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A chiral model related to the Einstein equation

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ABSTRACT. We construct some new rational solutions of the stationary axisymmetric Einstein equation.

0. Introduction

Our main objective in this paper is to construct a family of solutions of a field equation for $\sigma \in \mathrm{gl}(2, C[[t^{-1}, t, z]])$:

$$d * (td\sigma \cdot \sigma^{-1}) = 0, \tag{0.0}$$

where $*$ denotes the Hodge operator with a Lorentz metric $(dt)^2 - (dz)^2$ (i.e. $*dt = dz$, $*dz = dt$). This chiral model is the main part of the Einstein equation for a cyclindrical wave ansatz. Moreover, the equation of motion for the Ernst potential is written in a matrix form above. So the chiral model (0.0) is important in construction of exact vacuum gravitational fields, and much progress has been made on the inverse scattering method and universal Grassmann manifold approach [2], [3], [4], [5], [6].

Here, we seek solutions of (0.0) by a dressing method. Taking account of $d * (td \log t^s) = 0$, we consider an ansatz $\sigma = \tau \cdot \deg(t^{s_1}, t^{s_2})$ with $\tau \in GL(2, C[[t, z]])$ and $s_1, s_2 \in Z$. If σ satisfies (0.0) and c is a constant matrix, then $\sigma \cdot t^s$ and $c^{-1} \cdot \sigma \cdot c$ also satisfy (0.0). Hence we may assume that $s_1 \geq 0$ and $s_2 = 0$, without loss of generality. We are mainly concerned with this ansatz, and we investigate its solutions in a group-theoretic viewpoint.

Let $A = C[[t, z]]$. For $a \in A$, we set $\mathrm{ord}\, a = \sup\{k \in Z; a \in (At + Az)^k\}$. Let $\mathscr{A}$ denote an algebra $\{a = \sum a_n \lambda^n \in A[[\lambda, \lambda^{-1}]]; \mathrm{ord}\, a_n + n \geq 0\}$. If $\psi = \sum \psi_n \lambda^n \in \mathrm{gl}(2, \mathscr{A})$ and $\psi_0 \in GL(2, A)$, then ψ has a unique decomposition $\psi = \psi^- \cdot \psi^+$ with $\psi^- = 1 + \sum_{k<0} \psi_k^- \lambda^k$ and $\psi^+ = \sum_{k \geq 0} \psi_k^+ \lambda^k$ ([10]). We refer to this as the Birkhoff decomposition. Then we can construct a solution of (0.0) as follows.

THEOREM 0.0. *Let $s \in Z_+$, $\phi \in GL(2, C[[x]])$ and assume that $\phi_{12}, \phi_{21} \in$*

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$C[[x]]x^s$. *We set*

$$\psi = \begin{bmatrix} \phi_{11}(\xi) & \lambda^s \phi_{12}(\xi) \\ \lambda^{-s}\phi_{21}(\xi) & \phi_{22}(\xi) \end{bmatrix},$$

with $\xi = \lambda + 2z + t^2/\lambda$. *Let* $\psi = X_-^{-1} X_+$ *be the Birkhoff decomposition and set* $\tau = X_+(t, z, 0)$. *Then* $\sigma = \tau \cdot \mathrm{diag}(t^{2s}, 1)$ *is a solution of* (0.0).

If the entries of ϕ are polynomials, then we get easily X_+ by solving finite-dimensional linear algebraic equations over a field of rational functions $C(t, z)$ (see § 2). Consequently, we see that the entires of σ are rational functions of t and z.

In § 1, we give a proof of the theorem above and a characterization of our solutions. A main observation in our approach is to find their behavior at $t = 0$. In § 2, we construct some exact solutions of the Einstein vacuum field equations.

1. Ansatz

To begin with, we derive a field equation of our ansatz for (0.0). Let $\tau \in GL(2, C[[t, z]])$ and set $\sigma = \tau \cdot h$ with $h = \mathrm{diag}(t^s, 1)$. Then (0.0) is rewritten as:

$$\partial^i(t\partial_i \tau \cdot \tau^{-1}) + \partial_t(\tau S \tau^{-1}) = 0, \tag{1.0}$$

where $\partial^1 = \partial_1 = \partial_t$, $-\partial^2 = \partial_2 = \partial_z$, and $S = \mathrm{diag}(s, 0)$. Let ϑ denote $t\partial_t$. Then the equation above is equivalent to

$$\vartheta^2 \tau - \vartheta\tau \cdot \tau^{-1}\vartheta\tau - t^2(\partial_z^2 \tau - \partial_z \tau \cdot \tau^{-1}\partial_z \tau) + \vartheta\tau S - \tau S \tau^{-1}\vartheta\tau = 0. \tag{1.1}$$

