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First order partial differential equations on the curvature of 3-dimensional Heisenberg bundles

Dedicated to Professor Yoshihiro Tashiro on his 70th birthday

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Abstract. We study first order partial differential equations on the curvature of principal fibre bundles. We show that such differential equations are essentially exhausted by the one obtained from the Bianchi identity, and as one example, we express the differential equations in the case of 3-dimensional Heisenberg bundles in a geometric form. In the latter half of this paper, we study some algebraic properties concerning the Bianchi identity for 3-dimensional Heisenberg bundles. Several types of invariants and covariants naturally arise from studying this algebraic problem.

Introduction

"Prescribed curvature problem", i.e., the problem of characterizing "actual" curvature tensor fields (or forms) among the set of curvature like tensor fields (or forms), is one of the fundamental problem in differential geometry, and also in physics. In general, not all curvature like tensor fields are actually curvature, and several results are known at present concerning this problem for each geometric situation. For example, in a series of papers, Kazdan and Warner characterized the curvature functions on 2-dimensional manifolds from global viewpoints [8], [9], while local characterizations of curvatures are studied deeply in [3], [5], [6], [7], [13], etc.

If Ω is an actual curvature determined by a connection, the components of Ω must satisfy some partial differential equations. As a classically known example, in the context of principal G-bundles, the characteristic form $f(\Omega)$ corresponding to a G-invariant polynomial f is closed, and we may consider the equality $df(\Omega) = 0$ as a first order partial differential equation on Ω. It is also known that in the case of $SU(2)$-bundle over $\boldsymbol{R}^4$, the curvature like form Ω which satisfies some second order partial differential equations is an

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actual curvature, under some genericity condition on the pointwise value of Ω (cf. [13]).

In the present paper, we study the "local" prescribed curvature problem on principal fibre bundles, especially concerning the first order partial differential equations on curvatures. Let $P \to M$ be a principal bundle with a structure group G, and ω be a connection 1-form on P, which takes value in the Lie algebra $\mathfrak{g}$ of G (cf. [10; vol. I]). Then ω defines the curvature 2-form Ω on P by the structure equation

$$(S) \qquad \Omega = d\omega + 1/2 \cdot [\omega, \omega].$$

Since we consider only local characterization, we may pull back ω and Ω to the base manifold M, by using a suitable local cross section of P. If the Lie group G is abelian, then the above structure equation (S) is simply reduced to $\Omega = d\omega$, and hence, by Poincaré's lemma, a local $\mathfrak{g}$-valued 2-form Ω is an actual curvature if and only if it satisfies the first order partial differential equation $d\Omega = 0$. But, for general non-abelian Lie groups G, the situation is more complicated.

To obtain first order partial differential equations on general principal G-bundles, we differentiate the above structure equation (S). Then the Bianchi identity

$$(B) \qquad d\Omega = [\Omega, \omega]$$

follows, which involves the first derivatives of Ω. We cannot consider (B) itself as a differential equation on Ω because it also contains a connection form ω. But, we can obtain first order partial differential equations on Ω from (B) as follows. Let $A^p(M, \mathfrak{g})$ be the set of $\mathfrak{g}$-valued p-forms on M, and define a linear map

$$B_\Omega \colon A^1(M, \mathfrak{g}) \to A^3(M, \mathfrak{g})$$

by $B_\Omega(\alpha) = [\Omega, \alpha]$. We call B_Ω the Bianchi map. Then, from the identity (B), the form $d\Omega$ must be contained in the image of the map B_Ω if Ω is an actual curvature. In general, the map B_Ω is not surjective, and hence, some algebraic conditions are imposed on $d\Omega$, which may be considered as first order partial differential equations on Ω. Our first main purpose of this paper is to show that the "essential" first order partial differential equations on Ω are exhausted by the one obtained in this way. (For precise statements, see Theorem 1.1.) To prove this fact, we calculate the rank of the map determined by the 1-jet of the structure equation (S), under a pointwise genericity condition on the curvature Ω (cf. § 1).

Our next problem is to find all first order partial differential equations on Ω in an explicit form. But, for general Lie groups G, this is quite a difficult algebraic problem, in contrast to the abelian case which we explained above. In the present paper, as one example, we give a complete answer to this question in the case where the structure group G is the 3-dimensional Heisenberg group H_3. The structure of H_3 is very simple among non-abelian Lie groups, but in the standpoint of "prescribed curvature problem", it contains an interesting algebraic difficulty which is peculiar to this sort of problem. In the paper [7], DeTurck and Talvacchia already studied this problem in the case where the dimension of the base manifold is 3. For general dimensions, we show that first order partial differential equations are essentially exhausted by two types of equations: The first one is expressed as the closedness of characteristic forms as explained above, and the second one is a new type of non-linear equation on Ω, which appears only in the case $\dim M \geq 5$ (Theorem 2.3). We express this new differential equations in a simple geometric form by introducing a family of 5-dimensional subspaces of T_xM (Proposition 3.1 and Theorem 3.3).

The critical dimension 5 appeared in this context is of special interest for us, and some peculiar facts hold in several places of this paper if $\dim M = 5$. For example, only in this case, the Bianchi map B_Ω admits a one-dimensional unexpected kernel, which enables us to write down the defining equation of the image of B_Ω in a relatively simple way, because it is invariant under the action of the group $GL(5, \boldsymbol{R})$. (cf. Lemma 2.2, Proposition 3.1. For other phenomena, see §5.)

As stated above, in obtaining the first order partial differential equations, the Bianchi identity (or the Bianchi map) plays a fundamental role. In the latter half of this paper, we study some algebraic properties of the Bianchi map associated with 3-dimensional Heisenberg bundles. For these bundles, the essential part of B_Ω is simply reduced to the linear map

$$\varphi_F\colon V^* + V^* \to \bigwedge^3 V^*$$

defined by $\varphi_F(\alpha_1, \alpha_2) = F_1 \wedge \alpha_2 - \alpha_1 \wedge F_2$, where $V = T_xM$, $F = (F_1, F_2) \in \bigwedge^2 V^* + \bigwedge^2 V^*$ and $\alpha_1, \alpha_2 \in V^*$. (We denote the pointwise values of Ω and ω by F and α_i, respectively. For details, see §2.) If F is a generic element of $\bigwedge^2 V^* + \bigwedge^2 V^*$, this map is injective in the case $n \geq 6$, and this fact geometrically implies that two components of the connection 1-form ω are uniquely determined from the pointwise values of Ω and $d\Omega$. In §4, we explicitly write down this expression (the inverse formula of the map φ_F), whose denominators and numerators are the polynomials of Ω and $d\Omega$ with degree 6 (Proposition 4.1). In order to express this formula, we must introduce a flag $V^1 \subset V^4 \subset V^6 \subset V^n$ where $V^n = T_xM$, and the superscript indicates dimension.

In the final section of this paper, we consider the problem of characterizing "singular" curvatures from the standpoint of the Bianchi map in detail. Throughout § 1 ~ § 3, in determining the number of first order partial differential equations, or in obtaining the defining equation of the image of the Bianchi map, we consider only "generic" curvatures such that the Bianchi map takes the maximum rank. Hence, as one natural and important problem, it is desirable to characterize generic (or equivalently, singular) curvatures in the set of all curvature like forms. Roughly speaking, we can completely characterize such singular curvatures in terms of two concepts "reducibility" and "decomposability" of F. (For the precise statements, see Theorem 5.1.) On the other hand, by definition, singular curvatures constitute some algebraic sets of $\bigwedge^2 V^* + \bigwedge^2 V^*$, and as another characterization, we give the defining equations of these algebraic sets. Several new types of algebraic equations appear, including the invariants and the covariants of the group $GL(n, \boldsymbol{R}) \times GL(2, \boldsymbol{R})$ acting on the space $\bigwedge^2 V^* + \bigwedge^2 V^* \simeq \bigwedge^2 V^* \otimes \boldsymbol{R}^2$ (Theorem 5.2 and Proposition 5.11). We emphasize once again that the case $\dim M = 5$ has a special meaning in considering singular curvatures. In this case, generic pairs of 2-forms can be reduced to some normal form (Lemma 5.8), and this normal form plays one of the crucial roles in characterizing singular curvatures.

Finally, it should be remarked that first order partial differential equations are not in general enough to characterize "actual" curvatures, and it is necessary to study higher order partial differential equations on Ω. We will treat this problem in forthcoming papers.

1. First order partial differential equations on principal G-bundles

In this section, we show that first order partial differential equations on the curvature of principal G-bundles are exhausted essentially by the ones that are obtained from the Bianchi identity.

Let $P \to M$ be a principal G-bundle over an n-dimensional manifold M, and let $\mathfrak{g}$ be the Lie algebra of G. Let ω and Ω be $\mathfrak{g}$-valued connection 1-form on P and its curvature form, respectively. Then, they are related by the structure equation:

$$(S) \qquad \Omega = d\omega + 1/2 \cdot [\omega, \omega].$$

By applying the exterior differentiation d to (S), and using the formula $[[\omega, \omega], \omega] = 0$, we obtain the Bianchi identity

$$(B) \qquad d\Omega = [\Omega, \omega].$$

(For fundamental identities on $\mathfrak{g}$-valued forms, see for example [4].) Since partial differential equations essentially express the "local" property of unknown functions, we may restrict the problem to some open set of M where the bundle P is trivial, and we express this open set as M again. We fix a cross section $\sigma: M \to P$ and pull back the forms such as ω, Ω, $d\omega$, $d\Omega$ on P to M, and denote them by the same letters. Since the vertical value and the right translation of these forms are uniquely determined, we may consider the "prescribed curvature problem" on the base manifold M. In the following, we denote by $A^p(M, \mathfrak{g})$ the set of $\mathfrak{g}$-valued p-forms on M.

Now, using an element $\Omega \in A^2(M, \mathfrak{g})$, we define a linear map

$$B_\Omega: A^1(M, \mathfrak{g}) \to A^3(M, \mathfrak{g})$$

by $B_\Omega(\alpha) = [\Omega, \alpha]$ for $\alpha \in A^1(M, \mathfrak{g})$. Then, from the Bianchi identity (B), it is clear that the 3-form $d\Omega$ must be contained in the image of the map B_Ω if Ω is an actual curvature. For this reason, we call B_Ω the Bianchi map. It is easy to see that the property "$d\Omega \in \operatorname{Im} B_\Omega$" does not depend on the choice of a cross section of $P \to M$. When B_Ω is not surjective, we may say that the condition $d\Omega \in \operatorname{Im} B_\Omega$ (the Bianchi condition) is a first order partial differential equation on Ω because the defining equation of $\operatorname{Im} B_\Omega$ in $A^3(M, \mathfrak{g})$ contains the first derivatives of Ω. Actually, the map B_Ω is determined by a pointwise linear map

$$B_F: V^* \otimes \mathfrak{g} \to \bigwedge^3 V^* \otimes \mathfrak{g},$$

defined in the same way as above, where $V = T_xM$, and $F = \Omega_x \in \bigwedge^2 V^* \otimes \mathfrak{g}$. (In the following, we express $\mathfrak{g}$-valued 2-forms as F instead of Ω when the pointwise values of Ω are concerned.) The maximum rank of B_F, where F runs all over the space $\bigwedge^2 V^* \otimes \mathfrak{g}$, depends only on the Lie algebra $\mathfrak{g}$ and $n = \dim M$, and we denote this maximum rank by $r_n(\mathfrak{g})$. Clearly, the equality $\operatorname{rank} B_F = r_n(\mathfrak{g})$ holds for generic elements $F \in \bigwedge^2 V^* \otimes \mathfrak{g}$. Hence, if the pointwise value of Ω is generic, then the map B_F takes the maximum rank for any $x \in M$, and in particular, the number of first order partial differential equations obtained from the Bianchi condition $d\Omega \in \operatorname{Im} B_\Omega$ is equal to $1/6 \cdot n(n-1)(n-2) \times \dim \mathfrak{g} - r_n(\mathfrak{g})$, which is the codimension of the map B_F.

Next, we determine the essential number of all first order partial differential equations on the curvature Ω. To state the precise results, we use the following notation. First, we define the sets $J^p(\omega)$ and $J^p(\Omega)$ by

$$J^p(\omega) = \{p\text{-jets of } \mathfrak{g}\text{-valued 1-forms } \omega \text{ on } M\},$$

$$J^p(\Omega) = \{p\text{-jets of } \mathfrak{g}\text{-valued 2-forms } \Omega \text{ on } M\}$$

(the letters ω and Ω on the left hand sides possess only a symbolic meaning), and denote the elements of these spaces by $j^p(\omega)$ and $j^p(\Omega)$, respectively. Clearly, $J^p(\omega)$ and $J^p(\Omega)$ are differentiable manifolds, and it is easy to see that their dimensions are equal to $n\binom{n+p}{p} \times \dim \mathfrak{g}$ and $\binom{n}{2}\binom{n+p}{p} \times \dim \mathfrak{g}$. $\left(\text{Note that } \sum_{k=0}^{p}\binom{n+k-1}{k} = \binom{n+p}{p}.\right)$

The structure equation (S) naturally induces a quadratic map

$$Str^0 : J^1(\omega) \to J^0(\Omega)$$

because the pointwise value of Ω is uniquely determined by the 1-jet $j^1(\omega)$ of ω. And, by differentiating the equation (S), we naturally obtain the first prolongation of Str^0

$$Str^1 : J^2(\omega) \to J^1(\Omega),$$

which is also quadratic. (For details, see the proof of Theorem 1.1.) We may say that the defining equations of the image of Str^1 in $J^1(\Omega)$ are the first order partial differential equations on Ω, and the essential number of these equations is equal to the codimension of the map Str^1. We denote by $s_n(\mathfrak{g})$ the rank of Str^1_* (the differential of Str^1) at a generic point of $J^2(\omega)$, i.e., the maximum rank of the differential of the quadratic map Str^1. Then, the codimension of the map Str^1 is equal to

$$\dim J^1(\Omega) - s_n(\mathfrak{g}) = 1/2 \cdot n(n-1)(n+1) \times \dim \mathfrak{g} - s_n(\mathfrak{g}),$$

which depends only on the Lie algebra $\mathfrak{g}$ and the dimension of the manifold. Clearly, we have the inequality

$$\dim J^1(\Omega) - s_n(\mathfrak{g}) \geq 1/6 \cdot n(n-1)(n-2) \times \dim \mathfrak{g} - r_n(\mathfrak{g})$$

because the Bianchi condition $d\Omega \in \operatorname{Im} B_\Omega$ is the first order partial differential equation on Ω as explained before. Now, under the notation as above, our first main theorem is stated as follows.

THEOREM 1.1. *For any Lie algebra* $\mathfrak{g}$, *the equality*

$$\dim J^1(\Omega) - s_n(\mathfrak{g}) = 1/6 \cdot n(n-1)(n-2) \times \dim \mathfrak{g} - r_n(\mathfrak{g})$$

holds. In particular, essential first order partial differential equations on the curvature Ω *are exhausted by the Bianchi condition* $d\Omega \in \operatorname{Im} B_\Omega$ *for any principal G-bundle.*

PROOF. We prove this theorem by using a local coordinate system $(x_1, \cdots, x_n)$ of M. Let $\{e_1, \cdots, e_r\}$ be a basis of the Lie algebra $\mathfrak{g}$, and we put $[e_t, e_u] = \sum_s c^s_{tu} e_s$. Then, the components of a connection form ω and its curvature form $\Omega = d\omega + 1/2 \cdot [\omega, \omega]$ are locally expressed as

$$\omega\left(\frac{\partial}{\partial x_i}\right) = \sum_s \omega_{si} e_s,$$

$$\Omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \sum_s \Omega_{sij} e_s,$$

where

$$\Omega_{sij} = \omega_{sji} - \omega_{sij} + \sum_{tu} c^s_{tu} \omega_{ti} \omega_{uj},$$

and

$$\omega_{sij} = \frac{\partial \omega_{si}}{\partial x_j}.$$

We may use the components $\{\omega_{si}, \omega_{sij}\}$ and $\{\Omega_{sij}\}$ as local coordinates of the manifolds $J^1(\omega)$ and $J^0(\Omega)$, respectively. Clearly, the map Str^0 is locally expressed as

$$Str^0((\omega_{si}, \omega_{sij})) = (\Omega_{sij})$$

through the above equality on Ω_{sij}. Next, we differentiate the structure equation $\Omega = d\omega + 1/2 \cdot [\omega, \omega]$ with respect to x_k. Then, by putting

$$\omega_{sijk} = \frac{\partial^2 \omega_{si}}{\partial x_j \partial x_k} \qquad \text{and} \qquad \Omega_{sijk} = \frac{\partial \Omega_{sij}}{\partial x_k},$$

we have

$$\Omega_{sijk} = \omega_{sjik} - \omega_{sijk} + \sum_{tu} c^s_{tu}(\omega_{tik}\omega_{uj} + \omega_{ti}\omega_{ujk}),$$

and the map Str^1 is locally expressed as

$$Str^1((\omega_{si}, \omega_{sij}, \omega_{sijk})) = (\Omega_{sij}, \Omega_{sijk}),$$

which is quadratic if $\mathfrak{g}$ is not abelian. (As above, we may consider $\{\omega_{si}, \omega_{sij}, \omega_{sijk}\}$ and $\{\Omega_{sij}, \Omega_{sijk}\}$ as local coordinates of $J^2(\omega)$ and $J^1(\Omega)$.)

