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Determinantal varieties associated to rank two vector bundles on projective spaces and splitting theorems

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Abstract. Introducing determinantal varieties associated to rank two vector bundles on complex projective n-spaces $\mathbf{P}^n$ $(n \geq 4)$, we obtain two main splitting theorems for those bundles. By studying irreducible components of the Hilbert scheme of $\mathbf{P}^n$ containing those determinantal varieties it is shown that a rank two bundle E on $\mathbf{P}^n$ $(n = 4$ or $5)$ splits if and only if the first cohomology of the sheaf of endomorphisms of E vanishes. In addition, another cohomological criterion for the splitting of E is also shown using specific divisors of a determinantal variety X associated to E.

0. Introduction

(0.1) As for the splitting problems for rank two vector bundles on complex projective n-space $\mathbf{P}^n$, R. Hartshorne (cf. [14], [15], [33]) posed the following famous conjectures:

S_n: Every rank two algebraic vector bundle on $\mathbf{P}^n$ $(n \geq 7)$ splits into line bundles or the conjecture which is equivalent to S_n:

C_n: Every smooth closed subvariety of codimension 2 in $\mathbf{P}^n$ $(n \geq 7)$ is a complete intersection.

Later, H. Grauert and M. Schneider [11] tried to solve the following important problem. However there was a gap in their proof unfortunately.

GS: Every rank two unstable algebraic vector bundle on $\mathbf{P}^4$ is a direct sum of line bundles.

Though many mathematicians have tried to solve the conjectures S_n, C_n and the problem GS, e.g., W. Barth and A. Van de Ven [2], Z. Ran [31], Th. Peternell, J. Le-Potier and M. Schneider [29], ..., for almost thirty years, we have not had obtained any complete answers yet.

If we could solve the above conjectures or problem affirmatively, then it should bring us many useful results. For example:

1) It is well known that every algebraic vector bundle on the projective

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line $\mathbf{P}^1$ is a direct sum of line bundles [10]. Needless to say, it plays important roles in dealing with algebraic surfaces, algebraic threefolds, ..., e.g., in the classification of those algebraic varieties or in the minimal model theory of algebraic threefolds (cf. [25]), ... Therefore any profound solution to the splitting problems is likely to introduce important notions and useful tools in the study of higher-dimensional algebraic manifolds because they provide us with effective methods to deal with rational subvarieties $\mathbf{P}^n$ in higher-dimensional algebraic varieties.

2) Besides it is known that every topological vector bundle on $\mathbf{P}^2$ or $\mathbf{P}^3$ admits an algebraic structure [1]. It can be proved that a rank two algebraic vector bundle on $\mathbf{P}^n$ with $c_1^2 - 4c_2 \geq 0$ is always unstable. On the other hand, there exist many indecomposable topological rank two vector bundles with $c_1^2 - 4c_2 > 0$ on $\mathbf{P}^4$ (cf. [36], [40]). Consequently, if the *GS* were true, then we could observe that there exist many topological rank two vector bundles with no algebraic structures on $\mathbf{P}^4$, contrary to the 2- and 3-dimensional cases.

Thus it seems to be one of the most important tasks in the field of algebraic vector bundles that we manage to solve the above splitting problems. For a nice introduction to the above, see Van de Ven [41].

(0.2) In this paper, we shall introduce determinantal varieties associated to rank two vector bundles on projective varieties. It seems to be one of the difficulties for our attempt to solve the conjectures S_n, C_n or the problem *GS* that we have to handle higher-dimensional algebraic varieties. For instance, it sometimes happens that there remain a few cohomology groups of sheaves which we are not able to estimate because our objects are of high dimensions and might have non-reduced algebraic structures. The following are the reasons why we have introduced the determinantal varieties:

1) We would like to reduce our problems to the ones on lower-dimensional algebraic varieties without changing the essence of the problems.

2) We would like to construct new algebraic varieties enjoying useful algebro-geometric properties by using algebraic vector bundles.

This paper consists of the following six chapters.

1. Determinantal varieties associated to rank two bundles.
2. Topology of determinantal varieties.
3. Divisors on determinantal varieties.
4. Comparison theorems of cohomologies.
5. Vector bundles on determinantal varieties.
6. Geography of determinantal surfaces.

As a by-product of these investigations on determinantal varieties, we shall establish two main splitting theorems for rank two vector bundles on $\mathbf{P}^n$

($n \geq 4$) and we will continue to investigate the above splitting problems under these foundations in the sequel.

(0.3) In Chapter 1, we shall introduce determinantal varieties associated to rank two vector bundles on projective varieties and it will be shown that smooth determinantal varieties X associated to a rank two bundle E on a smooth projective variety S form a smooth family over an open subset of a Grassmann variety (Theorem 1). By definition, we see that X is a closed subvariety of S with $\dim X = \dim S/2$ if $\dim S$ is even (resp. $\dim X = (\dim S + 1)/2$ if $\dim S$ is odd). Moreover we find that X carries useful algebro-geometric structures which are inherited from those of E. In the following chapters, determinantal varieties associated to rank two vector bundles on projective spaces $S = \mathbf{P}^n$ ($n \geq 4$) are studied. Singular cohomology groups and homotopy groups of determinantal varieties X are calculated by means of the Lefschetz theorems in Chapter 2 (Propositions 1 and 2). In particular, we observe that X is simply connected. In Chapter 3, several distinguished divisors D, H, F, Z, Z^* on determinantal varieties X are introduced and their roles in algebro-geometric structures of X are shown. For example, it is found that $\mathrm{Pic}(X) = \mathbf{Z}D \oplus \mathbf{Z}H$, X is of general type in general and there exists a morphism $\varphi : X \to \mathbf{P}^m$. In the sequel, we shall find that Z, Z^*, and φ will play important roles in our attempt to solve the splitting problems. In Chapter 4, we shall show several comparison theorems between cohomologies $H^i(\mathbf{P}^n, S^r(E)(s))$ and $H^i(X, \mathcal{O}_X(rD + sD))$ $(r, s \in \mathbf{Z})$ (Theorems 3, 4, 6 and 7). There exists a canonical restriction homomorphism

$$\gamma_i : H^i(\mathbf{P}^n, S^r(E)(s)) \simeq H^i(P(E), \mathcal{O}(rD + sH)) \to H^i(X, \mathcal{O}_X(rD + sH)).$$

Hence studying the conditions for γ_i to be an isomorphism (resp. a surjection or an injection), we shall give the above comparison theorems. As applications of these comparison theorems, several interesting relations between algebro-geometric structures of X and those of E are revealed. For example, we find an isomorphism between $H^i(\mathbf{P}^n, \mathcal{E}nd(E))$ and $H^i(\mathcal{O}_X(2D - c_1H))$. Further we obtain the following necessary and sufficient condition (Theorem 9) for E to be a direct sum of line bundles in terms of the divisor Z^* on X, which is one of our splitting theorems:

The following are equivalent to each other.

1) E is a direct sum of line bundles.

2) Z^* is an effective divisor and numerically effective, i.e., $Z^* \cdot C \geq 0$ for every curve C in X.

3) Z^* is an effective divisor and it satisfies the following asymptotic conditions:

a) $\dim H^1(X, \mathcal{O}_X(-rZ^*)) \leq O(r^1)$ for sufficiently large integers r.

b) $\dim H^i(\mathcal{O}_X(-rZ^* - sH)) \leq P_i(s)$ for all positive integers r and s ($1 \leq i \leq \dim X - 1$), where $P_i(s)$ is a polynomial on s which is independent of r. In this way, we can reduce the splitting problem of E on $\mathbf{P}^n$ to the one on X of lower dimension. In Chapter 5, the tangent bundle T_X and the normal bundle $N_{X/\mathbf{P}^n}$ of X are studied. We shall show that there exists the following exact sequence (Theorem 10):

$$0 \to \mathcal{O}_X(D-F) \to \bigoplus^{m+1} \mathcal{O}_X(D) \to N_{X/\mathbf{P}^n} \to 0.$$

We can use this exact sequence and standard arguments to compute the Chern classes $c_i(T_X)$ ($0 \leq i \leq \dim X$) in terms of the Chern numbers $\{c_1, c_2\}$ of E and divisors D and H (Theorem 11). Consequently, the middle Betti number $b_{\dim X} = \dim_{\mathbf{C}} H^{\dim X}(X, \mathbf{C})$ of X is determined in terms of $\{c_1, c_2\}$ (Theorem 12). Further, we can show effective estimates of $\dim H^i(X, T_X)$ and $\dim H^i(X, N_{X/\mathbf{P}^n})$ ($0 \leq i \leq \dim X$) (Theorem 13) under certain conditions on E. We shall establish with these preliminaries the following splitting theorem (Theorem 15) which looks like the anologous one obtained by G. Kempf by studying some geometric structures of the Hilbert scheme of $\mathbf{P}^n$ at determinantal subvarieties. Let E be a rank two vector bundle on $\mathbf{P}^n$ ($n \geq 4$), P a 4- or 5-dimensional projective linear subspace of $\mathbf{P}^n$ and let $\bar{E}$ be the restriction of E to P.

Then the following are equivalent to each other:

1) E splits into line bundles.

2) $H^1(P, \mathcal{E}nd(\bar{E})) = 0$.

In the final Chapter 6, showing some examples of determinantal surfaces and their geometric structures, we shall describe the geography of determinantal surfaces and as a consequence, we shall give some new species in certain botanical gardens.

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1. Determinantal varieties associated to 2-bundles

(1.1) First we shall introduce determinantal schemes associated to 2-bundles. Let E be a rank 2 vector bundle on an n-dimensional projective variety S defined over an algebraically closed field k of arbitrary characteristic, $\pi : P(E) \to S$ the projective bundle associated to E over S and let L_E be the tautological line bundle on $P(E)$. We assume that E is very ample, i.e., L_E is a very ample line bundle (cf. [37], [38]). Let $n = 2m$ (resp. $n = 2m+1$). Then we can take a set of global sections $\{s_1, s_2, \dots, s_{m+1}\}$ of E satisfying the following conditions where D_i is the tautological divisor on $P(E)$ defined by s_i and $W(s_i)$ is the scheme of zeros on S of s_i $(1 \le i \le m+1)$:

(*)
1) $Y = D_1 \cap D_2 \cap \cdots \cap D_{m+1}$ is a closed subscheme of $P(E)$ of pure codimension $m+1$.
2) $W(s_1) \cap W(s_2) \cap \cdots \cap W(s_{m+1}) = \varnothing$.

Let X be a closed subscheme of S with the following defining equations:

$$s_i \wedge s_j = 0 \qquad \text{for all } i \text{ and } j \ (1 \le i \le j \le m+1).$$

Then Y and X are the closed subschemes defined locally as follows:

For an open subset U of S such that $E|U \simeq \bigoplus^2 \mathcal{O}_U$ and $s_i|U = (s_{i1}, s_{i2})$ where $s_{i1}, s_{i2} \in H^0(U, \mathcal{O}_U)$, X is defined on U by the equations

$$s_i \wedge s_j = s_{i1}s_{j2} - s_{i2}s_{j1} = 0 \qquad (1 \le i \le j \le m+1).$$

On the other hand, Y is defined on $\pi^{-1}(U) \simeq U \times \mathbf{P}^1$ by the equations:

$$s_{i1}X_1 + s_{i2}X_2 = 0 \qquad (1 \le i \le m+1),$$

where $\{X_1, X_2\}$ is a system of homogeneous coordinates of $\mathbf{P}^1$.

Then we have the following.

LEMMA 1. *In the above notation, Y is isomorphic to X through π. Hence X is a closed subscheme of S of pure codimension m.*

PROOF. Since $\pi : P(E) \to S$ is a $\mathbf{P}^1$-bundle and $\bigcap_i W(s_i) = \varnothing$, it is easily seen that π induces a bijection between Y and X. Hence the problem is to prove locally that π induces an isomorphism between Y and X. Let U be an affine open subset of S such that

1) $U \cap X \ne \varnothing$.
2) $E|U \simeq \bigoplus^2 \mathcal{O}_U$ and $s_i|U = (s_{i1}, s_{i2})$ $(s_{i1}, s_{i2} \in H^0(U, \mathcal{O}_U), 1 \le i \le m+1)$.

Since $\bigcap_i W(s_i) = \varnothing$, we may assume that $s_{12}(x) \ne 0$ for $x \in U$. Then Y has the following equations on $\pi^{-1}(U) \simeq U \times \mathbf{P}^1$:

$$g_i = s_{i1} + s_{i2}x = 0, \qquad x = X_2/X_1 \qquad (1 \le i \le m+1).$$

and X has the following equations on U:

$$s_1 \wedge s_i = s_{11}s_{i2} - s_{12}s_{i1} = 0 \qquad (2 \le i \le m+1).$$

Since $-s_1 \wedge s_i = s_{12}g_i - s_{i2}g_1 = s_{12}(g_i - g_1 s_{i2}/s_{12})$, the ideal $(g_1, \dots, g_{m+1})$ coincides with the ideal $(s_1 \wedge s_2, \dots, s_1 \wedge s_{m+1}, g_1)$ in $A[x]$, where A is the affine coordinate ring of U. Hence it yields an A-isomorphism:

$$A[x]/(g_1, \ldots, g_{m+1}) \simeq A/(s_1 \wedge s_2, \ldots, s_1 \wedge s_{m+1}),$$

which implies that Y is isomorphic to X through π. Since Y is a closed subscheme of $P(E)$ of pure dimension $n-m, X$ is a closed subscheme of S of pure codimension m.

(1.2) From now on we shall consider those closed subschemes X of S constructed as in (1.1). Although X is dependent on the choice of global sections of E subject to the conditions $(*)$, we make the following definition.

DEFINITION 1. We call closed subschemes X of S constructed as in (1.1) determinantal schemes associated to E.

(1.3) First we shall show that our determinantal schemes associated to E form a flat projective family over a Zariski open subset of a Grassmann variety.

For a set $s = \{s_1, \ldots, s_{m+1}\}$ of global sections of E which generates an $(m+1)$-dimensional linear subspace $\langle s_1, \ldots, s_{m+1}\rangle$ of $H^0(S, E)$, we put $Y_s = D_1 \cap \cdots \cap D_{m+1}$ a closed subsheme of $P(E)$, where D_i is the tautological divisor on $P(E)$ associated to s_i $(1 \leq i \leq m+1)$.

(1.3.1) If another set $t = \{t_1, \ldots, t_{m+1}\}$ spans the same linear subspace as s does in $H^0(S, E)$, then it is easily verified that $Y_s = Y_t$ and t satisfies the conditions $(*)$ in (1.1) if and only if s does. Hence Y_s is determined by the $(m+1)$-dimensional linear subspace $\langle s\rangle$ generated by s and so we say for simplicity that a linear subspace $\langle s\rangle$ satisfies the conditions $(*)$ if a basis of $\langle s\rangle$ does. Consider the following closed subsheme of $P(E) \times G$ where G is the Grassmann variety which parametrizes $(m+1)$-dimensional linear subspaces of $H^0(S, E)$:

$$\Gamma = \{(x, \langle s\rangle) \mid x \in Y_s, \langle s\rangle \in G\}$$

with projections $p : \Gamma \to P(E)$ and $q : \Gamma \to G$.