For $\tau \in GL(2, C[[t, z]])$, we set $\tau = \sum_{k \geq 0} \tau_k t^k$ with $\tau_k \in \mathfrak{gl}(2, C[[z]])$. If τ satisfies (1.1), we have

$$k^2 \tau_k + k\tau_k S - \tau_0 S \tau_0^{-1} k \tau_k + \langle \tau_i; i < k \rangle = 0,$$

where $\langle \tau_i; i < k \rangle \in \mathfrak{gl}(2, C[[z]])$ denotes an element which depends only on $\{\tau_i; i < k\}$. Putting

$$\begin{bmatrix} a_k & b_k \\ c_k & d_k \end{bmatrix} = \tau_0^{-1}\tau_k,$$

we see that

$$\begin{bmatrix} k^2 a_k & (k^2 - ks)b_k \\ (k^2 + ks)c_k & k^2 d_k \end{bmatrix} + \langle \tau_i; i < k \rangle = 0. \tag{1.2}$$

Therefore if s is not an integer, we see that all τ_k $(k > 0)$ are determined by τ_0. But since s is now a non-negative integer, the equation (1.2) becomes a constraint for τ_0 if $k = s > 0$. To avoid this difficulty, we introduce a special class of solutions. Let $\mathscr{B}$ denote the subalgebra of $\mathrm{gl}(2, C[[z]])$ consisting of elements whose (1, 2)-components are zero. Then it is easy to see that $\tau_i \in \mathscr{B}$ for $i < s$ if $\tau_0 \in \mathscr{B}$. Hence we have

LEMMA 1.1. *Let $s \in Z_+$ and $\tau_0 \in GL(2, C[[z]])$. We assume that* (1) $s > 0$, $\tau_0 \in \mathscr{B}$, *or* (2) $s = 0$. *Then τ_0 and the* (1, 2)-*component of τ_s uniquely determine a solution $\tau \in GL(2, C[[t, z]])$ of* (1.1).

This simple fact plays an important role in a characterization of our solutions.

PROOF OF THEOREM 0.0. Let $W_- = X_-H_-$ and $W_+ = X_+H_+$ with $H_- = \mathrm{diag}((1 + 2z/\lambda + t^2/\lambda^2)^{-s}, 1)$ and $H_+ = \mathrm{diag}((\lambda^2 + 2z\lambda + t^2)^s, 1)$. Then

$$w = W_-^{-1}W_+ = H_-^{-1}\psi H_+ = \begin{bmatrix} \xi^{2s}\phi_{11}(\xi) & \xi^s\phi_{12}(\xi) \\ \xi^s\phi_{21}(\xi) & \phi_{22}(\xi) \end{bmatrix}.$$

We note that $D_i\xi = 0$ $(i = 1, 2)$ for $D_1 = t\partial_t - \lambda\partial_z + 2\lambda\partial_\lambda$, $D_2 = t\partial_z - \lambda\partial_t$, and $\xi = \lambda + 2z + t^2/\lambda$. Hence $D_iw = 0$ $(i = 1, 2)$. Since $D_iW_+ = D_iW_- \cdot w$, we see that

$$D_iX_+ \cdot H_+ + X_+D_iH_+ = (D_iX_- \cdot H_- + X_-D_iH_-)H_-^{-1}X_-^{-1}X_+H_+,$$

$$D_iX_+ + X_+S_i = (D_iX_- + X_-S_i)X_-^{-1}X_+,$$

where $S_1 = D_1H_\pm \cdot H_\pm^{-1} = \mathrm{diag}(2s, 1)$ and $S_2 = D_2H_\pm \cdot H_\pm^{-1} = 0$. Hence

$$D_iX_+ \cdot X_+^{-1} + X_+S_iX_+^{-1} = D_iX_- \cdot X_-^{-1} + X_-S_iX_-^{-1}.$$

Therefore the both side terms of the equality above are independent of λ. Comparing the coefficients of λ^0, we have

$$t\partial_t\tau \cdot \tau^{-1} + \tau S_1\tau^{-1} = -\partial_zX_{-1} + S_1,$$

$$t\partial_z\tau \cdot \tau^{-1} = -\partial_tX_{-1},$$

where $\tau = X_+(t, z, 0)$. This implies that τ satisfies (1.0). Hence $\sigma = \tau \cdot \mathrm{diag}(t^{2s}, 1)$ is a solution of the chiral model (0.0).