Now, we determine the kernel of the differential of Str^1 at a generic point $j^2(\omega) = (\omega_{si}, \omega_{sij}, \omega_{sijk}) \in J^2(\omega)$. By considering the above equalities, the tangent vector

$$\begin{aligned}\alpha &= \sum \alpha_{si} \frac{\partial}{\partial \omega_{si}} + \sum \alpha_{sij} \frac{\partial}{\partial \omega_{sij}} + \sum \alpha_{sijk} \frac{\partial}{\partial \omega_{sijk}} \\ &= (\alpha_{si}, \alpha_{sij}, \alpha_{sijk})\end{aligned}$$

of $J^2(\omega)$ at $j^2(\omega)$ is contained in the kernel of Str^1_* if and only if

$$(*)\qquad \begin{aligned} &\alpha_{sji} - \alpha_{sij} + \sum_{tu} c^s_{tu}(\omega_{uj}\alpha_{ti} + \omega_{ti}\alpha_{uj}) = 0, \\ &\alpha_{sjik} - \alpha_{sijk} + \sum_{tu} c^s_{tu}(\omega_{uj}\alpha_{tik} + \omega_{tik}\alpha_{uj} + \omega_{ujk}\alpha_{ti} + \omega_{ti}\alpha_{ujk}) = 0. \end{aligned}$$

In the following, we determine the degree of freedom of α satisfying $(*)$ for generic $j^2(\omega)$. From the first equations of $(*)$, the component α_{sij} $(i > j)$ is uniquely determined by the values of α_{sij} $(i < j)$ and α_{si}. Similarly, since α_{sijk} is symmetric with respect to j and k, the component α_{sijk} is determined by the values α_{sijk} $(i \le j \le k)$, α_{sij} $(i \le j)$ and α_{si}, but not uniquely in this case. By putting

$$A_{sijk} = \alpha_{sjik} - \alpha_{sijk} + \sum_{tu} c^s_{tu}(\omega_{uj}\alpha_{tik} + \omega_{tik}\alpha_{uj} + \omega_{ujk}\alpha_{ti} + \omega_{ti}\alpha_{ujk}),$$

it is easy to see that this degree of freedom just comes from the equality

$$(**)\qquad A_{sijk} - A_{skji} + A_{skij} = 0,$$

which imposes some additional conditions on the components $(\alpha_{si}, \alpha_{sij})$. We rewrite this equality $(**)$ in a simple form in the following way. First, we have

$$\begin{aligned} A_{sijk} - A_{skji} + A_{skij} &= \sum c^s_{tu}(\omega_{uj}\alpha_{tik} + \omega_{tik}\alpha_{uj} + \omega_{ujk}\alpha_{ti} + \omega_{ti}\alpha_{ujk}) \\ &\quad - \sum c^s_{tu}(\omega_{uj}\alpha_{tki} + \omega_{tki}\alpha_{uj} + \omega_{uji}\alpha_{tk} + \omega_{tk}\alpha_{uji}) \\ &\quad + \sum c^s_{tu}(\omega_{ui}\alpha_{tkj} + \omega_{tkj}\alpha_{ui} + \omega_{uij}\alpha_{tk} + \omega_{tk}\alpha_{uij}) \\ &= \sum c^s_{tu}\{(\alpha_{ujk} - \alpha_{ukj})\omega_{ti} + (\alpha_{uki} - \alpha_{uik})\omega_{tj} + (\alpha_{uij} - \alpha_{uji})\omega_{tk}\} \\ &\quad + \sum c^s_{tu}\{(\omega_{tkj} - \omega_{tjk})\alpha_{ui} + (\omega_{tik} - \omega_{tki})\alpha_{uj} + (\omega_{tji} - \omega_{tij})\alpha_{uk}\} \\ &= 0. \end{aligned}$$

From the first equation in $(*)$, we have

$$\alpha_{sij} - \alpha_{sji} = \sum_{tu} c^s_{tu}(\omega_{uj}\alpha_{ti} + \omega_{ti}\alpha_{uj}),$$

and we substitute this equality into the above. Then, we have

$$\begin{aligned} &\sum c^s_{tu}\{(\omega_{tkj} - \omega_{tjk})\alpha_{ui} + (\omega_{tik} - \omega_{tki})\alpha_{uj} + (\omega_{tji} - \omega_{tij})\alpha_{uk}\} \\ &\quad + \sum c^s_{tu}c^u_{vw}\{(\omega_{wk}\alpha_{vj} + \omega_{vj}\alpha_{wk})\omega_{ti} + (\omega_{wi}\alpha_{vk} + \omega_{vk}\alpha_{wi})\omega_{tj} \\ &\quad + (\omega_{wj}\alpha_{vi} + \omega_{vi}\alpha_{wj})\omega_{tk}\} = 0. \end{aligned}$$

The coefficient of α_{ui} in this expression is equal to

$$\begin{aligned}
&\sum_t c^s_{tu}(\omega_{tkj} - \omega_{tjk}) + \sum_{tvw} c^s_{tw} c^w_{vu} \omega_{vk}\omega_{tj} + \sum_{tvw} c^s_{tv} c^v_{uw} \omega_{wj}\omega_{tk} \\
&\quad = \sum_t c^s_{tu}(\omega_{tkj} - \omega_{tjk}) + \sum_{tvw} c^s_{tw} c^w_{vu} \omega_{vk}\omega_{tj} + \sum_{tvw} c^s_{vw} c^w_{ut} \omega_{vk}\omega_{tj} \\
&\quad = \sum_t c^s_{tu}(\omega_{tkj} - \omega_{tjk}) + \sum_{tvw} c^s_{uw} c^w_{vt} \omega_{vk}\omega_{tj} \\
&\quad = \sum_t c^s_{tu}\left(\omega_{tkj} - \omega_{tjk} + \sum_{vw} c^t_{wv}\omega_{wj}\omega_{vk}\right) \\
&\quad = \sum_t c^s_{tu}\Omega_{tjk}.
\end{aligned}$$

(We used the Jacobi identity once in the above modification.) The coefficients of α_{uj} and α_{uk} can be calculated in the same way, and hence, the above equality is simplified as

$$\sum_t c^s_{tu}(\Omega_{tjk}\alpha_{ui} - \Omega_{tik}\alpha_{uj} + \Omega_{tij}\alpha_{uk}) = 0,$$

which is equivalent to $[\Omega, \alpha_0] = 0$, where $\alpha_0 = (\alpha_{si})$. Therefore, the degree of freedom of $\alpha = (\alpha_{si}, \alpha_{sij}, \alpha_{sijk})$, which is the dimension of Ker Str^1_*, is equal to

$$\begin{aligned}
&\{n + 1/2\cdot n(n+1) + 1/6\cdot n(n+1)(n+2)\} \times \dim \mathfrak{g} - r_n(\mathfrak{g}) \\
&\quad = 1/6\cdot n(n^2 + 6n + 11) \times \dim \mathfrak{g} - r_n(\mathfrak{g}),
\end{aligned}$$

because the equality $[\Omega, \alpha_0] = 0$ imposes $r_n(\mathfrak{g})$ conditions on α for a generic $j^2(\omega)$. (Note that the map $B_\Omega(\alpha_0) = [\Omega, \alpha_0]$ determined by $\Omega = Str^0(j^1(\omega))$ takes the maximum rank if $j^1(\omega)$ is a generic element in $J^1(\omega)$ because the map Str^0_* is surjective.) Therefore, we have

$$\begin{aligned}
s_n(\mathfrak{g}) &= \operatorname{rank} Str^1_* \quad \text{at } j^2(\omega) \\
&= \dim J^2(\omega) - \{1/6\cdot n(n^2 + 6n + 11) \times \dim \mathfrak{g} - r_n(\mathfrak{g})\} \\
&= \{1/2\cdot n(n+1)(n+2) - 1/6\cdot n(n^2 + 6n + 11)\} \times \dim \mathfrak{g} + r_n(\mathfrak{g}) \\
&= 1/6\cdot n(n-1)(2n+5) \times \dim \mathfrak{g} + r_n(\mathfrak{g}),
\end{aligned}$$

and hence the codimension of the map Str^1_* is equal to

$$\begin{aligned}
\dim J^1(\Omega) - s_n(\mathfrak{g}) &= 1/2\cdot n(n-1)(n+1) \times \dim \mathfrak{g} - 1/6\cdot n(n-1)(2n+5) \\
&\qquad \times \dim \mathfrak{g} - r_n(\mathfrak{g}) \\
&= 1/6\cdot n(n-1)(n-2) \times \dim \mathfrak{g} - r_n(\mathfrak{g}),
\end{aligned}$$

which proves the theorem. q.e.d.

REMARK. (1) Let f be an invariant polynomial of the Lie group G. Then, as stated in Introduction, the characteristic form $f(\Omega)$ on M is closed. (See [10; Vol. II].) We may consider this equality $df(\Omega)=0$ as a first order partial differential equation on Ω, and Theorem 1.1 implies that this equality follows essentially from the Bianchi condition $d\Omega \in \operatorname{Im} B_\Omega$. (And, in fact, the closedness of $f(\Omega)$ is proved in [10] by using only the Bianchi identity.)

(2) As stated in this proof, the degree of freedom on the expression of α_{sijk} comes from the equality (**) on A_{sijk}, and it is easy to see that this fact is equivalent to the exactness of the following natural complex (cf. [1]):

$$V^* \otimes S^2 V^* \to \bigwedge^2 V^* \otimes V^* \to \bigwedge^3 V^*.$$

The codimension $\dim J^1(\Omega) - s_n(\mathfrak{g})$ which is the essential number of first order partial differential equations may be also expressed as $1/6 \cdot n(n+1)(n-4) \times \dim \mathfrak{g} + k_n(\mathfrak{g})$, where $k_n(\mathfrak{g})$ is the dimension of the kernel of B_Ω for generic Ω. In the special case $n=4$, Mostow and Shnider [12] showed that the map B_Ω is the isomorphism if the Lie algebra $\mathfrak{g}$ is semi-simple and Ω is generic. Therefore, combining these results, we have

COROLLARY 1.2. *When* $\dim M = 4$ *and* $\mathfrak{g}$ *is semi-simple, there exists no first order partial differential equation on the curvature* Ω.

2. 3-dimensional Heisenberg bundles

Now, our next problem is to determine the rank $r_n(\mathfrak{g})$ (or equivalently, the rank $s_n(\mathfrak{g})$) for a given Lie algebra $\mathfrak{g}$, and to find the defining equations of the image of the map B_Ω. First, in this section, we determine the value $r_n(\mathfrak{g})$ when $\mathfrak{g}$ is the 3-dimensional Heisenberg Lie algebra. As stated in Introduction, prescribed curvature problem for this bundle is already studied in [7] in the case $\dim M = 3$.

Let H_3 be the 3-dimensional Heisenberg group:

$$H_3 = \left\{ \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; a, b, c \in \boldsymbol{R} \right\}.$$

Then, by putting

$$X_1 = \begin{bmatrix} 0 & 1 & 0 \\ & 0 & 0 \\ & & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 & 0 \\ & 0 & 1 \\ & & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 0 & 1 \\ & 0 & 0 \\ & & 0 \end{bmatrix},$$

$\{X_1, X_2, X_3\}$ forms a basis of the Lie algebra $\mathfrak{h}_3$ of H_3, and the bracket operations of $\mathfrak{h}_3$ are given by

$$[X_1, X_2] = X_3, \qquad [X_1, X_3] = [X_2, X_3] = 0.$$

Let $P \to M$ be a principal bundle with structure group H_3, and let ω (resp. Ω) be a connection (resp. curvature) form on P. As in § 1, we pull back the forms ω and Ω to M by a cross section of P, and denote by ω_i (resp. Ω_i) the X_i-component of ω (resp. Ω). Then the structure equation (S) for the 3-dimensional Heisenberg bundle is locally expressed as

$$(S) \qquad \begin{aligned} \Omega_1 &= d\omega_1, \\ \Omega_2 &= d\omega_2, \\ \Omega_3 &= d\omega_3 + \omega_1 \wedge \omega_2, \end{aligned}$$

and the Bianchi identity is

$$(B) \qquad \begin{aligned} d\Omega_1 &= 0, \\ d\Omega_2 &= 0, \\ d\Omega_3 &= \Omega_1 \wedge \omega_2 - \omega_1 \wedge \Omega_2. \end{aligned}$$

Our first purpose in this section is to prove the following theorem.

THEOREM 2.1. *For 3-dimensional Heisenberg bundles, the rank $r_n(\mathfrak{h}_3)$ and the essential number of first order partial differential equations* $\dim J^1(\Omega) - s_n(\mathfrak{h}_3)$ $(= 1/2 \cdot n(n-1)(n-2) - r_n(\mathfrak{h}_3))$ *on the curvature Ω are given in the following table, according as the dimension of the base manifold M.*

	$r_n(\mathfrak{h}_3)$	$\dim J^1(\Omega) - s_n(\mathfrak{h}_3)$
$n = 3$	1	2
$n = 4$	4	8
$n = 5$	9	21
$n \geq 6$	$2n$	$1/2 \cdot n(n^2 - 3n - 2)$

Since $d\Omega_1 = d\Omega_2 = 0$ for H_3-bundles, the X_1- and X_2-components of the image of the Bianchi map B_Ω defined in § 1 is zero. Hence, to prove this theorem, we have only to show the following lemma.

LEMMA 2.2. *Let V be an n-dimensional vector space, and $F = (F_1, F_2)$ be a pair of 2-forms on V. Then the maximum rank of the map*

$$\varphi_F \colon V^* + V^* \to \textstyle\bigwedge^3 V^*$$

defined by

$$\varphi_F(\alpha_1, \alpha_2) = F_1 \wedge \alpha_2 - \alpha_1 \wedge F_2, \qquad \alpha_1, \alpha_2 \in V^*$$

is given by

	rank φ_F
$n = 3$	1
$n = 4$	4
$n = 5$	9
$n \geq 6$	$2n$

.

REMARK. For 3-dimensional Heisenberg bundles, we may call φ_F the Bianchi map since φ_F is the essential part of B_F as explained above. (As before, in considering the pointwise problem, we express 2-forms as F instead of Ω.) It is clear that the Bianchi map φ_F takes the maximum rank for a generic F, and rank φ_F is not maximum if and only if F belongs to some algebraic set in $\bigwedge^2 V^* + \bigwedge^2 V^*$, consisting of singular elements. To determine the explicit defining equations of this algebraic set is another interesting algebraic problem, and we study this problem in § 5 in detail. (See Theorem 5.2 and Proposition 5.11.)

