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A finite element analysis for a thermal convection problem with the infinite Prandtl number

Hajime ITOH and Masahisa TABATA

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ABSTRACT. A finite element analysis for a thermal convection problem with the infinite Prandtl number is discussed. This problem is observed, for example, in the earth's mantle convection and in some industrial plants such as glass melting furnaces, where the Prandtl number is set to be infinite as a limit of high Prandtl numbers. A finite element scheme for the thermal convection problem with the infinite Prandtl number is presented. Error estimates of the finite element solution are established. Sample numerical simulation results are shown, which agree well with the theoretical ones.

1. Introduction

Numerical simulations in thermal convection problems for fluid with high or infinite Prandtl numbers have been a topic of interest for geophysicists because of its application to the analysis of mantle convection [8], [9], [15], [16], [19]. These problems appear also in some industrial plants such as glass melting furnaces. Since the inside temperature of the glass furnace goes up to about 1500 °C and direct measurement of physical quantities is quite difficult, numerical simulation in thermal convection is a powerful tool for analyzing phenomena in glass melting furnaces [11], [12], [14].

In the numerical simulation in the thermal convection problems, the Boussinesq equations are commonly used as governing equations, where the incompressible Navier-Stokes equations and energy equation are coupled with the term of buoyant force induced by thermal expansion and the term of thermal convection associated with the flow. When the kinematic viscosity of the fluid is much greater than the thermal diffusion, the phenomena can be approximated by equations in which the Prandtl number is assumed to be infinite as a limit of high Prandtl numbers. The system of equations obtained in this manner is a mathematical model of such thermal convection phenomena with the infinite Prandtl number, which is observed in the glass manufacturing

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process and so on. The model has also been used in the analysis of mantle movement in geophysics [9], [18].

The purpose of this paper is to present a finite element scheme for the thermal convection problem with the infinite Prandtl number and to give error estimates. To the best of our knowledge there is no literature discussing error estimates for this problem. As for the treatment of conventional thermal convection problems with finite Prandtl numbers, we refer to Boland et al. [1], [2] in which they showed error estimates of the velocity and the temperature for semi-discrete problems. Although our equations are simplified compared to the original thermal convection problems, our results are based on a full discrete problem and contain error estimates of the pressure. Our choice of finite element bases is best possible in the sense of H^1-norm for the velocity and L^2-norm for the pressure as well as temperature. See Remark 3.

The contents of the paper are as follows. In Section 2, governing equations of the thermal convection flows of fluids with the infinite Prandtl number are introduced. In Section 3, a finite element scheme for the problem is presented. In Section 4, stability properties and error estimates of the finite element solutions are analyzed. In section 5, sample numerical simulations are performed.

In what follows, Ω is a bounded domain in $\boldsymbol{R}^d$ with boundary $\partial\Omega$ for $d = 2, 3$, and $(0, T)$ is a time interval. $L^2(\Omega)$ is the space of square-integrable functions in Ω whose inner product and norm are denoted by $(\cdot,\cdot)$ and $\|\cdot\|_{0,\Omega}$, respectively. $L^2_0(\Omega)$ is the space of L^2-functions with zero mean value in Ω. $H^k(\Omega)$ is the Sobolev space of functions whose derivatives of order less than or equal to k lie in $L^2(\Omega)$. The norm and semi-norm are denoted by $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{k,\Omega}$, respectively. $H^1_0(\Omega)$ is the space of $H^1(\Omega)$-functions vanishing on the boundary $\partial\Omega$. $H^{-1}(\Omega)$ denotes the dual space of the Sobolev space $H^1_0(\Omega)$ whose norm is denoted by $\|\cdot\|_{-1,\Omega}$. The symbol $\langle\cdot,\cdot\rangle$ denotes the duality pairing. $C([0,T];X)$ is the space of continuous functions from $[0,T]$ to a Banach space X, whose norm is defined by

$$\|v\|_{C([0,T];X)} \equiv \max\{\|v(t)\|_X; t \in [0,T]\}.$$

$L^2(0,T;X)$ is the space of functions from $(0,T)$ to X satisfying

$$\|v\|_{L^2(0,T;X)} \equiv \left\{\int_0^T \|v(t)\|_X^2\, dt\right\}^{1/2} < +\infty.$$

$H^k(0,T;X)$ is the space of functions from $(0,T)$ to X satisfying

$$\|v\|_{H^k(0,T;X)} \equiv \left\{\sum_{j=0}^{k} \left\|\frac{\partial^j v}{\partial t^j}\right\|_{L^2(0,T;X)}^2\right\}^{1/2} < +\infty.$$

We employ the following three function spaces $V \equiv (H_0^1(\Omega))^d$, $Q \equiv L_0^2(\Omega)$ and $\Psi \equiv H_0^1(\Omega)$.

