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Certain maximal oscillatory singular integrals

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ABSTRACT. We prove the L^p-boundedness $1 < p < \infty$, for certain maximal oscillatory singular integral operators.

1. Introduction

Let $y = (y_1, \dots, y_k) \in \mathbb{R}^k$ and $\nabla = (\partial/\partial y_1, \dots, \partial/\partial y_k)$ be the gradient on $\mathbb{R}^k$. $K(y) \in C^1(\mathbb{R}^k \backslash \{0\})$ is said to be a Calderón-Zygmund kernel if there is an $A > 0$ such that

$$|K(y)| \le A|y|^{-k}; \quad |\nabla K(y)| \le A\, |y|^{-k-1}; \tag{1}$$

$$\int_{b \le |y| \le B} K(y)\, d\sigma(y) = 0 \quad \text{for } 0 < b < B < \infty. \tag{2}$$

For a multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k \cup \{0\}$, we write

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_k, \quad D^\alpha = (\partial/\partial y_1)^{\alpha_1} \cdots (\partial/\partial y_k)^{\alpha_k}.$$

Let $\mathscr{P}(y) = (P_1(y), \dots, P_n(y))$ be a polynomial mapping from $\mathbb{R}^k$ to $\mathbb{R}^n$, where each P_j, $j = 1, 2, \dots, n$, is a polynomial on $\mathbb{R}^k$. We define the degree of $\mathscr{P}(y)$ by $\deg(\mathscr{P}) = \max\{\deg(P_1), \deg(P_2), \dots, \deg(P_n)\}$.

The oscillatory singular integral $T_{\mathscr{P},\lambda} f(x)$ is defined by

$$T_{\mathscr{P},\lambda} f(x) = \int_{\mathbb{R}^k} e^{i\lambda\Phi(y)} K(y) f(x - \mathscr{P}(y))\, dy \tag{3a}$$

where $\Phi \in C^\infty(\mathbb{R}^k \backslash \{0\})$ is a real-valued function, $f \in s(\mathbb{R}^n)$, $\lambda \in \mathbb{R}$ and $K(y)$ is a Calderón-Zygmund kernel on $\mathbb{R}^k$. The maximal operator of $T_{\mathscr{P},\lambda}$ is defined by

$$T^*_{\mathscr{P},\lambda} f(x) = \sup_{\varepsilon > 0} \left| \int_{|y| > \varepsilon} e^{i\lambda\Phi(y)} K(y) f(x - \mathscr{P}(y))\, dy \right|. \tag{3b}$$

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For simplicity of the notation, we write

$$\text{(3c)} \qquad Tf(x) = T_{\mathscr{P},\lambda}f(x) \quad \text{if } n = k, \mathscr{P}(y) = (y_1, \dots, y_n) \quad \text{and} \quad \lambda = 1;$$
$$T_{\mathscr{P}}f(x) = T_{\mathscr{P},\lambda}f(x) \quad \text{if } \lambda = 0.$$

Similarly, we define $T^*f(x) = T^*_{\mathscr{P},\lambda}f(x)$ if $n = k$, $\mathscr{P}(y) = (y_1, \dots, y_n)$ and $\lambda = 1$; $T^*_{\mathscr{P}}f(x) = T^*_{\mathscr{P},\lambda}f(x)^0$ if $\lambda = 0$.

The significance and background of studying these operators $T_{\mathscr{P}}$ and T can be found in Stein's book [7] and in paper [5]. In particular, the following L^p-boundedness theorem was proved in [5].

THEOREM 1. *Let T be defined in* (3c). *Suppose that Φ satisfies*

$$\text{(4)} \qquad |D^\alpha \Phi(y)| \le C|y|^{a-|\alpha|} \quad \text{for } |\alpha| \le 3,$$

$$\text{(5)} \qquad \sum_{|\alpha|=2} |D^\alpha \Phi(y)| \ge C'|y|^{a-2}$$

where $a \neq 0$ is a fixed real number, C and C' are constants independent of $y \in \mathbb{R}^k \backslash \{0\}$. Then, for $1 < p < \infty$, there is a $C_p > 0$ such that

$$\|Tf\|_{L^p(\mathbb{R}^k)} \le C_p \|f\|_{L^p(\mathbb{R}^k)}.$$

We recall the following boundedness theorem.