PROOF. For the case $n = 3$, 4 and $n \geq 6$, we have only to find $F = (F_1, F_2)$ such that the rank of φ_F takes the values in the table because rank φ_F cannot exceed these values. For each case, by using a basis $\{e_1^*, \cdots, e_n^*\}$ of V^*, we put

$$n = 3 : F_1 = e_1^* \wedge e_2^*, \qquad F_2 = 0,$$

$$n = 4 : F_1 = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*, \qquad F_2 = 0,$$

$$n \geq 6 : F_1 = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*, \qquad F_2 = e_1^* \wedge e_5^* + e_2^* \wedge e_6^*.$$

Then, we can easily verify that the map φ_F is surjective in the case $n = 3$, 4, and injective in the case $n \geq 6$. Next, for the case $n = 5$, we put

$$F_1 = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*, \qquad F_2 = e_1^* \wedge e_4^* + e_3^* \wedge e_5^*.$$

Then, by direct calculations, we can show that rank $\varphi_F = 9$ with Ker $\varphi_F = \langle (e_3^*, -e_1^*) \rangle$. Hence, to complete the proof, we have only to show that the inequality dim Ker $\varphi_F \geq 1$ holds for any F in the case $n = 5$. For this purpose, we construct a canonical 1-dimensional kernel of φ_F in terms of F for generic F. First, using the volume form $\Phi = e_1^* \wedge \cdots \wedge e_5^*$, we define α_{1i}, $\alpha_{2i} \in \boldsymbol{R}$ $(1 \leq i \leq 5)$ by

$$\alpha_{1i}\Phi = F_1 \wedge F_1 \wedge (e_i \rfloor F_2),$$

$$\alpha_{2i}\Phi = -F_2 \wedge F_2 \wedge (e_i \rfloor F_1),$$

and put $\alpha_1 = \sum \alpha_{1i} e_i^*$, $\alpha_2 = \sum \alpha_{2i} e_i^*$. Then, we have $\varphi_F(\alpha_1, \alpha_2) = 0$. In fact, the $e_i^* \wedge e_j^* \wedge e_k^*$ component of $\alpha_1 \wedge F_2$ is equal to $\alpha_{1i}F_{2jk} - \alpha_{1j}F_{2ik} + \alpha_{1k}F_{2ij}$, and we have

$$\begin{aligned}(\alpha_{1i}F_{2jk} &- \alpha_{1j}F_{2ik} + \alpha_{1k}F_{2ij})\Phi \\ &= F_1 \wedge F_1 \wedge (F_{2jk}\cdot e_i \rfloor F_2 - F_{2ik}\cdot e_j \rfloor F_2 + F_{2ij}\cdot e_k \rfloor F_2) \\ &= -1/2\cdot F_1 \wedge F_1 \wedge \{e_i \rfloor e_j \rfloor e_k \rfloor (F_2 \wedge F_2)\}.\end{aligned}$$

On the other hand, as for the $e_i^* \wedge e_j^* \wedge e_k^*$ component of $\alpha_2 \wedge F_1$, we can show the equality

$$(\alpha_{2i}F_{1jk} - \alpha_{2j}F_{1ik} + \alpha_{2k}F_{1ij})\Phi = 1/2\cdot F_2 \wedge F_2 \wedge \{e_i \rfloor e_j \rfloor e_k \rfloor (F_1 \wedge F_1)\}$$

completely in the same way. Since any 6-forms automatically vanish on $\boldsymbol{R}^5$, we have

$$(F_1 \wedge F_1) \wedge \{e_j \rfloor e_k \rfloor (F_2 \wedge F_2)\} = 0,$$

$$\{e_i \rfloor (F_1 \wedge F_1)\} \wedge \{e_k \rfloor (F_2 \wedge F_2)\} = 0,$$

$$\{e_j \rfloor e_i \rfloor (F_1 \wedge F_1)\} \wedge (F_2 \wedge F_2) = 0,$$

and using these equalities, we have

$$\begin{aligned}F_1 \wedge F_1 \wedge \{e_i \rfloor e_j \rfloor e_k \rfloor (F_2 \wedge F_2)\} &= -\{e_i \rfloor (F_1 \wedge F_1)\} \wedge \{e_j \rfloor e_k \rfloor (F_2 \wedge F_2)\} \\ &= -\{e_j \rfloor e_i \rfloor (F_1 \wedge F_1)\} \wedge \{e_k \rfloor (F_2 \wedge F_2)\} \\ &= \{e_k \rfloor e_j \rfloor e_i \rfloor (F_1 \wedge F_1)\} \wedge F_2 \wedge F_2 \\ &= -F_2 \wedge F_2 \wedge \{e_i \rfloor e_j \rfloor e_k \rfloor (F_1 \wedge F_1)\},\end{aligned}$$

which shows that $\alpha_1 \wedge F_2 = \alpha_2 \wedge F_1$. Clearly $(\alpha_1, \alpha_2) \neq 0$ for generic F, and hence we have $\dim \operatorname{Ker} \varphi_F \geq 1$ for any F. q.e.d.

REMARK. The last inequality rank $\varphi_F \leq 9$ in the case $n = 5$ follows immediately from Proposition 3.1, where the existence of a non-trivial defining equation of $\operatorname{Im} \varphi_F$ is proved. This inequality can be also proved by using the results in §5. For details, see Remark after Lemma 5.8.

It is easy to see that the ring of invariant polynomials of the Lie group H_3 is generated by two elements with degree 1, and the corresponding characteristic forms are Ω_1 and Ω_2. Of course, we already know the closedness of the forms Ω_1 and Ω_2 by the Bianchi identity (B). These equations $d\Omega_1 = d\Omega_2 = 0$ contain $2\binom{n}{3} = 1/3 \cdot n(n-1)(n-2)$ independent first order partial differential equations on the components of Ω. And by subtracting this from the value in Theorem 2.1, we know that the number of the remaining first order partial differential equations is given by

	$\dim J^1(\Omega) - s_n(\mathfrak{h}_3) - 2\binom{n}{3}$
$n = 3$	0
$n = 4$	0
$n = 5$	1
$n \geq 6$	$1/6 \cdot n(n+2)(n-5)$

But, these numbers just coincide with the codimension of the image of the map φ_F in Lemma 2.2 because $\binom{n}{3} - 2n = 1/6 \cdot n(n+2)(n-5)$. Therefore, we have the following theorem, which may be considered as a refinement of Theorem 1.1 for 3-dimensional Heisenberg bundles.

THEOREM 2.3. *The essential first order partial differential equations on the curvature Ω of 3-dimensional Heisenberg bundles are exhausted by*

$$d\Omega_1 = d\Omega_2 = 0 \qquad \textit{for } n = 3, 4,$$

and

$$d\Omega_1 = d\Omega_2 = 0, \qquad d\Omega_3 \in \operatorname{Im} \varphi_{(\Omega_1, \Omega_2)} \qquad \textit{for } n \geq 5.$$

This result for the case $n = 3$ is also an immediate consequence of Proposition 2.4 in [7], where it is proved that a generic triple of 2-forms $(\Omega_1, \Omega_2, \Omega_3)$ with $d\Omega_1 = d\Omega_2 = 0$ is always a curvature of H_3-bundle over a 3-dimensional manifold. (Here, the term "generic" implies that the pointwise value of Ω is generic in a sense. For details, see [7; p. 34].)

Thus, our remaining problem for first order partial differential equations on Ω is to find the explicit defining equations of the map $\varphi_{(\Omega_1, \Omega_2)}$ in Lemma 2.2, which belongs to the problem of "Linear Algebra".

3. The Bianchi condition in the case $n \geq 5$

In this section, we give the explicit defining equations of the image of the map φ_F defined in Lemma 2.2 in a geometric form for $n \geq 5$. We first treat the case $n = 5$, which also plays a fundamental role for the general case $n \geq 6$. To state the results, we first prepare some notations.

Let V be a 5-dimensional real vector space, and we fix a volume form $\Phi \in \bigwedge^5 V^*$ throughout. Then, for any 4-form $\gamma \in \bigwedge^4 V^*$, the vector $\gamma^\# \in V$ is uniquely determined by the rule

$$\gamma^\# \rfloor \Phi = \gamma \in \bigwedge^4 V^*.$$

In this section, in the case $n = 5$, we say that the pair of 2-forms $F = (F_1, F_2) \in \bigwedge^2 V^* + \bigwedge^2 V^*$ is "*generic*" if

(1) three vectors $(F_1 \wedge F_1)^\#$, $(F_1 \wedge F_2)^\#$, $(F_2 \wedge F_2)^\#$ are linearly independent in V,

(2) the rank of the Bianchi map $\varphi_F \colon V^* + V^* \to \bigwedge^3 V^*$ is 9 (i.e., φ_F is of maximum rank. cf. Lemma 2.2).

We remark that such forms actually exist. For example, using a basis $\{e_1^*, \cdots, e_5^*\}$ of V^*, we put

$$F_1 = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*,$$

$$F_2 = e_1^* \wedge e_4^* + e_3^* \wedge e_5^*.$$

Then, with respect to the volume form $\Phi = e_1^* \wedge \cdots \wedge e_5^*$, we can easily check that

$$(F_1 \wedge F_1)^\# = 2e_5, \qquad (F_1 \wedge F_2)^\# = -e_4, \qquad (F_2 \wedge F_2)^\# = 2e_2,$$

and rank $\varphi_F = 9$. (See the proof of Lemma 2.2.) Therefore, "generic" forms constitute an open dense subset of $\bigwedge^2 V^* + \bigwedge^2 V^*$. (Actually, it is a complement of an algebraic set of $\bigwedge^2 V^* + \bigwedge^2 V^*$, and explicit defining equations of this algebraic set can be obtained immediately by using the results in Theorem 5.2.) Note that the genericity for the curvature (Ω_1, Ω_2) depends only on the pointwise 0-th jet of Ω, not on their derivatives, nor on the choice of the volume form of V. Now, the next propositon combined with Theorem 2.3 gives the complete answer to first order partial differential equations of Ω in the case $n = 5$. (In the following, we express the pointwise value of $d\Omega_3$ as G.)

Proposition 3.1. *Let $F = (F_1, F_2)$ be a generic element of $\bigwedge^2 V^* + \bigwedge^2 V^*$, where $V = \boldsymbol{R}^5$. Then, a 3-form $G \in \bigwedge^3 V^*$ is contained in the image of the Bianchi map $\varphi_F \colon V^* + V^* \to \bigwedge^3 V^*$ defined in Lemma 2.2 if and only if the following equality holds*:

$$G((F_1 \wedge F_1)^\#, \ (F_1 \wedge F_2)^\#, \ (F_2 \wedge F_2)^\#) = 0.$$

Note that the above equality is a non-trivial condition on G, and it does not depend on the choice of the volume form Φ. In particular, by this proposition, it follows that rank $\varphi_F \leq 9$ for generic (and hence, any) F in the case $n=5$ because $\dim \bigwedge^3 V^* = 10$. (cf. Lemma 2.2.) Geometrically, this proposition implies that the 3-form $d\Omega_3$ vanishes on the 3-dimensional subspace spanned by $(\Omega_1 \wedge \Omega_1)^\#$, $(\Omega_1 \wedge \Omega_2)^\#$, $(\Omega_2 \wedge \Omega_2)^\#$ at each point of M, and hence this condition may be considered as a first order partial differential equation on Ω_3.

To prove this proposition, we have only to show that the above equality holds in the case $G \in \operatorname{Im} \varphi_F$. In fact, since the above condition is a single equation on G and we already proved rank $\varphi_F = 9$ for generic F (Lemma 2.2), the converse part follows immediately. In order to prove the above equality on G, we first prepare the following lemma.

LEMMA 3.2. *Let* $F=(F_1, F_2)$ *be a generic element of* $\bigwedge^2 V^* + \bigwedge^2 V^*$, *where* $V=\boldsymbol{R}^5$. *If two vectors* v_1, $v_2 \in V$ *satisfy*

$$v_1 \rfloor (F_1 \wedge F_2) = v_2 \rfloor (F_2 \wedge F_2) = 0,$$

then two 1-*forms* $v_2 \rfloor F_1$ *and* $v_1 \rfloor F_2$ *are parallel in* V^*.

PROOF. Since the pair is generic and $\dim V = 5$, we may put

$$F_1 = \sum_{i<j} F_{ij} e_i^* \wedge e_j^*,$$

$$F_2 = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*,$$

in terms of a suitable basis $\{e_i^*\}$ of V^*. Then, from the condition $v_2 \rfloor (F_2 \wedge F_2) = 0$, we have $v_2 = k e_5$. Next, since $F_1 \wedge F_2$ is equal to

$$(F_{12} + F_{34}) e_{1234}^* + F_{35} e_{1235}^* + F_{45} e_{1245}^* + F_{15} e_{1345}^* + F_{25} e_{2345}^* \quad (\neq 0)$$

where $e_{ijkl}^* = e_i^* \wedge e_j^* \wedge e_k^* \wedge e_l^*$, we have

$$v_1 = l\{F_{25} e_1 - F_{15} e_2 + F_{45} e_3 - F_{35} e_4 + (F_{12} + F_{34}) e_5\}.$$

Hence,

$$\begin{aligned} k(v_1 \rfloor F_2) &= kl(F_{15} e_1^* + F_{25} e_2^* + F_{35} e_3^* + F_{45} e_4^*) \\ &= -kl(e_5 \rfloor F_1) \\ &= -l(v_2 \rfloor F_1), \end{aligned}$$

which proves the lemma. q.e.d.

REMARK. If we drop the genericity condition on F_1 and F_2, this lemma does not hold as the following example shows:

$$F_1 = F_2 = e_1^* \wedge e_2^*, \qquad v_1 = e_1 \qquad \text{and} \qquad v_2 = e_2.$$

PROOF OF PROPOSITION 3.1. We put

$$(F_1 \wedge F_1)^\# = v_0, \qquad (F_1 \wedge F_2)^\# = v_1, \qquad (F_2 \wedge F_2)^\# = v_2,$$

and show the equality

$$G(v_0, v_1, v_2) = 0$$

in the case G is expressed as $F_1 \wedge \alpha_2 - \alpha_1 \wedge F_2$ for some $\alpha_1, \alpha_2 \in V^*$. For this purpose, we have only to prove the equality

$$(F_1 \wedge \alpha_2)(v_0, v_1, v_2) = 0,$$

since the remaining second term also vanishes, as can be proved in the same way. First, from the definition, we have easily

$$v_0 \rfloor (F_1 \wedge F_1) = 0,$$

which is equivalent to $v_0 \rfloor F_1 = 0$. (Note that $\dim V = 5$ and $F_1 \wedge F_1 \neq 0$.) Thus, we have only to show the equality $F_1(v_1, v_2) = 0$. We evaluate the both sides of the following equality at the vector v_2.

$$0 = v_1 \rfloor (F_1 \wedge F_2) = (v_1 \rfloor F_1) \wedge F_2 + F_1 \wedge (v_1 \rfloor F_2).$$

Then, we have

$$0 = F_1(v_1, v_2) \cdot F_2 - (v_1 \rfloor F_1) \wedge (v_2 \rfloor F_2) + (v_2 \rfloor F_1) \wedge (v_1 \rfloor F_2) + F_2(v_1, v_2) \cdot F_1.$$

From Lemma 3.2, we have $(v_2 \rfloor F_1) \wedge (v_1 \rfloor F_2) = 0$, and since $v_2 \rfloor F_2 = 0$, the above equality implies the desired equality $F_1(v_1, v_2) = 0$. q.e.d.

REMARK. For $n = 5$, the general linear group $GL(5, \boldsymbol{R})$ acts canonically on the space $\bigwedge^2 V^* + \bigwedge^2 V^* + \bigwedge^3 V^*$. The expression

$$G((F_1 \wedge F_1)^\#, (F_1 \wedge F_2)^\#, (F_2 \wedge F_2)^\#)$$

may be considered as a polynomial on this space with total degree 7, and it is easy to see that this polynomial is the invariant of $GL(5, \boldsymbol{R})$, corresponding to the Schur function S_{33333}. This invariant is also expressed in the form

$$\sum_{\sigma, \tau, \rho \in \mathfrak{S}_5} \operatorname{sgn}(\sigma\tau\rho) F_{1\sigma(1)\sigma(2)} F_{1\sigma(3)\sigma(4)} F_{1\tau(1)\tau(2)} F_{2\tau(3)\tau(4)} F_{2\rho(1)\rho(2)} F_{2\rho(3)\rho(4)} G_{\sigma(5)\tau(5)\rho(5)},$$

up to the scalar multiplication by non-zero constants, where F_{1ij} and F_{2ij} are the components of F_1 and F_2. (For the definition of the Schur function and the meaning of the above summation, see [11], [2].) Since the map φ_F has some $GL(5, \boldsymbol{R})$-invariant property and the codimension of $\operatorname{Im} \varphi_F$ is 1, the defining equation of $\operatorname{Im} \varphi_F$ is an invariant of $GL(5, \boldsymbol{R})$, as expected.