In the rest of this section, we investigate τ_0 and the (1, 2)-component of τ_{2s} for the solution τ constructed in Theorem 0.0. We note that $GL(2, C[[x]]) = SL(2, C[[x]]) \cdot GL(1, C[[x]])$ and $\det\psi = \det\phi(\xi)$. Also the Birkhoff decomposition of an element f of $GL(1, \mathscr{A})$ is reduced to the Laurent decomposition of $\log f$. So it is enough to consider the case: $\phi \in SL(2, C[[x]])$.

Let $\psi = X_-^{-1}X_+$ be the Birkhoff decomposition. We set $N = X_-^{-1}$ and

$P = X_+^{-1}$. First we examine P_{12} and P_{22}. Because $\psi_{12} = \sum \partial_x^k \phi_{12}(2z + \lambda) \cdot t^{2k}\lambda^{s-k}/k!$, we see that ψ_{12} is holomorphic in $\lambda \bmod t^{2s+1}$. For an element $a = \sum a_n \lambda^n \in \sum C[[t, z]]\lambda^n$, we set $a_+ = \sum_{n\geq 0} a_n \lambda^n$. Then $\psi P = N$ implies that

$$(\psi_{11}P_{12})_+ + \psi_{12}P_{22} = 0 \qquad \bmod t^{2s+1}.$$

We expand $\psi_{11} = \sum \psi_{11,k}t^{2k}$, $P_{12} = \sum P_{12,k}t^{2k}$ and $\psi_{12}P_{22} = \sum \partial_x^k \phi(2z + \lambda) \cdot t^{2k}\lambda^{s-k}/k! \cdot P_{22} = \sum b_k t^{2k}$. Then $\psi_{11,k} = \partial_x^k \phi_{11}(2z + \lambda)\lambda^{-k}/k!$ and $b_k \in \lambda^{s-k}C[[z, \lambda]]$. Also we have

$$\sum_{0\leq j\leq k} \psi_{11,j}P_{12,k-j} + b_k = 0 \qquad (k = 0, \ldots, s).$$

Accordingly, we see that $P_{12,k} \in \lambda^{s-k}C[[z, \lambda]]$ by induction. Therefore, setting $c_k = \lim_{\lambda\to 0} b_k \lambda^{k-s} = \partial_x^k \phi_{12}(2z)/k! \cdot P_{22}(0, z, 0)$ and $p_k = \lim_{\lambda\to 0} P_{12,k}\lambda^{k-s}$, we have

$$\sum_{0\leq j\leq k} \partial_x^j \phi_{11}(2z)/j! \cdot p_{k-j} + c_k = 0.$$

From this, we can deduce an explicit expression for p_k.

LEMMA 1.2. $p_k = -(\phi_{11}\partial_x^k(\phi_{12}/\phi_{11})/k!)_{x=2z}$.

PROOF. If we put $t = 0$ and $\lambda = 0$, then we have $\phi_{11}p_0 + \phi_{12}P_{22} = 0$. Also $\psi_{21}P_{12} + \psi_{22}P_{22} = N_{22}$ implies that $\phi_{21}p_0 + \phi_{22}P_{22} = 1$. Since $\det \phi = 1$, we see that $p_0 = -\phi_{12}$ and $P_{22} = \phi_{11}$. Hence

$$\sum_{0\leq j\leq k} \partial_x^j \phi_{11} \cdot p_{k-j}/j! + \partial_x^k \phi_{12} \cdot \phi_{11}/k! = 0.$$

Therefore we have inductively

$$\phi_{11}p_k - \sum_{1\leq j\leq k} \partial_x^j \phi_{11} \cdot \phi_{11}{}^1 \partial_x^{k-j} v/(k-j)!j! + \partial_x^k \phi_{12}\phi_{11}/k! = 0,$$

where $v = \phi_{12}/\phi_{11}$. Since $\partial_x^k \phi_{12} = \partial_x^k(\phi_{11}v) = \sum_{0\leq j\leq k} \partial_x^j \phi_{11}\partial_x^{k-j} v k!/(k-j)!j!$, we see that

$$\phi_{11}p_k + \phi_{11}^2 \partial_x^k v/k! = 0.$$

Hence we complete the proof by induction.