THEOREM 2. (see [7]) *There is a constant C_p independent of the coefficients of $\mathscr{P}(y)$ such that $\|T^*_{\mathscr{P}}f\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(\mathbb{R}^n)}$.*

As a supplement of the paper [5], *in this paper we will prove the following result.*

THEOREM A. *Let $\Phi \in C^\infty(\mathbb{R}^k \backslash \{0\})$ be a real-valued function satisfying*

$$\text{(7)} \qquad \sum_{|\alpha|=m} |D^\alpha \Phi(y)| \ge C'|y|^{a-m}$$

$$\text{(8)} \qquad |D^\alpha \Phi(y)| \le C|y|^{a-|\alpha|} \quad \text{for } |\alpha| = 0 \quad \text{and} \quad m+1,$$

for some fixed $m \ge 2$, $a \neq 0$. Suppose that $\deg(\mathscr{P})$ *is less than m. Then, for $1 < p < \infty$, there is a constant $C_p > 0$ independent of λ and the coefficients of $\mathscr{P}(y)$, such that $\|T^*_{\mathscr{P},\lambda}f\|_{L^p(\mathbb{R}^n)} \le C_p \|f\|_{L^p(\mathbb{R}^n)}$.*

REMARKS.

(i) A constant function does not satisfy condition (7), but in this case the operator $T_{\mathscr{P},\lambda}$ contains no oscillatory factor in its integral so that it is reduced to certain singular integral of Calderón-Zygmund type, whose boundedness is well-known.

(ii) When $k=1$ and $\Phi(y)=|y|^a$, if a is not a non-negative integer, then $\Phi(y)$ satisfies (7) and (8) for all integers m. If a is a non-negative integer, then Φ satisfies (7) and (8) for $m=a-1$, unless $a=0$ and $a=1$. When $a=0$, again $T_{\mathscr{P},\lambda}$ is reduced to the case in (i). When $a=1$, $T_{\mathscr{P},\lambda}$ is known to be unbounded in L^p.

(iii) When $k>1$ and $\Phi(y)=|y|^a$, $a\neq 0$, clearly Φ satisfies (7) and (8) for some integer m, since a simple calculation shows

$$\left(\sum_{|\alpha|=2}|D^\alpha\Phi(y)|^2\right)^{1/2}=|a|\{(a-1)^2+(n-1)\}^{1/2}|y|^{a-2}.$$

In particular, if a is not a positive even integer then $\Phi(y)$ satisfies (7) and (8) for any positive integer m. For simplicity, we explain the reasoning by considering an even integer m. Let Δ be the Laplace operator. By spherical coordinates, it is easy to see that $|\Delta^{m/2}\Phi(y)|=C|y|^{a-m}$. Thus there exist a constant $c_m\in(0,1)$ and a multi-index α with $|\alpha|=m$ such that $|D^\alpha\Phi(y)|\geq c_m|y|^{a-m}$. This shows (7). A direct computation can show that Φ satisfies (8).

(iv) Other functions satisfying (7) and (8) are easily available (see [5]).

By the above (ii) and (iii) in Remarks, we easily obtain the following corollary of Theorem A.

Corollary. *Suppose that $\Phi(y)=|y|^a$ is not a polynomial of $y=(y_1,\dots,y_n)$ then $\|T^*_{\mathscr{P},\lambda}f\|_p\leq C_p\|f\|_p$ for any $\mathscr{P}(y)$, where C_p is a constant independent of the coefficients of $\mathscr{P}(y)$.*

Thus our Theorem A is also an extension of Theorem 2. We notice that in Theorem A, the Calderón-Zygmund kernel $K(y)$ must be a smooth function on $\mathbb{R}^k\backslash\{0\}$. Now we will consider a kernel with a certain roughness. Let S^{k-1} be the unit sphere in $\mathbb{R}^k$, $k\geq 2$, with induced Lebesgue measure $d\sigma$. For any $y\neq 0$, let $y'=y/|y|$ so that $y'\in S^{k-1}$. Suppose that $\Omega(y)=\Omega(y')$ is a function in $L^q(S^{k-1})$, $q>1$, that satisfies the mean zero property

$$\int_{S^{k-1}}\Omega(y')\,d\sigma(y')=0. \tag{9}$$

The rough kernel which shall be studied is defined by $R(y)=\Omega(y)|y|^{-k}$. The L^p boundedness for the singular intergral operator $R*f(x)$ and its varieties were well-studied by many authors (see [2], [4], [1], [3], [6], etc). One of these results is the following theorem.

THEOREM 3. *The maximal integral operator*

$$\sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} R(y)f(x-\mathscr{P}(y))\,dy\right|$$

is bounded in $L^p(\mathbb{R}^n)$ *and the bound is independent of the coefficients of* $\mathscr{P}$.

