setminus E(\xi)} u(x) \text{ exists and is finite for any } a > 1$$

and

$$\lim_{j \to \infty} j^{p-1} C_{1,p,p-n}(E(\xi) \cap B_j(\xi); B(\xi, 1)) = 0. \tag{9}$$

If $x \in T_\gamma(\xi)$ and $2^{-j} \leqq |x - \xi| < 2^{-j+2}$, then $B(x, x_n/2) \subset B_j(\xi)$. Since $2^{-j\gamma} \leqq |x - \xi|^\gamma < x_n$ for $x \in T_\gamma(\xi)$, Lemma 7 gives

$$j^{p-1} C_{1,p,p-n}(B(x, x_n/4); B(\xi, 1)) \geqq M_1 j^{p-1} [\log(1/x_n)]^{1-p} \geqq M_2$$

for some positive constants M_1 and M_2. Hence it follows from (9) that there is j_0 such that $B(x, x_n/4) \not\subset E(\xi)$, so that there exists $y(x) \in B(x, x_n/4)\setminus E(\xi)$, whenever $x \in T_\gamma(\xi) \cap B(\xi, 2^{-j_0})$. Since u is monotone on R^n_+,

$$|u(x) - u(y)|^p \leqq M x_n^{p-n} \int_{B(x,x_n/2)} |\nabla u(z)|^p dz$$

for $y \in B(x, x_n/4) \subset R^n_+$ (see [5]). Thus we infer that

$$\lim_{x \to \xi, x \in T_\gamma(\xi)} |u(x) - u(y(x))| = 0,$$

from which it follows that

$$\lim_{x\to\xi, x\in T_\gamma(\xi)} u(x) = \lim_{x\to\xi, x\in T_\gamma(\xi)} u(y(x)).$$

Note that there is $a > 0$ such that $y(x) \in T_h(\xi, a)$ if $x \in T_\gamma(\xi)$. Hence the limit on the left exists and is finite. Now, in view of Corollary 2, we see that $E = A_f \cup A_{h,f}$ has all the required properties.

4 Proofs of Theorems 2 and 3

First suppose u satisfies (3) with g satisfying (2). If we set $f = |\nabla \bar{u}|$ as before, then

$$\begin{aligned}
&\int_{B(0,1)} |y|^{1-n} f(y)dy \\
&\quad \leqq \left(\int_{B(0,1)} |y|^{p'(1-n)}[g(|y|)|y_n|^{p-n}]^{-p'/p}dy\right)^{1/p'} \left(\int_{B(0,1)} f(y)^p g(|y|)|y_n|^{p-n}dy\right)^{1/p} \\
&\quad \leqq M\left(\int_0^1 g(r)^{-p'/p}r^{-1}dr\right)^{1/p'} \left(\int_{B(0,1)} f(y)^p g(|y|)|y_n|^{p-n}dy\right)^{1/p} < \infty.
\end{aligned}$$

Now let $\omega(r) = r^{p-n}$. Then $h_\omega(r) = [\log(1/r)]^{1-p}$ for $0 < r < 2^{-1}$ as before. Since

$$M \geqq \int_r^{\sqrt{r}} g(t)^{-p'/p}t^{-1}dt \geqq g(r)^{-p'/p}\log(1/\sqrt{r}),$$

we see that $[h_\omega(r)]^{-1} \leqq Mg(r)$. Hence (3) implies

$$\limsup_{r\to 0}\, [h_\omega(r)]^{-1} \int_{B_+(0,r)} f(y)^p y_n^{p-n}dy \leqq \limsup_{r\to 0} \int_{B_+(0,r)} f(y)^p g(|y|)y_n^{p-n}dy = 0.$$

Thus $0 \notin A_f \cup A_{h_\omega,f}$, and the proof of Theorem 1 implies that u has a finite T_∞-limit at the origin.

Next suppose u satisfies (4). Since $g(|\xi - x|) \leqq g(x_n)$ for $\xi \in \partial R^n_+$ and $x \in R^n_+$, the above considerations show that

$$\int_{B(\xi,1)} |\xi - y|^{1-n} f(y)dy < \infty.$$

In the present case, let $\omega(r) = g(r)r^{p-n}$. Then $h_\omega(0) > 0$ by (2). Hence

$$\lim_{r\to 0}\, [h_\omega(r)]^{-1} \int_{B_+(\xi,r)} f(y)^p g(y_n)y_n^{p-n}dy = 0$$

for every $\xi \in \partial R^n_+$, so that

$$A_f = A_{h_\omega, f} = \varnothing.$$

For $\xi \in \partial R^n_+$ and $h = h_\omega$, consider $E(\xi)$ as in Lemma 5; here note that $T_h(\xi, a) = R^n_+$ for large a. With the aid of Lemma 7, we infer that

$$B(x, x_n/4) \not\subset E(\xi) \qquad \text{whenever } x \in B_+(\xi, 2^{-j_0}) \text{ for some } j_0.$$

Thus, as in the proof of Theorem 1, we see that u has a finite limit at ξ.

5 Remarks

Now we give some remarks on our theorems.

REMARK 1. In this paper, we have assumed that $p > n - 1$. In this connection, we remark the following: if u is harmonic in $B_+(0, N)$ and satisfies

$$\int_{B_+(0,N)} |\nabla u(x)|^p x_n^{p-n} dx < \infty \tag{10}$$

for $1 \leqq p \leqq n - 1$, then u is constant.