Next we examine P_{11} and P_{21}. Assume that $s > 0$. Then $\psi_{11}P_{11} + \psi_{12}P_{21} = N_{11}$ implies that if $t = 0$ and $\lambda = 0$, then

$$\psi_{11}P_{11} = 1, \qquad (\psi_{11}P_{11})_k = 0, \quad (0 < k < s),$$

$$(\psi_{11}P_{11})_s + \phi_{12}P_{21} = 0,$$

where $(\cdots)_k$ denotes the coefficient of λ^k. Hence for $k \leq s$,

$$\partial_\lambda^k P_{11} = \partial_\lambda^k(\psi_{11}P_{11}/\psi_{11}) = \partial_\lambda^k(\psi_{11}P_{11})/\psi_{11} + \psi_{11}P_{11}\partial_\lambda^k(1/\psi_{11}).$$

In particular for $k < s$,

$$\partial_\lambda^k P_{11} = \partial_\lambda^k(1/\psi_{11}).$$

Since $\partial_\lambda^s(\psi_{11}P_{11})/s! + \phi_{12}P_{12} = 0$, we have

$$\psi_{11}\partial_\lambda^s P_{11}/s! + A + \phi_{12}P_{21} = 0,$$

where $A = \sum_{j=1}^{s} \partial_\lambda^j \psi_{11}\partial_\lambda^{s-j}(1/\psi_{11})/j!(s-j)! = -\psi_{11}\partial_\lambda^s(1/\psi_{11})/s!$.

Also $\psi_{21}P_{11} + \psi_{22}P_{21} = N_{21}$ implies that $(\phi_{21}P_{11})_s + \psi_{22}P_{21} = 0$. Therefore

$$\phi_{21}\partial_\lambda^s P_{11}/s! + B + \psi_{22}P_{21} = 0,$$

with $B = \sum_{0<j\leq s} \partial_\lambda^j \phi_{21}\partial_\lambda^{s-j}(1/\psi_{11})/j!(s-j)! = \partial_\lambda^s(\phi_{21}/\psi_{11})/s! - \phi_{21}\partial_\lambda^s(1/\psi_{11})/s!$.

Using $\det \phi = 1$, we see that

$$P_{21} = \phi_{21}A - \psi_{11}B = -\psi_{11}\partial_\lambda^s(\phi_{21}/\phi_{11})/s!.$$

In the case $s = 0$, setting $t = \lambda = 0$, we have $\psi_{11}P_{11} + \psi_{12}P_{21} = 1$, $\psi_{12}P_{11} + \psi_{22}P_{21} = 0$. Therefore $P_{21} = -\psi_{21}$ and $P_{11} = \psi_{22}$.

Since $P(t, z, 0) = \tau^{-1}$, we obtain

THEOREM 1.3. *Let $s \in Z_+$ and $\phi \in SL(2, C[[x]])$ with $\phi_{12}, \phi_{21} \in C[[x]]x^s$. Let τ be the solution of* (1.1) *constructed from ϕ by the group-theoretic method. Then*

Case (1) $s > 0$:

$$\tau(0, z) = \begin{bmatrix} \phi_{11} & 0 \\ \phi_{11}\partial_x^s(\phi_{21}/\phi_{11})/s! & 1/\phi_{11} \end{bmatrix}_{x=2x},$$

$$\tau_{12} = t^{2s}\phi_{11}\partial_x^s(\phi_{12}/\phi_{11})/s!|_{x=2z} + o(t^{2s}),$$

Case (2) $s = 0$:

$$\tau(0, z) = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}_{x=2x}$$

COROLLARY 1.4. *If ϕ is symmetric, so is the solution $\sigma = \tau \cdot \operatorname{diag}(t^{2s}, 1)$.*

PROOF. Note that ${}^t\sigma$ is also a solution of (0.0). Let $\chi = {}^t\sigma \operatorname{diag}(t^{-2s}, 1)$. Then $\chi_{12} = \tau_{21}t^{2s}$ and $\chi_{21} = \tau_{12}t^{-2s}$. Hence Lemma 1.1 implies that $\chi = \tau$.

Also we can state a characterization of the solutions obtained by our group-theoretic method.

PROPOSITION 1.5. *Let $s \in Z_+$, $\tau \in GL(2, C[[t, z]])$, and let $\sigma = \tau \cdot \operatorname{diag}(t^{2s}, 1)$ be a solution of the chiral model* (0.0). *If* (1) $s > 0$ *and* $\tau_0 \in \mathscr{B}$, *or if* (2) $s = 0$, *then τ is constructed from a suitable $\phi \in GL(2, C[[x]])$ by the method as in Theorem* 0.0.