Next, under these preliminaries, we consider the general case $n \geq 6$. In this case, we can express the differential equations on Ω in a geometric form as in Proposition 3.1 by introducing a family of 5-dimensional subspaces of tangent spaces. We first fix a 5-dimensional subspace W of $V = \boldsymbol{R}^n$ and the volume form of W. And next, we restrict the forms F_1, F_2, G to this subspace W, which we denote by F_1^W, F_2^W, G^W, respectively. Then, from Proposition 3.1, it is clear that the equality

$$G^W((F_1^W \wedge F_1^W)^\#, (F_1^W \wedge F_2^W)^\#, (F_2^W \wedge F_2^W)^\#) = 0$$

holds if $G \in \bigwedge^3 V^*$ is contained in the image of φ_F. (Note that the above equality does not depend on the choice of the volume form of W, as before.) If W runs all over the 5-dimensional subspaces of V, the 3-vectors

$$(F_1^W \wedge F_1^W)^\# \wedge (F_1^W \wedge F_2^W)^\# \wedge (F_2^W \wedge F_2^W)^\#$$

span a subspace of $\bigwedge^3 V$ which is determined by F_1 and F_2 independently on the choice of the volume form. In the following, in the case $n \geq 6$, we say that the pair of 2-forms $F = (F_1, F_2)$ is "*generic*" if

(1) the dimension of the above subspace of $\bigwedge^3 V$ takes a maximum value,
(2) the Bianchi map φ_F is injective.

(Note that these conditions are natural generalizations of the corresponding genericity conditions in the case $n = 5$ defined before.) Clearly, generic pairs F constitute an open dense subset of $\bigwedge^2 V^* + \bigwedge^2 V^*$. Now, our main result for general n (≥ 5) is the following.

THEOREM 3.3. *Let $F = (F_1, F_2) \in \bigwedge^2 V^* + \bigwedge^2 V^*$ be a generic element. Then, $G \in \bigwedge^3 V^*$ is contained in the image of the Bianchi map φ_F: $V^* + V^* \to \bigwedge^3 V^*$ if and only if*

$$G^W((F_1^W \wedge F_1^W)^\#, (F_1^W \wedge F_2^W)^\#, (F_2^W \wedge F_2^W)^\#) = 0$$

for any 5-dimensional subspace W of V.

PROOF. The case $n = 5$ is already proved in Proposition 3.1. In the following, we consider the case $n \geq 6$. In this case, since the codimension of $\operatorname{Im} \varphi_F$ is equal to $\binom{n}{3} - 2n = 1/6 \cdot n(n+2)(n-5)$ (cf. Lemma 2.2), we have only to show that the 3-vectors

$$(*) \qquad (F_1^W \wedge F_1^W)^\# \wedge (F_1^W \wedge F_2^W)^\# \wedge (F_2^W \wedge F_2^W)^\#$$

span a $1/6 \cdot n(n+2)(n-5)$-dimensional subspace of $\bigwedge^3 V$ when W runs all over 5-dimensional subspaces of V. And for this purpose, we have only to find one pair F satisfying this property because the dimension of this subspace spanned by $(*)$ cannot exceed the value $1/6 \cdot n(n+2)(n-5)$.

In the following, we divide the proof into two cases $n = 6$ and $n \geq 7$. First, we treat the case $n = 6$. Using a basis $\{e_1^*, \cdots, e_6^*\}$ of V^*, we put

$$F_1 = e_1^* \wedge e_2^* + e_5^* \wedge e_6^*,$$

$$F_2 = e_1^* \wedge e_3^* + e_2^* \wedge e_4^*.$$

Then, it is easy to see that φ_F is injective. In the case $n = 6$, the value $1/6 \cdot n(n+2)(n-5)$ is equal to 8, and we will show that 3-vectors $(*)$ span an 8-dimensional subspace of $\bigwedge^3 V$. We restrict the forms F_1, F_2 to the subspace W spanned by the following five vectors

$$v_1 = e_1 + a_1 e_6,$$

$$\cdots\cdots$$

$$v_5 = e_5 + a_5 e_6,$$

where $a_1 \sim a_5$ are real parameters that may be considered as a local coordinate system of the Grassmann manifold, consisting of all 5-dimensional subspaces of V. Let $\{\alpha_1, \cdots, \alpha_5\}$ be a basis of W^*, which is the dual of $\{v_1, \cdots, v_5\}$. Then, in terms of $\{\alpha_i\}$, the forms F_1^W, F_2^W are expressed as

$$F_1^W = \alpha_1 \wedge \alpha_2 - (a_1\alpha_1 + \cdots + a_4\alpha_4) \wedge \alpha_5,$$

$$F_2^W = \alpha_1 \wedge \alpha_3 + \alpha_2 \wedge \alpha_4,$$

and hence, we have

$$F_1^W \wedge F_1^W = -2(a_3\alpha_{1235} + a_4\alpha_{1245}),$$

$$F_1^W \wedge F_2^W = a_2\alpha_{1235} - a_1\alpha_{1245} - a_4\alpha_{1345} + a_3\alpha_{2345},$$

$$F_2^W \wedge F_2^W = -2\alpha_{1234},$$

where $\alpha_{1235} = \alpha_1 \wedge \alpha_2 \wedge \alpha_3 \wedge \alpha_5$ etc. Then, by using the volume form $\Phi = \alpha_1 \wedge \cdots \wedge \alpha_5$, we have

$$(F_1^W \wedge F_1^W)^\# = -2(a_4 v_3 - a_3 v_4),$$

$$(F_1^W \wedge F_2^W)^\# = a_3 v_1 + a_4 v_2 - a_1 v_3 - a_2 v_4,$$

$$(F_2^W \wedge F_2^W)^\# = -2v_5.$$

We express the 3-vector $(*)$ in terms of the basis $\{e_i\}$. Then, after straightforward calculations, we have

$$\begin{aligned}
&1/4\cdot(F_1^W \wedge F_1^W)^{\#} \wedge (F_1^W \wedge F_2^W)^{\#} \wedge (F_2^W \wedge F_2^W)^{\#} \\
&\quad = -a_1a_3a_5e_{346} - a_1a_3e_{345} - a_2a_4a_5e_{346} - a_2a_4e_{345} \\
&\qquad + a_3^2a_5e_{146} + a_3^2e_{145} + a_3a_4a_5(e_{246} - e_{136}) \\
&\qquad + a_3a_4(e_{245} - e_{135}) - a_4^2a_5e_{236} - a_4^2e_{235},
\end{aligned}$$

where $e_{346} = e_3 \wedge e_4 \wedge e_6$ etc. Hence, if the space W varies according as the value of $a_1 \sim a_5$, the 3-vectors $(*)$ span the 8-dimensional subspace

$$\langle e_{135} - e_{245}, e_{136} - e_{246}, e_{145}, e_{146}, e_{235}, e_{236}, e_{345}, e_{346}\rangle \subset \bigwedge^3 V,$$

and hence, this completes the proof of the theorem in the case $n = 6$.

Next, we consider the general case $n \geq 7$. In this case, we prove the theorem completely in the same way as above, but a tremendous amount of calculations is required. First, we put

$$\begin{aligned}
F_1 &= e_1^* \wedge e_2^* + e_5^* \wedge e_6^*, \\
F_2 &= e_1^* \wedge e_3^* + e_4^* \wedge e_7^*,
\end{aligned}$$

and consider the 5-dimensional subspace W of V spanned by

$$\begin{aligned}
v_1 &= e_1 + a_{16}e_6 + \cdots + a_{1n}e_n, \\
&\cdots\cdots\cdots\cdots\cdots \\
v_5 &= e_5 + a_{56}e_6 + \cdots + a_{5n}e_n,
\end{aligned}$$

where $\{a_{ij}\}$ may be considered as a local coordinate system of the Grassmann manifold consisting of all 5-dimensional subspaces of V. We take the same procedure as in the case of $n = 6$. Then, by using the volume form $\Phi = \alpha_1 \wedge \cdots \wedge \alpha_5$, we finally have

$$\begin{aligned}
(F_1^W \wedge F_1^W)^{\#} &= 2(-a_{46}v_3 + a_{36}v_4), \\
(F_1^W \wedge F_2^W)^{\#} &= (a_{26}a_{37} - a_{27}a_{36})v_1 + (a_{17}a_{36} - a_{16}a_{37} + a_{46})v_2 \\
&\quad + (a_{16}a_{27} - a_{17}a_{26} + a_{57})v_3 - a_{26}v_4 - a_{37}v_5, \\
(F_2^W \wedge F_2^W)^{\#} &= 2(-a_{57}v_2 + a_{27}v_5).
\end{aligned}$$

By expressing the vectors v_i in terms of $e_1 \sim e_n$, the above equalities become

$$\begin{aligned}
(F_1^W \wedge F_1^W)^{\#} &= 2\{-a_{46}e_3 + a_{36}e_4 + (a_{36}a_{47} - a_{37}a_{46})e_7 + \cdots \\
&\quad + (a_{36}a_{4n} - a_{3n}a_{46})e_n\},
\end{aligned}$$

$$
\begin{aligned}
(F_1^W \wedge F_2^W)^\# = {} & (a_{26}a_{37} - a_{27}a_{36})e_1 + (a_{17}a_{36} - a_{16}a_{37} + a_{46})e_2 \\
& + (a_{16}a_{27} - a_{17}a_{26} + a_{57})e_3 - a_{26}e_4 - a_{37}e_5 \\
& + (a_{36}a_{57} - a_{37}a_{56})e_6 + (a_{27}a_{46} - a_{26}a_{47})e_7 \\
& + \{(a_{26}a_{37} - a_{27}a_{36})a_{18} + (a_{17}a_{36} - a_{16}a_{37} + a_{46})a_{28} \\
& + (a_{16}a_{27} - a_{17}a_{26} + a_{57})a_{38} - a_{26}a_{48} - a_{37}a_{58}\}e_8 + \cdots \\
& + \{(a_{26}a_{37} - a_{27}a_{36})a_{1n} + (a_{17}a_{36} - a_{16}a_{37} + a_{46})a_{2n} \\
& + (a_{16}a_{27} - a_{17}a_{26} + a_{57})a_{3n} - a_{26}a_{4n} - a_{37}a_{5n}\}e_n, \\
(F_2^W \wedge F_2^W)^\# = {} & 2\{-a_{57}e_2 + a_{27}e_5 + (a_{27}a_{56} - a_{26}a_{57})e_6 \\
& + (a_{27}a_{58} - a_{28}a_{57})e_8 + \cdots + (a_{27}a_{5n} - a_{2n}a_{57})e_n\}.
\end{aligned}
$$

Now, in this situation, we show that the 3-vectors $(*)$ span the $1/6 \cdot n(n+2)(n-5)$-dimensional subspace of $\bigwedge^3 V$ generated by the following vectors:

$e_{123} + e_{247} - e_{356}$,	e_{23i} $(i = 4 \sim n)$,	e_{3ij} $(6 \leq i < j \leq n)$,
$e_{12i} - e_{56i}$ $(i = 4, 7 \sim n)$,	e_{24i} $(i = 5, 6, 8 \sim n)$,	e_{45i} $(i = 8 \sim n)$,
$e_{13i} - e_{47i}$ $(i = 5, 6, 8 \sim n)$,	e_{25i} $(i = 7 \sim n)$,	e_{46i} $(i = 8 \sim n)$,
e_{14i} $(i = 5, 6, 8 \sim n)$,	e_{2ij} $(6 \leq i < j \leq n)$,	e_{4ij} $(8 \leq i < j \leq n)$,
e_{15i} $(i = 7 \sim n)$,	e_{34i} $(i = 5, 6, 8 \sim n)$,	e_{5ij} $(7 \leq i < j \leq n)$,
e_{1ij} $(6 \leq i < j \leq n)$,	e_{35i} $(i = 7 \sim n)$,	e_{ijk} $(6 \leq i < j < k \leq n)$.

But actually, it is difficult to write down all 3-vectors $(*)$ explicitly. And we calculate only several parts of them. First, we calculate 3-vectors in $(*)$ whose coefficients are equal to $a_{26}a_{37}a_{46}a_{57}$. By considering each term of $(F_i^W \wedge F_j^W)^\#$, it is easy to see that the desired vectors are contained in the part

$$
\begin{aligned}
4(-a_{46}e_3 - a_{37}a_{46}e_7) \wedge (a_{26}a_{37}e_1 + a_{46}e_2 + a_{57}e_3 - a_{26}e_4 - a_{37}e_5) \\
\wedge (-a_{57}e_2 - a_{26}a_{57}e_6).
\end{aligned}
$$

Hence, they are equal to $4a_{26}a_{37}a_{46}a_{57}(e_{123} + e_{247} - e_{356})$. Thus, the 3-vector $e_{123} + e_{247} - e_{356}$ is contained in the subspace spanned by $(*)$. We continue this procedure for remaining 3-vectors listed up above. We omit the detailed calculations, and in the following, we only list up the monomials of a_{ij} by which we can extract the above 3-vectors:

$$\begin{array}{ll}
e_{124}-e_{456}: a_{26}a_{36}a_{37}a_{57}, & e_{12i}-e_{56i}: a_{27}a_{36}^2a_{4i}a_{57}, \\
e_{135}+e_{457}: a_{26}a_{27}a_{37}a_{46}, & e_{136}+e_{467}: a_{26}^2a_{37}a_{46}a_{57}, \\
e_{13j}-e_{47j}: a_{27}^2a_{36}a_{46}a_{5j}, & e_{145}: a_{26}a_{27}a_{36}a_{37}, \\
e_{146}: a_{26}^2a_{36}a_{37}a_{57}, & e_{14j}: a_{27}^2a_{36}^2a_{5j}, \\
e_{15i}: a_{27}^2a_{36}^2a_{4i}, & e_{16i}: a_{27}^2a_{36}^2a_{4i}a_{56}, \\
e_{1ij}: a_{27}^2a_{36}^2a_{4i}a_{5j}, & e_{234}: a_{16}a_{27}a_{36}a_{57}, \\
e_{235}: a_{27}a_{46}^2, & e_{236}: a_{26}a_{46}^2a_{57}, \\
e_{237}: a_{36}a_{47}a_{57}^2, & e_{23j}: a_{27}a_{46}^2a_{5j}, \\
e_{245}: a_{27}a_{36}a_{46}, & e_{246}: a_{36}^2a_{57}^2, \\
e_{24j}: a_{36}a_{3j}a_{57}^2, & e_{25i}: a_{27}a_{3i}a_{46}^2, \\
e_{26i}: a_{36}^2a_{4i}a_{57}^2, & e_{27j}: a_{27}a_{3j}a_{46}^2a_{57}, \\
e_{2jk}: a_{27}a_{3k}a_{46}^2a_{5j}, & e_{345}: a_{26}a_{27}a_{46}, \\
e_{346}: a_{26}^2a_{46}a_{57}, & e_{34j}: a_{2j}a_{36}a_{57}^2, \\
e_{357}: a_{27}^2a_{46}^2, & e_{35j}: a_{27}a_{2j}a_{46}^2, \\
e_{367}: a_{27}^2a_{46}^2a_{56}, & e_{36j}: a_{26}^2a_{46}a_{4j}a_{57}, \\
e_{37j}: a_{27}^2a_{46}^2a_{5j}, & e_{3jk}: a_{26}a_{27}a_{46}a_{4k}a_{5j}, \\
e_{45j}: a_{1j}a_{27}^2a_{36}^2, & e_{46j}: a_{2j}a_{36}^2a_{57}^2, \\
e_{4jk}: a_{1k}a_{27}^2a_{36}^2a_{5j}, & e_{57j}: a_{27}^2a_{3j}a_{46}^2, \\
e_{5jk}: a_{27}a_{2j}a_{3k}a_{46}^2, & e_{67j}: a_{27}^2a_{3j}a_{46}^2a_{56}, \\
e_{6jk}: a_{1j}a_{27}^2a_{36}^2a_{4k}a_{56}, & e_{ijk}: a_{1j}a_{27}^2a_{36}^2a_{4i}a_{5k}.
\end{array}$$

In this list, the range of the indices is understood to be

$$7 \le i \le n, \quad 8 \le j \le n, \quad 7 \le i < j \le n, \quad 8 \le j < k \le n, \quad 7 \le i < j < k \le n,$$

if the subscript of 3-vectors e_{***} contains "i", "j", "ij", "jk", or "ijk", respectively. q.e.d.