2. Governing equations for thermal convection phenomena with the infinite Prandtl number

We formulate the governing equations and discuss the numerical analysis of the problems of finding a velocity field $u : (0, T) \to V$, a pressure field $p : (0, T) \to Q$ and a temperature field $\theta : (0, T) \to \Psi$ that satisfy the following equations in $\Omega \times (0, T)$:

$$-2\nabla \otimes D(u(t)) + \nabla p(t) = E\theta(t), \tag{1}$$

$$\nabla \cdot u(t) = 0, \tag{2}$$

$$\frac{\partial \theta}{\partial t}(t) + \mathrm{Ra}\, u(t) \cdot \nabla\theta(t) = \Delta\theta(t) + f(t), \tag{3}$$

subject to the initial condition

$$\theta(0) = \theta^0, \tag{4}$$

where $\nabla \otimes D(v)$ is the stress divergence term defined by

$$[\nabla \otimes D(v)]_i \equiv \sum_{j=1}^{d} \frac{\partial}{\partial x_j} D_{ij}(v), \quad D_{ij}(v) \equiv \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right),$$

and $E \in \boldsymbol{R}^d$ is a unit vector, (0, 1) or (0, 0, 1). Ra is the Rayleigh number and $f : (0, T) \to H^{-1}(\Omega)$ represents a source term. The system of equations (1)–(3) is a model of thermal convection phenomena with the infinite Prandtl number. These equations are derived from a thermal convection model described by the Boussinesq equations:

$$\frac{\partial u}{\partial t}(t) + u(t) \cdot \nabla u(t) = -\mathrm{Pr}\ \mathrm{Ra}\, \nabla p(t) + 2\mathrm{Pr}\, \nabla \otimes D(u(t)) + \mathrm{Pr}\ \mathrm{Ra}\, E\theta(t), \tag{5}$$

$$\nabla \cdot u(t) = 0, \tag{6}$$

$$\frac{\partial \theta}{\partial t}(t) + u(t) \cdot \nabla\theta(t) = \Delta\theta(t) + f(t), \tag{7}$$

supplemented by initial conditions,

$$u(0) = u^0, \quad \theta(0) = \theta^0,$$

where Pr denotes the Prandtl number. In (5) the last term represents the buoyant force induced by thermal expansion. The coefficient of the pressure gradient term is expressed by the product of nondimensional numbers Pr and Ra because the gradient of the reference pressure is considered to be the same order as the buoyant force.

Since we consider the case of fluids with the infinite Prandtl number, letting Pr tend to infinity implies

$$0 = -\mathrm{Ra}\,\nabla p(t) + 2\nabla \otimes D(u(t)) + \mathrm{Ra}\,E\theta(t),$$

$$\nabla \cdot u(t) = 0,$$

$$\frac{\partial \theta}{\partial t}(t) + u(t) \cdot \nabla\theta(t) = \Delta\theta(t) + f(t).$$

Scaling the velocity u by the Rayleigh number and denoting the scaled velocity by the same symbol u, we obtain equations (1)–(3).

3. A finite element scheme for the thermal convection problem with the infinite Prandtl number

We consider a variational formulation of (1)–(3) in the following way:

$$a(u(t), v) + b(v, p(t)) = (E\theta(t), v), \qquad \forall v \in V,\ t \in (0, T), \tag{8}$$

$$b(u(t), q) = 0, \qquad \forall q \in Q,\ t \in (0, T), \tag{9}$$

$$\left(\frac{\partial \theta}{\partial t}(t), \psi\right) + \mathrm{Ra}\,c_1(u(t), \theta(t), \psi) + c_0(\theta(t), \psi) = \langle f(t), \psi\rangle,$$

$$\forall \psi \in \Psi,\ t \in (0, T). \tag{10}$$

Here a is a bilinear form on $V \times V$ and b is a bilinear form on $V \times Q$ defined by

$$a(v, w) \equiv 2\sum_{i,j=1}^{d} \int_\Omega D_{ij}(v) D_{ij}(w)\,dx, \quad b(v, q) \equiv -\sum_{i=1}^{d} \int_\Omega \frac{\partial v_i}{\partial x_i} q\,dx,$$

respectively. c_1 is a trilinear form on $V \times \Psi \times \Psi$ and c_0 is a bilinear form on $\Psi \times \Psi$ defined by

$$c_1(v, \psi, \varphi) \equiv \frac{1}{2}\sum_{i=1}^{d} \int_\Omega v_i \left(\frac{\partial \psi}{\partial x_i}\varphi - \psi\frac{\partial \varphi}{\partial x_i}\right) dx, \quad c_0(\psi, \varphi) \equiv \sum_{i=1}^{d} \int_\Omega \frac{\partial \psi}{\partial x_i}\frac{\partial \varphi}{\partial x_i}\,dx,$$

respectively. We use the same notation to represent inner products in $L^2(\Omega)$ and $(L^2(\Omega))^d$,

$$(\psi, \varphi) \equiv \int_\Omega \psi\varphi\, dx, \qquad \psi, \varphi \in L^2(\Omega),$$

$$(v, w) \equiv \sum_{i=1}^{d} \int_\Omega v_i w_i\, dx, \qquad v, w \in (L^2(\Omega))^d,$$

which will not cause any confusion.

REMARK 1. If $v \in V$ satisfies $\operatorname{div} v = 0$, then we have

$$c_1(v, \psi, \varphi) = \int_\Omega (v \cdot \nabla\psi)\varphi\, dx.$$

Preparing ourself for the case where the incompressibility is satisfied only approximately, we employ the trilinear form c_1 defined above to ensure that $c_1(v, \psi, \psi) = 0$ for any functions $v \in V$ and $\psi \in \Psi$.