The proof of Theorem 3 (*actually a more general version*) *can be found in* [6], *where one considers* $R(y)=b(|y|)\Omega(y)|y|^{-k}$ *with* $b\in L^\infty(0,\infty)$ *and* $\Omega\in H^1(S^{k-1})$.

Instead of conditions (7) *and* (8), *we require the phase function* $\Phi(y)$ *is a homogeneous function that satisfies*

$$\Phi(ty)=t^a\Phi(y)\quad \text{for } t>0 \quad \text{and some } a\neq 0;, \tag{10}$$

$$\Phi(y')\in L^\infty(S^{k-1}) \quad \text{and} \quad \int_{S^{k-1}}|\Phi(y')|^{-\delta}\,d\sigma(y')\le C_\Phi<\infty. \tag{11}$$

with some $\delta>0$.

THEOREM B. *Let* $\Omega(y')\in L^q(S^{k-1})$, $q>1$, *satisfy* (9). *Suppose that* Φ *is a function satisfying* (10) *and* (11), *where either the index* $a\neq 0$ *is not a positive integer or* a *is a positive integer larger than* $\deg(\mathscr{P})$. *Then the maximal singular integral*

$$I^*_{\mathscr{P}}f(x)=\sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} e^{i\Phi(y)}R(y)f(x-\mathscr{P}(y))\,dy\right|$$

is bounded in $L^p(\mathbb{R}^n)$. Moreover, the operator norm is independent of the coefficients of $\mathscr{P}$.

2. Some known lemmas

In this section we list several known lemmas which will be used in the proofs of the theorems.

LEMMA 1. *Suppose that* $\Psi\in C_0^1(\mathbb{R}^n)$, ϕ *is real-valued and for some* $m\ge 1$

$$\sum_{|\alpha|=m}|D^\alpha\phi(x)|\ge 1$$

throughout the support of Ψ. *Then*

$$\left|\int_{\mathbb{R}^n} e^{i\lambda\phi(x)}\Psi(x)\,dx\right|\le C_m(\phi)\lambda^{-1/m}(\|\Psi\|_\infty+\|\nabla\Psi\|_1),$$

and the constant $C_m(\phi)$ is independent of λ and Ψ, and remains bounded as long as the C^{m+1} norm of ϕ is bounded.

Lemma 1 *is a slightly stronger version of Proposition* 5 *in Chapter VIII of* [7]. *The proof given by Stein in* [7] *can be used here with a little modification.*

LEMMA 2. *Let the maximal function $M_{\mathscr{P}}f$ be defined by*

$$M_{\mathscr{P}}f(x) = \sup_{r>0} r^{-k}\left|\int_{|y|<r} f(x-\mathscr{P}(y))\,dy\right|.$$

Then $\|M_{\mathscr{P}}f\|_{L^p(\mathbb{R}^n)} \le C_p\|f\|_{L^p(\mathbb{R}^n)}$, where C is a constant independent of the coefficients of $\mathscr{P}(y)$. The proof of Lemma 2 *can be found on page* 485 *in* [7].

For the rough kernel $R(y) = |y|^{-k}\Omega(y')$, we define the maximal operator $\mu^*_{\mathscr{P}}(f)(x)$ by

$$\mu^*_{\mathscr{P}}(f)(x) = \sup_{j\in\mathbb{Z}} \int_{2^{j-1}\le|y|<2^j} |R(y)f(x-\mathscr{P}(y))|\,dy. \tag{16}$$

Then the following lemma is Theorem 7.4 *in* [6].

LEMMA 3. *The operator $\mu^*_{\mathscr{P}}$ is bounded in $L^p(\mathbb{R}^n)$ for $p\in(1,\infty)$. Furthermore, the bound for the operator norm is independent of the coefficients of $\mathscr{P}$.*

3. Proof of Theorem A

We will only prove the case $a>0$ since the proof for $a<0$ is similar; see also the proof of Theorem B. Without loss of generality, we may assume $\lambda>0$. let Ψ be a non-negative C^∞ radial function satisfying

$$\operatorname{supp}(\Psi) \subseteq \{y\in\mathbb{R}^k : 1/2<|y|<2\}$$

and

$$\sum_{j=-\infty}^{\infty} \Psi(2^{-j}y) \equiv 1 \quad \text{for all } y\neq 0.$$

Now for any fixed $\lambda>0$, choose an integer N such that $2^N \cong \lambda^{-1/a}$. We let

$$\Psi_j(y) = \Psi(2^{-j}y),\quad \eta(y) = 1-\sum_{j=N+1}^{\infty}\Psi_j(y),\quad \Omega_0(y) = e^{i\lambda\Phi(y)}K(y)\eta(y) \quad \text{and}$$