REMARK. (1) We must divide the above proof into two cases $n = 6$ and $n \ge 7$ because the pair of 2-forms

$$F_1 = e_1^* \wedge e_2^* + e_5^* \wedge e_6^*,$$

$$F_2 = e_1^* \wedge e_3^* + e_2^* \wedge e_4^*,$$

which we used in the former part of the proof generates only 20-dimensional subspace of $\bigwedge^3 V$ in the case $n = 7$, though the codimension of $\operatorname{Im} \varphi_F$ is equal to $1/6 \cdot n(n+2)(n-5) = 21$.

(2) In the case $n \geq 6$, if we fix a 5-dimensional subspace V^5 of $T_x M$, then the curvature Ω naturally determines a flag

$$V^3 \subset V^5 \subset T_x M,$$

under a pointwise genericity condition on Ω, and the above theorem implies that all first order partial differential equations on Ω can be described by considering all such flags. This situation has some resemblance to the curvatures of Riemannian manifolds where the curvatures are completely determined by their sectional curvatures that are decided by 2-dimensional subspaces of $T_x M$.

4. The inverse formula of the Bianchi map $\varphi_F: V^* + V^* \to \bigwedge^3 V^*$

In the rest of this paper, we state several algebraic properties concerning the Bianchi map $\varphi_F: V^* + V^* \to \bigwedge^3 V^*$ associated with 3-dimensional Heisenberg bundles, which is defined in §2. In Lemma 2.2, we proved that the map φ_F is one-to-one in the case $n \geq 6$, and admits a 1-dimensional non-trivial kernel in the case $n = 5$ for generic $F = (F_1, F_2) \in \bigwedge^2 V^* + \bigwedge^2 V^*$. Hence, if $n \geq 6$, the pair of 1-forms (α_1, α_2) is uniquely determined from F and the image $G = \varphi_F(\alpha_1, \alpha_2) \in \bigwedge^3 V^*$, which renders geometrically that the $\langle X_1, X_2 \rangle$-components of the connection 1-form on principal H_3-bundles are uniquely determined from the curvature 2-forms Ω_1, Ω_2 and the exterior derivative $d\Omega_3$. In this section, we give the inverse formula of the map φ_F explicitly for both cases $n \geq 6$ and $n = 5$. But the expressions of the inverse formulas are not so simple as in the case of standard inverse matrices of linear isomorphisms. First, in the case $n \geq 6$, we prove the following proposition.

PROPOSITION 4.1. (*The inverse formula of* φ_F.) *Assume* $n \geq 6$, *and let* $\{e_1, \cdots, e_n\}$ *be a basis of* V. *Then, the following equalities hold if* $G \in \bigwedge^3 V^*$ *is expressed as* $G = F_1 \wedge \alpha_2 - \alpha_1 \wedge F_2$.

$$\begin{aligned}
&\alpha_1(e_1) \cdot \{(e_1 \rfloor F_1) \wedge F_2 \wedge F_2\}_{1234} \wedge \{(e_1 \rfloor F_2) \wedge F_1 \wedge F_1\}_{1234} \\
&\quad = -2\{(e_1 \rfloor F_1) \wedge (e_1 \rfloor G) \wedge F_2\}_{1234} \wedge \{(e_1 \rfloor F_2) \wedge F_1 \wedge F_1\}_{1234} \in \textstyle\bigwedge^2 V^*, \\
&\alpha_2(e_1) \cdot \{(e_1 \rfloor F_1) \wedge F_2 \wedge F_2\}_{1234} \wedge \{(e_1 \rfloor F_2) \wedge F_1 \wedge F_1\}_{1234} \\
&\quad = 2\{(e_1 \rfloor F_1) \wedge F_2 \wedge F_2\}_{1234} \wedge \{(e_1 \rfloor F_2) \wedge (e_1 \rfloor G) \wedge F_1\}_{1234} \in \textstyle\bigwedge^2 V^*.
\end{aligned}$$

(In these expressions, the form $\{\cdots\}_{1234}$ means the interior product $e_4 \rfloor e_3 \rfloor e_2 \rfloor e_1 \rfloor \{\cdots\}$.)

PROOF. We substitute $G = F_1 \wedge \alpha_2 - \alpha_1 \wedge F_2$ into the expression $(e_1 \rfloor F_1) \wedge (e_1 \rfloor G) \wedge F_2$. Then, it is equal to

$$\begin{aligned}
&(e_1 \rfloor F_1) \wedge (e_1 \rfloor F_1) \wedge \alpha_2 \wedge F_2 + \alpha_2(e_1)(e_1 \rfloor F_1) \wedge F_1 \wedge F_2 \\
&\qquad - \alpha_1(e_1)(e_1 \rfloor F_1) \wedge F_2 \wedge F_2 + (e_1 \rfloor F_1) \wedge \alpha_1 \wedge (e_1 \rfloor F_2) \wedge F_2 \\
&\quad = \alpha_2(e_1)(e_1 \rfloor F_1) \wedge F_1 \wedge F_2 - \alpha_1(e_1)(e_1 \rfloor F_1) \wedge F_2 \wedge F_2 \\
&\qquad + (e_1 \rfloor F_1) \wedge \alpha_1 \wedge (e_1 \rfloor F_2) \wedge F_2.
\end{aligned}$$

The following two equalities are easy to check:

$$2e_1 \rfloor \{(e_1 \rfloor F_1) \wedge \alpha_1 \wedge (e_1 \rfloor F_2) \wedge F_2\} = \alpha_1(e_1) \cdot e_1 \rfloor \{(e_1 \rfloor F_1) \wedge F_2 \wedge F_2\},$$

$$2e_1 \rfloor \{(e_1 \rfloor F_1) \wedge F_1 \wedge F_2\} + e_1 \rfloor \{(e_1 \rfloor F_2) \wedge F_1 \wedge F_1\} = 0,$$

and from these equalities, we have

$$2\{(e_1 \rfloor F_1) \wedge \alpha_1 \wedge (e_1 \rfloor F_2) \wedge F_2\}_{1234} = \alpha_1(e_1)\{(e_1 \rfloor F_1) \wedge F_2 \wedge F_2\}_{1234}$$

and

$$\{(e_1 \rfloor F_1) \wedge F_1 \wedge F_2\}_{1234} \wedge \{(e_1 \rfloor F_2) \wedge F_1 \wedge F_1\}_{1234} = 0.$$

Hence, we have the equality

$$\begin{aligned}
&-2\{(e_1 \rfloor F_1) \wedge (e_1 \rfloor G) \wedge F_2\}_{1234} \wedge \{(e_1 \rfloor F_2) \wedge F_1 \wedge F_1\}_{1234} \\
&\quad = -2\alpha_2(e_1)\{(e_1 \rfloor F_1) \wedge F_1 \wedge F_2\}_{1234} \wedge \{(e_1 \rfloor F_2) \wedge F_1 \wedge F_1\}_{1234} \\
&\qquad + 2\alpha_1(e_1)\{(e_1 \rfloor F_1) \wedge F_2 \wedge F_2\}_{1234} \wedge \{(e_1 \rfloor F_2) \wedge F_1 \wedge F_1\}_{1234} \\
&\qquad - 2\{(e_1 \rfloor F_1) \wedge \alpha_1 \wedge (e_1 \rfloor F_2) \wedge F_2\}_{1234} \wedge \{(e_1 \rfloor F_2) \wedge F_1 \wedge F_1\}_{1234} \\
&\quad = \alpha_1(e_1)\{(e_1 \rfloor F_1) \wedge F_2 \wedge F_2\}_{1234} \wedge \{(e_1 \rfloor F_2) \wedge F_1 \wedge F_1\}_{1234}.
\end{aligned}$$

The second equality in this proposition can be proved completely in the same way. q.e.d.

REMARK. (1) We consider the pair of 2-forms

$$F_1 = e_1^* \wedge e_2^* + e_3^* \wedge e_5^*,$$

$$F_2 = e_1^* \wedge e_4^* + e_3^* \wedge e_6^*,$$

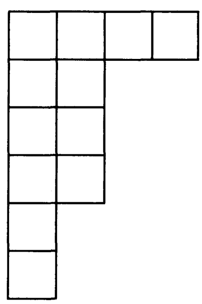
where $\{e_1^*, \cdots, e_n^*\}$ is the dual basis. Then, the form

$$(*) \qquad \{(e_1 \rfloor F_1) \wedge F_2 \wedge F_2\}_{1234} \wedge \{(e_1 \rfloor F_2) \wedge F_1 \wedge F_1\}_{1234}$$

is equal to $4e_5^* \wedge e_6^* \neq 0$, which implies that the 2-form $(*)$ is non-zero for generic pairs $F = (F_1, F_2) \in \bigwedge^2 V^* + \bigwedge^2 V^*$. Hence, from the equalities in

Proposition 4.1, the values $\alpha_1(e_1)$ and $\alpha_2(e_1)$ are uniquely determined from F_1, F_2 and $G = \varphi_F(\alpha_1, \alpha_2)$. By changing the order of $\{e_i\}$ suitably, we can replace the first vector e_1 by an arbitrary e_i, and thus we obtain the desired inverse formula of φ_F, having the above $(*)$ as a typical denominator. Note that this inverse formula essentially depends only on the flag $V^1 \subset V^4 \subset V^6 \subset V$, determined by $V^1 = \langle e_1 \rangle$, $V^4 = \langle e_1, \cdots, e_4 \rangle$ and $V^6 = \langle e_1, \cdots, e_6 \rangle$, but not on the basis $\{e_i\}$ itself. In addition, there exist many ways to express $\alpha_i(e_j)$ in terms of F and G by considering different flags. This implies implicitly that there is an algebraic relation between F and G, which is nothing but the equality stated in Theorem 3.3.

(2) In this inverse formula, the coefficient of $e_5^* \wedge e_6^*$ in the denominator $(*)$ is a polynomial on the space $\bigwedge^2 V^* + \bigwedge^2 V^*$ with total degree 6, which is the generator of the $GL(V)$-invariant subspace of $S^6(\bigwedge^2 V^* + \bigwedge^2 V^*)^*$ corresponding to the Schur function S_{422211}. We can write down it by using the method in [2] with the aid of computers, and as a result, it is expressed as a sum of 240 monomials of the components of F_1 and F_2. The corresponding Young diagram



indicates that the above flag $V^1 \subset V^4 \subset V^6 \subset V$ naturally appears in the expression of this inverse formula.

(3) If we use the flag $V^1 \subset V^2 \subset V^6 \subset V$ where $V^2 = \langle e_1, e_2 \rangle$ instead of the above, then we can formally prove the equality

$$\begin{aligned} &\alpha_1(e_1) \cdot \{(e_1 \lrcorner F_1) \wedge F_2 \wedge F_2\}_{12} \wedge \{(e_1 \lrcorner F_2) \wedge F_1 \wedge F_1\}_{12} \\ &\qquad = -2\{(e_1 \lrcorner F_1) \wedge (e_1 \lrcorner G) \wedge F_2\}_{12} \wedge \{(e_1 \lrcorner F_2) \wedge F_1 \wedge F_1\}_{12} \in \textstyle\bigwedge^6 V^*, \end{aligned}$$

completely in the same way as Proposition 4.1. But, in this case, it is easy to see that the 6-form

$$\{(e_1 \lrcorner F_1) \wedge F_2 \wedge F_2\}_{12} \wedge \{(e_1 \lrcorner F_2) \wedge F_1 \wedge F_1\}_{12}$$

reduces identically to zero, and hence, this equality does not serve as the inverse formula. We also note that the 2-form $(*)$ is always equal to zero in the case $n \leq 5$, and hence the above inverse formula is useful only in the range $n \geq 6$.

By this proposition, we can express the $\langle X_1, X_2 \rangle$-components of the connection 1-form ω in terms of Ω_1, Ω_2 and $d\Omega_3$, which may be considered as a sort of algebraic rigidity on the connection. (Compare the result of Tsarev [13] for the case of $SU(2)$-bundles over $\boldsymbol{R}^4$, where the connection is completely determined by the curvature. See also [12].) By substituting this inverse formula into the structure equations $\Omega_1 = d\omega_1$ and $\Omega_2 = d\omega_2$, we can theoretically obtain the second order partial differential equations on the curvature Ω. But, unfortunately, it is almost impossible to write down them explicitly. Note that in the case of $n \geq 6$, actual curvatures are completely characterized in terms of first and second order partial differential equations under a genericity condition on the pointwise value of Ω on account of the following lemma, which is essentially stated in [7].

LEMMA 4.2. *Let $\Omega = \{\Omega_1, \Omega_2, \Omega_3\}$ be a $\mathfrak{h}_3$-valued 2-form on an n-dimensional manifold M $(n \geq 3)$. Assume that there exist 1-forms ω_1 and ω_2 such that*

$$\begin{aligned} \Omega_1 &= d\omega_1, \\ \Omega_2 &= d\omega_2, \\ d\Omega_3 &= \Omega_1 \wedge \omega_2 - \omega_1 \wedge \Omega_2. \end{aligned}$$

Then, Ω is an actual curvature determined by a connection.

This lemma is easy to prove by applying Poincaré's lemma on the form $\Omega_3 - \omega_1 \wedge \omega_2$. By this lemma, if 1-forms ω_1, ω_2 determined uniquely by Ω_1, Ω_2 and $d\Omega_3$ satisfy the first two equalities $\Omega_1 = d\omega_1$, $\Omega_2 = d\omega_2$, then Ω is an actual curvature. This implies that first and second order partial differential equations are sufficient to characterize actual curvatures for generic cases if $n \geq 6$.

Next, we give the inverse formula of φ_F in the case $n = 5$. In this case (α_1, α_2) is not uniquely determined from F_1, F_2 and G because φ_F always admits a non-trivial 1-dimensional kernel. The result is expressed in the following slightly complicated form.

Using the volume form $\Phi = e_1^* \wedge \cdots \wedge e_5^*$, we define s_{1i}, s_{2i}, $m_{ij} \in \boldsymbol{R}$ by

$$\begin{aligned} s_{1i}\Phi &= 1/2 \cdot F_1 \wedge F_1 \wedge (e_i \rfloor F_2), \\ s_{2i}\Phi &= -1/2 \cdot F_2 \wedge F_2 \wedge (e_i \rfloor F_1), \\ m_{ij}\Phi &= 1/2 \cdot \{(e_i \rfloor F_1) \wedge (e_j \rfloor F_2) + (e_j \rfloor F_1) \wedge (e_i \rfloor F_2)\} \wedge G. \end{aligned}$$

(Note that $m_{ij} = m_{ji}$.) Then, the inverse formula in the case $n = 5$ is expressed in the following form.