(1.3.2) For any $\langle s\rangle \in G$, we observe that $Y_s = q^{-1}(\langle s\rangle)$ is mapped to a closed subscheme of S isomorphically via π if and only if Y_s does not contain any fiber for π. Hence if we denote by L_s the restriction of $L = \pi^*(M)$ to Y_s where M is an ample line bundle on S, then π induces an isomorphism on Y_s if and only if L_s is ample.

Let $V = \{\langle s\rangle \in G \mid Y_s \text{ is of pure dimension } n-m\}$. Then since q is a proper morphism, V is a Zariski open subset of G and moreover $q : q^{-1}(V) \to V$ is faithfully flat because the closed immersion: $q^{-1}(V) \to P(E) \times V$ is a regular immersion. Thus if we put $U = \{\langle s\rangle \in V | (\pi \circ p)^*(L) | Y_s = L_s \text{ is ample}\}$, then U is an open subset of G and it coincides with the set $\{\langle s\rangle \in G \mid \langle s\rangle \text{ satisfies the conditions } (*) \text{ in } (1.1)\}$.

(1.3.3) We denote by $X_s = \pi(Y_s)$ the determinantal scheme associated to a point $\langle s\rangle \in U$ and put $\xi = \pi \times 1 : P(E) \times G \to S \times G$. Then ξ induces an isomorphism of $\Gamma_U = q^{-1}(U)$ to a closed subscheme Ξ of $S \times U$ such that

$$\Xi = \{(x, \langle s\rangle) \in S \times U \mid x \in X_s\}.$$

Hence if we denote also by q the second projection of $\Xi \subset S \times U \to U$ for simplicity, then q is a faithfully flat projective morphism and we see $X_s = q^{-1}(\langle s\rangle)$ for all point $\langle s\rangle \in U$.

(1.3.4) In addition to the conditions $(*)$ in (1.1), we consider the following:

1) $Y_s = D_1 \cap D_2 \cap \cdots \cap D_{m+1}$ is a smooth closed subscheme of $P(E)$

$(*)'$ of pure codimension $m+1$,

2) $W(s_1) \cap W(s_2) \cap \cdots \cap W(s_{m+1}) = \varnothing$,

and put $U' = \{\langle s\rangle \in G \mid \langle s\rangle$ satisfies the above conditions $(*)'\}$. Then by an argument similar to the above, we see that there exists a closed subscheme Ξ' of $S \times U'$ such that the second projection $q : \Xi' \to U$ is a smooth projective morphism. Hence smooth determinantal schemes associated to E form a smooth family over an open subset U' of G.

Summing up the above, we obtain the following.

THEOREM 1. *Let the notation be as above.*

1) $U' = \{\langle s\rangle \in G \mid \langle s\rangle$ satisfies the conditions $(*)'$ in $(1.1)\}$ *is a Zariski open subset of* G.

2) *There exists a closed subscheme* Ξ' *of* $S \times U'$ *such that the second projection* $q : \Xi' \subset S \times U' \to U'$ *is faithfully flat and* $X_s = q^{-1}(\langle s\rangle)$ *for any* $\langle s\rangle \in U'$. *Thus smooth determinantal schemes associated to* E *form a smooth family over an open subset of* G *and hence they are diffeomorphic to each other.*

2. Topology of determinantal varieties

(2.1) From now on we shall consider smooth determinantal varieties associated to 2-bundles on projective spaces.

Let E be a very ample rank two bundle on an n-dimensional projective space $\mathbf{P}^n$ defined over the complex number field $\mathbf{C}$ and X a smooth determinantal variety associated to E which is defined by $(m+1)$-global sections $\{s_1, s_2, \ldots, s_{m+1}\}$ of E (cf. (1.1) and (1.3.3)). First we shall study some topological properties of determinantal varieties X. By Theorem 1, smooth determinantal varieties are diffeomorphic to each other.

(2.2) By the weak Lefschetz theorem, we get the following on singular cohomologies of X:

$$H^i(P(E), \mathbf{Z}) \simeq H^i(X, \mathbf{Z}) \qquad \text{for } i \leq n-(m+1),$$

$H^i(P(E), \mathbf{Z}) \to H^i(X, \mathbf{Z})$ is injective and the cokernel has no torsion elements for $i = n - m$.

Since the cohomology ring $H^*(P(E), \mathbf{Z})$ coincides with $H^*(\mathbf{P}^n, \mathbf{Z})[\xi]/(\xi^2 - c_1(E)\xi + c_2(E))$ where ξ is the cohomology class of L_E and $c_i(E)$ is the i-th Chern number of E ($i = 1, 2$), we have that $H^{2i+1}(X, \mathbf{Z}) = 0$ ($1 \leq 2i+1 \leq n-(m+1)$) and $H^{2i}(X, \mathbf{Z}) = \mathbf{Z}H^i \oplus \mathbf{Z}H^{i-1}D$ ($0 \leq 2i \leq n-(m+1)$) where H is the cohomology class of the restriction of hyperplanes of $\mathbf{P}^n$ and D is the cohomology class of the restriction of L_E to X through π. Combining the above with the Hodge decomposition theorem and the Serre duality theorem, we obtain the following, where $H^{p,q} = \dim H^q(X, \Omega_X^p)$ is the Hodge (p, q)-component of X.

PROPOSITION 1. *Let the notation be as above.*

1) *For $i \leq n-(m+1)$, we have $H^i(X, \mathbf{Z}) = 0$ (i : odd) and $H^i(X, \mathbf{Z}) = \mathbf{Z}H^{i/2} \oplus \mathbf{Z}H^{i/2-1}D$ ($i \geq 2$: even).*

2) *For $p+q \leq n-(m+1)$, $H^{p,q} = 0$ if $p+q$ is odd or if $p+q$ is even and $p \neq q$.*

3) *For $2p \leq n-(m+1)$, $H^{p,p} = \mathbf{C}H^p \oplus \mathbf{C}H^{p-1}D$.*

4) *For $p+q \geq n-(m-1)$, $H^{p,q} = 0$ if $p+q$ is odd or if $p+q$ is even and $p \neq q$.*

5) *For $2p \geq n-(m-1)$, $H^{p,p} = \mathbf{C}H^p \oplus \mathbf{C}H^{p-1}D$.*

(2.3) COROLLARY 1. 1) *If $n \geq 3$, then $H^1(X, \mathbf{Z}) = 0$. Hence we have $H^{1,0} = H^{0,1} = 0$, i.e., the irregularity $q(X) = 0$.*

2) *If $n \geq 5$, then $H^2(X, \mathbf{Z}) = \mathbf{Z}H \oplus \mathbf{Z}D$. Hence we have $H^{2,0} = 0$ and $H^{1,1} = \mathbf{C}H \oplus \mathbf{C}D$.*

We shall study the cohomology group $H^{n-m}(X, \mathbf{C})$ in (5.2.3) later.

(2.4) By the Lefschetz theorem again, we get the following on homotopy groups of X:

$$\pi_i(P(E)) \simeq \pi_i(X) \qquad \text{for } i \leq n-(m+1),$$

where $\pi_i(X)$ is the i-th homotopy group of X.

On the other hand, there exists the following exact sequence of homotopy groups:

$$\to \pi_i(\mathbf{P}^1) \to \pi_i(P(E)) \to \pi_i(\mathbf{P}^n) \to \pi_{i-1}(\mathbf{P}^1) \to .$$

Hence $\pi_1(P(E)) = 0, \pi_2(P(E)) = \mathbf{Z} \oplus \mathbf{Z}$, and $\pi_i(P(E)) \simeq \pi_i(\mathbf{P}^1)$ for $3 \leq i \leq$

$2n-1$ because $\pi_i(\mathbf{P}^n)=0$ *for* $i\leq 2n, i\neq 2$ and $\pi_2(\mathbf{P}^n)\simeq \mathbf{Z}$. Thus we obtain the following.

PROPOSITION 2. 1) *If* $n\geq 3$, *then* $\pi_1(X)=0$, *i.e., X is simply connected.*
2) *If* $n\geq 5$, *then* $\pi_2(X)\simeq \mathbf{Z}\oplus\mathbf{Z}$.
3) *If* $n\geq 7$, *then* $\pi_i(X)\simeq \pi_i(\mathbf{P}^1)$ *for* $3\leq i\leq n-(m+1)$.

3. Divisors on determinantal varieties

(3.1) In this section, we shall study several distinguished divisors on X which will play important roles in studying several geometric structures of X. From now on we assume $n\geq 3$.

As usual we denote

$A^i(X)$(resp.$A_i(X)$): the Chow group of codimension i cycles (resp. dimension i cycles),
$N^i(X)$(resp.$N_i(X)$) $=\{A^i(X)/\text{Numerical equivalence}\}\otimes\mathbf{R}$ (resp. $\{A_i(X)/\text{Numerical equivalence}\}\otimes\mathbf{R}$)
$\mathrm{Pic}(X)=A^1(X)$: the Picard group of X,
$N^1(X)=\{\mathrm{Pic}(X)/\text{Numerical equivalence}\}\otimes\mathbf{R}$,
$\rho=\dim N^1(X)$: the Picard number of X,
$NA(X)$: the Cone of ample divisors of X (= convex cone generated by classes of ample divisors of X in $N^1(X)$),
$\overline{NA}(X)$: the Cone of nef divisors of X (= closed convex cone generated by classes of numerically effective divisors of X in $N^1(X)$),
$NE^1(X)$: the Cone of effective divisors of X (= convex cone generated by classes of effective divisors of X in $N^1(X)$),
$\overline{NE}^1(X)$: the Cone of pseudo-effective divisors of X (= closure of the cone of effective divisors of X in $N^1(X)$),
$NE_1(X)$: the Cone of effective curves of X (= convex cone generated by classes of effective curves of X in $N_1(X)$),
$\overline{NE}_1(X)$: the Cone of pseudo-effective curves of X (= closure of the cone of effective curves of X in $N_1(X)$),
c_i: the i-th Chern number of E ($i=1,2$).

(3.2) Divisors H and D on X.

(3.2.1) Let H be the restriction of a hyperplane of $\mathbf{P}^n$ to X and D the restriction of a tautological divisor of $P(E)$ to X through the isomorphism π. Then as we have observed in (2.2), H and D are linearly independent elements of $\mathrm{Pic}(X)$ and $\mathrm{Pic}(X)=\mathbf{Z}H\oplus\mathbf{Z}D$ for $n\geq 5$. Let K_X be the canonical divisor of X. Then by the adjunction formula, we see

$$K_X=(m-1)D+\{c_1-(n+1)\}H.$$

When $n=4$, our determinantal varieties are smooth projective surfaces with $H^{2,0}=H^0(X,K_X)=H^0(X,\mathcal{O}_X(D+(c_1-5)H))$. Hence if $c_1\geq 5$, then $H^{2,0}\neq 0$. In addition, $H^{2,0}(D_1\cap D_2)=0$ by the Lefschetz theorem where $\{D_i\}$ $(1\leq i\leq 3)$ is a set of tautological divisors which determines X and hence $\mathrm{Pic}(X)=\mathbf{Z}H\oplus\mathbf{Z}H$ if $c_1\geq 5$ by the following result due to B. G. Moishezon [23].

LEMMA (Moishezon). *Let Y be a 3-dimensional non-singular variety embedded in a projective space $\mathbf{P}^m$ and X a general hyperplane section of Y. Then each linear equivalence class of divisors on X is cut out by some linear equivalence class of divisors on Y, i.e.,* $\mathrm{Pic}(Y)\simeq\mathrm{Pic}(X)$ *if and only if one of the following two conditions is satisfied:*

1) $b_2(Y)=b_2(X)$.
2) $h^{2,0}(X)>h^{2,0}(Y)$.

Moreover in the cases $c_1\leq 4$, we shall prove in Section 4 (4.4.9) that (c_1,c_2) coincides with one of the following pairs of natural numbers: $(2,1),(3,2),(4,3),(4,4)$ and that E is isomorphic to either one of the following rank two bundles: $\mathcal{O}(1)\oplus\mathcal{O}(1),\mathcal{O}(1)\oplus\mathcal{O}(2),\mathcal{O}(1)\oplus\mathcal{O}(3),\mathcal{O}(2)\oplus\mathcal{O}(2)$, respectively. Hence we observe that $H^0(X,K_X)\neq 0$ except for the case $E\simeq\bigoplus^2\mathcal{O}(1)$ by Theorem 3 in (4.1.1) and that X is isomorphic to a rational scroll $F_1=P(\mathcal{O}(0)\oplus\mathcal{O}(1))$ in the case $E\simeq\bigoplus^2\mathcal{O}(1)$ by (6.1.4) 3) (cf. [18]). Therefore we find also in the case $n=4$ that $\mathrm{Pic}(X)=\mathbf{Z}H\oplus\mathbf{Z}D$ for a general determinantal variety X.

Let L be a divisor of X which is numerically equivalent to 0. Then since $q(X)=0$, rL is rationally equivalent to 0 for some positive integer r. On the other hand, $\mathrm{Pic}(X)$ has no torsion elements and hence it follows that numerical equivalence coincides with rational equivalence for divisors on X.

(3.2.2) Let us define the following polynomial functions $\{p_i=p_i(c_1,c_2),$ $q_i=q_i(c_1,c_2)\}$ $(i=0,1,\dots)$ inductively as follows:

$$p_0=1,\qquad q_0=0,$$
$$p_{i+1}=c_1p_i+q_i,\qquad q_{i+1}=-c_2p_i.$$

Then it is easily calculated that for every integer $i\geq 0$,

$$p_{i+2}-c_1p_{i+1}+c_2p_i=0,$$

and

$$p_i=\sum_{k=0}^{[i/2]}(-1)^k\binom{i-k}{k}c_1^{i-2k}c_2^k,$$

$$q_i=\sum_{k=0}^{[i-1/2]}(-1)^k\binom{i-1-k}{k}c_1^{i-1-2k}c_2^{k+1},$$

where $[a]$ $\left(\text{resp. } \binom{a}{b}\right)$ stands for the largest integer which is less than or equal to a real number a (resp. the binomial coefficient of integers a and b).

Using the relation $D^2 - c_1HD + c_2H^2 = 0$ in $A^2(X)$, where c_i is the i-th Chern number of E $(i = 1, 2)$, the intersection number $H^{n-m-i}D^i$ is calculated as follows:

$$H^{n-m-i}D^i = p_{m+i}(c_1, c_2) \qquad (0 \leq i \leq n-m).$$

In particular, we have $\deg X = H^{n-m} = p_m(c_1, c_2)$.

For example,

$$n = 4. \qquad \deg X = c_1^2 - c_2, \qquad HD = c_1^3 - 2c_1c_2, \qquad D^2 = c_1^4 - 3c_1^2c_2 + c_2^2.$$

$$n = 5. \quad \deg X = c_1^2 - c_2, \quad H^2D = c_1^3 - 2c_1c_2, \quad HD^2 = c_1^4 - 3c_1^2c_2 + c_2^2,$$
$$D^3 = c_1^5 - 4c_1^3c_2 + 3c_1c_2^2.$$

Summing up the above, we obtain the following.