For this, we first show that the extension $\bar{u}$ is harmonic in $B(0, N)$. If $\varphi \in C_0^\infty(B(0, N))$ and $\varepsilon > 0$, then we have by Green's formula

$$\begin{aligned}\int_{\{x:|x_n|>\varepsilon\}} \bar{u} \Delta \varphi dx &= \int u(x', \varepsilon)\left\{-\frac{\partial \varphi}{\partial x_n}(x', \varepsilon) + \frac{\partial \varphi}{\partial x_n}(x', -\varepsilon)\right\} dx' \\ &\quad + \int \frac{\partial u}{\partial x_n}(x', \varepsilon)\{\varphi(x', \varepsilon) + \varphi(x', -\varepsilon)\} dx' = I_1 + I_2.\end{aligned}$$

Note that

$$u(x', \varepsilon) = u(x', a) - \int_\varepsilon^a (\partial/\partial x_n) u(x', x_n) dx_n, \qquad (0 < \varepsilon, a < \sqrt{N^2 - |x'|^2})$$

so that

$$|u(x', \varepsilon)| \leqq |u(x', a)| + M\left(\int_\varepsilon^a |(\partial/\partial x_n) u(x', x_n)|^p x_n^{p-n} dx_n\right)^{1/p}.$$

Hence, by (10) we see that $\int_{\{x':|x'|<N'\}} |u(x', \varepsilon)|^p dx'$ is bounded when $0 < \varepsilon < a$ $(0 < a < N - N')$, which in turn implies

$$\lim_{\varepsilon \to 0} I_1 = 0.$$

Since $p - n \leqq -1$, (10) implies

$$\liminf_{\varepsilon \to 0} \int_{\{x' : |x'| < N'\}} |\nabla u(x', \varepsilon)|^p dx' = 0,$$

which gives

$$\liminf_{\varepsilon \to 0} I_2 = 0.$$

Now it follows that

$$\int \bar{u} \Delta \varphi dx = 0$$

and thus $\bar{u}$ is harmonic in $B(0, N)$. The above considerations also show that

$$\int_{\{x' : |x'| < N\}} |\nabla \bar{u}(x', 0)|^p dx' = 0.$$

Thus $\bar{u}$ is constant on $B(0, N) \cap \partial R^n_+$, say, $\bar{u} = C$ on $B(0, N) \cap \partial R^n_+$. This implies that the function

$$u^*(x) = \begin{cases} u(x', x_n) & \text{if } x \in B_+(0, N), \\ 2C - u(x', -x_n) & \text{if } x \in B_-(0, N), \end{cases}$$

is also harmonic in $B(0, N)$. Thus

$$u(x', -x_n) = 2C - u(x', -x_n)$$

and hence $u = C$ on $B_+(0, N)$.

REMARK 2. Let $p > n - 1$. If $E \subset \partial R^n_+$ and $C_{n/p, p}(E) = 0$, then we can find a harmonic function u satisfying (1) such that

$$\lim_{x \to \xi, x \in R^n_+} u(x) = \infty \qquad \text{for every } \xi \in E$$

(see [9, Theorem 2] and [12, Remark 3]).

REMARK 3. In Theorem 2, if g does not satisfy (2), then there exists a monotone function u which satisfies (3) but fails to have a finite T_∞-limit at the origin.

In fact, letting

$$G(r) = \int_r^2 g(t)^{-1/(p-1)} t^{-1} dt,$$

we consider

$$u(x) = \log [G(|x|)/G(1)]$$

for $x \in B(0, 1)$; set $u = 0$ outside $B(0, 1)$. Then u is monotone on R^n_+, as was pointed out in Example 3. Since $|\nabla u(x)| = -G'(|x|)/G(|x|)$ for $x \in B_+(0, 1)$, we have

$$\int_{R^n_+} |\nabla u(x)|^p g(|x|) x_n^{p-n} dx = M \int_0^1 [G(r)^{-1} g(r)^{-1/(p-1)} r^{-1}]^p g(r) r^{p-n} r^{n-1} dr$$
$$= M \int_0^1 G(r)^{-p} [-G'(r)] dr < \infty,$$

but

$$\lim_{x \to 0} u(x) = \infty.$$

REMARK 4. For any g considered in Theorem 2, we can find a monotone function u which satisfies (3) but fails to have a finite limit at the origin.

For this purpose, we modify the function in Remark 3 as follows: let $\mathbf{e}_j = (2^{-j}, 0, \dots, 0)$ and consider

$$u_j(x) = \begin{cases} \log \dfrac{\log (1/|x - \mathbf{e}_j|)}{\log (1/r_j)} & \text{on } B_+(\mathbf{e}_j, r_j), \\ 0 & \text{elsewhere.} \end{cases}$$

Set

$$u(x) = \sum_{j=1}^{\infty} u_j(x),$$

where $\{r_j\}$ is a sequence of positive numbers satisfying $r_j < 2^{-j-2}$ and

$$\sum_{j=1}^{\infty} g(2^{-j}) [\log (1/r_j)]^{1-p} < \infty.$$

Since $\{B_+(\mathbf{e}_j, r_j)\}$ is disjoint, we see that u is monotone on R^n_+. Moreover,

$$\limsup_{x \to 0, x \in R^n_+} u(x) = \infty$$

and

$$\int_{R^n_+} |\nabla u(x)|^p g(|x|) x_n^{p-n} dx \leqq M \sum_{j=1}^{\infty} g(2^{-j}) \int |\nabla u_j(x)|^p x_n^{p-n} dx$$
$$= M \sum_{j=1}^{\infty} g(2^{-j}) \int_0^{r_j} [\log (1/t)]^{-p} t^{-1} dt$$
$$= M \sum_{j=1}^{\infty} g(2^{-j}) [\log (1/r_j)]^{1-p} < \infty.$$

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