Proposition 4.3. *(The inverse formula of φ_F.) Assume $n = 5$ and $G = \varphi_F(\alpha_1, \alpha_2)$ for some $\alpha_1 = \sum \alpha_{1i} e_i^*$, $\alpha_2 = \sum \alpha_{2i} e_i^* \in V^*$. Then, α_{1i} and α_{2i} are expressed as*

$$\alpha_{1i} = -\frac{m_{ii}}{2s_{2i}} + k_i s_{1i},$$

$$\alpha_{2i} = \frac{m_{ii}}{2s_{1i}} + k_i s_{2i},$$

where $\{k_i\}_{1 \le i \le 5}$ are real numbers satisfying

$$k_i - k_j = \frac{(s_{1i}s_{2j} + s_{1j}s_{2i})(s_{1j}s_{2j}m_{ii} + s_{1i}s_{2i}m_{jj}) - 4s_{1i}s_{1j}s_{2i}s_{2j}m_{ij}}{2s_{1i}s_{1j}s_{2i}s_{2j}(s_{1i}s_{2j} - s_{1j}s_{2i})}.$$

Proof. We first show the following equality

$$(**) \qquad \begin{vmatrix} s_{1i} & s_{2j} \\ \alpha_{1i} & \alpha_{2j} \end{vmatrix} + \begin{vmatrix} s_{1j} & s_{2i} \\ \alpha_{1j} & \alpha_{2i} \end{vmatrix} = 2m_{ij}.$$

To prove this, we substitute the vector e_j to the equality

$$\alpha_2 \wedge F_1 \wedge F_1 \wedge (e_i \rfloor F_2) = 0.$$

Then, we have

$$\begin{aligned} 0 &= e_j \rfloor \{\alpha_2 \wedge F_1 \wedge F_1 \wedge (e_i \rfloor F_2)\} \\ &= \alpha_{2j} \cdot F_1 \wedge F_1 \wedge (e_i \rfloor F_2) - 2\alpha_2 \wedge (e_j \rfloor F_1) \wedge F_1 \wedge (e_i \rfloor F_2) - F_{2ij} \cdot \alpha_2 \wedge F_1 \wedge F_1, \end{aligned}$$

and from this equality, we have

$$\begin{aligned} 1/2 \cdot \alpha_{2j} \cdot F_1 \wedge F_1 \wedge (e_i \rfloor F_2) &= \alpha_2 \wedge (e_j \rfloor F_1) \wedge F_1 \wedge (e_i \rfloor F_2) \\ &\quad + 1/2 \cdot F_{2ij} \cdot \alpha_2 \wedge F_1 \wedge F_1. \end{aligned}$$

In the same way, we can prove

$$\begin{aligned} 1/2 \cdot \alpha_{2i} \cdot F_1 \wedge F_1 \wedge (e_j \rfloor F_2) &= \alpha_2 \wedge (e_i \rfloor F_1) \wedge F_1 \wedge (e_j \rfloor F_2) \\ &\quad + 1/2 \cdot F_{2ji} \cdot \alpha_2 \wedge F_1 \wedge F_1. \end{aligned}$$

Adding these two equalities, we have

$$\begin{aligned} (s_{1i}\alpha_{2j} + s_{1j}\alpha_{2i})\Phi &= 1/2 \cdot \{\alpha_{2j} \cdot F_1 \wedge F_1 \wedge (e_i \rfloor F_2) + \alpha_{2i} \cdot F_1 \wedge F_1 \wedge (e_j \rfloor F_2)\} \\ &= \alpha_2 \wedge (e_j \rfloor F_1) \wedge F_1 \wedge (e_i \rfloor F_2) + \alpha_2 \wedge (e_i \rfloor F_1) \wedge F_1 \wedge (e_j \rfloor F_2) \\ &= \{(e_i \rfloor F_1) \wedge (e_j \rfloor F_2) + (e_j \rfloor F_1) \wedge (e_i \rfloor F_2)\} \wedge F_1 \wedge \alpha_2. \end{aligned}$$

Similarly, we have

$$(s_{2j}\alpha_{1i} + s_{2i}\alpha_{1j})\Phi = \{(e_i \rfloor F_1) \wedge (e_j \rfloor F_2) + (e_j \rfloor F_1) \wedge (e_i \rfloor F_2)\} \wedge \alpha_1 \wedge F_2,$$

which combined with the above proves the desired equality (**).

Now, we put $i = j$ in (**). Then, after a slight modification, we have

$$\frac{\alpha_{1i}}{s_{1i}} + \frac{m_{ii}}{2s_{1i}s_{2i}} = \frac{\alpha_{2i}}{s_{2i}} - \frac{m_{ii}}{2s_{1i}s_{2i}},$$

and we express this value as k_i. As a result, we have

$$\alpha_{1i} = -\frac{m_{ii}}{2s_{2i}} + k_i s_{1i},$$

$$\alpha_{2i} = \frac{m_{ii}}{2s_{1i}} + k_i s_{2i}.$$

In addition, we substitute these equalities into (**). Then the desired equality on $k_i - k_j$ follows immediately. q.e.d.

REMARK. (1) Clearly, the above inverse formula contains one free parameter, as we already know from Lemma 2.2. In addition, if $G = 0$, then we have $m_{ij} = 0$ and $k_i = k_j$. Hence, this inverse formula also gives the expression of the canonical 1-dimensional kernel of the map φ_F, which we showed during the proof of Lemma 2.2.

(2) We put

$$s_1 = \sum s_{1i} e_i^* \qquad \text{and} \qquad s_2 = \sum s_{2i} e_i^*.$$

Then the equality

$$4s_1 \wedge s_2 \wedge G = -G((F_1 \wedge F_1)^\#, (F_1 \wedge F_2)^\#, (F_2 \wedge F_2)^\#)\Phi \in \bigwedge^5 V^*$$

holds, where the vectors $(F_i \wedge F_j)^\#$ are defined in terms of the volume form Φ. By this equality, we get another expression for the defining equation of $\operatorname{Im} \varphi_F$.

5. Characterization of singular elements of $\bigwedge^2 V^* + \bigwedge^2 V^*$

In this final section, we prove the theorems which characterize "singular" (and consequently, "generic") elements $F = (F_1, F_2) \in \bigwedge^2 V^* + \bigwedge^2 V^*$ from the standpoint of Lemma 2.2. In this section, we say that F is "*singular*" if the Bianchi map $\varphi_F: V^* + V^* \to \bigwedge^3 V^*$ is not of maximum rank. To state the precise results, we first prepare two notions on F.

We say that $F=(F_1, F_2)$ satisfies condition (R_k) $(k=3, 4, 5, \cdots)$ if there exists a k-dimensional subspace W^* of V^* such that F_1, $F_2 \in \bigwedge^2 W^*$, and F satisfies condition (D) if there exists a pair of real numbers $(k, l) \neq (0, 0)$ such that the 2-form $kF_1 + lF_2$ is decomposable. These two conditions are enough to characterize singular elements. Under these preliminaries, we have the following theorem.

THEOREM 5.1. *Let $F=(F_1, F_2)$ be an element of $\bigwedge^2 V^* + \bigwedge^2 V^*$. Then, F is singular if and only if the following conditions are satisfied.*

The case $n=3: F_1 = F_2 = 0$.
The case $n=4: F$ satisfies condition (R_3).
The case $n=5: F$ satisfies condition (R_4) or (D).
The case $n \geq 6: F$ satisfies condition (R_5) or (D).

By definition, singular elements are characterized in terms of some polynomial relations on the components of F_1 and F_2 that are the minor determinants of the matrix corresponding to φ_F. But these relations may be expressed in a simpler geometric form (i.e., polynomials with lower degree), and to find these polynomials is in general a hard algebraic problem. The following theorem answers to this problem in the case of $n=4$ and 5.

THEOREM 5.2. *An element $F=(F_1, F_2) \in \bigwedge^2 V^* + \bigwedge^2 V^*$ is singular if and only if*

$n=4$: $\quad F_1 \wedge F_1 = F_1 \wedge F_2 = F_2 \wedge F_2 = 0.$

$n=5$: $\quad F_1 \wedge F_1 \wedge (v \rfloor F_2) = F_2 \wedge F_2 \wedge (v \rfloor F_1) = 0 \quad$ *for any $v \in V$,*

or

$$\begin{vmatrix} \langle f_{11} \wedge f_{22}, \alpha \wedge \beta \rangle & 2\langle f_{11} \wedge f_{12}, \alpha \wedge \beta \rangle \\ 2\langle f_{12} \wedge f_{22}, \alpha \wedge \beta \rangle & \langle f_{11} \wedge f_{22}, \alpha \wedge \beta \rangle \end{vmatrix} = 0$$

for any α, $\beta \in V^*$, where $f_{ij} = (F_i \wedge F_j)^{\#} \in V$, and $\langle\ ,\ \rangle$ is the natural pairing of $\bigwedge^2 V$ and $\bigwedge^2 V^*$. (We fix a volume form of $V = \boldsymbol{R}^5$ throughout.)

Note that the above conditions are equivalent to three polynomial relations of the components of F_1 and F_2 with degree 2, 3 and 8 respectively, if we rewrite them by using a basis of V.

To prove these theorems, we must prepare several lemmas. We first give three lemmas concerning conditions $(R_3) \sim (R_5)$. In contrast to the case of a single 2-form, it is slightly difficult to characterize the reducibility of (F_1, F_2) to a low dimensional subspace of V^* in terms of polynomial relations.

LEMMA 5.3. *A pair of 2-forms $F = (F_1, F_2)$ satisfies condition (R_3) if and only if*

$$F_1 \wedge F_1 = F_1 \wedge F_2 = F_2 \wedge F_2 = 0.$$

PROOF. Clearly, we have only to show the "if" part. The case $F_1 = F_2 = 0$ is trivial, and we assume $F_1 \neq 0$. Then, from the condition $F_1 \wedge F_1 = 0$, the form F_1 is expressed as $F_1 = \alpha_1 \wedge \alpha_2$ for some linearly independent 1-forms α_1 and α_2. Then, from the condition $F_1 \wedge F_2 = 0$, we can express F_2 as $F_2 = \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2$, and from the condition $F_2 \wedge F_2 = 0$, it follows that α_1, α_2, β_1, β_2 are linearly dependent, which proves the lemma. q.e.d.

LEMMA 5.4. *A pair of 2-forms $F = (F_1, F_2)$ satisfies condition (R_4) if and only if*

$$F_1 \wedge F_1 \wedge F_1 = F_1 \wedge F_1 \wedge F_2 = F_1 \wedge F_2 \wedge F_2 = F_2 \wedge F_2 \wedge F_2 = 0$$

and

$$F_1 \wedge F_1 \wedge (v \rfloor F_2) = F_2 \wedge F_2 \wedge (v \rfloor F_1) = 0 \qquad \textit{for any } v \in V.$$

PROOF. Considering the degree of the above forms, we know that the "only if" part of this lemma holds trivially. We prove the "if" part. Assume $F_1 \wedge F_1 = F_2 \wedge F_2 = 0$. Then F_1, F_2 are expressed as $F_1 = \alpha_1 \wedge \alpha_2$ and $F_2 = \alpha_3 \wedge \alpha_4$ for some $\alpha_i \in V^*$, and hence the existence of the 4-dimensional subspace W^* follows immediately. Hence, by the symmetry of F_1 and F_2, we may assume $F_1 \wedge F_1 \neq 0$. Then, from the condition $F_1 \wedge F_1 \wedge F_1 = 0$, the form F_1 is expressed as $F_1 = \alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4$ for some linearly independent 1-forms α_i. Then, using the condition $F_1 \wedge F_1 \wedge (v \rfloor F_2) = 0$ for any $v \in V$, we can easily show that $F_2 \in \langle \alpha_i \wedge \alpha_j \rangle_{1 \leq i < j \leq 4}$, and the lemma follows. q.e.d.

REMARK. (1) We may drop the conditions "$F_1 \wedge F_1 \wedge F_2 = F_1 \wedge F_2 \wedge F_2 = 0$" in this lemma. In fact, as the above proof shows, these conditions follow from the remaining conditions automatically. We add these one in order to express the conditions on F in a form which is invariant under the natural group action of $GL(2, \boldsymbol{R})$ on the space $\bigwedge^2 V^* + \bigwedge^2 V^* = \bigwedge^2 V^* \otimes \boldsymbol{R}^2$.

(2) Two types of conditions in this lemma are actually necessary as the following two examples show:

$$F_1 = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*, \qquad F_2 = e_1^* \wedge e_5^*,$$

and

$$F_1 = e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + e_5^* \wedge e_6^*, \qquad F_2 = 0.$$

It is easy to see that the former satisfies only the first condition, the latter satisfies only the second condition, and both pairs cannot be reduced to a 4-dimensional subspace of V^*.

LEMMA 5.5. *A pair of 2-forms $F = (F_1, F_2)$ satisfies condition (R_5) if and only if*

$$F_1 \wedge F_1 \wedge F_1 = F_1 \wedge F_1 \wedge F_2 = F_1 \wedge F_2 \wedge F_2 = F_2 \wedge F_2 \wedge F_2 = 0$$

and

$$\{v \rfloor w \rfloor (F_1 \wedge F_1)\} \wedge \{v \rfloor w \rfloor (F_2 \wedge F_2)\} = 0 \qquad \textit{for any } v, w \in V.$$

PROOF. We first prove the "only if" part. The first equality follows immediately from the fact $\dim W^* = 5$. To prove the second equality, we may assume $v = e_1$ and $w = e_2$, where $\{e_1, \cdots, e_5\}$ is a basis of W. Then, for distinct indices $i \sim l$, the value

$$\{e_1 \rfloor e_2 \rfloor (F_1 \wedge F_1)\} \wedge \{e_1 \rfloor e_2 \rfloor (F_2 \wedge F_2)\}(e_i, e_j, e_k, e_l)$$

is equal to zero because at least one of $i \sim l$ is 1 or 2.

Now, we prove the "if" part. If $F_1 \wedge F_1 = F_2 \wedge F_2 = 0$, then as in the proof of Lemma 5.4, there exists a 4-dimensional subspace W^* of V^* such that $F_1, F_2 \in \bigwedge^2 W^*$. Next, assume $F_1 \wedge F_1 \neq 0$. Then, from the condition $F_1 \wedge F_1 \wedge F_1 = 0$, we have $F_1 = \alpha_1 \wedge \alpha_2 + \alpha_3 \wedge \alpha_4$ for some linearly independent 1-forms α_i. Then, from the condition $F_1 \wedge F_1 \wedge F_2 = 2\alpha_1 \wedge \cdots \wedge \alpha_4 \wedge F_2 = 0$, the 2-form F_2 is expressed as $F_2 = \alpha_1 \wedge \beta_1 + \cdots + \alpha_4 \wedge \beta_4$ for some β_i. In this situation, using the condition $F_2 \wedge F_2 \wedge F_2 = 0$, we can easily show that $\dim\langle \alpha_1, \cdots, \alpha_4, \beta_1, \cdots, \beta_4 \rangle \leq 6$. If the dimension of this space is equal to 6, we may assume that the six forms $\alpha_1, \cdots, \alpha_4, \beta_1, \beta_2$ or $\alpha_1, \cdots, \alpha_4, \beta_1, \beta_3$ are independent on account of the symmetry of β_i. In the first case, we put $v = e_1$ and $w = e_2$, where $\{e_1, \cdots, e_n\}$ is a basis of V satisfying $\alpha_i(e_j) = \delta_{ij}$. Then, we have

$$v \rfloor w \rfloor (F_1 \wedge F_1) = -2\alpha_3 \wedge \alpha_4,$$

$$v \rfloor w \rfloor (F_2 \wedge F_2) \equiv 2\beta_1 \wedge \beta_2 \qquad (\mathrm{mod}\ \alpha_3, \alpha_4),$$

and hence $\{v \rfloor w \rfloor (F_1 \wedge F_1)\} \wedge \{v \rfloor w \rfloor (F_2 \wedge F_2)\} \neq 0$, which contradicts the assumption. In the second case, by putting $v = e_1$ and $w = e_3$, we have the contradiction completely in the same way, and hence, 2-forms F_1 and F_2 belong to the exterior product of the space $\langle \alpha_1, \cdots, \alpha_4, \beta_1, \cdots, \beta_4 \rangle$ with dimension ≤ 5. q.e.d.

REMARK. As in the case of Lemma 5.4, two types of conditions in this lemma are indispensable. In fact, the pair of forms

$$F_1 = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*, \qquad F_2 = e_1^* \wedge e_5^* + e_3^* \wedge e_6^*$$

satisfy only the first condition, and the pair

$$F_1 = e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + e_5^* \wedge e_6^*, \qquad F_2 = 0$$

satisfy only the second condition. Clearly, these pairs cannot be reduced to a 5-dimensional subspace of V^*.

Next, we prepare two lemmas concerning the kernel of the Bianchi map φ_F, which play an important role in characterizing singular elements. To state the result, we define a new condition on F. We say that F satisfies condition (N) if there exist 1-forms α_1, α_2, β_1, β_2, β_3 such that

$$F_1 = \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2,$$
$$F_2 = \alpha_1 \wedge \beta_3 - \alpha_2 \wedge \beta_1.$$

Clearly, condition (N) implies condition (R_5).

LEMMA 5.6. *Let $F = (F_1, F_2)$ be an element of $\bigwedge^2 V^* + \bigwedge^2 V^*$. Then the map φ_F: $V^* + V^* \to \bigwedge^3 V^*$ admits a non-trivial kernel if and only if F satisfies condition (N) or (D).*

PROOF. First, assume that F_1 and F_2 are expressed as

$$F_1 = \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2, \qquad F_2 = \alpha_1 \wedge \beta_3 - \alpha_2 \wedge \beta_1.$$

Then the pair (α_1, α_2) belongs to the kernel of φ_F because

$$F_1 \wedge \alpha_2 - \alpha_1 \wedge F_2 = \alpha_1 \wedge \beta_1 \wedge \alpha_2 + \alpha_1 \wedge \alpha_2 \wedge \beta_1 = 0.$$

If $(\alpha_1, \alpha_2) = 0$, then $F_1 = F_2 = 0$, and the map φ_F also admits a non-trivial kernel. Next, assume that $kF_1 + lF_2$ is decomposable. Then, it is expressed as $\alpha \wedge \beta$ with $\alpha \neq 0$, and it is easily checked that the map φ_F admits a non-trivial kernel $(l\alpha, -k\alpha)$. (Actually, in this case, we have $\dim \operatorname{Ker} \varphi_F \geq 2$ as we shall prove later.)