We now discretize (8)–(10). For the spatial discretization, we use the finite element method. Let $\mathfrak{I}_h$ be a triangulation of $\bar{\Omega}$, where h denotes the maximum diameter of all elements $K \in \mathfrak{I}_h$. Let $V_h \subset V$, $Q_h \subset Q$ and $\Psi_h \subset \Psi$ be finite dimensional subspaces. For the time discretization, we introduce a time increment $\tau > 0$. The discrete solution is defined on the time discretization $t = n\tau$, $n = 0, \ldots, N_T \equiv \left[\frac{T}{\tau}\right]$. We employ the backward Euler method for the time derivative. Thus the discrete problem is to find $\{u_h^n, p_h^n, \theta_h^n\}_{n=0}^{N_T} \subset V_h \times Q_h \times \Psi_h$ such that

$$a(u_h^n, v_h) + b(v_h, p_h^n) = (E\theta_h^n, v_h), \qquad \forall v_h \in V_h,\ n = 0, \ldots, N_T, \tag{11}$$

$$b(u_h^n, q_h) = 0, \qquad \forall q_h \in Q_h,\ n = 0, \ldots, N_T, \tag{12}$$

$$(D_\tau\theta_h^{n-1}, \psi_h) + \mathrm{Ra}\, c_1(u_h^{n-1}, \theta_h^n, \psi_h) + c_0(\theta_h^n, \psi_h) = \langle f_h^n, \psi_h\rangle,$$
$$\forall \psi_h \in \Psi_h,\ n = 1, \ldots, N_T, \tag{13}$$

where D_τ is the forward difference operator defined by

$$D_\tau \psi_h^k \equiv \frac{\psi_h^{k+1} - \psi_h^k}{\tau},$$

and $f_h^n \in H^{-1}(\Omega)$ is an approximation to $f(n\tau)$. The initial condition is given as an approximation to θ^0 in such a way that $\theta_h^0 \in \Psi_h$. The system of equations (11)–(13) is a finite element scheme for the thermal convection problem for the infinite Prandtl number. It is clear from the estimate (15) in the next section that the problem (11)–(13) is uniquely solvable for any $\theta_h^0 \in \Psi_h$ and $f_h^n \in H^{-1}(\Omega)$, $n = 1, \ldots, N_T$.

Corresponding to the norms of $C([0,T];X)$ and $L^2(0,T;X)$, we define the two norms below:

$$\|v_h\|_{l^\infty(X)} \equiv \max\{\|v_h^n\|_X; n = 0,\dots,N_T\}, \quad \|v_h\|_{l^2(X)} \equiv \left\{\tau\sum_{n=1}^{N_T}\|v_h^n\|_X^2\right\}^{1/2}$$

for discrete functions $v_h = \{v_h^n\}_{n=0}^{N_T}$ with values in a Hilbert space X.

4. Analyses of the finite element scheme

Throughout this section, let $(u_h^n, p_h^n, \theta_h^n; n = 0,\dots,N_T)$ be the finite element solution to (11)–(13) and let (u,p,θ) be the exact solution to (8)–(10) and (4). The symbol C stands for various positive constants independent of h and τ, which appear in the estimates treated below.

4.1. Stability analysis

We here discuss the stability of the finite element scheme (11)–(13). In the analysis of the mixed finite element method, the uniform inf-sup condition [6] plays a key role. We suppose that there exists a constant $\beta > 0$ independent of h such that

$$\inf_{q_h\in Q_h}\sup_{v_h\in V_h}\frac{b(v_h,q_h)}{\|q_h\|_{0,\Omega}\|v_h\|_{1,\Omega}} \ge \beta. \tag{14}$$

This condition is satisfied for the P2/P1 element, for example. See Remark 3.

Lemma 1. *Suppose* (14) *is satisfied. Then we have*

$$\|u_h^n\|_{1,\Omega}, \|p_h^n\|_{0,\Omega} \le C\|\theta_h^n\|_{0,\Omega}, \quad n = 0,\dots,N_T.$$

Proof. Equations (11) and (12) are regarded as the finite element approximation of the Stokes problem with external force $E\theta_h^n$. Therefore the results are well known [6]. □

The following theorem shows the stability of the finite element scheme (11)–(13).

Theorem 1. *Suppose* (14) *is satisfied. Then we have*

$$\|u_h\|_{l^\infty((H^1)^d)}, \|p_h\|_{l^\infty(L^2)}, \|\theta_h\|_{l^\infty(L^2)\cap l^2(H^1)} \le C\{\|\theta_h^0\|_{0,\Omega} + \|f_h\|_{l^2(H^{-1})}\}. \tag{15}$$

Proof. Substituting θ_h^n into ψ_h in (13), and taking Remark 1 into account, we have

$$(D_\tau\theta_h^{n-1}, \theta_h^n) + |\theta_h^n|_{1,\Omega}^2 = \langle f_h^n, \theta_h^n\rangle, \quad n = 1,\dots,N_T. \tag{16}$$

The right-hand side of (16) is estimated as follows:

$$\begin{aligned}\langle f_h^n, \theta_h^n \rangle &\le \frac{1}{2C_1}\|\theta_h^n\|_{1,\Omega}^2 + \frac{C_1}{2}\|f_h^n\|_{-1,\Omega}^2 \\ &\le \frac{1}{2}|\theta_h^n|_{1,\Omega}^2 + \frac{C_1}{2}\|f_h^n\|_{-1,\Omega}^2,\end{aligned}$$

where $C_1 > 0$ is a constant appearing in the inequality

$$\|\psi\|_{1,\Omega}^2 \le C_1|\psi|_{1,\Omega}^2, \quad \forall\psi \in \Psi,$$

which is derived from the Poincaré inequality. Using the inequality $\frac{1}{2}\beta^2 - \frac{1}{2}\alpha^2 \le (\beta - \alpha)\beta$, we have

$$\|\theta_h^n\|_{0,\Omega}^2 - \|\theta_h^{n-1}\|_{0,\Omega}^2 + \tau|\theta_h^n|_{1,\Omega}^2 \le C_1\tau\|f_h^n\|_{-1,\Omega}^2. \tag{17}$$