2. Applications

Let $\phi \in GL(2, C[x])$ with $\phi_{12}, \phi_{21} \in C[x]x^s$. Define ψ as in Theorem 0.0 and let $\psi = X_-^{-1}X_+$ be the Birkhoff decomposition. We set

$$X_- = 1_2 + \sum_{i<0} X_i\lambda^i$$

and

$$\psi = \sum_{-m\le i\le m} \psi_i\lambda^i.$$

Since the entires of $X_- = X_+\psi^{-1}$ and ψ^{-1} are Laurent polynomials in λ, we see that $X_i = 0$ for $i < -m$. Also $X_+ = X_-\psi = \sum(X_-\psi)_i\lambda^i$ implies that $(X_-\psi)_i = 0$ for $i < 0$. Therefore we have

$$\psi_{-i} + \sum_{1\le j\le m} X_{-j}\psi_{j-i} = 0, \qquad i = 1, 2, \dots, m. \tag{2.0}$$

Hence for a solution X_{-i}, $i = 1, \dots, m$ of the linear algebraic equation (2.0), setting

$$\tau = \psi_0 + \sum_{1\le j\le m} X_{-j}\psi_j,$$

we get a solution $\sigma = \tau\cdot\mathrm{diag}(t^{2s}, 1)$ for the chiral model (0.0).

In the rest of this section, we construct some vacuum gravitational fields. Let $s = 1$ and $g(\rho, z) = \sigma(i\rho, z)$. Then g satisfies

$$d(\rho * dg\cdot g^{-1}) = 0,$$

where $*d\rho = dz$, $*dz = -d\rho$. Hence if we can solve

$$\partial_\rho \log f = -1/\rho + \mathrm{Tr}(U^2 - V^2)/4\rho, \qquad \partial_z \log f = \mathrm{Tr}(UV)/2\rho$$

with $U = \rho\partial_\rho g\cdot g^{-1}$ and $V = \rho\partial_z g\cdot g^{-1}$, we obtain a stationary axially symmetric Einstein field:

$$ds^2 = g_{ab}dx^a dx^b - f(d\rho^2 + dz^2),$$

(cf. [3]).

Example 2.0. Let $u = ax + bx^2 + cx^3$ and $\phi = \begin{bmatrix} 1 & u \\ u & 1+u^2 \end{bmatrix} \in GL(2, C[x])$. Then we have the following:

$$g_{ab} = h_{ab}/f,$$

$$f = 1 + b^2\rho^4 - 2c^2\rho^6 + c^4\rho^{12} + 12bc\rho^4 z + 36c^2\rho^4 z^2,$$

$$h_{11} = -\rho^2 + c^2\rho^8,$$

$$h_{12} = h_{21} = -\rho^2(a - 3c\rho^2 + b^2c\rho^6 - ac^2\rho^6 + 3c^3\rho^8 + 4bz + 8bc^2\rho^6 z + 12cz^2 + 24c^3\rho^6 z^2),$$

$$h_{22} = (-1 - a\rho + 4c\rho^3 - b^2\rho^4 + ac\rho^4 - 2c^2\rho^6 - c^3\rho^9 - 4b\rho z - 8bc\rho^4 z - 12c\rho z^2 - 24c^2\rho^4 z^2)(-1 + a\rho - 4c\rho^3 - b^2\rho^4 + ac\rho^4 - 2c^2\rho^6 + c^3\rho^9 + 4b\rho z - 8bc\rho^4 z + 12c\rho z^2 - 24c^2\rho^4 z^2).$$

Finally we consider the Weyl's static axially symmetric solution:

$$ds^2 = e^\nu dt^2 - \rho^2 e^{-\nu} d\varphi^2 - e^{\gamma-\nu}(d\rho^2 + dz^2).$$

Then the field equations are

(a) $(\rho\partial_\rho)^2\nu + \rho^2\partial_z^2\nu = 0$,

(b) $\gamma_\rho = \rho((\partial_\rho\nu)^2 - (\partial_z\nu)^2)/2$, $\gamma_z = \rho\partial_\rho\nu\cdot\partial_z\nu$.

Here we notice that the equation (a) is the axially symmetric Laplace equation. Then our proof of Theorem 0.0 implies

PROPOSITION 2.1. *For $v \in C[[x]]$, the constant term ν of the Laurent expansion of $v(\lambda + 2z - \rho^2/\lambda)$ is a solution of* (a).