Now, we show the converse part. Assume that φ_F admits a non-trivial kernel (α_1, α_2), i.e., $F_1 \wedge \alpha_2 - \alpha_1 \wedge F_2 = 0$.

(i) The case α_1, α_2 are linearly independent. In this case, from the above assumption, we have $\alpha_1 \wedge \alpha_2 \wedge F_1 = \alpha_1 \wedge \alpha_2 \wedge F_2 = 0$, and hence F_1 and F_2 are expressed as

$$F_1 = \alpha_1 \wedge \bar{\beta}_1 + \alpha_2 \wedge \beta_2, \qquad F_2 = \alpha_1 \wedge \beta_3 + \alpha_2 \wedge \bar{\beta}_4.$$

Then, we have

$$F_1 \wedge \alpha_2 - \alpha_1 \wedge F_2 = -\alpha_1 \wedge \alpha_2 \wedge (\bar{\beta}_1 + \bar{\beta}_4) = 0,$$

and hence $\bar{\beta}_1 + \bar{\beta}_4 = p\alpha_1 + q\alpha_2$ for some p, $q \in \boldsymbol{R}$. Then, by putting $\beta_1 = \bar{\beta}_1 - p\alpha_1$ $(= q\alpha_2 - \bar{\beta}_4)$, we obtain the desired expressions. (These expressions can be directly obtained by using a generalization of Cartan's lemma stated in [1; p. 473].)

(ii) The case α_1, α_2 are linearly dependent. In this case, $k\alpha_1 + l\alpha_2 = 0$ for some $(k, l) \neq (0, 0)$. By the symmetry, we may assume $l \neq 0$. Then, we have

$$\begin{aligned} 0 = F_1 \wedge \alpha_2 - \alpha_1 \wedge F_2 &= -k/l \cdot F_1 \wedge \alpha_1 - \alpha_1 \wedge F_2 \\ &= -1/l \cdot (kF_1 + lF_2) \wedge \alpha_1, \end{aligned}$$

and hence $kF_1 + lF_2 = \alpha_1 \wedge \beta$ for some β. q.e.d.

REMARK. In the case of $n = 4$, the map $\varphi_F\colon V^* + V^* \to \bigwedge^3 V^*$ clearly admits a non-trivial kernel. Hence, any F satisfies condition (N) or (D). The pair of 2-forms

$$F_1 = e_1^* \wedge e_3^* + e_2^* \wedge e_4^*, \qquad F_2 = e_1^* \wedge e_4^* - e_2^* \wedge e_3^*$$

satisfies (N), but not (D) because

$$(kF_1 + lF_2) \wedge (kF_1 + lF_2) = -2(k^2 + l^2)e_1^* \wedge \cdots \wedge e_4^* \neq 0$$

for $(k, l) \neq (0, 0)$. Conversely, the pair of 2-forms

$$F_1 = F_2 = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*$$

satisfies only (D). In fact, if F_1 and F_2 are expressed as

$$F_1 = \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2, \qquad F_2 = \alpha_1 \wedge \beta_3 - \alpha_2 \wedge \beta_1$$

for some α_i, β_i, then the map φ_F admits a non-trivial kernel (α_1, α_2). But, in this case, the kernel of φ_F must be in the form (γ, γ) $(\gamma \in V^*)$, and hence, we have $\alpha_1 = \alpha_2$. Therefore, F_1 is decomposable, which is a contradiction.

LEMMA 5.7. *Let $F = (F_1, F_2)$ be an element of $\bigwedge^2 V^* + \bigwedge^2 V^*$. Then the map $\varphi_F\colon V^* + V^* \to \bigwedge^3 V^*$ admits a kernel with dimension ≥ 2 if and only if F satisfies condition (R_4) or (D).*

PROOF. We first prove the "if" part. Assume that there exists a 4-dimensional subspace W^* of V^* such that $F_1, F_2 \in \bigwedge^2 W^*$. Then, by Lemma 2.2, the rank of the restricted map $\varphi_F\colon W^* + W^* \to \bigwedge^3 W^*$ is at most 4, and since $\dim(V^* + V^*) - \dim(W^* + W^*) = 2n - 8$, the rank of the original map $\varphi_F\colon V^* + V^* \to \bigwedge^3 V^*$ is at most $(2n - 8) + 4 < 2n - 2$. Next, assume $kF_1 + lF_2$ is expressed as $\alpha_1 \wedge \alpha_2 \neq 0$. Then, it is easy to see that the pairs of forms $(l\alpha_1, -k\alpha_1)$, $(l\alpha_2, -k\alpha_2)$ are in the kernel of φ_F and hence $\dim \operatorname{Ker} \varphi_F \geq 2$. If $kF_1 + lF_2 = 0$, then the pair of 1-forms (β_1, β_2) with $k\beta_1 + l\beta_2 = 0$ belongs to the kernel of φ_F, and hence we also have $\dim \operatorname{Ker} \varphi_F \geq 2$.

Now, conversely, assume that φ_F admits a kernel with dimension ≥ 2. First, if φ_F admits a non-trivial kernel of type $(p\alpha, q\alpha)$, then as we showed in the proof of Lemma 5.6, the 2-form $qF_1 - pF_2$ is decomposable. Next, we divide the remaining situation into three cases according as the type of the kernel. In the following, we assume that the 1-forms $\alpha_1, \cdots, \alpha_4$ are linearly independent.

(i) When (α_1, α_2) and (α_3, α_4) belong to the kernel of φ_F. In this case, from the proof of Lemma 5.6, we have

$$F_1 = \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2 = \alpha_3 \wedge \gamma_1 + \alpha_4 \wedge \gamma_2,$$

$$F_2 = \alpha_1 \wedge \beta_3 - \alpha_2 \wedge \beta_1 = \alpha_3 \wedge \gamma_3 - \alpha_4 \wedge \gamma_1,$$

for some β_i, γ_i. Then, by Cartan's lemma, we have β_i, $\gamma_i \in \langle \alpha_1, \cdots, \alpha_4 \rangle$ and hence the space $W^* = \langle \alpha_1, \cdots, \alpha_4 \rangle$ satisfies the desired property.

(ii) When (α_1, α_2) and $(\alpha_3, p_1\alpha_1 + p_2\alpha_2 + p_3\alpha_3)$ belong to the kernel of φ_F. As above, the forms F_1 and F_2 are expressed as

$$(*) \qquad \begin{aligned} F_1 &= \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2, \\ F_2 &= \alpha_1 \wedge \beta_3 - \alpha_2 \wedge \beta_1. \end{aligned}$$

We multiply the 1-forms α_1 and α_2 to the equality $F_1 \wedge (p_1\alpha_1 + p_2\alpha_2 + p_3\alpha_3) - \alpha_3 \wedge F_2 = 0$. Then, we have

$$(\beta_1 + p_3\beta_2) \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 = (\beta_3 - p_3\beta_1) \wedge \alpha_1 \wedge \alpha_2 \wedge \alpha_3 = 0,$$

and hence $\beta_1 \equiv -p_3\beta_2$, $\beta_3 \equiv -p_3^2\beta_2 \pmod{\alpha_1, \alpha_2, \alpha_3}$. In particular, we have F_1, $F_2 \in \bigwedge^2 \langle \alpha_1, \alpha_2, \alpha_3, \beta_2 \rangle$.

(iii) When (α_1, α_2) and $(p_1\alpha_1 + p_2\alpha_2, p_3\alpha_1 + p_4\alpha_2)$ belong to the kernel of φ_F. By using the above equality $(*)$, we have

$$\begin{aligned} 0 &= F_1 \wedge (p_3\alpha_1 + p_4\alpha_2) - (p_1\alpha_1 + p_2\alpha_2) \wedge F_2 \\ &= \{(p_1 - p_4)\beta_1 + p_3\beta_2 + p_2\beta_3\} \wedge \alpha_1 \wedge \alpha_2. \end{aligned}$$

Hence, we have $(p_1 - p_4)\beta_1 + p_3\beta_2 + p_2\beta_3 \in \langle \alpha_1, \alpha_2 \rangle$. Since (α_1, α_2) and $(p_1\alpha_1 + p_2\alpha_2, p_3\alpha_1 + p_4\alpha_2)$ are not parallel, it follows that one of $p_1 - p_4$, p_2, p_3 is not zero. Hence, we have $\dim\langle \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3 \rangle \leq 4$. q.e.d.

REMARK. We consider the pair of forms

$$F_1 = e_1^* \wedge e_3^* + e_4^* \wedge e_5^*, \qquad F_2 = e_1^* \wedge e_2^*.$$

Then, it is easy to see that $\operatorname{Ker} \varphi_F = \langle (e_1^*, 0), (e_2^*, 0) \rangle$, and hence the case "$\dim \operatorname{Ker} \varphi_F = 2$" actually occurs if $n \geq 5$. On the contrary, if F satisfies condition (R_4), we have $\dim \operatorname{Ker} \varphi_F \geq 4$ as we showed in the above proof.

In the special case $n=5$, we have the following lemma, which may be considered as one of the normal forms of pairs of 2-forms on $\boldsymbol{R}^5$.

LEMMA 5.8. *Assume $n=5$. Then, any pair of 2-forms $F=(F_1,F_2)\in \bigwedge^2 V^* + \bigwedge^2 V^*$ satisfies condition (N) or (D).*

PROOF. In a different situation, we already proved in [2; p. 38] that for any F_1 and F_2, there exist linearly independent 1-forms α_1 and α_2 such that $\alpha_1 \wedge \alpha_2 \wedge F_1 = \alpha_1 \wedge \alpha_2 \wedge F_2 = 0$. Hence, we have

$$F_1 = \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2,$$

$$F_2 = \alpha_1 \wedge \beta_3 + \alpha_2 \wedge \beta_4,$$

for some β_i. Since $\dim V = 5$, we may assume $\beta_4 \in \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3 \rangle$ by the symmetry, and we express $\beta_4 = a_1\alpha_1 + a_2\alpha_2 + b_1\beta_1 + b_2\beta_2 + b_3\beta_3$. We divide the proof into two cases.

(i) The case $b_1 \neq 0$. By putting $\bar{\alpha}_1 = \alpha_1 + b_3\alpha_2$, $\bar{\beta}_1 = \beta_1 + a_1/b_1 \cdot \alpha_1$ and $\bar{\beta}_2 = \beta_2 - b_3\bar{\beta}_1$, 2-forms F_1 and F_2 are expressed as

$$F_1 = \bar{\alpha}_1 \wedge \bar{\beta}_1 + \alpha_2 \wedge \bar{\beta}_2,$$

$$F_2 = \bar{\alpha}_1 \wedge \beta_3 + \alpha_2 \wedge (p\bar{\beta}_1 + q\bar{\beta}_2),$$

where $p = b_1 + b_2 b_3$ and $q = b_2$. If $p = 0$, then the form $qF_1 - F_2$ is equal to $\bar{\alpha}_1 \wedge (q\bar{\beta}_1 - \beta_3)$, which is decomposable. If $p \neq 0$, the above expressions are deformed into

$$F_1 = 1/p \cdot \bar{\alpha}_1 \wedge (p\bar{\beta}_1 + q\bar{\beta}_2) + (q/p \cdot \bar{\alpha}_1 - \alpha_2) \wedge (-\bar{\beta}_2),$$

$$F_2 = 1/p \cdot \bar{\alpha}_1 \wedge (p\beta_3 + pq\bar{\beta}_1 + q^2\bar{\beta}_2) - (q/p \cdot \bar{\alpha}_1 - \alpha_2) \wedge (p\bar{\beta}_1 + q\bar{\beta}_2),$$

and thus F satisfies condition (N).

(ii) The case $b_1 = 0$. In this case, by putting $\bar{\beta}_3 = \beta_3 - a_1\alpha_2$, we have

$$F_1 = \alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2,$$

$$F_2 = \alpha_1 \wedge \bar{\beta}_3 + \alpha_2 \wedge (b_2\beta_2 + b_3\bar{\beta}_3).$$

If $b_2 = 0$, then the form $F_2 = (\alpha_1 + b_3\alpha_2) \wedge \bar{\beta}_3$ is decomposable, and if $b_3 = 0$, then the form $b_2F_1 - F_2 = \alpha_1 \wedge (b_2\beta_1 - \bar{\beta}_3)$ is decomposable. If $b_2 \neq 0$ and $b_3 \neq 0$, then the above expressions are deformed into

$$F_1 = (\alpha_1 + b_3\alpha_2) \wedge 1/b_3 \cdot \beta_2 + b_2\alpha_1 \wedge (1/b_2 \cdot \beta_1 - 1/b_2 b_3 \cdot \beta_2),$$

$$F_2 = (\alpha_1 + b_3\alpha_2) \wedge (b_2/b_3 \cdot \beta_2 + \bar{\beta}_3) - b_2\alpha_1 \wedge 1/b_3 \cdot \beta_2$$

that are the desired expressions. q.e.d.

REMARK. It is easy to see that the pair of 2-forms

$$F_1 = e_1^* \wedge e_3^* + e_4^* \wedge e_5^*,$$
$$F_2 = e_1^* \wedge e_2^*$$

does not satisfy condition (N), and the pair of 2-forms

$$F_1 = e_1^* \wedge e_3^* + e_2^* \wedge e_4^*,$$
$$F_2 = e_1^* \wedge e_5^* - e_2^* \wedge e_3^*$$

does not satisfy condition (D). Hence both cases actually occur. But, the above proof shows that generic pairs of 2-forms $F = (F_1, F_2)$ satisfy condition (N), which may be considered as a normal form of F. On the other hand, pairs satisfying condition (D) are contained in some algebraic set of $\bigwedge^2 V^* + \bigwedge^2 V^*$, as the next lemma shows. We also remark that in the case $n = 5$, the inequality rank $\varphi_F \le 9$ in Lemma 2.2 follows directly from Lemma 5.6 and Lemma 5.8.

LEMMA 5.9. *Assume $n = 5$, and let $F = (F_1, F_2)$ be an element of $\bigwedge^2 V^* + \bigwedge^2 V^*$. If F satisfies condition (D), then with respect to any volume form of V, the following equality holds for any α, $\beta \in V^*$.*

$$\begin{vmatrix} \langle f_{11} \wedge f_{22}, \alpha \wedge \beta \rangle & 2\langle f_{11} \wedge f_{12}, \alpha \wedge \beta \rangle \\ 2\langle f_{12} \wedge f_{22}, \alpha \wedge \beta \rangle & \langle f_{11} \wedge f_{22}, \alpha \wedge \beta \rangle \end{vmatrix} = 0.$$

($f_{ij} = (F_i \wedge F_j)^{\#} \in V$, and $\langle\ ,\ \rangle$ is the natural pairing of $\bigwedge^2 V$ and $\bigwedge^2 V^*$.)