LEMMA 2. *Let the notation be as above.*

1) *The numerical equivalence coincides with rational equivalence for divisors on X and so we have* $\mathrm{Pic}(X) \otimes \mathbf{R} = N^1(X)$. *Moreover* $\mathrm{Pic}(X) = \mathbf{Z}H \oplus \mathbf{Z}D$, *provided* $n \geq 5$. *If* $n = 4$, *then* $\mathrm{Pic}(X) = \mathbf{Z}H \oplus \mathbf{Z}D$ *for a general determinantal surfaces X.*

2) *The intersection number* $H^{n-m-i}D^i$ *is equal to* $p_{m+i}(c_1, c_2)$ $(0 \leq i \leq n-m)$.

3) $K_X = (m-1)D + (c_1 - n - 1)H$.

(3.3) Divisor F on X.

(3.3.1) We shall introduce a specific divisor F of determinantal varieties. There exists a canonical exact sequence on $P(E)$:

$$0 \to \mathcal{O}_{P(E)}(\pi^*(c_1(E))) \otimes L_E^{-1} \to \pi^*(E) \to L_E \to 0.$$

Hence if we denote by $E|X$ the restriction of E to X, then we obtain the following exact sequence:

$$0 \longrightarrow \mathcal{O}_X(F) \xrightarrow{\alpha} E|X \xrightarrow{\beta} \mathcal{O}_X(D) \longrightarrow 0,$$

where F is a divisor on X which is rationally equivalent to $c_1H - D$, i.e., $F = c_1H - D \in Pic(X)$. The homomorphisms α, β are described locally as follows. Let U be an affine open subset of $\mathbf{P}^n$ such that $E|U \simeq \bigoplus^2 \mathcal{O}_U$ and for $1 \leq i \leq m+1, s_i|U = (s_{i1}, s_{i2})$ $(s_{ik} \in H^0(U, \mathcal{O}_U), k = 1, 2)$. Here we may assume $s_{12}(x) \neq 0$ for $x \in U$. Then $\beta(a, b) = a - (s'_{11}/s'_{12})b$ for $(a, b) \in H^0(U, E|X)$ and $\alpha(c)$

$= ((s'_{11}/s'_{12})c, c)$ for $c \in H^0(U, \mathcal{O}_U)$, where s' stands for the restriction of a regular function s on U to X.

Let $s_i|X$ be the restriction of s_i to X. Then $\beta \circ (s_i|X)$ is a zero map and hence every $s_i|X$ factors through $\mathcal{O}_X(F)$, i.e., it defines a global section of $\mathcal{O}_X(F)$ which we shall denote also by $s_i|X$ for simplicity. Then $s_i|X$ is described on U as follows: $\mathcal{O}_U \ni 1 \to s'_{i2} \in \mathcal{O}_U(F)$. If $s_i|X$ is a nonzero section, then it defines an effective divisor F_i of X that is linearly equivalent to F with the following defining equations in $\mathbf{P}^n$:

$$F_i : s_i = 0, s_j \wedge s_k = 0 \qquad \text{for } j \neq i, k \neq i \qquad (1 \leq j, k \leq m+1).$$

(3.3.2) Let $I = \{i | s_i|X \neq 0\}$. Since every F_i is a member of the complete linear system $|F|$ of F and $\bigcap_{i=1}^{m+1} W(s_i) = \varnothing$, the set $\{F_i\}$ $(i \in I)$ generates a linear subsystem $\mathcal{L}$ of $|F|$ without base points:

$$\mathcal{L} = (F_k, L), \qquad L = \sum \mathbf{C} f_i \text{ (a defining module of } \mathcal{L}\text{)},$$

where f_i is a rational function of X satisfying $F_i = F_k + (f_i)$ for all $i \in I$. Let φ be the morphism associated to $\mathcal{L}$.

We shall show that $\mathcal{L}$ is an m-dimensional linear system and that φ is a surjective morphism of X to $\mathbf{P}^m$. In particular, it follows that every $s_i|X$ is not zero, i.e., $I = \{1, \dots, m+1\}$.

1) Since $F = c_1 H - D$, we have the relation $F^2 - c_1 HF + c_2 H^2 = 0$ in $A^2(X)$. Using the relation, the intersection numbers $H^{n-m-i}F^i$ are calculated as follows:

$$a)\ n = 2m. \qquad H^{m-i}F^i = c_2^i p_{m-i}(c_1, c_2) \qquad (0 \leq i \leq m).$$

$$b)\ n = 2m+1. \qquad H^{m+1-i}F^i = c_2^i p_{m-i}(c_1, c_2) \qquad (0 \leq i \leq m),$$

$$F^{m+1} = 0.$$

2) Suppose that S is a surface contained in a fiber of φ. Since $\mathcal{O}_S(F) \simeq \mathcal{O}_S$, we have the exact sequence from (3.3.1):

$$0 \to \mathcal{O}_S \to E|S \to \mathcal{O}_S(D) \to 0.$$

Let $\sigma : \tilde{S} \to S$ be the desingularization of S. Then we have

$$0 = c_2(\sigma^*(E|S)) = \sigma^*(c_2(E|X)) = c_2 \sigma^*(H)^2.$$

However since c_2 is positive and $\sigma^*(H)^2 > 0$, it yields a contradiction. Hence every fiber of φ is of dimension ≤ 2.

3) $\dim \mathcal{L} = m$ and $\varphi : X \to \mathbf{P}^m$ is surjective. Indeed, if $n = 2m$, then we see $\dim X = m$ and $F^m = c_2^m > 0$ by 1), which imply that $\dim \mathcal{L} = m$ and φ is

a surjective morphism. If $n = 2m+1$, then $\dim X = m+1$. Since the dimension of every fiber of $\varphi \leq 1$ by 2), it follows that $\dim \mathscr{L} = m$ and φ is surjective.

(3.3.3) Let U be an affine open subset of $\mathbf{P}^n$ with $E|U \simeq \bigoplus^2 \mathcal{O}_U$. Assume that $s_i|U = (s_{i1}, s_{i2})$ $(1 \leq i \leq m+1)$ and $s_{12}(x) \neq 0$ for all $x \in U$. Since $s'_{i1}s'_{12} = s'_{i2}s'_{11}$ and $s_i|X \neq 0$ $(1 \leq i \leq m+1)$, we see $s'_{i2} \neq 0$. Hence if we choose a rational function $f_i = s'_{i2}/s'_{12}$ on X, then it is easily checked that f_i does not depend on the choice of an affine open subset satisfying the above conditions and the trivialization of E and moreover it enjoys the following equality:

$$F_i = F_1 + (f_i) \qquad (1 \leq i \leq m+1).$$

Let $\mathscr{L} = (F_1, L)$ be the m-dimensional linear system with the defining module $L = \sum \mathbf{C} f_i$ and let $\varphi : X \to \mathbf{P}^m$ be the morphism associated with $\mathscr{L}$. If $n = 2m$, then φ is a generically finite morphism with fibers of dimension ≤ 1. If $n = 2m+1$, then φ is a faithfully flat morphism with fibers of dimension $= 1$. Thus in both cases, F is 1-ample in the sense of A. Sommese [35]. Hence we get the following vanishing: For all positive integers r,

$$H^q(X, \mathcal{O}_X(rF) \otimes \Omega_X^p) = 0 \qquad \text{for } p+q \geq n-m+2,$$

$$H^q(X, \mathcal{O}_X(-rF)) = 0 \qquad \text{for } 0 \leq q \leq n-m-2.$$

Summing up the above, we obtain the following.

THEOREM 2. 1) *There exists the following exact sequence:*

$$0 \to \mathcal{O}_X(F) \to E|X \to \mathcal{O}_X(D) \to 0,$$

where $F (= c_1 H - D$ *in* $\mathrm{Pic}(X))$ *is an effective divisor on* X.

2) *The intersection number* $H^{m-i}F^i$ *is calculated as follows:*

a) $n = 2m$. $H^{m-i}F^i = c_2^i p_{m-i}(c_1, c_2)$ $(0 \leq i \leq m)$.

b) $n = 2m+1$. $H^{m+1-i}F^i = c_2^i p_{m-i}(c_1, c_2)$ $(0 \leq i \leq m), F^{m+1} = 0$.

3) *There exists an m-dimensional sublinear system* $\mathscr{L} = (F_1, L)$ *of the complete linear system* $|F|$ *which is free from base points. Let* $\varphi : X \to \mathbf{P}^m$ *be the morphism defined by* $\mathscr{L}$. *Then we have:*

a) $n = 2m$. φ *is a generically finite morphism with fibers of dimension* ≤ 1.

b) $n = 2m+1$. φ *is a faithfully flat morphism with fibers of dimension* 1. *Hence* F *is* 1*-ample in the sense of A. Sommese and it turns out that for any positive integer* r,

$$H^q(X, \mathcal{O}_X(rF) \otimes \Omega_X^p) = 0 \qquad \textit{for } p+q \geq n-m+2,$$

$$H^q(X, \mathcal{O}_X(-rF)) = 0 \qquad \textit{for } q \leq n-m-2.$$

(3.4) $\overline{NE}^1(X)$.

(3.4.1) When $n \geq 4$, we have $\mathrm{Pic}(X) = \mathbf{Z}H \oplus \mathbf{Z}D$ as we have shown in (3.2.1). A divisor Z is called effective if $H^0(X, \mathcal{O}(Z)) \neq 0$. From now on we assume $n \geq 4$. Let us study the cone $\overline{NE}^1(X)$ of pseudo-effective divisors on X. Since it is a closed convex cone in $N^1(X) \simeq \bigoplus^2 \mathbf{R}$, it has two boundaries, i.e., half lines which we shall call the extremal rays of the cone $\overline{NE}^1(X)$.

For every positive integer r, let us put

$$a(r) = \max\{s \mid rD - sH \text{ is effective}\},$$

$$b(r) = \min\{s \mid sH - rD \text{ is effective}\}.$$

Then $a(r)$ and $b(r)$ $(r \in \mathbf{N})$ satisfy the following:

1) $a(r+s) \geq a(r) + a(s)$,
2) $a(r)/r \leq D^{n-m}/D^{n-m-1}H = p_n(c_1, c_2)/p_{n-1}(c_1, c_2)$,
3) $b(r+s) \leq b(r) + b(s)$,
4) $b(r)/r \geq (c_1^2 - c_2)/c_1$ if n is even and $b(r)/r \geq c_1$ if n is odd.

DEFINITION 2. $\theta_1 = \sup\{a(r)/r\}$ and $\theta_2 = \inf\{b(r)/r\}$.

Then $\ell_1 = \mathbf{R}_{\geq 0}[D - \theta_1 H]$ and $\ell_2 = \mathbf{R}_{\geq 0}[\theta_2 H - D]$ are the two extremal rays of $\overline{NE}^1(X)$ where $\mathbf{R}_{\geq 0} = \{r \in \mathbf{R} \mid r \geq 0\}$ and $[G]$ stands for the class of a divisor G in $N^1(X)$.

If n is odd, then $b(r) - rc_1 \geq 0$ and hence $b(r)H - rD = (b(r) - rc_1)H + rF$ is a linear combination of H and F with non-negative integers as coefficients. Thus $\ell_2 = \mathbf{R}_{\geq 0}[F]$, i.e., $\theta_2 = c_1$.

(3.4.2) We shall prove that every member of $|F|$ is irreducible when $n \geq 5$ is odd. For that purpose, we shall prepare an elementary criterion for an effective divisor to be irreducible.

Let V be a closed convex cone in a Euclidean space $\mathbf{R}^\rho$. A half line $\ell = \mathbf{R}_{\geq 0}v$ $(v \in V)$ is called an *extremal ray* of V if $v_1 + v_2 \in \ell$ for $v_1, v_2 \in V$ implies that $v_1 \in \ell$ and $v_2 \in \ell$.

LEMMA 3. *Let F be an effective divisor on a smooth projective variety X. Assume that $N^1(X)$ has a basis $\{D_1, \dots, D_\rho\}$ $(D_i \in \mathrm{Pic}(X))$ satisfying the following.*

1) *Every divisor D of X is expressed in $N^1(X)$ as follows:*

$$D = \sum_{i=1}^{\rho} n_i D_i \qquad \text{with } n_i \in \mathbf{Z} \qquad (1 \leq i \leq \rho).$$

2) *$F = \sum_{i=1}^{\rho} m_i D_i$ with $\gcd(\mathrm{m}_1, \dots, \mathrm{m}_\rho) = 1$.*
3) *$\mathbf{R}_{\geq 0}[F]$ is an extremal ray of $\overline{NE}^1(X)$.*

Then every member of the complete linear system $|F|$ is irreducible.

PROOF. Let $F' \in |F|$ and assuming that F' is not irreducible, let $F' = \sum_{k=1}^{r} \alpha_k F_k$ where a) every F_k is a prime divisor of X and $\alpha_k \in \mathbf{N}$ and b) $r \geq 2$ or $\alpha_1 \geq 2$ if $r = 1$. Then since $\mathbf{R}_{\geq 0}[F]$ is an extremal ray of $\overline{NE}^1(X)$, every $F_k \in \mathbf{R}_{\geq 0}[F]$, i.e., $F_k \equiv \beta_k F$ (numerical equiv.) with $\beta_k \in \mathbf{Q}$ $(1 \leq k \leq r)$. Thus we get $1 = \sum_{k=1}^{r} \alpha_k \beta_k$, which implies $0 < \beta_k < 1$. Moreover let $F_i = \sum_{j=1}^{\rho} n_{ij} D_j$ $(n_{ij} \in \mathbf{Z})$. Then we have $n_{kj} = \beta_k m_j$ $(1 \leq k \leq r, 1 \leq j \leq \rho)$. Since $\gcd(m_1, \dots, m_\rho) = 1$, there are integers $\{n_j\}$ $(1 \leq j \leq \rho)$ satisfying $\sum_{j=1}^{\rho} m_j n_j = 1$, from which follows $\beta_k = \sum_{j=1}^{\rho} n_{kj} n_j$ for all k. Thus we get a contradiction.

(3.4.3) Accordingly we obtain the following.

PROPOSITION 3. *If $n \geq 5$ is odd, then every member of $|F|$ is irreducible.*

PROOF. By Lemma 2, we find that $\{H, D\}$ is a basis of $N^1(X)$ enjoying the conditions in Lemma 3 because $F = c_1 H - D$ in $\mathrm{Pic}(X)$ and $\mathbf{R}_{\geq 0}[F]$ is an extremal ray of $\overline{NE}^1(X)$ for odd $n \geq 5$ by (3.4.1). Thus it follows that every member of $|F|$ is irreducible by Lemma 3.

(3.5) Divisors Z and Z^* on X.