Summing up (17) from $n = 1$ through k $(\le N_T)$, we obtain

$$\|\theta_h^k\|_{0,\Omega}^2 + \tau\sum_{n=1}^{k}|\theta_h^n|_{1,\Omega}^2 \le \|\theta_h^0\|_{0,\Omega}^2 + C_1\tau\sum_{n=1}^{k}\|f_h^n\|_{-1,\Omega}^2. \tag{18}$$

From (18) and Lemma 1, we obtain (15). □

4.2. Error analysis

Here we discuss error estimates for the finite element solution to equations (11)–(13). Suppose that a family of triangulations $\{\mathfrak{I}_h\}_h$ is regular, i.e., the non-degenerate condition is satisfied and the maximum side length tends to zero. Let P_k be the space of all polynomials of degree less than or equal to k. We define finite element spaces V_h, Q_h and Ψ_h by $V_h = \{v_h \in V; v_h|_K \in P_{k(u)}, K \in \mathfrak{I}_h\}$, $Q_h = \{q_h \in Q; q_h|_k \in P_{k(p)}, K \in \mathfrak{I}_h\}$ and $\Psi_h = \{\psi_h \in \Psi; \psi_h|_K \in P_{k(\theta)}, K \in \mathfrak{I}_h\}$, where $k(u)$, $k(p)$, and $k(\theta)$ are non-negative integers. In what follows, we suppose the uniform inf-sup condition (14) is satisfied.

LEMMA 2. *Suppose that* $u \in C([0,T];(H^{k(u)+1}(\Omega))^d \cap V)$, $p \in C([0,T]; H^{k(p)+1}(\Omega) \cap Q)$ *and* $\theta \in C([0,T];L^2(\Omega))$, *and that* (u,p,θ) *is a solution. Then we have*

$$\begin{aligned}&\|u(n\tau) - u_h^n\|_{1,\Omega}, \|p(n\tau) - p_h^n\|_{0,\Omega} \\ &\qquad \le C(h^{k(u)}|u(n\tau)|_{k(u)+1,\Omega} + h^{k(p)+1}|p(n\tau)|_{k(p)+1,\Omega} + \|\theta(n\tau) - \theta_h^n\|_{0,\Omega}),\end{aligned} \tag{19}$$

for $n = 0, \ldots, N_T$.

PROOF. We introduce $\hat{u}_h^n \in V_h$ and $\hat{p}_h^n \in Q_h$, $n = 0, \ldots, N_T$, defined by

$$a(\hat{u}_h^n, v_h) + b(v_h, \hat{p}_h^n) = (E\theta(n\tau), v_h), \qquad \forall v_h \in V_h, \tag{20}$$

$$b(\hat{u}_h^n, q_h) = 0, \qquad \forall q_h \in Q_h, \tag{21}$$

Subtracting (11) and (12) from (20) and (21) respectively, we obtain from Lemma 1

$$\|\hat{u}_h^n - u_h^n\|_{1,\Omega}, \|\hat{p}_h^n - p_h^n\|_{0,\Omega} \le C_1 \|\theta(n\tau) - \theta_h^n\|_{0,\Omega} \tag{22}$$

with a constant $C_1 > 0$. On the other hand, since $(\hat{u}_h^n, \hat{p}_h^n)$ is nothing but the Stokes projection [6] of $(u(n\tau), p(n\tau))$ to $V_h \times Q_h$, there exists a constant $C_2 > 0$ independent of h *and* τ such that

$$\begin{aligned} &\|u(n\tau) - \hat{u}_h^n\|_{1,\Omega}, \|p(n\tau) - \hat{p}_h^n\|_{0,\Omega} \\ &\quad \le C_2(h^{k(u)}|u(n\tau)|_{k(u)+1,\Omega} + h^{k(p)+1}|p(n\tau)|_{k(p)+1,\Omega}). \end{aligned} \tag{23}$$

Combining (22) and (23), we obtain (19). □

We now define the Poisson projection $P_h : \Psi \ni \psi \to P_h\psi \in \Psi_h$ by

$$c_0(P_h\psi, \psi_h) = c_0(\psi, \psi_h), \quad \forall \psi_h \in \Psi_h. \tag{24}$$

REMARK 2. The following results are well known [3].

(i) It holds that

$$\|P_h\psi\|_{1,\Omega} \le C\|\psi\|_{1,\Omega}, \quad \forall \psi \in \Psi.$$

and

$$\|\psi - P_h\psi\|_{1,\Omega} \le Ch^{k(\theta)}|\psi|_{k(\theta)+1,\Omega}, \quad \forall \psi \in \Psi \cap H^{k(\theta)+1}(\Omega).$$

(ii) Suppose the Poisson problem is regular in the sense that $\Delta\psi \in L^2(\Omega)$ and $\psi \in H_0^1(\Omega)$ imply $\psi \in H^2(\Omega)$, e.g., the case where Ω is a convex polygonal domain in $\mathbf{R}^2$. Then we have

$$\|\psi - P_h\psi\|_{0,\Omega} \le Ch^{k(\theta)+1}|\psi|_{k(\theta)+1,\Omega}, \quad \forall \psi \in \Psi \cap H^{k(\theta)+1}(\Omega).$$

The following theorem shows an error estimate for the finite element solution to (11)–(13).