$\Omega_j(y) = e^{i\lambda\Phi(y)}K(y)\Psi_j(y)$ for $j=1,2,\ldots.$ We also write

$$\Delta(y) = \sum_{j=N+1}^{\infty} e^{i\lambda\Phi(y)}\Psi_j(y)K(y).$$

Then

$$T^*_{\mathscr{P},\lambda}f(x) \le \sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} \Omega_0(y)f(x-\mathscr{P}(y))\,dy\right| + \sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} \Delta(y)f(x-\mathscr{P}(y))\,dy\right|$$
$$= I_1 f(x) + I_2 f(x).$$

Noting that $\operatorname{supp}\Omega_0 \subseteq \{y \in \mathbb{R}^k : |y| \le 2^{N+1}\}$ and $\eta(y) \equiv 1$ for $|y| < 2^N$, we can decompose

$$\Omega_0(y) = K(y)\chi_{\{|y|<2^N\}}(y) + K(y)(e^{i\lambda\Phi(y)} - 1)\chi_{\{|y|<2^N\}}(y)$$
$$+ e^{i\lambda\Phi(y)}K(y)\eta(y)\chi_{\{2^N \le |y| \le 2^{N+1}\}}(y).$$

Then we easily see that

$$I_1 f(x) \le J_1 f(x) + J_2 f(x) + J_3 f(x).$$

Here

$$J_1 f(x) = \sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} K(y)\chi_{\{|y|<2^N\}}(y)f(x-\mathscr{P}(y))\,dy\right|$$
$$= \sup_{\varepsilon>0}\left|\int_{\varepsilon<|y|<2^N} K(y)f(x-\mathscr{P}(y))\,dy\right|$$
$$\le \sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} K(y)f(x-\mathscr{P}(y))\,dy\right| + \left|\int_{|y|>2^N} K(y)f(x-\mathscr{P}(y))\,dy\right|.$$

By Theorem 2 we know that $J_1 f(x)$ is bounded in $L^p(\mathbb{R}^n)$. Next, by (8) it is easy to see

$$J_2 f(x) \le \int_{|y|<2^N} |K(y)|\,|e^{i\lambda\Phi(y)} - 1|\,|f(x-\mathscr{P}(y))|\,dy$$
$$\le C\lambda \int_{|y|<2^N} |y|^{-k+a}|f(x-\mathscr{P}(y))|\,dy$$
$$\le C\sum_{i=-\infty}^{N} \lambda \int_{2^{i-1}\le|y|<2^i} |y|^{-k+a}\,|f(x-\mathscr{P}(y))|\,dy$$
$$\le C\lambda \sum_{i=-\infty}^{N} 2^{ai}2^{-ik}\int_{|y|<2^i} |f(x-\mathscr{P}(y))|\,dy \le C\lambda 2^{aN} M_{\mathscr{P}}|f|(x).$$

So by the choice of N and Lemma 2, we easily see $\|J_2 f\|_p \le C_p\|f\|_p$. Finally,

$$J_3 f(x) \le \int_{2^{N-1}\le|y|\le 2^{N+1}} |y|^{-k}\,|f(x-\mathscr{P}(y))|\,dy \le C\,M_{\mathscr{P}}|f|(x).$$

This proves the L^p-boundedness for $I_1 f$. It remains to prove the L^p boundedness for $\sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} \Delta(y) f(x-\mathscr{P}(y))\,dy\right|$.

By the support condition of Ω_j, we know $\operatorname{supp}\Delta(y) \subseteq \{y, |y| > 2^N\}$. Thus

$$\sup_{\varepsilon>0}\left|\int_{|y|>\varepsilon} \Delta(y) f(x-\mathscr{P}(y))\,dy\right| = \sup_{\varepsilon\geq 2^N}\left|\int_{|y|>\varepsilon} \Delta(y) f(x-\mathscr{P}(y))\,dy\right|$$
$$= \sup_{\varepsilon>2^N}\left|\sum_{j=N+1}^{\infty}\int_{|y|>\varepsilon} \Omega_j(y) f(x-\mathscr{P}(y))\,dy\right|.$$

For any $\varepsilon \geq 2^N$ choose an integer $\nu \geq N+1$ such that $2^{\nu-1} \leq \varepsilon < 2^\nu$. Thus

$$\left|\sum_{j=N+1}^{\infty}\int_{|y|>\varepsilon} \Omega_j(y) f(x-\mathscr{P}(y))\,dy\right|$$
$$\leq \left|\sum_{j=N+1}^{\infty}\int_{|y|\geq 2^\nu} \Omega_j(y) f(x-\mathscr{P}(y))\,dy\right|$$
$$+\left|\sum_{j=N+1}^{\infty}\int_{\varepsilon<|y|<2^\nu} \Omega_j(y) f(x-\mathscr{P}(y))\,dy\right|$$
$$= Af(x) + Bf(x).$$