(3.5.1) For simplicity let us put $a = a(1)$ and define the following effective divisor:

$$Z = D - aH.$$

Then we see $0 \leq a < c_1$ by definition. The effective divisor Z will play an important role in the splitting problem of E in the sequel.

The intersection numbers $Z^i H^{n-m-i}$ $(1 \leq i \leq n-m)$ are calculated as follows:

$$\begin{aligned} Z^i H^{n-m-i} &= \sum_{k=0}^{i} (-a)^k p_{m+i-k} \\ &= \sum_{k=0}^{i} (-1)^k (c_1 - a)^{i-k} c_2^k p_{m-k}. \end{aligned}$$

LEMMA 4. *If $\dim |rZ| = 0$ for every positive integer r, then $a(r) = ra$ for all $r \in \mathbf{N}$ and so $\theta_1 = a$. Hence $\ell_1 = \mathbf{R}_{\geq 0}[Z]$ and every member of $|Z|$ is irreducible.*

PROOF. For every $r \in \mathbf{N}, rZ = (rD - a(r)H) + (a(r) - ra)H$. If $a(r) > ra$ for some r, then we get $\dim |rZ| \geq 1$, which contradicts our assumption. Hence it follows that $a(r) = ra$ for all $r \in \mathbf{N}$ and so $\theta_1 = a$. Since $\mathbf{R}_{\geq 0}[Z]$ is an extremal ray of $\overline{NE}^1(X)$ and $Z = D - aH$, every member of $|Z|$ is irreducible by Lemma 3.

(3.5.2) Moreover we define the following divisor which we call the dual divisor of Z:

$$Z^* = D - (c_1 - a)H = aH - F.$$

Then it satisfies the following properties.

1) $Z + Z^* = 2D - c_1 H = D - F$.

2) If $2a \geq c_1$, then $Z^* = Z + (2a - c_1)H$ is effective.

3) $Z \cdot Z^* = (D - aH)(D - (c_1 - a)H) = -(a^2 - c_1 a + c_2)H^2$ as a cycle of codimension 2 in X.

4) The intersection numbers for $0 \leq i \leq n - m$,

$$\begin{aligned} Z^{*i} H^{n-m-i} &= \sum_{k=0}^{i} (-1)^k a^{i-k} c_2^k p_{m-k} \\ &= \sum_{k=0}^{i} (-1)^k (c_1 - a)^k p_{m+i-k}. \end{aligned}$$

The divisor Z^* will also play an important role in the splitting problem of E in the sequel (cf. Theorem 9).

4. Comparison theorems of cohomologies

In this chapter we shall show several comparison theorems between cohomologies $H^i(X, \mathcal{O}_X(rD + sH))$ and $H^i(\mathbf{P}^n, S^r(E)(s))$ and their applications.

Recall the following commutative diagram:

$$\begin{array}{ccc} P(E) & \supset & Y = D_1 \cap D_2 \cap \cdots \cap D_{m+1} \\ {\scriptstyle\pi} \downarrow & & \downarrow \\ \mathbf{P}^n & \supset & X. \end{array}$$

Since Y is a closed subscheme of $P(E|X)$, there exists the following sequence of line bundles:

$$\mathcal{O}_{P(E)}(rD + sH) \to \mathcal{O}_{P(E|X)}(rD + sH) \to \mathcal{O}_Y(rD + sH),$$

which gives rise to a sequence of cohomologies:

$$H^i(\mathcal{O}_{P(E)}(rD + sH)) \xrightarrow{\alpha_i} H^i(\mathcal{O}_{P(E|X)}(rD + sH)) \xrightarrow{\beta_i} H^i(\mathcal{O}_Y(rD + sH)).$$

We shall study the maps $\gamma_i = \beta_i \circ \alpha_i$ in order to compare $H^i(\mathcal{O}_{P(E)}(rD + sH))$ with $H^i(\mathcal{O}_Y(rD + sH))$.

$Y_k := D_1 \cap \cdots \cap D_k$ is a complete intersection in $P(E)$ for every integer k $(1 \leq k \leq m + 1)$ and $Y_0 := P(E)$. Since Y coincides with Y_{m+1}, we have a tower of closed subschemes

$$P(E) = Y_0 \supset Y_1 \supset \cdots \supset Y_{m+1} = Y,$$

which we shall use to obtain our comparison theorems.

According to the values of r and s, we shall procced to study the map $\gamma_i : H^i(\mathcal{O}_{P(E)}(rD - sH)) \to H^i(\mathcal{O}_Y(rD - sH))$ as follows:

1) $r = 1,\ s \geq 0$ 2) $2 \leq r \leq m+1,\ s \geq 0$ 3) $r \geq m+2,\ s \geq 0$ 4) $r \leq 0,\ s \geq 0$

(4.1) $r = 1, s \geq 0$.

(4.1.1) Consider the canonical exact sequence

$$(*)_1 \qquad 0 \to \mathcal{O}_{P(E)}(-sH) \to \mathcal{O}_{P(E)}(D - sH) \to \mathcal{O}_{Y_1}(D - sH) \to 0,$$

from which we obtain the following exact sequence:

$$\to H^i(\mathcal{O}_{P(E)}(-sH)) \to H^i(\mathcal{O}_{P(E)}(D - sH)) \to H^i(\mathcal{O}_{Y_1}(D - sH)) \to .$$

Since $H^i(\mathcal{O}_{P(E)}(-sH)) = H^i(\mathcal{O}_{\mathbf{P}^n}(-s)) = 0$ for either $0 \leq i \leq n-1,\ s > 0$ or $1 \leq i \leq n,\ s = 0$, the map $\delta_i^1 : H^i(\mathcal{O}_{P(E)}(D - sH)) \to H^i(\mathcal{O}_{Y_1}(D - sH))$ is an isomorphism for either $0 \leq i \leq n-2, s > 0$ or $1 \leq i \leq n-1, s = 0$.

Similarly, consider the exact sequences

$$(*)_2 \qquad 0 \to \mathcal{O}_{Y_1}(-sH) \to \mathcal{O}_{Y_1}(D - sH) \to \mathcal{O}_{Y_2}(D - sH) \to 0,$$

$$0 \to \mathcal{O}_{P(E)}(-D - sH) \to \mathcal{O}_{P(E)}(-sH) \to \mathcal{O}_{Y_1}(-sH) \to 0.$$

Taking cohomologies, they give rise to the exact sequences

$$\to H^i(\mathcal{O}_{Y_1}(-sH)) \to H^i(\mathcal{O}_{Y_1}(D - sH)) \to H^i(\mathcal{O}_{Y_2}(D - sH)) \to,$$

$$\to H^i(\mathcal{O}_{P(E)}(-D - sH)) \to H^i(\mathcal{O}_{P(E)}(-sH)) \to H^i(\mathcal{O}_{Y_1}(-sH)) \to,$$

from which we obtain $H^i(\mathcal{O}_{Y_1}(-sH)) = 0$ for either $0 \leq i \leq n-1,\ s > 0$ or $1 \leq i \leq n,\ s = 0$ because $H^i(\mathcal{O}_{P(E)}(-D - sH)) = 0$ for $0 \leq i \leq n+1$. Hence $\delta_i^2 : H^i(\mathcal{O}_{Y_1}(D - sH)) \to H^i(\mathcal{O}_{Y_2}(D - sH))$ is isomorphic for either $0 \leq i \leq n-2,\ s > 0$ or $1 \leq i \leq n-1, s = 0$.

Inductively, take the exact sequences

$$(*)_k \qquad 0 \to \mathcal{O}_{Y_{k-1}}(-sH) \to \mathcal{O}_{Y_{k-1}}(D - sH) \to \mathcal{O}_{Y_k}(D - sH) \to 0,$$

$$0 \to \mathcal{O}_{Y_{k-1}}(-D - sH) \to \mathcal{O}_{Y_{k-1}}(-sH) \to \mathcal{O}_{Y_k}(-sH) \to 0.$$

We prove $H^i(\mathcal{O}_{Y_k}(-sH)) = 0$ for either $0 \leq i \leq n-k,\ s > 0$ or $1 \leq i \leq n-k$, $s = 0$ by induction on k. If $k = 1$, then it is verified already. Since Y_{k-1} is a complete intersection of very ample divisors $\{D_1, \ldots, D_{k-1}\}$ in $P(E)$ of dimension $n-k+2$ and $D+sH$ is ample, it follows that $H^i(\mathcal{O}_{Y_{k-1}}(-D-sH)) = 0$ for $0 \le i \le n-k+1$. Thus we obtain $H^i(\mathcal{O}_{Y_k}(-sH)) \simeq H^i(\mathcal{O}_{Y_{k-1}}(-sH)) = 0$ for either $0 \le i \le n-k$, $s > 0$ or $1 \le i \le n-k$, $s = 0$ from the exact sequence $(*)_k$ and the induction hypothesis. Hence we see that the map $\delta_i^k : H^i(\mathcal{O}_{Y_{k-1}}(D-sH)) \to H^i(\mathcal{O}_{Y_k}(D-sH))$ is isomorphic for either $0 \le i \le n-k$, $s > 0$ or $1 \le i \le n-k$, $s = 0$.

Since $\gamma_i = \prod_{k=1}^{m+1} \delta_i^k$, it turns out that $\gamma_i : H^i(\mathcal{O}_{P(E)}(D-sH)) \to H^i(\mathcal{O}_Y(D-sH))$ is an isomorphism for either $0 \le i \le n-m-1$, $s > 0$ or $1 \le i \le n-m-1$, $s = 0$. The above arguments also state that γ_i is injective for $i = n-m$ and $\delta_0^k : H^0(\mathcal{O}_{Y_{k-1}}(D)) \to H^0(\mathcal{O}_{Y_k}(D))$ is surjective with $\operatorname{Ker} \delta_0 = \mathbf{C}$ for every k, thereby $\gamma_0 : H^0(\mathcal{O}_{P(E)}(D)) \to H^0(\mathcal{O}_Y(D))$ is surjective with $\operatorname{Ker} \gamma_0 = \bigoplus^{m+1} \mathbf{C}$. Therefore we get the following comparison theorem.

Theorem 3. *The canonical restriction map* $\gamma_i : H^i(\mathcal{O}_{P(E)}(D-sH)) \to H^i(\mathcal{O}_Y(D-sH))$ *is an isomorphism for either* $0 \le i \le n-m-1, s > 0$ *or* $1 \le i \le n-m-1$, $s = 0$ *and injective for* $i = n-m$, $s \ge 0$. *Moreover* $\gamma_0 : H^0(\mathcal{O}_{P(E)}(D)) \to H^0(\mathcal{O}_Y(D))$ *is surjective with* $\operatorname{Ker} \gamma_0 = \bigoplus^{m+1} \mathbf{C}$.

(4.1.2) We shall show some corollaries of Theorem 3. The vector bundle E is unstable with respect to the line bundle $\mathcal{O}_{\mathbf{P}^n}(1)$ if and only if $H^0(E(-s)) \neq 0$ for $s \ge c_1/2$ (c_1: even) (resp. $s \ge (c_1+1)/2$ (c_1: odd)) (cf. [27]). Recall that the integer $a = a(1)$ is the maximal integer such that $Z = D - aH$ is effective. Since $H^0(\mathcal{O}_{P(E)}(D-sH) \simeq H^0(\mathbf{P}^n, E(-s))$ and $H^0(\mathcal{O}_Y(D-sH)) \simeq H^0(\mathcal{O}_X(D-sH))$ for all positive integers s, E is unstable if and only if $a \ge c_1/2$ (c_1: even) (resp. $a \ge (c_1+1)/2$ (c_1: odd)) by Theorem 3.

Corollary 2. *E is unstable with respect to $\mathcal{O}_{\mathbf{P}^n}(1)$ if and only if $a \ge c_1/2$ (c_1: even) (resp. $a \ge (c_1+1)/2$ (c_1: odd)).*

We shall recall the following lemmas to show a further result. One is the following vanishing theorem [39] that is generalized to ample vector bundles with any rank on $\mathbf{P}^n$ by L. Manivel [20].

Lemma 5 (Sumihiro). *Let E be a rank two bundle which is k-ample (resp. generated by global sections) and L a line bundle which is generated by global sections (resp. k-ample) on $\mathbf{P}^n$. Then we have $H^q(\mathbf{P}^n, S^r(E) \otimes L \otimes \Omega_{\mathbf{P}^n}^p) = 0$ for $p+q \ge n+2+k, r \ge 1$.*

The other is the following lemma on higher direct images of tensor product of tautological line bundles which is established easily using the duality theorem for π (cf. [12]). Let $\pi : P(E) \to S$ be the projective bundle associated to a vector bundle E of rank r on a smooth projective variety S and D a tautological divisor of $P(E)$.

LEMMA 6. *For a positive integer p, we have*

1) $R^i\pi_*(\mathcal{O}_{P(E)}(-pD)) = 0$ *for either* $0 \le i \le r-2$ *or* $i = r-1$, $1 \le p \le r-1$.

2) *If* $p \ge r$, *then* $R^i\pi_*(\mathcal{O}_{P(E)}(-pD)) = S^{p-r}(E^*) \otimes \mathcal{O}_S(-c_1(E))$ *for* $i = r-1$.

With these preliminaries, we shall show the following.

COLLORARY 3. *The restriction map* $H^0(\mathbf{P}^n, E) \to H^0(X, E|X)$ *is surjective and* $\dim H^0(X, \mathcal{O}_X(F)) = m+1$.