THEOREM 2. *Suppose* $u \in C([0,T]; (H^{k(u)+1}(\Omega))^d \cap V) \cap H^1(0,T; (H^1(\Omega))^d)$, $p \in C([0,T]; H^{k(p)+1}(\Omega) \cap Q)$ *and* $\theta \in C([0,T]; H^{k(\theta)+1}(\Omega) \cap \Psi) \cap H^2(0,T; H^{-1}(\Omega))$, *and that* (u,p,θ) *is a solution. Then we have*

$$\begin{aligned} &\|u - u_h\|_{l^\infty((H^1)^d)}, \|p - p_h\|_{l^\infty(L^2)}, \|\theta - \theta_h\|_{l^\infty(L^2) \cap l^2(H^1)} \\ &\quad \le C\{\|\theta_h^0 - P_h\theta(0)\|_{0,\Omega} + \|f - f_h\|_{l^2(H^{-1})} + h^k + \tau\}, \end{aligned} \tag{25}$$

where $k \equiv \min\{k(u), k(p)+1, k(\theta)\}$.

PROOF. In the following C_i, $i = 1, \dots, 7$, are positive constants independent of h *and* τ. We denote by $\hat{\theta}_h^n \equiv P_h\theta(n\tau)$ the Poisson projection at $t = n\tau$, $n = 0, \dots, N_T$, and by $e_h^n \equiv \theta_h^n - \hat{\theta}_h^n$ the error between the finite element solution and their projection. From (10), (13), and (24) we derive

$$(D_\tau e_h^{n-1}, \psi_h) + \mathrm{Ra}\, c_1(u_h^{n-1}, e_h^n, \psi_h) + c_0(e_h^n, \psi_h) = \langle \varepsilon_1^n + \varepsilon_2^n + \varepsilon_3^n, \psi_h \rangle$$

where ε_i^n, $i = 1, 2, 3$, are defined by

$$\varepsilon_1^n \equiv \frac{\partial \theta}{\partial t}(n\tau) - D_\tau \hat{\theta}_h^{n-1}, \quad \varepsilon_2^n \equiv \mathrm{Ra}\{c_1(u(n\tau), \theta(n\tau), \cdot) - c_1(u_h^{n-1}, \hat{\theta}_h^n, \cdot)\}, \quad \varepsilon_3^n \equiv f_h^n - f(n\tau). \tag{26}$$

From (18) we have for $n = 1, \ldots, N_T$,

$$\|e_h^n\|_{0,\Omega}^2 + \tau \sum_{l=1}^{n} |e_h^l|_{1,\Omega}^2 \le \|e_h^0\|_{0,\Omega}^2 + C_1 \tau \sum_{l=1}^{n} \sum_{i=1}^{2} \|\varepsilon_i^l\|_{-1,\Omega}^2. \tag{27}$$

We then estimate each term of the right-hand side of (27). Since

$$\begin{aligned} \varepsilon_1^n &= \left\{ \frac{\partial \theta}{\partial t}(n\tau) - \frac{1}{\tau}(\theta(n\tau) - \theta((n-1)\tau)) \right\} + \frac{1}{\tau}(I - P_h)\{\theta(n\tau) - \theta((n-1)\tau)\} \\ &= \frac{1}{\tau} \int_{(n-1)\tau}^{n\tau} \{s - (n-1)\tau\} \frac{\partial^2 \theta}{\partial t^2}(s)\, \mathrm{d}s + \frac{1}{\tau} \int_{(n-1)\tau}^{n\tau} (I - P_h) \frac{\partial \theta}{\partial t}(s)\, \mathrm{d}s, \end{aligned}$$

we obtain, from Remark 2,

$$\|\varepsilon_1^n\|_{-1,\Omega} \le \sqrt{\frac{\tau}{3}} \left\| \frac{\partial^2 \theta}{\partial t^2} \right\|_{L^2(I_n; H^{-1}(\Omega))} + \frac{C_2 h^{k(\theta)}}{\sqrt{\tau}} \left\| \frac{\partial \theta}{\partial t} \right\|_{L^2(I_n; H^{k(\theta)+1}(\Omega))}, \tag{28}$$

where $I_n \equiv ((n-1)\tau, n\tau)$. On the other hand,

$$\begin{aligned} \varepsilon_2^n = \mathrm{Ra}\{&c_1(u(n\tau) - u((n-1)\tau), \theta(n\tau), \cdot) + c_1(u((n-1)\tau, \theta(n\tau) - \hat{\theta}_h^n, \cdot) \\ &+ c_1(u((n-1)\tau) - u_h^{n-1}, \hat{\theta}_h^n, \cdot)\}, \end{aligned}$$