By the support condition of Ω_j, it is easy to see that

$$Bf(x) \leq \sum_{j=N}^{\infty}\int_{2^{\nu-1}<|y|<2^\nu} |\Omega_j(y)|\,|f(x-\mathscr{P}(y))|\,dy$$
$$= \sum_{j=\nu-1}^{\nu+1}\int_{2^{\nu-1}<|y|<2^\nu} |\Omega_j(y)|\,|f(x-\mathscr{P}(y))|\,dy \leq C\,M_{\mathscr{P}}|f|(x).$$

Thus we have $\|Bf\|_p \leq C_p\|f\|_p$. To estimate $Af(x)$, we notice that, by the support condition of Ω_j,

$$Af(x) = \left|\sum_{j=\nu+1}^{\infty}\int_{|y|\geq 2^\nu} \Omega_j(y) f(x-\mathscr{P}(y))\,dy\right| + C\,M_{\mathscr{P}} f(x)$$
$$\leq \left|\sum_{j=\nu+1}^{\infty}\int_{\mathbb{R}^k} \Omega_j(y) f(x-\mathscr{P}(y))\,dy\right|$$
$$+\left|\sum_{j=\nu+1}^{\infty}\int_{|y|<2^\nu} \Omega_j(y) f(x-\mathscr{P}(y))\,dy\right| + C\,M_{\mathscr{P}} f(x)$$
$$\leq \left|\sum_{j=\nu+1}^{\infty}\int_{\mathbb{R}^k} \Omega_j(y) f(x-\mathscr{P}(y))\,dy\right| + C\,M_{\mathscr{P}} f(x).$$

The last inequality above follows from the fact that the support condition of Ω_j implies

$$\int_{|y|<2^v} \Omega_j(y) f(x - \mathscr{P}(y))\, dy = 0$$

for all $j \geq v+1$. This proves

$$\sup_{\varepsilon \geq 2^N} \left| \sum_{j=N+1}^{\infty} \int_{|y|\geq\varepsilon} \Omega_j(y) f(x - \mathscr{P}(y))\, dy \right|$$
$$\leq C M_{\mathscr{P}} f(x) + \sup_{v\geq N} \left| \sum_{j=v+1}^{\infty} \int_{\mathbf{R}^k} \Omega_j(y) f(x - \mathscr{P}(y))\, dy \right|.$$

Let

$$\sup_{v\geq N} \left| \sum_{j=v+1}^{\infty} \int_{\mathbf{R}^k} \Omega_j(y) f(x - \mathscr{P}(y))\, dy \right| = G(f)(x).$$

To prove the theorem, it suffices to show that for $1 < p < \infty$,

$$\|G(f)\|_{L^p(\mathbf{R}^n)} \leq C_p \|f\|_{L^p(\mathbf{R}^n)}.$$

In fact, if we write

$$\Omega_j * f(x) = \int_{\mathbf{R}^k} \Omega_j(y) f(x - \mathscr{P}(y))\, dy,$$

then

$$G(f)(x) = \sup_{v\geq N} \left| \sum_{j=1}^{\infty} \Omega_{j+v} * f(x) \right| \leq \sum_{j=1}^{\infty} \sup_{v\geq N} |\Omega_{v+j} * f(x)|.$$

It is easy to see $\sup_{v\geq N} |\Omega_{j+v} * f(x)| \leq C M_{\mathscr{P}} f(x)$, where C is a constant independent of N and j. So

$$\| \sup_{v\geq N} |\Omega_{j+v} * f| \|_p \leq C_p \|f\|_p, \quad 1 < p < \infty \tag{17}$$

with C_p independent of j, N, f and the coefficients of $\mathscr{P}(y)$. On the other hand,

$$\sup_{v\geq N} |\Omega_{j+v} * f(x)| \leq \sum_{v=N}^{\infty} |\Omega_{j+v} * f(x)|.$$

Thus

$$\|\sup_{\nu\geq N} |\Omega_{j+\nu} * f(x)|\,\|_{L^2(\mathbb{R}^n)} \leq \sum_{\nu=N}^{\infty} \|\Omega_{j+\nu} * f\|_{L^2(\mathbb{R}^n)}$$

$$\leq C\sum_{\nu=N}^{\infty} \|\hat{f}\|_2\, \|\hat{\Omega}_{j+\nu}\|_\infty \leq C\|f\|_2 \sum_{\nu=N}^{\infty} \|\hat{\Omega}_{j+\nu}\|_\infty.$$