Also setting $\bar\partial = (\partial_\rho + i\partial_z)/2$, we can rewrite (b) as

$$\bar\partial\gamma = \rho(\bar\partial\nu)^2.$$

For $\phi(x) = x + 2m$, we set $\psi = \phi(\lambda + 2z + t^2/\lambda)$. Let $\psi = X_-^{-1}X_+$ be the Birkhoff decomposition. Since $X_-^{-1} = \psi X_+^{-1}$, we have $\lambda X_-^{-1} \in C[[t, z, \lambda]]$. So we can set $X_-^{-1} = 1 - a/\lambda$ with $a \in [[t, z]]$. Then $X_-\psi \in C[[t, z, \lambda]]$ implies that $a^2 + 2(z + m) + t^2 = 0$. Hence $a = -(z + m) + \sqrt{(z + m)^2 - t^2}$, and

$$X_+(t, z, 0) = 2(z + m) + a = (z + m) + \sqrt{(z + m)^2 - t^2}.$$

Let $\mu = (z + m) + \sqrt{(z + m)^2 + \rho^2}$. Then $\nu = -\log\mu$ is a solution of (a). We note that $\partial_z\mu = 2\mu^2/(\mu^2 + \rho^2)$ and $\partial_\rho\mu = 2\mu\rho/(\mu^2 + \rho^2)$. Thus

$$\bar\partial \log\mu = \frac{1}{\rho - i\mu}.$$

It is now ready to construct a generalized multi-Schwarzschild solution (cf. [3]). Let m_i $(i = 1, \ldots, n)$ be distinct real numbers. We set $\mu_i = (z + m_i) + \sqrt{(z + m_i)^2 + \rho^2}$ and $\nu_i = \log\mu_i$. Then for real numbers a_i $(i = 1, \ldots, n)$,

$$v = \sum_{i=1}^{n} a_i v_i$$

is a solution of (a). A direct computation shows that

$$2\bar{\partial} \log \mu - \bar{\partial} \log(\mu^2 + \rho^2) = \frac{\rho}{(\rho - i\mu)^2},$$

$$\bar{\partial} \log(\mu_i - \mu_j) = \frac{\rho}{(\rho - i\mu_i)(\rho - i\mu_j)}.$$

Hence

$$\gamma = \sum_{i=1}^{n} a_i^2 \log\left(\frac{\mu_i^2}{\mu_i^2 + \rho^2}\right) + \sum_{i<j} 2a_i a_j \log(\mu_i - \mu_j)$$

satisfies $\bar{\partial}\gamma = \rho(\bar{\partial} v)^2$.

References

[1] F. J. Ernst, New formulation of the axially symmetric gravitational field problem II, Phys. Rev. **168**, 1415–1417 (1968).

[2] V. A. Belinsky and V. E. Zakharov, Integration of the Einstein equations by means of the inverse scattering problem technique, Sov. Phys. J.E.T.P. **48**, 985–994 (1978).

[3] V. A. Belinsky and V. E. Zakharov, Stationary gravitational solitons with axial symmetry, Sov. Phys. J.T.E.P. **50**, 1–9 (1979).

[4] R. Geroch, A method for generating new solutions of Einstein's equations II, J. Math. Phys. **13**, 394–404 (1972).

[5] K. Nagatomo, Formal power series solutions of the stationary axisymmetric vacuum Einstein equations, Osaka J. Math. **25**, 49–70 (1988).

[6] K. Nagatomo, The Ernst equation as a motion on a universal Grassmann manifold, Comm. Math. Phys. **122**, 439–453 (1989).

[7] K. Nagatomo, Explicit description of ansatz E_n for the Ernst equation in general relativity, J. Math. Phys. **30**, 1100–1102 (1989).

[8] Y. Nakamura, Symmetries of stationary axially symmetric vacuum Einstein equations and the new family of exact solutions, J. Math. Phys. **20**, 606–609 (1983).

[9] Y. Nakamura, On a linearization of the stationary axially symmetric Einstein equations, Class. Quantum Grav. **4**, 437–440 (1987).

[10] K. Takasaki, A new approach to the self-dual Yang-Mills equations II, Saitama Math. J. **3**, 11–40 (1985).

[11] L. Witten, Static axially symmetric solutions of self-dual $SU(2)$ gauge fields in Euclidean four-dimensional space, Phys. Rev. D **19**, 718–720 (1979).

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