and so we obtain, from Remark 2,

$$\begin{aligned} \|\varepsilon_2^n\|_{-1,\Omega} \le\ & C_3\{\|u(n\tau) - u((n-1)\tau)\|_{1,\Omega} \|\theta(n\tau)\|_{1,\Omega} \\ &+ \|u((n-1)\tau)\|_{1,\Omega} \|(I - P_h)\theta(n\tau)\|_{1,\Omega} \\ &+ \|u((n-1)\tau) - u_h^{n-1}\|_{1,\Omega} \|\hat{\theta}_h^n\|_{1,\Omega}\} \\ \le\ & C_4 \left\{ \sqrt{\tau} \left\| \frac{\partial u}{\partial t} \right\|_{L^2(I_n; (H^1(\Omega))^d)} \|\theta\|_{C([0,T]; H^1(\Omega))} \right. \\ &+ h^{k(\theta)} \|u\|_{C([0,T]; (H^1(\Omega))^d)} \|\theta\|_{C([0,T]; H^{k(\theta)+1}(\Omega))} \\ &\left. + \|u((n-1)\tau) - u_h^{n-1}\|_{1,\Omega} \|\theta\|_{C([0,T]; H^1(\Omega))} \right\}. \end{aligned} \tag{29}$$

Substituting (26), (28) and (29) into (27) and using $n\tau \leq T$, we obtain

$$\|e_h^n\|_{0,\Omega}^2 \leq \|\theta_h^0 - \hat{\theta}_h^0\|_{0,\Omega}^2 + C_5\Bigg[\tau^2\|\theta\|_{H^2(0,T;H^{-1}(\Omega))}^2 + h^{2k(\theta)}\left\|\frac{\partial\theta}{\partial t}\right\|_{L^2(0,T;H^{k(\theta)+1}(\Omega))}^2$$
$$+ \tau^2\|u\|_{H^1(0,T;(H^1(\Omega))^d)}^2\|\theta\|_{C([0,T];H^1(\Omega))}^2$$
$$+ h^{2k(\theta)}\|u\|_{C([0,T];(H^1(\Omega))^d)}^2\|\theta\|_{C([0,T];H^{k(\theta)+1}(\Omega))}^2$$
$$+ \left\{\tau\sum_{l=1}^n \|u((l-1)\tau) - u_h^{l-1}\|_{1,\Omega}^2\right\}\|\theta\|_{C([0,T];H^1(\Omega))}^2$$
$$+ \tau\sum_{l=1}^n \|f(l\tau) - f_h^l\|_{-1,\Omega}^2\Bigg], \tag{30}$$

for $n = 1, \dots, N_T$. Using the triangular inequality $\|\theta_h^n - \theta(n\tau)\|_{0,\Omega} \leq \|e_h^n\|_{0,\Omega} + \|\hat{\theta}_h^n - \theta(n\tau)\|_{0,\Omega}$, combining (30) with Lemma 2, and applying the discrete Gronwall inequality [7], we obtain

$$\|\theta_h^n - \theta(n\tau)\|_{0,\Omega}^2 \leq C_6\Bigg\{\|\theta_h^0 - \hat{\theta}_h^0\|_{0,\Omega}^2 + \tau\sum_{l=1}^n \|f_h^l - f(l\tau)\|_{-1,\Omega}^2$$
$$+ \tau^2 + h^{2k(\theta)} + h^{2k(u)} + h^{2k(p)+2}\Bigg\}\exp(C_7 n\tau), \tag{31}$$

for $n = 1, \dots, N_T$. Combining (31) with Lemma 2, we obtain (25). □

When the Poisson problem is regular in the sense of Remark 2 (ii), the error estimate can be refined as follows.

COROLLARY 1. *Besides the assumptions of Theorem 2, suppose the Poisson problem is regular and $u \in C([0,T];(L^\infty(\Omega))^d)$. Then we have*

$$\|u - u_h\|_{l^\infty((H^1)^d)}, \|p - p_h\|_{l^\infty(L^2)}, \|\theta - \theta_h\|_{l^\infty(L^2)\cap l^2(H^1)}$$
$$\leq C\{\|\theta_h^0 - P_h\theta(0)\|_{0,\Omega} + \|f - f_h\|_{l^2(H^{-1})} + h^k + \tau\}, \tag{32}$$

where $k \equiv \min\{k(u), k(p)+1, k(\theta)+1\}$.

PROOF. As $u \in C([0,T];(L^\infty(\Omega))^d)$ and $\operatorname{div} u = 0$, we have

$$c_1(u((n-1)\tau), \theta(n\tau) - \hat{\theta}_h^n, \psi_h) \leq C_1\|u((n-1)\tau)\|_{0,\infty,\Omega}\|(I - P_h)\theta(n\tau)\|_{0,\Omega}\|\psi_h\|_{1,\Omega},$$

where $c_1(\cdot,\cdot,\cdot)$ is the trilinear form defined in Remark 1, $\|\cdot\|_{0,\infty,\Omega}$ denotes the L^∞-norm of measurable functions over Ω and $C_1 > 0$ is a constant independent of h *and* τ. In the proof of Theorem 2, the fourth term in the bracket of the right-hand side of (30) can be replaced by

$$h^{2k(\theta)+2}\|u\|^2_{C([0,T];(L^\infty(\Omega))^d)}\|\theta\|^2_{C([0,T];H^{k(\theta)+1}(\Omega))},$$

which implies (32). □

REMARK 3. Suppose that the domain Ω is bounded by a convex polygon in $\boldsymbol{R}^2$. When the P2/P1/P1 interpolations are used for the velocity/pressure/temperature fields, i.e., quadratic interpolation for the velocity, linear interpolations for the pressure and the temperature are employed, which are known to satisfy the uniform inf-sup condition [13], we obtain the following error estimates from (32) with $k(u) = 2$, $k(p) = 1$, and $k(\theta) = 1$:

$$\begin{aligned}&\|u - u_h\|_{l^\infty((H^1)^d)}, \|p - p_h\|_{l^\infty(L^2)}, \|\theta - \theta_h\|_{l^\infty(L^2)\cap l^2(H^1)}\\&\quad \le C\{\|\theta_h^0 - P_h\theta(0)\|_{0,\Omega} + \|f - f_h\|_{l^2(H^{-1})} + h^2 + \tau\}.\end{aligned} \tag{33}$$