We note that

$$\hat{\Omega}_j(\xi) = 2^{jk}\int_{\mathbb{R}^k} e^{i\lambda\{\Phi(2^jy)-\lambda^{-1}\langle\mathscr{P}(2^jy),\xi\rangle\}} K(2^jy)\Psi(y)\,dy.$$

Let $\phi(y) = 2^{-ja}\{\Phi(2^jy) - \lambda^{-1}\langle\xi,\mathscr{P}(2^jy)\rangle\}$. If $|\alpha| = m \geq 2$ then $D^\alpha\phi(y) = 2^{j(m-a)}(D^\alpha\Phi)(2^jy)$, since $\deg(\mathscr{P}) \leq m-1$. Thus by (7), for $1/2 < |y| \leq 2$, we have

$$\sum_{|\alpha|=m} |D^\alpha\phi(y)| \geq C'2^{j(m-a)}|2^jy|^{a-m} \geq C.$$

Similarly, for $|y| \leq 2$ and $|\alpha| = m+1$, by (8) we have $|D^\alpha\phi(y)| \leq C$. Invoking Lemma 1 and (1), we obtain that $\|\hat{\Omega}_j\|_\infty \leq C2^{-ja/m}\lambda^{-1/m}$ for all j. Thus by the choice of N, we have

$$\sum_{\nu=N}^{\infty} \|\hat{\Omega}_{j+\nu}\|_\infty \leq 2^{-ja/m}\lambda^{-1/m}\sum_{\nu=N}^{\infty} 2^{-a\nu/m} \leq C2^{-ja/m}.$$

This proves

$$\left\|\sup_{\nu\geq N} |\Omega_{j+\nu} * f|\right\|_2 \leq C2^{-ja/m}\|f\|_2. \tag{18}$$

By (17) and (18) and interpolation for any $1 < p < \infty$, we have

$$\left\|\sup_{nu\geq N} |\Omega_{j+\nu} * f|\right\|_p \leq C2^{-j\delta}\|f\|_p \quad \text{for some } \delta > 0.$$

Therefore we prove that $\|G(f)\|_p \leq C_p\|f\|_p$ for $1 < p < \infty$. From the above proof, we can see that the constant C_p in the last inequality is independent of all the essential variables. The theorem is proved.

4. Proof of Theorem B

The proof of Theorem B is essentially the same as that of Theorem A. For the sake of completeness, we will prove the case $a < 0$. Let Ψ and

Ψ_j be the same as in the proof of Theorem A. Let

$$\eta(y) = \sum_{j=1}^{\infty} \Psi_j(y), \quad \Omega_\infty(y) = e^{i\Phi(y)} R(y)\eta(y),$$

$$\Omega_j(y) = e^{i\Phi(y)} R(y)\Psi_j(y), \quad \Delta(y) = \sum_{j=-\infty}^{0} e^{i\Phi(y)} R(y)\Psi_j(y).$$

Then

$$I^*_{\mathscr{P}} f(x) \leq \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} \Omega_\infty(y) f(x-\mathscr{P}(y))\, dy \right| + \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} \Delta(y) f(x-\mathscr{P}(y))\, dy \right|$$
$$= \tilde{I}_1 f(x) + \tilde{I}_2 f(x).$$

Noting that $\operatorname{supp} \Omega_\infty \subseteq \{y \in \mathbb{R}^k : |y| \geq 2^{-3}\}$ and $\eta(y) \equiv 1$ for $|y| > 2$, we can decompose

$$\Omega_\infty(y) = R(y)\chi_{\{|y|>2\}}(y) + R(y)(e^{i\Phi(y)} - 1)\chi_{\{|y|>2\}}(y)$$
$$+ e^{i\Phi(y)} R(y)\eta(y)\chi_{\{2^{-3} \leq |y| < 2\}}(y).$$

Then we easily see that

$$\tilde{I}_1 f(x) \leq \tilde{J}_1 f(x) + \tilde{J}_2 f(x) + \tilde{J}_3 f(x).$$

Here

$$\tilde{J}_1 f(x) = \sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} R(y)\chi_{\{|y|>2\}}(y) f(x-\mathscr{P}(y))\, dy \right|.$$