PROOF. 1) First we prove that the restriction map is surjective. For the purpose, we show by induction on k and i that for every k $(1 \le k \le m+1)$ and i $(0 \le i \le k-1)$,

$$H^{k-i}(P(E), \pi^*(E) \otimes \mathcal{O}_{D_1 \cap \cdots \cap D_i}(-(k-i)D)) = 0.$$

It is obvious in case $k = 1$ by Lemma 6. Hence we assume that it holds for $k-1$. Now we check the vanishing $H^k(P(E), \pi^*(E) \otimes \mathcal{O}(-kD)) = 0$. By Lemma 6, it follows that

$$\begin{aligned} H^k(P(E), \pi^*(E) \otimes \mathcal{O}(-kD)) &\simeq H^{k-1}(\mathbf{P}^n, E \otimes S^{k-2}(E^*) \otimes \mathcal{O}(-c_1)) \\ &= H^{k-1}(\mathbf{P}^n, E^* \otimes S^{k-2}(E^*)) \simeq H^{n-k+1}(\mathbf{P}^n, E \otimes S^{k-2}(E) \otimes K_{\mathbf{P}^n}). \end{aligned}$$

Since there is an exact sequence

$$0 \to S^{k-3}(E) \otimes \mathcal{O}(c_1) \to E \otimes S^{k-2}(E) \to S^{k-1}(E) \to 0,$$

the vanishing follows from

$$H^{n-k+1}(\mathbf{P}^n, S^{k-3}(E)(c_1) \otimes K_{\mathbf{P}^n}) = 0, \qquad H^{n-k+1}(\mathbf{P}^n, S^{k-1}(E) \otimes K_{\mathbf{P}^n}) = 0,$$

which hold by Lemma 5. Thus we assume that it holds for $i-1$ in case k. Consider the following exact sequence:

$$\begin{aligned} 0 &\to \pi^*(E) \otimes \mathcal{O}_{D_1 \cap \cdots \cap D_{i-1}}(-D_i - (k-i)D) \to \pi^*(E) \otimes \mathcal{O}_{D_1 \cap \cdots \cap D_{i-1}}(-(k-i)D) \\ &\to \pi^*(E) \otimes \mathcal{O}_{D_1 \cap \cdots \cap D_i}(-(k-i)D) \to 0. \end{aligned}$$

Then it gives rise to the exact sequence:

$$\begin{aligned} &\to H^{k-i}(\pi^*(E) \otimes \mathcal{O}_{D_1 \cap \cdots \cap D_{i-1}}(-(k-i)D)) \\ &\to H^{k-i}(\pi^*(E) \otimes \mathcal{O}_{D_1 \cap \cdots \cap D_i}(-(k-i)D)) \\ &\to H^{k-i+1}(\pi^*(E) \otimes \mathcal{O}_{D_1 \cap \cdots \cap D_{i-1}}(-(k-(i-1))D)) \to \end{aligned}$$

Since $H^{k-i}(\pi^*(E) \otimes \mathcal{O}_{D_1 \cap \cdots \cap D_{i-1}}(-(k-i)D)) = 0$ and $H^{k-(i-1)}(\pi^*(E) \otimes \mathcal{O}_{D_1 \cap \cdots \cap D_{i-1}}(-(k-(i-1))D)) = 0$ by the induction hypothesis, it follows that $H^{k-i}(\pi^*(E) \otimes \mathcal{O}_{D_1 \cap \cdots \cap D_i}(-(k-i)D)) = 0$.

In particular we have $H^1(P(E), \pi^*(E) \otimes \mathcal{O}_{D_1 \cap \cdots \cap D_{k-1}}(-D_k)) = 0$ $(1 \le k \le m+1)$. Thus the restriction map $H^0(\pi^*(E) \otimes \mathcal{O}_{D_1 \cap \cdots \cap D_{k-1}}) \to H^0(\pi^*(E) \otimes \mathcal{O}_{D_1 \cap \cdots \cap D_k})$ $(1 \le k \le m+1)$ is surjective and it implies the surjectivity of the restriction map $H^0(P(E), \pi^*(E)) = H^0(\mathbf{P}^n, E) \to H^0(Y, \pi * (E)) = H^0(X, E|X)$.

2) Consider the following commutative diagram in which the middle vertical map is surjective by 1):

$$\begin{array}{ccccccc}
0 \longrightarrow & \operatorname{Ker}\gamma_0 & \longrightarrow & H^0(P(E), \mathcal{O}(D)) & \xrightarrow{\gamma_0} & H^0(Y, \mathcal{O}(D)) & \longrightarrow \\
& \downarrow & & \downarrow & & & \\
& \bigoplus^{m+1}\mathbf{C} & & H^0(\mathbf{P}^n, E) & & & \\
& \downarrow & & \downarrow & & & \\
0 \longrightarrow & H^0(X, \mathcal{O}_X(F)) & \longrightarrow & H^0(X, E|X) & \longrightarrow & H^0(X, \mathcal{O}(D)) & \longrightarrow
\end{array}$$

Thus we find that $\dim H^0(X, \mathcal{O}_X(F)) \le m+1$. Accordingly, since we have shown $\dim H^0(X, \mathcal{O}_X(F)) \ge m+1$ in Theorem 2, we get $\dim H^0(X, \mathcal{O}_X(F)) = m+1$.

(4.1.3) REMARK 1. From Theorem 3, we have:

1) a is the maximal integer such that $H^0(\mathbf{P}^n, E(-a)) \ne 0$.

2) The trace of the complete linear system $|D - aH|$ on $P(E)$ to X is complete.

Let $t \in H^0(E(-a))$ be a nonzero section and D' the effective divisor on $P(E)$ associated to t. Then D' is irreducible by virtue of 1). Since the trace is complete, we find that every member of $|Z|$ is a cut of some D' to Y. Hence a general member of $|Z|$ is irreducible and every member of $|Z|$ has the following defining equations in $\mathbf{P}^n$:

$$t \wedge s_i = 0, \qquad s_j \wedge s_k = 0 \qquad (1 \le i, j, k \le m+1).$$

If U is an affine open subset of $\mathbf{P}^n$ such that $E|U \simeq \bigoplus^2 \mathcal{O}_U$ and $s_1|U \ne 0$, then they coincide with the equations on U:

$$t \wedge s_1 = 0, \qquad s_1 \wedge s_i = 0 \qquad (2 \le i \le m+1).$$

(4.2) $2 \le r \le m+1, s \ge 0$.

(4.2.1) We shall show the following in this case.

THEOREM 4. *Let j, l $(0 \le j < l)$ be integers such that for every p $(1 \le p \le r-1)$,*

$$H^i(\mathbf{P}^n, S^p(E)(-s)) = 0 \qquad \text{for } j \le i \le l-p+1.$$

Then the restriction map $H^i(\mathcal{O}_{P(E)}(rD-sH)) \to H^i(\mathcal{O}_X(rD-sH))$ *is isomorphic for* $j \le i \le \min(l-r+1, n-m-1)$, *surjective for* $i = j-1$ *and injective for* $i = \min(l-r+1, n-m-1)+1$.

Proof. First we prove the following by induction on q $(1 \le q \le r-1)$ and k $(0 \le k \le m-r+q+1)$:

$$H^i(\mathcal{O}_{Y_k}(qD-sH)) = 0 \qquad \text{for } j \le i \le \min(l-q+1, n-m+r-q-1).$$

As we have seen in (4.1.1), $H^i(\mathcal{O}_{P(E)}(D-sH)) \simeq H^i(\mathcal{O}_{Y_k}(D-sH))$ for either $0 \le i \le n-k, s>0$ or $1 \le i \le n-k, s=0$. (When $s=0$, our assumption insures that $j \ge 1$.) If $q=1$, then $n-k \ge n-(m-r+2) = n-m+r-2$. Thus we observe $H^i(\mathcal{O}_{Y_k}(D-sH)) = 0$ for $j \le i \le \min(l, n-m+r-2)$ from the assumption, which is our claim for $q=1$. If $k=0$, then it is easily deduced from our assumption for all q. Assuming $q \ge 2$ and $k \ge 1$, consider the exact sequence

$$\to H^i(\mathcal{O}_{Y_{k-1}}((q-1)D-sH)) \to H^i(\mathcal{O}_{Y_{k-1}}(qD-sH)) \to H^i(\mathcal{O}_{Y_k}(qD-sH)) \to .$$

Since $H^i(\mathcal{O}_{Y_{k-1}}((q-1)D-sH)) = 0$ for $j \le i \le \min(l-q+2, n-m+r-q)$ by the induction hypothesis, the restriction map: $H^i(\mathcal{O}_{Y_{k-1}}(qD-sH)) \to H^i(\mathcal{O}_{Y_k}(qD-sH))$ is isomorphic for $j \le i \le \min(l-q+1, n-m+r-q-1)$. Hence we can prove $H^i(\mathcal{O}_{Y_k}(qD-sH)) = 0$ for $j \le i \le \min(l-q+1, n-m+r-q-1)$ by the induction hypothesis on k.

Taking $q = r-1$, we have the following vanishing:

$$H^i(\mathcal{O}_{Y_k}((r-1)D-sH)) = 0 \quad \text{for } 0 \le k \le m \text{ and } j \le i \le \min(l-r+2, n-m).$$

Consequently the canonical map: $H^i(\mathcal{O}_{Y_{k-1}}(rD-sH)) \to H^i(\mathcal{O}_{Y_k}(rD-sH))$ is isomorphic provided $1 \le k \le m+1$ and $j \le i \le \min(l-r+1, n-m-1)$. Hence the restriction map $H^i(\mathcal{O}_{P(E)}(rD-sH)) \to H^i(\mathcal{O}_X(rD-sH))$ is an isomorphism for $j \le i \le \min(l-r+1, n-m-1)$. The remaining assertions are easily checked.

(4.2.2) Since there exists an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^n} \to \mathcal{E}nd(E) \to S^2(E)(-c_1) \to 0,$$

we have canonical isomorphisms $H^i(\mathcal{E}nd(E)) \simeq H^i(S^2(E)(-c_1))$ for $1 \le i \le n$ and $\dim H^0(\mathcal{E}nd(E)) = \dim H^0(S^2(E)(-c_1)) + 1$. Thus as for the canonical maps $H^i(\mathcal{E}nd(E)) \to H^i(\mathcal{O}_X(2D-c_1H))$, we have the following.

Corollary 4. *The canonical map* $H^i(\mathcal{E}nd(E)) \to H^i(\mathcal{O}_X(2D-c_1H))$ *is an isomorphism for* $1 \le i \le n-m-1$ *and* $\dim H^0(\mathcal{E}nd(E)) =$

$\dim H^0(\mathcal{O}_X(2D - c_1H)) + 1$. *Hence E is unstable with respect to $\mathcal{O}_{\mathbf{P}^n}(1)$ if and only if $H^0(\mathcal{O}_X(2D - c_1H)) \neq 0$.*

Proof. Since $H^i(E(-c_1)) \simeq H^{n-i}(E \otimes K_{\mathbf{P}^n}) = 0$ for $0 \leq i \leq n-2$ by Le-Potier vanishing theorem [30], the restriction map $H^i(\mathcal{O}_{P(E)}(2D - c_1H)) \to H^i(\mathcal{O}_X(2D - c_1H))$ is isomorphic for $0 \leq i \leq n-m-1$ by Theorem 4. Therefore the canonical map $H^i(\mathcal{E}nd(E)) \to H^i(\mathcal{O}_X(2D - c_1H))$ is isomorphic for $1 \leq i \leq n-m-1$ and $\dim H^0(\mathcal{E}nd(E)) = \dim H^0(\mathcal{O}_X(2D - c_1H)) + 1$. In addition E is unstable with respect to $\mathcal{O}_{\mathbf{P}^n}(1)$ if and only if E is not simple. Hence E is unstable if and only if $H^0(\mathcal{O}_X(2D - c_1H)) \neq 0$.

(4.2.3) We shall show the following concerning the irreducibility of every member of the complete linear system $|Z|$.

Theorem 5. *If $n \geq 5$ is odd, then every member of $|Z|$ is irreducible.*

Proof. Assuming $Z = Z_1 + Z_2$ (Z_i being an effective divisor), we shall derive a contradiction. Let $Z_i = \alpha_i D + \beta_i H$ ($a_i, \beta_i \in \mathbf{Z}$) in $\mathrm{Pic}(X)$. Since $Z = D - aH$, we have relations

$$1 = \alpha_1 + \alpha_2, \qquad -a = \beta_1 + \beta_2.$$

We may assume $\alpha_1 \leq \alpha_2$ and so $\alpha_1 \leq 0$.

1) Case $\alpha_1 = 0$. Since Z_1 is effective, we have $\beta_1 > 0$. Thus $Z - H = (\beta_1 - 1)H + Z_2$ is effective, which contradicts our choice that a is the maximal integer such that $D - aH$ is effective.

2) Case $\alpha_1 < 0$. Since $F^{m+1} = 0$ by Theorem 3, Z_1 is expressed as

$$Z_1 = \alpha_1 D + \beta_1 H = (\alpha_1 c_1 + \beta_1)H + (-\alpha_1)F \qquad \text{with } \alpha_1 c_1 + \beta_1 \geq 0,$$

which implies that $Z - F = 2D - (c_1 + a)H$ is effective, i.e., $H^0(\mathcal{O}_X(2D - (c_1 + a)H) \neq 0$. In particular, E is unstable, i.e., $a \geq c_1/2$ (c_1 : even) (resp. $\geq (c_1 + 1)/2 (c_1$: odd)) by Corollary 4.

Let $s \in H^0(\mathbf{P}^n, E)$ be a section such that $W(s)$ (the scheme of zeros of s) is a smooth closed subscheme of codimension 2 in $\mathbf{P}^n$ which represents a second Chern class $c_2(E)$ of E and consider the exact sequence:

$$0 \to \mathcal{O}_{\mathbf{P}^n} \to E \to \mathcal{J} \otimes \mathcal{O}_{\mathbf{P}^n}(c_1) \to 0,$$

where $\mathcal{J}$ is the defining ideal of $W(s)$. Tensoring it with $E^*(-a)$, we get $H^0(\mathcal{J} \otimes E(-a)) \simeq H^0(E \otimes E^*(-a)) \simeq H^0(S^2(E)(-(c_1 + a)) = H^0(\mathcal{O}_X(2D - (c_1 + a)H)) \neq 0$ by Theorem 4 because $H^i(E^*(-a)) \simeq H^{n-i}(E(a) \otimes K_{\mathbf{P}^n}) = 0$ for $0 \leq i \leq n-2$ by Le-Potier vanishing theorem. Let $t \in H^0(\mathcal{J} \otimes E(-a))$ be a nonzero section, i.e., a section of $E(-a)$ vanishing on $W(s)$. Then $W(t)$ does not contain any effective divisor of $\mathbf{P}^n$ and so it represents $c_2(E(-a)) = (c_2 - ac_1 + a^2)H^2$. Therefore it induces the inequality $c_2 \leq c_2 - ac_1 + a^2$ from the inclusion $W(s) \subseteq W(t)$. Since a is positive, we get $a \geq c_1$, which contradicts $0 \leq a < c_1$.

(4.3) $r \geq m+2, s \geq 0$.

(4.3.1) We can prove the following by an argument similar to the one above.

THEOREM 6. *Let j, l $(0 \leq j < l)$ be integers such that for every integer p $(0 \leq p \leq m)$,*

$$H^i(\mathbf{P}^n, S^{r-1-m+p}(E)(-s)) = 0 \qquad \textit{for } j \leq i \leq l-p.$$

Then the restriction map $H^i(\mathcal{O}_{P(E)}(rD - sH)) \to H^i(\mathcal{O}_Y(rD - sH))$ is isomorphic for $j \leq i \leq l-m-1$, surjective for $i = j-1$ and injective for $i = l-m$.

PROOF. We prove the following by induction on q $(0 \leq q \leq m)$ and k $(0 \leq k \leq q)$:

$$H^i(\mathcal{O}_{Y_k}((r-1-m+q)D - sH)) = 0 \qquad \text{for } j \leq i \leq l-q.$$

If $k = 0$, then the above is nothing but our assumption and so it is verified for $q = 0$. Consider the exact sequence

$$\to H^i(\mathcal{O}_{Y_{k-1}}((r-1-m+q-1)D - sH))$$

$$\to H^i(\mathcal{O}_{Y_{k-1}}((r-1-m+q)D - sH)) \to$$

$$\to H^i(\mathcal{O}_{Y_k}((r-1-m+q)D - sH)) \to$$

Since $H^i(\mathcal{O}_{Y_{k-1}}((r-1-m+q-1)D - sH)) = 0$ for $j \leq i \leq l-q+1$ by the induction hypothesis, the restriction map $H^i(\mathcal{O}_{Y_{k-1}}((r-1-m+q)D - sH)) \to H^i(\mathcal{O}_{Y_k}((r-1-m+q)D - sH))$ is isomorphic for $j \leq i \leq l-q$. Thus we have $H^i(\mathcal{O}_{Y_k}((r-1-m+q)D - sH)) = 0$ for $j \leq i \leq l-q$ because $H^i(\mathcal{O}_{Y_{k-1}}((r-1-m+q)D - sH)) = 0$ by the induction hypothesis on k.