Furthermore, if we take $\theta_h^0 = P_h\theta(0)$ and $f_h^n = f(n\tau)$ $(n = 1, \dots, N_T)$ when $f \in C([0,T]; H^{-1}(\Omega))$, we have

$$\|u - u_h\|_{l^\infty((H^1)^d)}, \|p - p_h\|_{l^\infty(L^2)}, \|\theta - \theta_h\|_{l^\infty(L^2)\cap l^2(H^1)} \le C(h^2 + \tau).$$

If $\theta^0 \in H^2(\Omega)$ and $f \in C([0,T]; H^2(\Omega))$, we can take $\theta_h^0 = I_h\theta^0$ and $f_h^n = I_h f(n\tau)$, $n = 1, \dots, N_T$, where I_h is the interpolation operator. In this case the same error estimates are also obtained.

REMARK 4. The stability and the error analysis discussed above can be extended straightforwardly to the case where the velocity and the temperature satisfy inhomogeneous boundary conditions.

5. Numerical results

In this section we perform numerical experiments to certify the theoretical result discussed in the previous section. A sample problem is defined in the square $(-1,1) \times (-1,1)$ and in the time interval (0, 3.2). The Rayleigh number Ra is taken to be 100. The source term f of the energy equation and the initial and the boundary conditions of the temperature are given so that the solutions of equations (1)–(4) are as follows:

$$\begin{aligned}u =&(0.4t^2(x_1^2 - 1)^2x_2(x_2^2 - 1), -0.4t^2x_1(x_1^2 - 1)(x_2^2 - 1)^2),\\ p =&t^2(0.48x_1^5x_2 - 3.2x_1^3x_2 + 4x_1x_2 + 1.6x_1^3x_2^3 - 1.6x_1x_2^3),\\ \theta =&t^2(0.48x_1^5 - 4.8x_1^3 + 8x_1 + 9.6x_1^3x_2^2 - 14.4x_1x_2^2 + 2.4x_1x_2^4).\end{aligned}$$

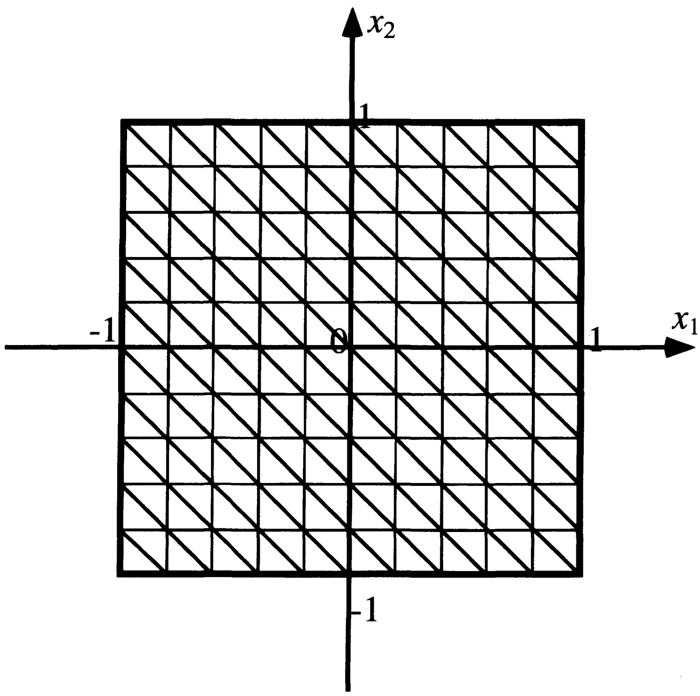

Figure 1: Domain Ω and its subdivision for $N = 10$ (case (B)).

Table 1. Discretization parameters in each case.

Case	N	Element Diameter h	Node Number Velocity	Node Number Pressure Temperature
(A)	5	0.5657	121	36
(B)	10	0.2828	441	121
(C)	20	0.1414	1681	441
(D)	40	0.0707	6561	1681

As for spatial discretization, uniform triangular elements are adopted. The domain is divided into a union of $N \times N \times 2$ triangles. With Figure 1 we illustrate a subdivision of the domain when $N = 10$. We use the P2/P1/P1 finite element approximation to the velocity/pressure/temperature fields.