By Theorem 3 we know that $\tilde{J}_1 f$ is bounded in $L^p(\mathbb{R}^n)$. Next, by (10) it is easy to see

$$\tilde{J}_2 f(x) \leq \int_{|y|\geq 1} |R(y)|\, |e^{i\Phi(y)} - 1|\, |f(x-\mathscr{P}(y))|\, dy$$
$$\leq C \int_{|y|\geq 1} |y|^{-k+a}\, |\Phi(y')\Omega(y') f(x-\mathscr{P}(y))|\, dy.$$

Thus, noting that $\Phi(y') \in L^\infty(S^{k-1})$, we have

$$\tilde{J}_2 f(x) \leq C_\Phi \sum_{i=1}^{\infty} 2^{ia} \int_{2^{i-1} \leq |y| < 2^i} |y|^{-k}\, |\Omega(y')|\, |f(x-\mathscr{P}(y))|\, dy$$
$$\leq C \sum_{i=1}^{\infty} 2^{ai} \mu^*_{\mathscr{P}}(f)(x).$$

So by Lemma 3 and the fact $a < 0$, we easily see $\|\tilde{J}_2 f\|_p \le C_p \|f\|_p$. Also it is easy to see

$$\tilde{J}_3 f(x) \le \int_{2^{-3} \le |y| \le 2} |y|^{-k} |\Omega(y') f(x - \mathscr{P}(y))| \, dy \le C\mu^*_{\mathscr{P}}(f)(x).$$

This proves the L^p-boundedness for $\tilde{I}_1 f(x)$. It remains to prove the L^p boundedness for $\sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} \Delta(y) f(x - \mathscr{P}(y)) \, dy \right|$.

By the support condition of Ω_j, we know $\operatorname{supp} \Delta(y) \subseteq \{y, |y| < 2\}$. Thus

$$\begin{aligned}
\sup_{\varepsilon>0} \left| \int_{|y|>\varepsilon} \Delta(y) f(x - \mathscr{P}(y)) \, dy \right| &= \sup_{0<\varepsilon<2} \left| \int_{|y|>\varepsilon} \Delta(y) f(x - \mathscr{P}(y)) \, dy \right| \\
&= \sup_{0<\varepsilon<2} \left| \sum_{j=-\infty}^{0} \int_{|y|>\varepsilon} \Omega_j(y) f(x - \mathscr{P}(y)) \, dy \right|.
\end{aligned}$$

For any $\varepsilon \in (0, 2)$, choose an integer $\nu \le 1$ such that $2^{\nu-1} \le \varepsilon < 2^\nu$. Thus

$$\begin{aligned}
&\left| \sum_{j=-\infty}^{0} \int_{|y|>\varepsilon} \Omega_j(y) f(x - \mathscr{P}(y)) \, dy \right| \\
&\quad \le \left| \sum_{j=-\infty}^{0} \int_{2>|y|\ge 2\nu} \Omega_j(y) f(x - \mathscr{P}(y)) \, dy \right| \\
&\quad\quad + \left| \sum_{j=-\infty}^{0} \int_{\varepsilon<|y|<2^\nu} \Omega_j(y) f(x - \mathscr{P}(y)) \, dy \right| \\
&\quad = \tilde{A}f(x) + \tilde{B}f(x).
\end{aligned}$$

By the support condition of Ω_j, it is easy to see that

$$\tilde{B}f(x) \le \sum_{j=\nu-1}^{\nu+1} \int_{2^{\nu-1} \le |y| < 2^\nu} |\Omega_j(y)| \, |f(x - \mathscr{P}(y))| \, dy \le C\mu^*_{\mathscr{P}}(|f|)(x).$$

To estimate $\tilde{A}f(x)$, we notice that, by the support condition of Ω_j,

$$\begin{aligned}
\tilde{A}f(x) &= \left| \sum_{j=\nu-1}^{0} \int_{|y|\ge 2^\nu} \Omega_j(y) f(x - \mathscr{P}(y)) \, dy \right| + C\mu^*_{\mathscr{P}}(f)(x) \\
&\le \left| \sum_{j=\nu-1}^{0} \int_{\mathbb{R}^n} \Omega_j(y) f(x - \mathscr{P}(y)) \, dy \right| + C\mu^*_{\mathscr{P}}(f)(x).
\end{aligned}$$