Hence taking $q = m$, we have $H^i(\mathcal{O}_{Y_k}((r-1)D - sH)) = 0$ for $j \leq i \leq l-m$ and $0 \leq k \leq m$, which implies that the restriction map $H^i(\mathcal{O}_{P(E)}(rD - sH)) \to H^i(\mathcal{O}_X(rD - sH))$ is isomorphic for $j \leq i \leq l-m-1$, surjective for $i = j-1$ and injective for $i = l-m$.

(4.3.2) Assume that $E \simeq \mathcal{O}(\alpha) \oplus \mathcal{O}(\beta)$ $(0 < \alpha \leq \beta)$ is a direct sum of line bundles. Then it is obvious that $\beta = a$ and $\alpha = c_1 - a$. In this case we can easily prove the following by Theorems 3, 4 and 6 because $H^i(\mathbf{P}^n, S^r(E)(-s)) = 0$ for $1 \leq i \leq n-1$ and $H^0(\mathbf{P}^n, S^r(E(-a))(-s)) = 0$ for $s > 0$.

COROLLARY 5. 1) $\theta_1 = a$, *i.e.*, $\ell_1 = \mathbf{R}_{\geq 0}[Z]$.

2) $H^i(X, \mathcal{O}_X(rD - sH)) = 0$ *for* $1 \le i \le n - m - 1$, $1 \le r \le m + 1$, $s \ge 0$ *and* $H^i(X, \mathcal{O}_X(rD - sH)) = 0$ *for* $1 \le i \le n - m - 2$, $r \ge m + 2$, $s \ge 0$.

(4.4) Next we shall deal with the cases $-rD + sH$ with $r \ge 0$ and $s \ge 0$.

(4.4.1) Let us adopt the following convention:
1) For a coherent sheaf $G, H^i(\mathbf{P}^n, G) = 0$ for $i \le -1$.
2) For a vector bundle $G, S^p(G)$ is a zero sheaf for $p \le -1$.

THEOREM 7. *Let* j, l $(0 \le j \le l - m + 1)$ *be integers such that for every integer* p $(0 \le p \le m)$,

$$H^i(\mathbf{P}^n, S^{r-1+m-p}(E^*)(-c_1 + s)) = 0 \qquad \textit{for } j - 1 \le i \le l - p.$$

Then we get the vanishing $H^i(\mathcal{O}_{Y_k}(-(r + 1 + m - q)D + sH)) = 0$ *for* $j \le i \le l - q + 1$ *and* $0 \le q \le m, 0 \le k \le q$. *Hence the canonical restriction map:* $H^i(\mathcal{O}_{P(E)}(-rD + sH)) \to H^i(\mathcal{O}_Y(-rD + sH))$ *is isomorphic for* $j \le i \le l - m$, *surjective for* $i = j - 1$ *and injective for* $i = l - m + 1$.

PROOF. We prove the vanishing by induction on q and k. If $r + 1 + m - q \ge 2$, then by Lemma 6 and our assumption, we have

$$H^i(\mathcal{O}_{P(E)}(-(r + 1 + m - q)D + sH)) \simeq H^{i-1}(\mathbf{P}^n, S^{r-1+m-q}(E^*)(-c_1 + s)) = 0$$

for $j \le i \le l - q + 1$. If $r + 1 + m - q = 1$, then $H^i(\mathcal{O}_{P(E)}(-D + sH)) = 0$ for all i. Hence if $k = 0$, then the vanishing holds for all q. Assuming $q \ge 1$ and $k \ge 1$, consider the exact sequence

$$\begin{aligned}
&\to H^i(\mathcal{O}_{Y_{k-1}}(-(r + 1 + m - (q - 1))D + sH)) \\
&\to H^i(\mathcal{O}_{Y_{k-1}}(-(r + 1 + m - q)D + sH)) \\
&\to H^i(\mathcal{O}_{Y_k}(-(r + 1 + m - q)D + sH)) \\
&\to H^{i+1}(\mathcal{O}_{Y_{k-1}}(-(r + 1 + m - (q - 1))D + sH)).
\end{aligned}$$

Then $H^i(\mathcal{O}_{Y_{k-1}}(-(r + 1 + m - (q - 1))D + sH)) = 0$ for $j \le i \le l - q + 2$ by the induction hypothesis, from which it follows that the restriction map $H^i(\mathcal{O}_{Y_{k-1}}(-(r + 1 + m - q)D + sH)) \to H^i(\mathcal{O}_{Y_k}(-(r + 1 + m - q))D + sH))$ is isomorphic for $j \le i \le l - q + 1$. Hence we get $H^i(\mathcal{O}_{Y_k}(-(r + 1 + m - q))D + sH)) = 0$ for $j \le i \le l - q + 1$ by the induction hypothesis. Thus our claim is proved by an argument similar to the one above.

(4.4.2) We shall show several corollaries of Theorem 7.
First consider the case $r = 0$.
1) For an integer p $(0 \le p \le m)$, we have by Serre duality

$$H^i(\mathbf{P}^n, S^{m-1-p}(E^*)(-c_1 + s)) \simeq H^{n-i}(\mathbf{P}^n, S^{m-1-p}(E)(c_1 - s) \otimes K_{\mathbf{P}^n}).$$

Hence if $s < c_1$ (resp. $s = c_1$), then $H^i(\mathbf{P}^n, S^{m-1-p}(E^*)(-c_1+s)) = 0$ for $0 \le i \le n-2$ (resp. $1 \le i \le n-2$) by Lemma 5. Thus for $s < c_1$, the restriction map: $H^i(\mathcal{O}_{P(E)}(sH)) \to H^i(\mathcal{O}_Y(sH))$ is isomorphic for $0 \le i \le n-m-2$ by Theorem 7, which implies that $H^i(\mathcal{O}_X(sH)) = 0$ for $1 \le i \le n-m-2$ and $H^0(\mathcal{O}_X(sH)) = H^0(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(s))$, i.e., X is s-normal. Similarly for $s = c_1$, we have that the restriction map $H^i(\mathcal{O}_{(P(E)}(s)) \to H^i(\mathcal{O}_X(sH))$ is isomorphic for $2 \le i \le n-m-2$ and surjective for $i = 1$, which implies $H^i(\mathcal{O}_X(sH)) = 0$ for $1 \le i \le n-m-2$.

2) If E is a direct sum of line bundles, then $H^i(\mathbf{P}^n, S^{m-1-p}(E^*)(-c_1+s)) = 0$ for $1 \le i \le n-1$ and arbitrary s. Thus we get $H^i(\mathcal{O}_X(sH)) = 0$ for $1 \le i \le n-m-1$ and arbitrary s.

Summing up the above, we obtain the following. For simplicity, we denote the line bundle $\mathcal{O}_X(sH)$ by $\mathcal{O}_X(s)$.

COROLLARY 6. 1) *The restriction map* $H^0(\mathcal{O}_{\mathbf{P}^n}(s)) \to H^0(\mathcal{O}_X(s))$ *is isomorphic, i.e., X is s-normal for $s < c_1$ and* $H^i(\mathcal{O}_X(s)) = 0$ *for* $1 \le i \le n-m-2$, $0 \le s \le c_1$.

2) *If E is a direct sum of line bundles, then* $H^i(\mathcal{O}_X(s)) = 0$ *for* $1 \le i \le n-m-1$ *and arbitrary s.*

(4.4.3) $r \ge 1$.

Since $H^i(\mathbf{P}^n, S^{r-1+m-p}(E^*)(-c_1+s) \simeq H^{n-i}(\mathbf{P}^n, S^{r-1+m-p}(E)(c_1-s) \otimes K_{\mathbf{P}^n})$, we have from Lemma 5 for every integer p ($0 \le p \le m$):

1) $r = 1$. $H^i(\mathbf{P}^n, S^{r-1+m-p}(E^*)(-c_1+s)) = 0$ for $1 \le i \le n-2$ and $s \le c_1$ (resp. for $0 \le i \le n-2$ if $s < c_1$).

2) $r \ge 2$. $H^i(\mathbf{P}^n, S^{r-1+m-p}(E^*)(-c_1+s)) = 0$ for $0 \le i \le n-2$ and $s \le c_1$.

Thus we get

1)$'$ The restriction map $H^i(\mathcal{O}_{P(E)}(-D+sH)) \to H^i(\mathcal{O}_X(-D+sH))$ is isomorphic for $2 \le i \le n-m-2$ and surjective for $i = 1$ if $s \le c_1$ (resp. for $0 \le i \le n-m-2$ if $s < c_1$). Hence we have $H^i(\mathcal{O}_X(-D+sH)) = 0$ for $1 \le i \le n-m-2$ and $s \le c_1$ (resp. for $0 \le i \le n-m-2$ if $s < c_1$).

2)$'$ The restriction map $H^i(\mathcal{O}_{P(E)}(-rD+sH)) \to H^i(\mathcal{O}_X(-rD+sH))$ is isomorphic for $0 \le i \le n-m-2$ and $r \ge 2$, $s \le c_1$. In addition $H^i(\mathcal{O}_{P(E)}(-rD+sH)) \simeq H^{i-1}(\mathbf{P}^n, S^{r-2}(E^*)(-c_1+s)) \simeq H^{n-i+1}(\mathbf{P}^n, S^{r-2}(E)(c_1-s) \otimes K_{\mathbf{P}^n}) = 0$ in the following cases: $r \ge 3$, $0 \le i \le n-1$, $s \le c_1$ or $r = 2$, $s < c_1$, $0 \le i \le n$ or $r = 2$, $s = c_1$, $0 \le i \le n-1$, $i \ne 1$ and meanwhile, $H^1(\mathcal{O}_{P(E)}(-2D+c_1H)) = \mathbf{C}$. Hence we obtain that $H^i(\mathcal{O}_X(-rD+sH)) = 0$ for either $r \ge 3$, $0 \le i \le n-m-1$, $s \le c_1$ or $r = 2$, $s = c_1$, $0 \le i \le n-m-1$, $i \ne 1$ and $H^1(\mathcal{O}_X(-2D+c_1H)) = \mathbf{C}$. In particular since $-2D+c_1H = F-D$, it follows that the exact sequence $0 \to \mathcal{O}_X(F) \to E|X \to \mathcal{O}_X(D) \to 0$ is the unique non-trivial extension of $\mathcal{O}_X(D)$ by $\mathcal{O}_X(F)$. To this end, it is enough to prove that the above exact sequence does not split. When n is odd, it is easily seen as follows. If $E|X \simeq \mathcal{O}_X(F) \otimes \mathcal{O}_X(D)$, the $\mathcal{O}_X(F)$ is ample. However this contradicts $F^{m+1} = 0$. In the general case, it is verified in the following way. Since the above exact sequence does not split if and only if $H^0(E|X \otimes \mathcal{O}_X(-D)) = 0$ because $H^0(\mathcal{O}_X(F-D)) = 0$, it suffices to show that $H^0(E|X \otimes \mathcal{O}_X(-D)) = 0$. For the purpose, we prove the vanishing $H^i(P(E), \pi^*(E) \otimes \mathcal{O}_{P(E)}(-rD)) = 0$ for $0 \le i \le n-1$ and $r \ge 2$. Indeed by considering the tower of closed subschemes $\{Y_k\}$ $(1 \le k \le m+1)$, we can deduce $H^i(\pi^*(E) \otimes \mathcal{O}(-rD)|Y_k) = 0$ for $0 \le k \le m+1$, $0 \le i \le n-k-1$ and $r \ge 1$ from the vanishing and $H^i(P(E), \pi^*(E) \otimes \mathcal{O}_{P(E)}(-D)) = 0$ for all i. By Lemma 6, it follows that $H^i(P(E), \pi^*(E) \otimes \mathcal{O}_{P(E)}(-rD)) \simeq H^{i-1}(\mathbf{P}^n, E \otimes S^{r-2}(E^*)(-c_1)) \simeq H^{n-i+1}(\mathbf{P}^n, E \otimes S^{r-2}(E) \otimes K_{\mathbf{P}^n})$. Since E is of rank two, there exists an exact sequence for every integer r

$$0 \to S^{r-1}(E) \otimes \mathcal{O}_{\mathbf{P}^n}(c_1) \to E \otimes S^r(E) \to S^{r+1}(E) \to 0.$$

Accordingly we get that $H^i(P(E), \pi^*(E) \otimes \mathcal{O}_{P(E)}(-rD)) = 0$ for $0 \le i \le n-1$ and $r \ge 2$ by Lemma 5.

We shall consider the case where E splits into line bundles.

3) If E is a direct sum of line bundles, then we have $H^i(\mathbf{P}^n, S^{r-1+m-p}(E^*)(-c_1+s)) = 0$ for $1 \le i \le n-1$ and for all r and s. Thus we get:

3)$'$ The restriction map $H^i(\mathcal{O}_{P(E)}(-rD+sH)) \to H^i(\mathcal{O}_X(-rD+sH))$ is isomorphic for $2 \le i \le n-m-1$ and surjective for $i=1$. Since $H^i(\mathcal{O}_{P(E)}(-rD+sH)) \simeq H^{i-1}(\mathbf{P}^n, S^{r-2}(E^*)(-c_1+s)) = 0$ for $2 \le i \le n$, we see $H^i(\mathcal{O}_X(-rD+sH)) = 0$ for $2 \le i \le n-m-1$ and for all r and s. Moreover $H^1(\mathcal{O}_{P(E)}(-rD+sH)) \simeq H^0(\mathbf{P}^n, S^{r-2}(E^*)(-c_1+s)) = 0$ if $s < a + (r-1) \cdot (c_1 - a)$. Thus $H^1(\mathcal{O}_X(-rD+sH)) = 0$ for all r and $s < a + (r-1)(c_1-a)$.