Computation is performed for $N = 5$ (Case(A)), 10 (Case(B)), 20 (Case(C)) and 40 (Case(D)) using the finite element scheme presented in Section 3. The discretization parameters in each case are shown in Table 1. Results of the

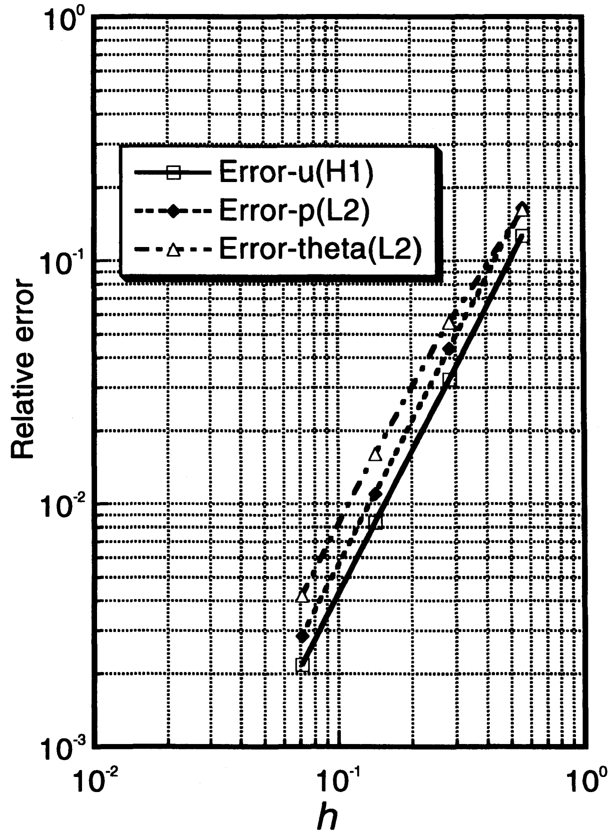

Figure 2: The relative error versus element diameter when $\tau = h^2$.

computation are evaluated in relative errors between the exact solution and the computational results:

$$\mathrm{Error} - u(H^1) \equiv \frac{\|u - u_h\|_{l^\infty((H^1)^d)}}{\|u\|_{l^\infty((H^1)^d)}},$$

$$\mathrm{Error} - p(L^2) \equiv \frac{\|p - p_h\|_{l^\infty(L^2)}}{\|p\|_{l^\infty(L^2)}},$$

$$\mathrm{Error} - \theta(L^2) \equiv \frac{\|\theta - \theta_h\|_{l^\infty(L^2)}}{\|\theta\|_{l^\infty(L^2)}}.$$

We first set the time increment $\tau = h^2$. The relations between the element lengths and the relative errors are shown in Figure 2. The slopes calculated from the results in cases (C) and (D) for velocity, pressure and temperature in

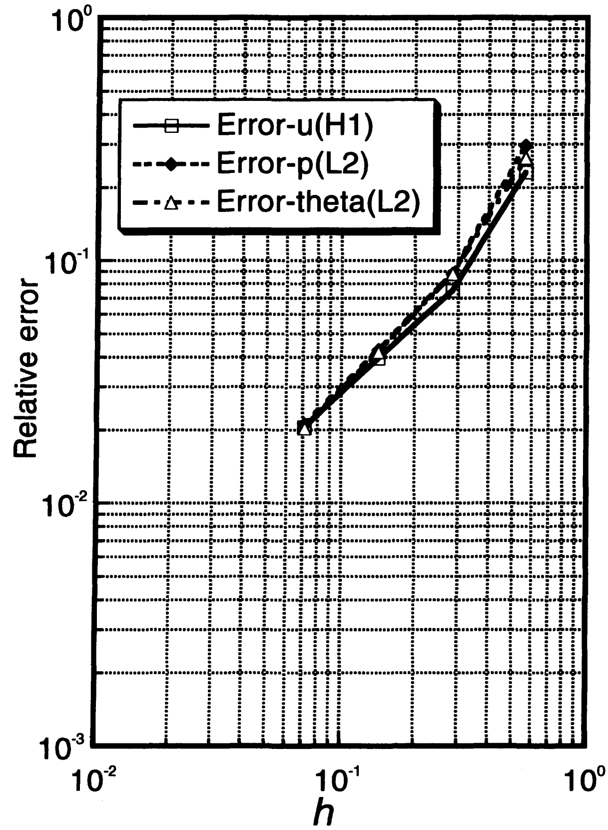

Figure 3: The relative error versus element diameter when $\tau = h$.

Figure 2 are 1.944, 1.946 and 1.939, respectively. The numerical results agree well in the error order $O(h^2)$ derived from Corollary 1 with $k(u) = 2$, $k(p) = 1$, $k(\theta) = 1$ and $\tau = h^2$.

We next set $\tau = h$. The relations between the element lengths and the relative errors are shown in Figure 3. The slopes calculated from the results in cases (C) and (D) for velocity, pressure and temperature are 0.942, 1.041 and 1.016, which also agree in the theoretical error order $O(\tau + h^2) = O(h)$.

6. Conclusions

We established an error analysis of a finite element scheme for the thermal convection problem with the infinite Prandtl number. In our model there exists no time derivative for velocity, which makes it easy to obtain the error estimate for the pressure. As a numerical experiment, sample simulation has been performed, and the simulation results agreed well with the theoretical results.

The Rayleigh number, which is contained in the convection term in the energy equation, becomes high if so does the Grashof number. In such case, it is known that some "upwind' technique is necessary for the stable computation [5], [13], [17]. We are planning to investigate some upwind and stabilized methods [4], [10], for flows with high Rayleigh numbers with the infinite Prandtl number.

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Asahi Glass Co. Ltd., Research Center
1150, Hazawacho, Kanagawa-ku
Yokohama 221-8755, Japan
and
Departement of Mathematics
Faculty of Science
Hiroshima University
Higashi-hiroshima 739-8526, Japan

Graduate School of Mathematics
Kyushu University
Hakozaki, Higashi-ku
Fukuoka 812-8581, Japan