The last inequality above follows from the fact that the support condition of Ω_j

implies

$$\int_{|y|<2^{\nu}} \Omega_j(y) f(x - \mathscr{P}(y))\, dy = 0$$

for all $j \geq \nu + 2$. This proves

$$\begin{aligned}
\sup_{\varepsilon \geq 0} &\left| \int_{|y|>\varepsilon} \Delta(y) f(x - \mathscr{P}(y))\, dy \right| \\
&\leq \sup_{\varepsilon < 2} \left| \sum_{j=-\infty}^{0} \int_{|y| \geq \varepsilon} \Omega_j(y) f(x - \mathscr{P}(y))\, dy \right| \\
&\leq C\mu^*_{\mathscr{P}}(f)(x) + \sup_{\nu \leq 1} \left| \sum_{j=\nu-1}^{0} \int_{\mathbb{R}^k} \Omega_j(y) f(x - \mathscr{P}(y))\, dy \right|.
\end{aligned}$$

Let

$$\sup_{\nu \leq 1} \left| \sum_{j=\nu-1}^{0} \int_{\mathbb{R}^k} \Omega_j(y) f(x - \mathscr{P}(y))\, dy \right| = \tilde{G}(f)(x).$$

To prove the theorem, it suffices to show for $1 < p < \infty$,

$$\|\tilde{G}(f)\|_{L^p(\mathbb{R}^n)} \leq C_p \|f\|_{L^p(\mathbb{R}^n)}.$$

In fact, we write

$$\Omega_j * f(x) = \int_{\mathbb{R}^k} \Omega_j(y) f(x - \mathscr{P}(y))\, dy.$$

Then

$$\tilde{G}(f)(x) = \sup_{\nu \leq 1} \left| \sum_{j=-\infty}^{0} \Omega_{j+\nu} * f(x) \right| \leq \sum_{j=-\infty}^{0} \sup_{\nu \leq 1} |\Omega_{\nu+j} * f(x)|.$$

It is easy to see $\sup_{\nu} |\Omega_{j+\nu} * f(x)| \leq C\mu^*_{\mathscr{P}}(f)(x)$.

So

$$\| \sup_{\nu \leq 1} |\Omega_{j+\nu} * f| \|_p \leq C_p \|f\|_p, \quad 1 < p < \infty. \tag{18}$$

On the other hand,

$$\sup_{\nu \leq 1} |\Omega_{j+\nu} * f(x)| \leq \sum_{v=-\infty}^{1} |\Omega_{j+v} * f(x)|.$$

Thus

$$\|\sup_{v\le 1}|\Omega_{j+v} * f(x)\|_{L^2(\mathbb{R}^n)} \le \sum_{v=-\infty}^{1} \|\Omega_{j+v} * f\|_{L^2(\mathbb{R}^n)}$$

$$\le C \sum_{v=-\infty}^{1} \|\hat{f}\|_2 \|\hat{\Omega}_{j+v}\|_\infty \le C\|f\|_2 \sum_{v=-\infty}^{1} \|\hat{\Omega}_{j+v}\|_\infty,$$

where

$$|\hat{\Omega}_j(\xi)| = \left|\int_{S^{k-1}} \Omega(y') \int_0^\infty e^{i\{t^a 2^{aj}\Phi(y') - \langle \mathcal{P}(2^j t y'), \xi\rangle\}} t^{-1} \Psi(t)\, dt\, d\sigma(y')\right|$$

$$\le \int_{S^{k-1}} |\Omega(y')| I_j(\Phi(y'))\, d\sigma(y')$$

and

$$I_j(\Phi(y')) = \left|\int_0^\infty e^{i\{t^a 2^{aj}\Phi(y') - \langle \mathcal{P}(2^j t y'), \xi\rangle\}} \Psi(t) t^{-1}\, dt\right|.$$

Defining a function $\tilde{\Psi}$ by $\tilde{\Psi}(t) = \Psi(t)$ if $t \ge 0$ and $\tilde{\Psi}(t) = 0$ if $t < 0$, we have

$$I_j(\Phi(y')) \le C \min\{1, |2^{aj}\Phi(y')|^{-1/m}\},$$

by Lemma 1. Thus we let $I_j(\Phi(y')) \le C|2^{aj}\Phi(y')|^{-\delta/q'}$, where $0 < \delta < q'/m$. Then using Hölder's inequality and the condition on Φ, we have

$$|\hat{\Omega}_j(\xi)| \le C_\Phi 2^{-\delta aj/q'} \|\Omega\|_q.$$

This shows

$$\sum_{v=-\infty}^{1} \|\hat{\Omega}_{j+v}\|_\infty \le C2^{-\delta aj/q'} \sum_{v=-\infty}^{1} 2^{-\delta av/q'} \le C2^{-\delta aj/q'},$$

since $a < 0$. Now using (18) and interpolation, we obtain the L^p boundedness of $\tilde{G}(f)$. The theorem is proved.

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