3)$''$ Recall that an effective divisor $Z^* = D - (c_1 - a)H$ in (3.5.2) is the dual divisor of Z. Then we obtain the following from the above:

$$H^1(\mathcal{O}_X(-rZ^*)) = 0 \qquad \text{for all } r \ge 1 \quad \text{if } 2a > c_1,$$

$$\dim H^1(\mathcal{O}_X(-rZ^*)) = r-1 \qquad \text{for all } r \ge 1 \quad \text{if } 2a = c_1.$$

$$H^i(\mathcal{O}_X(-rZ^*-sH)) = 0 \quad \text{for all positive integers } r \text{ and } s \ (1 \le i \le n-m-1).$$

Indeed since $r(c_1-a) < a + (r-1)(c_1-a)$ if $2a > c_1$, it follows that $H^1(-rZ^*) = 0$ for all $r \ge 1$ from 3)$'$. In addition if $2a = c_1$, then we find that $Z^* = Z$ and $\mathcal{O}_Z(Z) \simeq \mathcal{O}_Z$ by Remark (4.1.3) 2). Moreover it follows that $H^1(\mathcal{O}_Z) = 0$ from 3)$'$ provided $n \ge 5$ and the restriction map $H^0(\mathcal{O}_X(K_X)) \to H^0(\mathcal{O}_Z(K_X))$ is surjective from Corollary 6, 2) provided $n = 4$. Hence it is easily shown that

$\dim H^1(-rZ^*) = r-1$ for all $r \geq 1$ by using the exact sequence

$$0 \to \mathcal{O}_X(-(r+1)Z) \to \mathcal{O}_X(-rZ) \to \mathcal{O}_Z \to 0.$$

The third claim is checked similarly.

Summing up the above, we get the following.

COROLLARY 7. 1) *$H^i(\mathcal{O}_X(-D+sH) = 0$ for $0 \leq i \leq n-m-2$ and $s < c_1$ (resp. for $1 \leq i \leq n-m-2$ and $s \leq c_1$).*

2) *$H^i(\mathcal{O}_X(-rD+sH)) = 0$ for either $r \geq 3, 0 \leq i \leq n-m-1, s \leq c_1$ or $r = 2$, $s = c_1$, $0 \leq i \leq n-m-1, i \neq 1$ and $H^1(\mathcal{O}_X(-2D+c_1H)) = \mathbf{C}$. In particular, we see that the exact sequence $0 \to \mathcal{O}_X(F) \to E|X \to \mathcal{O}_X(D) \to 0$ is the unique non-trivial extension of $\mathcal{O}_X(D)$ by $\mathcal{O}_X(F)$.*

3) *If E is a direct sum of line bundles, then the following hold for the effective divisor $Z^* = D-(c_1-a)H$,*

$$H^1(\mathcal{O}_X(-rZ^*)) = 0 \qquad \text{for all } r \geq 1 \quad \text{if } 2a > c_1,$$

$$\dim H^1(\mathcal{O}_X(-rZ^*)) = r-1 \qquad \text{for all } r \geq 1 \quad \text{if } 2a = c_1.$$

$$H^i(\mathcal{O}_X(-rZ^*-sH)) = 0 \quad \text{for all positive integers } r \text{ and } s \ (1 \leq i \leq n-m-1).$$

(4.4.4) *As an application of the vanishing $H^0(\mathcal{O}_X(-D+sH)) = 0$ $(s < c_1)$, we shall derive a criterion for the irreducibility of every member of $|F|$ in the case $n = 2m$ is even (cf. (3.4.3) Proposition 3).*

COROLLARY 8. *When $\dim|rZ| = 0$ for every positive integer r, the following are equivalent.*

1) *$|F-Z| = \varnothing$.*

2) *Every member of $|F|$ is irreducible.*

PROOF. Since the implication 2) $\Rightarrow$ 1) is obvious, it suffices to show the implication 1) $\Rightarrow$ 2). Let $F_i = \alpha_i D + \beta_i H$ $(\alpha_i, \beta_i \in \mathbf{Z}, i = 1,2)$ be effective divisors such that $F_1 + F_2 \in |F|$. Then we have $\alpha_1 + \alpha_2 = -1$, $\beta_1 + \beta_2 = c_1$. We may suppose $\alpha_1 < 0 \leq \alpha_2$. We derive a contradiction in the following cases.

(1) $\alpha_1 = -1$. Since $\alpha_2 = 0$ and $\beta_2 > 0$, it follows that $F_1 = -D+\beta_1 H$ with $\beta_1 = c_1 - \beta_2 < c_1$, which contradicts $H^0(\mathcal{O}_X(-D+sH) = 0$ $(s < c_1)$.

(2) $\alpha_1 \leq -2$. In this case, $\alpha_2 = -1-\alpha_1 \geq 1$. From our assumption $\dim|rZ| = 0$ for every positive integer r, it turns out by Lemma 4 that $\beta_2 \geq -\alpha_2 a$ and so $\beta_1 = c_1 - \beta_2 \leq c_1 + \alpha_2 a$. On the other hand, we have $F_1 = (-1-\alpha_2)D + \beta_1 H = (-D+c_1H) - \alpha_2(D-aH) - (c_1 + \alpha_2 a - \beta_1)H = F - \alpha_2 Z - (c_1 + \alpha_2 a - \beta_1)H$, which contradicts $|F-Z| = \varnothing$.

(4.4.5) As for a cohomological criterion for numerically effective divisors, we shall show the following which might be well known (cf. A. Moriwaki [26] for a cohomological criterion for big and numerically effective divisors and also [45]).

THEOREM 8. *Let X be a d-dimensional $(d \geq 2)$ smooth projective variety defined over an algebraically closed field k of characteristc 0, D an effective divisor on X and let H be an ample divisor on X. Then the following are equivalent to each other.*

1) *D is a numerically effective divisor.*
2) a) $\dim H^1(\mathcal{O}_X(-rD)) \leq O(r^1)$ *for sufficiently large integers r.*
   b) $\dim H^i(\mathcal{O}_X(-rD - sH)) \leq P_i(s)$ *for all positive integers r, s $(1 \leq i \leq d-1)$, where $P_i(s)$ is a polynomial with respect to s which is independent of r.*

Here we shall give a proof of Theorem 8 for the sake of completeness.

PROOF. Since the implication: 1) $\Rightarrow$ 2) is obvious by considering a tower of closed subschemes consisting of complete intersections by very ample divisors $\in |mH|$ and by the Kodaira vanishing theorem, it is enough to prove the converse. We prove that $D \cdot C \geq 0$ for every irreducible curve C in X.

(1) First we assume that C is a smooth curve. Consider the exact sequences

$$0 \to H^0(I_C \otimes \mathcal{O}_X(mH)) \to H^0(\mathcal{O}_X(mH)) \to H^0(\mathcal{O}_C \otimes \mathcal{O}_X(mH)) \to,$$

$$\to H^0(I_C \otimes \mathcal{O}_X(mH)) \to H^0(I_C/I_C^2 \otimes \mathcal{O}_X(mH)) \to H^1(I_C^2 \otimes \mathcal{O}_X(mH)) \to,$$

where I_C is the defining ideal sheaf of C and take a sufficiently large integer m so that $I_C/I_C^2 \otimes \mathcal{O}_X(mH)$ is generated by global sections and $H^1(I_C^2 \otimes \mathcal{O}_X(mH)) = 0$. Then it follows from the above exact sequences that there exists a smooth member Y of the complete linear system $|mH|$ which contains C. If we denote by D' (resp. H') the restriction of D to Y (resp. H to Y), then we observe also that D' satisfies the condition a) and condition b) in 2) with respect to H'. Hence iterating this process, we may assume that X is a smooth surface which contains C and that D is an effective divisor satisfying the condition $\dim H^1(\mathcal{O}_X(-rD)) \leq O(r^1)$ for sufficiently large integers r.

(2) Let $D = N + E, E = \sum n_i E_i$ $(n_i \in \mathbf{Q}_{\geq 0}, E_i$ being an irreducible curve) be the Zariski decomposition of D, where N is a numerically effective divisor, the intersection number $N \cdot E_i = 0$ for all i and the intersection matrix $(E_i \cdot E_j)$ is negative definite (cf. [45]). After multiplying a suitable positive integer, we may assume that E is an integral divisor. We prove that if $E \neq 0$, then

$$\dim H^1(\mathcal{O}_X(-rD)) \geq \tfrac{1}{2}(-E^2)r^2 + (\text{a linear form on } r) \text{ for sufficiently large integers } r.$$

Hence if D satisfies the above cohomological condition, then E must be 0, i.e., $D = N$ is a numerically effective divisor. In particular, we obtain $D \cdot C \geq 0$.

Consider the exact sequence

$$0 \to \mathcal{O}_X(-rN - rE) \to \mathcal{O}_X(-rN) \to \mathcal{O}_{rE}(-rN) \to 0.$$

Then it induces the following exact sequence:

$$\to H^0(\mathcal{O}_X(-rN)) \to H^0(\mathcal{O}_{rE}(-rN)) \to H^1(\mathcal{O}_X(-rN-rE)) \to H^1(\mathcal{O}_X(-rN)) \to .$$

Since $\dim H^0(\mathcal{O}_X(-rN)) \leq 1$ and $\dim H^1(\mathcal{O}_X(-rN)) \leq O(r^1)$ for $r \gg 0$, we have only to establish

$$\dim H^0(\mathcal{O}_{rE}(-rN)) \geq \tfrac{1}{2}(-E^2)r^2 + (\text{a linear form on } r) \qquad \text{for } r \gg 0.$$

(3) We prove this by induction on the number of irreducible components of E. First assume that $E = pE_1$ for some positive integer p.

For every positive integer i, consider the exact sequence

$$0 \to \mathcal{O}_{E_1}(-rN - (rp - i)E_1) \to \mathcal{O}_{(rp-i+1)E_1}(-rN) \to \mathcal{O}_{(rp-i)E_1}(-rN) \to 0,$$

which gives rise to the exact sequence

$$\begin{aligned} 0 &\to H^0(\mathcal{O}_{E_1}(-(rp - i)E_1 - rN) \to H^0(\mathcal{O}_{(rp-i+1)E_1}(-rN)) \\ &\to H^0(\mathcal{O}_{(rp-i)E_1}(-rN)) \to H^1(\mathcal{O}_{E_1}(-(rp - i)E_1 - rN)) \to . \end{aligned}$$

By Serre duality, we have

$$H^1(\mathcal{O}_{E_1}(-(rp - i))E_1 - rN)) \simeq H^0(\mathcal{O}_{E_1}((rp - i + 1)E_1 + rN + K_X)).$$

Hence if $(rp - i + 1)E_1^2 + rNE_1 + K_XE_1 = (rp - i + 1)E_1^2 + K_XE_1 < 0$, *i.e.*, $1 \leq i \leq rp - \lambda$, where λ is the smallest integer that is greater than or equal to $K_XE_1/(-E_1^2)$, then $H^1(\mathcal{O}_{E_1}(-(rp - i)E_1 - rN)) = 0$ and so we get by the Riemann-Roch theorem,

$$\begin{aligned} H^0(\mathcal{O}_{E_1}(-(rp - i))E_1 - rN)) &= -(rp - i)E_1^2 - g(E_1) + 1, \\ g(E_1) &= 1 + \tfrac{1}{2}(E_1^2 + E_1K_X). \end{aligned}$$

Thus it follows from the above exact sequence that

$$\dim H^0(\mathcal{O}_{(rp-i+1)E_1}(-rN)) = \dim H^0(\mathcal{O}_{(rp-i)E_1}(-rN)) - (rp - i)E_1^2 - g(E_1) + 1.$$

Therefore we obtain

$$\dim H^0(\mathcal{O}_{rpE_1}(-rN)) = \dim H^0(\mathcal{O}_{\lambda E_1}(-rN)) + \sum_{i=1}^{rp-\lambda}(-(rp-i)E_1^2 - g(E_1)+1)$$

$$= \tfrac{1}{2}(-E_1^2)r^2p^2 + (\tfrac{1}{2}E_1^2 - g(E_1)+1)rp + \{\tfrac{1}{2}\lambda(\lambda-1)E_1^2 - \lambda(-g(E_1)+1)\}$$

$$+ \dim H^0(\mathcal{O}_{\lambda E_1}(-rN))$$

$$\geq \tfrac{1}{2}(-E^2)r^2 + (\text{a linear form}).$$

(4) $E = n_1E_1 + E'$ where E' does not contain E_1 as an irreducible component. Considering the usual exact sequence

$$0 \to \mathcal{O}_{rE}(-rN) \to \mathcal{O}_{rn_1E_1}(-rN) \oplus \mathcal{O}_{rE'}(-rN) \to \mathcal{O}_{rn_1E_1 \cap rE'}(-rN) \to 0,$$

we get the exact sequence

$$0 \to H^0(\mathcal{O}_{rE}(-rN)) \to H^0(\mathcal{O}_{rn_1E_1}(-rN)) \oplus H^0(\mathcal{O}_{rE'}(-rN))$$

$$\to H^0(\mathcal{O}_{rn_1E_1 \cap rE'}(-rN)) \to .$$

Hence the following inequality holds:

$$\dim H^0(\mathcal{O}_{rn_1E_1}(-rN)) + \dim H^0\mathcal{O}_{rE'}(-rN) - \dim H^0(\mathcal{O}_{rn_1E_1 \cap rE'}(-rN))$$

$$\leq \dim H^0(\mathcal{O}_{rE}(-rN).$$

So far, we have the following for sufficiently large integers r:

(a) $\dim H^0(\mathcal{O}_{rn_1E_1}(-rN)) \geq \frac{1}{2}(-(n_1E_1)^2)r^2 + (\text{a linear form})$ as we have shown in 1).

(b) $\dim H^0\mathcal{O}_{rE'}(-rN) \geq \frac{1}{2}(-(E')^2)r^2 + (\text{a linear form})$ by the induction hypothesis.

(c) $\dim H^0(\mathcal{O}_{rn_1E_1 \cap rE'}(-rN)) = n_1E_1E'r^2$ since $rn_1E_1 \cap rE'$ is a 0-dimensional closed subscheme.

Therefore we can establish the desired inequality

$$\dim H^0(\mathcal{O}_{rE}(-rN) \geq \tfrac{1}{2}(-E^2)r^2 + (\text{a linear form}).$$

(5) Now we deal with the case where C is a singular curve. Let p be a singular point of C, $f : X' \to X$ the blowing-up of X with center p and let $E \simeq \mathbf{P}^{d-1}$ be the exceptinal divisor of f. Then there exists a positive integer m such that $H' = mf^*(H) - E$ is ample. Here we show that $f^*(D)$ satisfies the condition a) and the condition b) in 2) with respect to the ample divisor H'. It is easily seen that the condition a) is fulfilled because $H^1(\mathcal{O}_{X'}(f^*(-rD)) \simeq H^1(\mathcal{O}_X(-rD))$ for all integers r because $f_*(\mathcal{O}_{X'}) = \mathcal{O}_X$ and $R^qf_*(\mathcal{O}_{X'}) = 0$ $(q \geq 1)$. Further it is observed by the fundamental theorem for the proper

morphism f that for every positive integer s,

$$f_*(\mathcal{O}_{X'}(sE)) = \mathcal{O}_X, \qquad R^q f_*(\mathcal{O}_{X'}(sE)) = 0 \qquad (1 \le q \le d-2)$$

and $R^{d-1}f_*(\mathcal{O}_{X'}(sE))$ is a skyscraper sheaf supported at p with $\dim R^{d-1}f_*(\mathcal{O}_{X'}(sE))_p = P(s)$, where $P(s)$ is a polynomial function with respect to s. Consider the spectral sequence $E_2^{p,q} = H^p(\mathcal{O}_X(-rD - msH) \otimes R^q f_*(\mathcal{O}_{X'}(sE))$ for f. Then we have

$$E_\infty^{p,q} = 0 \quad (1 \le q \le d-2), \qquad E_\infty^{d-1,0} = E_2^{d-1,0}, \qquad E_\infty^{0,d-1} \subseteq E_2^{0,d-1}.$$

Hence $H^i(\mathcal{O}_{X'}(f^*(-rD - msH) + sE))$ is isomorphic to $H^i(\mathcal{O}_X(-rD - msH))$ for $1 \le i \le d-2$ and $H^{d-1}(\mathcal{O}_{X'}(f^*(-rD - msH) + sE)) \subseteq H^{d-1}(\mathcal{O}_X(-rD - msH)) \oplus H^0(R^{d-1}f_*(\mathcal{O}_{X'}(sE))$. Therefore we can show that the condition b) is also fulfilled, i.e., there exists a set of polynomials $\{P_i'(s)\ (1 \le i \le d-1)\}$ which are independent of r such that $\dim H^i(\mathcal{O}_{X'}(-rf^*(D) - sH')) \le P_i'(s)$ $(1 \le i \le d-1)$ for all positive integers r and s.