PROOF. First, we consider the natural group action of $GL(2, \boldsymbol{R})$ on the space $\bigwedge^2 V^* + \bigwedge^2 V^* = \bigwedge^2 V^* \otimes \boldsymbol{R}^2$. We put

$$\bar{F}_1 = pF_1 + qF_2,$$
$$\bar{F}_2 = rF_1 + sF_2,$$

with $\Delta = ps - qr \neq 0$, and $\bar{f}_{ij} = (\bar{F}_i \wedge \bar{F}_j)^{\#}$. Then we have

$$\bar{f}_{11} = p^2 f_{11} + 2pq f_{12} + q^2 f_{22},$$
$$\bar{f}_{12} = pr f_{11} + (ps + qr) f_{12} + qs f_{22},$$
$$\bar{f}_{22} = r^2 f_{11} + 2rs f_{12} + s^2 f_{22}.$$

And hence

$$\bar{f}_{11} \wedge \bar{f}_{22} = \Delta\{2pr f_{11} \wedge f_{12} + (ps + qr) f_{11} \wedge f_{22} + 2qs f_{12} \wedge f_{22}\},$$
$$\bar{f}_{11} \wedge \bar{f}_{12} = \Delta\{p^2 f_{11} \wedge f_{12} + pq f_{11} \wedge f_{22} + q^2 f_{12} \wedge f_{22}\},$$
$$\bar{f}_{12} \wedge \bar{f}_{22} = \Delta\{r^2 f_{11} \wedge f_{12} + rs f_{11} \wedge f_{22} + s^2 f_{12} \wedge f_{22}\}.$$

Using these expressions, we can prove the equality

$$(\#) \qquad \begin{vmatrix} \langle \bar{f}_{11} \wedge \bar{f}_{22}, \alpha \wedge \beta \rangle & 2\langle \bar{f}_{11} \wedge \bar{f}_{12}, \alpha \wedge \beta \rangle \\ 2\langle \bar{f}_{12} \wedge \bar{f}_{22}, \alpha \wedge \beta \rangle & \langle \bar{f}_{11} \wedge \bar{f}_{22}, \alpha \wedge \beta \rangle \end{vmatrix}$$

$$= \Delta^4 \begin{vmatrix} \langle f_{11} \wedge f_{22}, \alpha \wedge \beta \rangle & 2\langle f_{11} \wedge f_{12}, \alpha \wedge \beta \rangle \\ 2\langle f_{12} \wedge f_{22}, \alpha \wedge \beta \rangle & \langle f_{11} \wedge f_{22}, \alpha \wedge \beta \rangle \end{vmatrix}$$

for any α, $\beta \in V^*$ after simple calculations. Hence, to prove the lemma, we may replace F_1 and F_2 by $\bar{F}_1$ and $\bar{F}_2$. In particular, we may assume that $\bar{F}_1$ is decomposable. Then, in this case, we have $\bar{f}_{11} = 0$, and the above determinant is clearly equal to zero, which proves the lemma. q.e.d.

REMARK. The above equality $(\#)$ shows that the determinant

$$\begin{vmatrix} \langle f_{11} \wedge f_{22}, \alpha \wedge \beta \rangle & 2\langle f_{11} \wedge f_{12}, \alpha \wedge \beta \rangle \\ 2\langle f_{12} \wedge f_{22}, \alpha \wedge \beta \rangle & \langle f_{11} \wedge f_{22}, \alpha \wedge \beta \rangle \end{vmatrix}$$

is the $GL(2, \boldsymbol{R})$-invariant of the space $\bigwedge^2 V^* + \bigwedge^2 V^* = \bigwedge^2 V^* \otimes \boldsymbol{R}^2$ with degree 8. As we show later, this expression is a non-trivial condition on F. It should be remarked that in the case of $n \geq 6$, the similar results in this lemma hold if we fix a 5-dimensional subspace W, its volume form, and restrict several forms and vectors to W. (See Proposition 5.11.)

We prove one more lemma concerning condition (D).

LEMMA 5.10. *Let $F = (F_1, F_2)$ be an element of $\bigwedge^2 V^* + \bigwedge^2 V^*$. If F satisfies condition (D), then the following equality holds for any $v_1 \sim v_4 \in V$.*

$$\{(v_1 \rfloor F_1) \wedge F_2 \wedge F_2\}_{1234} \wedge \{(v_1 \rfloor F_2) \wedge F_1 \wedge F_1\}_{1234} = 0 \in \textstyle\bigwedge^2 V^*.$$

(The form $\{\cdots\}_{1234}$ implies the interior product $v_4 \rfloor v_3 \rfloor v_2 \rfloor v_1 \rfloor \{\cdots\}$.)

PROOF. We prove this lemma in a similar method as in Lemma 5.9. As above, we put

$$\bar{F}_1 = pF_1 + qF_2,$$

$$\bar{F}_2 = rF_1 + sF_2,$$

with $\Delta = ps - qr \neq 0$. Then, we have

$$\bar{F}_1 \wedge \bar{F}_1 = p^2 F_1 \wedge F_1 + 2pqF_1 \wedge F_2 + q^2 F_2 \wedge F_2,$$

$$\bar{F}_2 \wedge \bar{F}_2 = r^2 F_1 \wedge F_1 + 2rsF_1 \wedge F_2 + s^2 F_2 \wedge F_2,$$

and hence

$$\begin{aligned}(v_1 \lrcorner \bar{F}_1) \wedge \bar{F}_2 \wedge \bar{F}_2 = {} & pr^2(v_1 \lrcorner F_1) \wedge F_1 \wedge F_1 + 2prs(v_1 \lrcorner F_1) \wedge F_1 \wedge F_2 \\ & + ps^2(v_1 \lrcorner F_1) \wedge F_2 \wedge F_2 + qr^2(v_1 \lrcorner F_2) \wedge F_1 \wedge F_1 \\ & + 2qrs(v_1 \lrcorner F_2) \wedge F_1 \wedge F_2 + qs^2(v_1 \lrcorner F_2) \wedge F_2 \wedge F_2,\end{aligned}$$

and

$$\begin{aligned}(v_1 \lrcorner \bar{F}_2) \wedge \bar{F}_1 \wedge \bar{F}_1 = {} & p^2r(v_1 \lrcorner F_1) \wedge F_1 \wedge F_1 + 2pqr(v_1 \lrcorner F_1) \wedge F_1 \wedge F_2 \\ & + q^2r(v_1 \lrcorner F_1) \wedge F_2 \wedge F_2 + p^2s(v_1 \lrcorner F_2) \wedge F_1 \wedge F_1 \\ & + 2pqs(v_1 \lrcorner F_2) \wedge F_1 \wedge F_2 + q^2s(v_1 \lrcorner F_2) \wedge F_2 \wedge F_2.\end{aligned}$$

Using the equalities

$$\{(v_1 \lrcorner F_1) \wedge F_1 \wedge F_1\}_{1234} = \{(v_1 \lrcorner F_2) \wedge F_2 \wedge F_2\}_{1234} = 0,$$

$$\{(v_1 \lrcorner F_1) \wedge F_1 \wedge F_2\}_{1234} = -1/2\{(v_1 \lrcorner F_2) \wedge F_1 \wedge F_1\}_{1234},$$

and

$$\{(v_1 \lrcorner F_2) \wedge F_1 \wedge F_2\}_{1234} = -1/2\{(v_1 \lrcorner F_1) \wedge F_2 \wedge F_2\}_{1234},$$

we have

$$\begin{aligned}\{(v_1 \lrcorner \bar{F}_1) \wedge \bar{F}_2 \wedge \bar{F}_2\}_{1234} = {} & s\Delta\{(v_1 \lrcorner F_1) \wedge F_2 \wedge F_2\}_{1234} \\ & - r\Delta\{(v_1 \lrcorner F_2) \wedge F_1 \wedge F_1\}_{1234}\end{aligned}$$

and

$$\begin{aligned}\{(v_1 \lrcorner \bar{F}_2) \wedge \bar{F}_1 \wedge \bar{F}_1\}_{1234} = {} & -q\Delta\{(v_1 \lrcorner F_1) \wedge F_2 \wedge F_2\}_{1234} \\ & + p\Delta\{(v_1 \lrcorner F_2) \wedge F_1 \wedge F_1\}_{1234}.\end{aligned}$$

Thus, we obtain the equality

$$\begin{aligned}& \{(v_1 \lrcorner \bar{F}_1) \wedge \bar{F}_2 \wedge \bar{F}_2\}_{1234} \wedge \{(v_1 \lrcorner \bar{F}_2) \wedge \bar{F}_1 \wedge \bar{F}_1\}_{1234} \\ & \quad = \Delta^3\{(v_1 \lrcorner F_1) \wedge F_2 \wedge F_2\}_{1234} \wedge \{(v_1 \lrcorner F_2) \wedge F_1 \wedge F_1\}_{1234}.\end{aligned}$$

Hence, as in the proof of Lemma 5.9, we may assume that F_1 is decomposable, i.e., $F_1 \wedge F_1 = 0$ in order to prove the lemma. And, in this case, the equality clearly holds. q.e.d.

REMARK. The expression appeared in this lemma is nothing but the one appeared in Proposition 4.1, which corresponds to the denominator of the inverse formula. It is the $GL(2, \boldsymbol{R})$-invariant of the space $\bigwedge^2 V^* + \bigwedge^2 V^* = \bigwedge^2 V^* \otimes \boldsymbol{R}^2$ with degree 6. We also remark that this expression identically vanishes in the case $n \leq 5$, as we explained in Remark (3) after Proposition 4.1.

Now, under these preliminaries, we prove Theorem 5.1 and Theorem 5.2, simultaneously. In Theorem 5.1, the case $n = 3$ is almost trivial, and the case $n \geq 6$ follows immediately from Lemma 5.6 and Lemma 5.8 because F is singular if and only if φ_F admits a non-trivial kernel. (Note that, as stated before, condition (N) implies condition (R_5).)

In the case $n = 4$, we prove that the following three conditions are equivalent:

(i) $F_1 \wedge F_1 = F_1 \wedge F_2 = F_2 \wedge F_2 = 0$.
(ii) F satisfies condition (R_3).
(iii) F is singular.

The equivalence of (i) and (ii) follows from Lemma 5.3. Next, assume that F satisfies the condition (ii). We take a basis $\{e_1^*, e_2^*, e_3^*, e_4^*\}$ of V^* such that $W^* = \langle e_1^*, e_2^*, e_3^* \rangle$. Then, it is easy to see that the image of the map φ_F is contained in the space $\langle e_1^* \wedge e_2^* \wedge e_3^*, e_4^* \wedge F_1, e_4^* \wedge F_2 \rangle$, and hence we have rank $\varphi_F \leq 3$, which implies that F is singular. Conversely, assuming that F is singular, we show the equalities $F_1 \wedge F_1 = F_1 \wedge F_2 = F_2 \wedge F_2 = 0$. If $F_1 \wedge F_1 \neq 0$, then the form F_1 is expressed as $F_1 = e_1^* \wedge e_2^* + e_3^* \wedge e_4^*$ with respect to some basis $\{e_i^*\}$, and it is easy to check that φ_F is onto in this situation. Hence, we have $F_1 \wedge F_1 = 0$, and in the same way, we have $F_2 \wedge F_2 = 0$. If $F_1 \wedge F_2 \neq 0$, we may express $F_1 = e_1^* \wedge e_2^*$ and $F_2 \equiv k e_3^* \wedge e_4^* \pmod{e_1^*, e_2^*}$ with $k \neq 0$. In this situation, we can also easily show that φ_F is surjective, which is a contradiction. Therefore, we have $F_1 \wedge F_1 = F_1 \wedge F_2 = F_2 \wedge F_2 = 0$.

Finally, we show the theorems in the case $n = 5$. In this case, we consider the following five conditions on F:

(i) $\dim \operatorname{Ker} \varphi_F \geq 2$ (i.e., F is singular).
(ii) F satisfies condition (R_4).
(iii) F satisfies condition (D).
(iv) $F_1 \wedge F_1 \wedge (v \rfloor F_2) = F_2 \wedge F_2 \wedge (v \rfloor F_1) = 0$ for any $v \in V$.
(v) $\begin{vmatrix} \langle f_{11} \wedge f_{22}, \alpha \wedge \beta \rangle & 2\langle f_{11} \wedge f_{12}, \alpha \wedge \beta \rangle \\ 2\langle f_{12} \wedge f_{22}, \alpha \wedge \beta \rangle & \langle f_{11} \wedge f_{22}, \alpha \wedge \beta \rangle \end{vmatrix} = 0$ for any $\alpha, \beta \in V^*$ and for some (and hence, any) volume form of V.

We already proved that F satisfies the condition (i) if and only if it satisfies (ii) or (iii) by Lemma 5.7, and the condition (ii) is equivalent to (iv) by Lemma 5.4. (Note that the first equalities in Lemma 5.4 is automatically satisfied in the case $n = 5$.) In addition, from Lemma 5.9, the condition (iii) implies (v). Hence, to complete the proof, we have only to show that the condition (v) implies (iii) in the case where (iv) does not hold. In this situation, under the condition (v), we assume that there exists a vector $v_0 \in V$ such that

$$F_1 \wedge F_1 \wedge (v_0 \rfloor F_2) \neq 0 \qquad \text{or} \qquad F_2 \wedge F_2 \wedge (v_0 \rfloor F_1) \neq 0.$$

If the form $kF_1 + lF_2$ is not decomposable for any $(k, l) \neq (0, 0)$, then we have by Lemma 5.8

$$F_1 = \alpha_1 \wedge \alpha_3 + \alpha_2 \wedge \alpha_4,$$

$$F_2 = \alpha_1 \wedge \alpha_5 - \alpha_2 \wedge \alpha_3$$

for some α_i. If the forms $\alpha_1 \sim \alpha_5$ are linearly dependent, then the forms F_1 and F_2 can be reduced to a 4-dimensional subspace W^*, which contradicts our assumption that the above vector v_0 exists. (cf. Lemma 5.4.) Hence, the above five 1-forms α_i form a basis of V^*. We denote by $\{e_1, \cdots, e_5\}$ the dual basis. Then, with respect to the volume form $\alpha_1 \wedge \cdots \wedge \alpha_5$, we have $f_{11} = -2e_5$, $f_{12} = e_3$, $f_{22} = 2e_4$. Hence, by putting $\alpha \wedge \beta = \alpha_4 \wedge \alpha_5$, we have

$$\begin{vmatrix} \langle f_{11} \wedge f_{22}, \alpha \wedge \beta \rangle & 2\langle f_{11} \wedge f_{12}, \alpha \wedge \beta \rangle \\ 2\langle f_{12} \wedge f_{22}, \alpha \wedge \beta \rangle & \langle f_{11} \wedge f_{22}, \alpha \wedge \beta \rangle \end{vmatrix} = 16 \neq 0,$$

which contradicts the condition (v). Therefore, there exists a pair $(k, l) \neq (0, 0)$ such that $kF_1 + lF_2$ is decomposable, which completes the proof in the case $n = 5$. q.e.d.

In the general case $n \geq 6$, it is hard to characterize singular elements F only in terms of polynomial relations of F_1 and F_2. The following proposition gives the partial answer to this problem.

PROPOSITION 5.11. *Assume $n \geq 6$, and let $F = (F_1, F_2) \in \bigwedge^2 V^* + \bigwedge^2 V^*$ be a singular element. Then one of the following two cases* (a) *or* (b) *occurs.*

(a) $F_1 \wedge F_1 \wedge F_1 = F_1 \wedge F_1 \wedge F_2 = F_1 \wedge F_2 \wedge F_2 = F_2 \wedge F_2 \wedge F_2 = 0$, *and* $\{v \rfloor w \rfloor (F_1 \wedge F_1)\} \wedge \{v \rfloor w \rfloor (F_2 \wedge F_2)\} = 0$ *for any* $v, w \in V$.

(b) $\{(v_1 \rfloor F_1) \wedge F_2 \wedge F_2\}_{1234} \wedge \{(v_1 \rfloor F_2) \wedge F_1 \wedge F_1\}_{1234} = 0 \in \bigwedge^2 V^*$, *where* $v_1 \sim v_4 \in V$ *and* $\{\cdots\}_{1234} = v_4 \rfloor v_3 \rfloor v_2 \rfloor v_1 \rfloor \{\cdots\}$, *and*

$$\begin{vmatrix} \langle f_{11} \wedge f_{22}, \alpha \wedge \beta \rangle & 2\langle f_{11} \wedge f_{12}, \alpha \wedge \beta \rangle \\ 2\langle f_{12} \wedge f_{22}, \alpha \wedge \beta \rangle & \langle f_{11} \wedge f_{22}, \alpha \wedge \beta \rangle \end{vmatrix} = 0 \quad \textit{for any } \alpha, \beta \in W^*,$$

where W is any 5-dimensional subspace of V and $f_{ij} = (F_i^W \wedge F_j^W)^\# \in W$.

Conversely, if F satisfies the conditions in (a), *then F is singular.*

This proposition follows immediately from Theorem 5.1 (the case $n \geq 6$), Lemma 5.5, Lemma 5.9 (and its Remark), Lemma 5.10, and we omit the details. It is easy to see that the pair

$$F_1 = e_1^* \wedge e_3^* + e_2^* \wedge e_4^*,$$

$$F_2 = e_1^* \wedge e_5^* - e_2^* \wedge e_3^*$$

belongs to the case (a), but not to (b), and conversely, the pair

$$F_1 = e_1^* \wedge e_2^* + e_3^* \wedge e_4^* + e_5^* \wedge e_6^*,$$

$$F_2 = e_3^* \wedge e_4^* + e_5^* \wedge e_6^*,$$

which is also singular, belongs to the case (b), not to (a). Hence, both cases in this proposition actually occur. At present, we do not know whether the conditions in (b) are sufficient to say that F is singular.

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