(6) Let $f : X' \to X$ be a succession of blowing-ups of X with points as centers such that the proper transform C' of C by f is a smooth curve and let $\{E_i\}$ $(1 \le i \le n)$ be the exceptional divisors of f. Then we observe from (5) that there is a set of positive integers $\{m, m_1, \dots, m_n\}$ such that $H' = mf^*(H) - \sum_{i=1}^{i=n} m_i E_i$ is an ample divisor on X' and $f^*(D)$ satisfies the condition a) and the condition b) in 2) with respect to H'. Therefore it follows from (1) that $D \cdot C = f^*(D) \cdot C' \ge 0$.

(4.4.6) Combining Corollary 7 with Theorem 8, we obtain the following criterion for a rank two bundle E to be a direct sum of line bundles.

THEOREM 9. *Let E be a very ample rank two bundle on $\mathbf{P}^n$ $(n \ge 4)$ and let $Z^* = D - (c_1 - a)H$ which is the dual divisor of Z. Then the following are equivalent to each other.*

1) *E splits into line bundles.*
2) *Z^* is effective and numerically effective, i.e., $Z^* \in \overline{NA}(X)$.*
3) *Z^* is effective and it satisfies the following conditions:*
   a) *$\dim H^1(-rZ^*) \le O(r^1)$ for sufficiently large integers r.*
   b) *$\dim H^i(\mathcal{O}_X(-rZ^* - sH)) \le P_i(s)$ for all positive integers r and s $(1 \le i \le n-m-1)$, where $P_i(s)$ is a polynomial with respect to s which is independent of r.*

PROOF. By Corollary 7 and Theorem 8, we have only to prove the implication: 2) $\Rightarrow$ 1). Assume that Z^* is numerically effective. Then we have $a^2 - c_1 a + c_2 = 0$ because $Z \cdot Z^* = -(a^2 - c_1 a + c_2)H^2$ as a cycle of codimension 2 in X by (3.5.2) 3). On the other hand, let s be a nonzero global section of $E(-a)$. Then there exists an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^n} \xrightarrow{s} E(-a) \longrightarrow \mathscr{I} \otimes \mathcal{O}_{\mathbf{P}^n}(-(2a-c_1)) \longrightarrow 0,$$

where $\mathscr{I}$ is the defining ideal of the zero locus $W(s)$ of s. Since $\deg W(s) = a^2 - c_1 a + c_2 = 0$, we have $W(s) = \varnothing$, from which it follows that E is a direct sum of line bundles.

(4.4.7) When $n = 4$ or 5, let I_X be the defining ideal of a determinantal subvariety X in $\mathbf{P}^n$. Then I_X has the following resolution by vector bundles.

LEMMA 7. *In the above notation, there exists an exact sequence*

$$0 \to E^*(-c_1) \to \bigoplus^3 \mathcal{O}_{\mathbf{P}^n}(-c_1) \to I_X \to 0.$$

PROOF. Let $\{s_1, s_2, s_3\}$ be a set of global sections of E which defines a determinantal subvariety X. Then we can define homomorphisms

$$\alpha : \bigoplus^3 \mathcal{O}_{\mathbf{P}^n} \ni e_i \wedge e_j \to s_i \wedge s_j \in \bigwedge^2 E \qquad (1 \le i < j \le 3),$$

$$\beta : E^* \ni f \to f(s_3)e_1 \wedge e_2 - f(s_2)e_1 \wedge e_3 + f(s_1)e_2 \wedge e_3 \in \bigoplus^3 \mathcal{O}_{\mathbf{P}^n},$$

where $\{e_i \wedge e_j\}$ is a basis of $\bigoplus^3 \mathcal{O}_{\mathbf{P}^n}$. Then $\alpha : \bigoplus^3 \mathcal{O}_{\mathbf{P}^n} \to I_X \otimes \mathcal{O}(c_1)$ is surjective and β is injective. Hence in order to prove our assertion, it suffices to show that the following sequence is exact:

$$0 \to E^* \xrightarrow{\beta} \bigoplus^3 \mathcal{O}_{\mathbf{P}^n} \xrightarrow{\alpha} I_X \otimes \mathcal{O}(c_1) \to 0.$$

The problem is local. Let $U = \mathrm{Spec}(A)$ be an affine open subset of $\mathbf{P}^n$ such that $E|U \simeq \bigoplus_{i=1}^2 \mathcal{O}_U d_i$ and $s_i \mid U = s_{i1}d_1 + s_{i2}d_2$ ($s_{ij} \in A, 1 \le i \le 3, 1 \le j \le 2$).

a) For any $f \in \Gamma(U, E^*)$, we have $f(s_i) = s_{i1}f(d_1) + s_{i2}f(d_2)$. Hence it follows that

$$(\alpha \circ \beta)(f) = f(s_3)s_1 \wedge s_2 - f(s_2)s_1 \wedge s_3 + f(s_1)s_2 \wedge s_3$$

$$= \det \begin{bmatrix} s_{11} & s_{12} & f(s_1) \\ s_{21} & s_{22} & f(s_2) \\ s_{31} & s_{32} & f(s_3) \end{bmatrix} = 0,$$

which implies $\alpha \circ \beta = 0$.

b) We assume that $s_{11} \neq 0$ on U. Let $t_1 = s_{22} - (s_{12}/s_{11})s_{21}$ and $t_2 = s_{32} - (s_{12}/s_{11})s_{31}$. Then it follows that

$$s_1 \wedge s_2 = s_{11}t_1, \qquad s_1 \wedge s_3 = s_{11}t_2, \qquad s_2 \wedge s_3 = s_{21}t_2 - s_{31}t_1.$$

Suppose that $xe_1 \wedge e_2 + ye_1 \wedge e_3 + ze_2 \wedge s_3$ $(x, y, z \in A)$ is an element of $\mathrm{Ker}\,\alpha$.

Since $0 = t_1(xs_{11} - zs_{31}) + t_2(ys_{11} + zs_{21})$, there is an element $w \in A$ such that $xs_{11} - zs_{31} = wt_2$ and $ys_{11} + zs_{21} = -wt_1$ because $\{t_1, t_2\}$ is a regular system of parameters of X on U. Thus it is observed that $\operatorname{Ker}\alpha$ is a vector bundle of rank two. Since $E^* \subseteq \operatorname{Ker}\alpha$ and $c_1(\operatorname{Ker}\alpha) = -c_1$, E^* is isomorphic to $\operatorname{Ker}\alpha$.

(4.4.8) As an application of Lemma 7, we obtain the following results on the first cohomology groups $H^1(X, \mathcal{O}_X(s))$ $(s \in \mathbf{Z})$ for $n = 4$ or 5 (cf. Corollary 6). Moreover we shall find that Lemma 7 will play an important role in the proof of another splitting theorem (cf. Theorem 15).

COROLLARY 9. *When* $n = 4$ or 5, *we have*

1) $H^1(X, \mathcal{O}_X(s)) = 0$ *for* $s \geq 2c_1 - 5$.

2) *If* $H^1(\mathbf{P}^n, E(k)) = 0$ *for an integer* $k \in \mathbf{Z}$, *then* $H^1(X, \mathcal{O}_X(c_1 - k - 5)) = 0$.

3) *E is a direct sum of line bundles if and only if* $H^1(X, \mathcal{O}_X(s)) = 0$ *for all* $s \in \mathbf{Z}$.

Since the verification consists of arguments similar to that above, we leave it to the reader.

(4.4.9) As was announced in (3.2.1), we shall prove here that if E is a very ample rank two bundle with $c_1 \leq 4$ on $\mathbf{P}^4$, then it is isomorphic to a direct sum of line bundles.

1) To this end, we shall recall the following:
First it follows from (3.2.2) that

a) $c_i > 0$ $(i = 1, 2)$, $DH = c_1(c_1^2 - 2c_2) > 0$, $D^2 = c_1^4 - 3c_1^2c_2 + c_2^2 > 0$.
Secondly, we have:

b) Schwartzenberger conditions.

$$S_3^2 : c_1c_2 \equiv 0 \pmod 2, \quad S_4^2 : c_2(c_2 + 1 - 3c_1 - 2c_1^2) \equiv 0 \pmod{12}.$$

Hence the following (c_1, c_2) are the only possible pairs of positive integers subject to the above conditions provided $c_1 \leq 4$:

$$(c_1, c_2) = (2, 1),\ (3, 2),\ (4, 3),\ (4, 4)$$

In all cases, since $c_1^2 - 4c_2 \geq 0$, we find that E is unstable. Thus it follows that $(c_1 + \sqrt{c_1^2 - 4c_2})/2 \leq a < c_1$ because $a \geq c_1/2$ and $a^2 - c_1a + c_2 \geq 0$.

2) Consequently we observe that $a = 1$ (resp. 2 or 3) if $(c_1, c_2) = (2, 1)$ (resp. $(3, 2)$ or $(4, 3)$). In these cases, it is calculated directly that $a^2 - c_1a + c_2 = 0$. Hence we see that E is a direct sum of line bundles. Moreover when $(c_1, c_2) = (4, 4)$, we see that a is equal to either 2 or 3. When $a = 2$, E splits into line bundles for the same reason. Hence suppose $a = 3$. Consider an exact sequence

$$0 \to \mathcal{O}_{\mathbf{P}^4} \to E(-a) \to \mathcal{I} \otimes \mathcal{O}_{\mathbf{P}^4}(-2) \to 0,$$

where $\mathcal{I}$ is the defining ideal sheaf of the zero locus $W(s)$ of a nonzero global section s of $E(-a)$. Since $\deg W(s) = a^2 - c_1 a + c_2 = 1$, $W(s)$ is a linear plane of $\mathbf{P}^4$, from which we obtain $H^1(\mathcal{I} \otimes \mathcal{O}(n)) = H^2(\mathcal{I} \otimes \mathcal{O}(n)) = 0$ for all integers n because the canonical homomorphism $H^0(\mathcal{O}(n)) \to H^0(\mathcal{O}_{W(s)}(n))$ is surjective and $H^1(\mathcal{O}_{W(s)}(n)) = 0$ for all integers n. Hence it follows from the above exact sequence that $H^1(E(n)) = H^2(E(n)) = 0$ for all integers n. Thus we find that E splits into line bundles and that this case does not occur. (cf. [27])

COROLLARY 10. *If E is a very ample rank two bundle with $c_1 \leq 4$ on $\mathbf{P}^4$, then E splits into line bundles.*

5. Vector bundles on determinantal varieties

In this section, we shall investigate several properties of some vector bundles on determinantal varieties, say the normal bundle of X in $\mathbf{P}^n$ or the tangent bundle of X which will play important roles in the study of Hilbert schemes, the deformation spaces of X, the number of moduli of X and the geography of X. From the point of view of our splitting problems, we will concern ourselves later with the study of Hilbert schemes and the geography of determinantal varieties.

(5.1) Normal bundle $N_{X/\mathbf{P}^n}$ of X in $\mathbf{P}^n$.

(5.1.1) Recall that our determinantal variety X is located in the diagram.

$$\begin{array}{ccc} P(E) & \supset & Y = D_1 \cap \cdots \cap D_{m+1} \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ \mathbf{P}^n & \supset & X. \end{array}$$

Restricting the exact sequence

$$0 \to \pi^*(\Omega^1_{\mathbf{P}^n}) \to \Omega^1_{P(E)} \to \Omega^1_{P(E)/\mathbf{P}^n} \to 0$$

to Y, we get an exact sequence

$$0 \to \pi^*(\Omega^1_{\mathbf{P}^n}) \mid Y \to \Omega^1_{P(E)} \mid Y \to \mathcal{O}_Y(c_1 H - 2D) \to 0.$$

On the other hand, there exist usual exact sequences

$$0 \to \mathcal{I}(Y)/\mathcal{I}(Y)^2 \to \Omega^1_{P(E)} \mid Y \to \Omega^1_Y \to 0,$$

$$0 \to \mathcal{I}(X)/\mathcal{I}(X)^2 \to \Omega^1_{\mathbf{P}^n} \mid X \to \Omega^1_X \to 0.$$

where $\mathscr{I}(Y)$ (resp. $\mathscr{I}(X)$) is the defining ideal of Y in $P(E)$ (resp. the defining ideal of X in $\mathbf{P}^n$). Since Y is isomorphic to X via the morphism π, we may identify $\pi^*(\Omega^1_{\mathbf{P}^n}) \mid Y$ with $\Omega^1_{\mathbf{P}^n} \mid X$. Thus combining the above, we obtain the following diagram of exact sequences:

$$(*)\qquad \begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & \mathscr{I}(X)/\mathscr{I}(X)^2 & \to & \Omega^1_{\mathbf{P}^n} \mid X & \to & \Omega^1_X & \to & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \alpha} & & \downarrow & & \\ 0 & \to & \mathscr{I}(Y)/\mathscr{I}(Y)^2 & \to & \Omega^1_{P(E)} \mid Y & \to & \Omega^1_Y & \to & 0 \\ & & \downarrow{\scriptstyle \beta} & & \downarrow & & & & \\ & & \mathscr{O}_Y(c_1H - 2D) & \simeq & \mathscr{O}_Y(c_1H - 2D) & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

where α is the composite of the isomorphism $\Omega^1_{\mathbf{P}^n} \mid X \simeq \pi^*(\Omega^1_{\mathbf{P}^n}) \mid Y$ with the injection $\pi^*(\Omega^1_{\mathbf{P}^n}) \mid Y \to \Omega^1_{P(E)} \mid Y$. Since α maps $\mathscr{I}(X)/\mathscr{I}(X)^2$ to $\mathscr{I}(Y)/\mathscr{I}(Y)^2$, it induces a surjection $\beta : \mathscr{I}(Y)/\mathscr{I}(Y)^2 \to \mathscr{O}_Y(c_1H - 2D)$ by the snake lemma.

(5.1.2) As for the surjection $\beta : \mathscr{I}(Y)/\mathscr{I}(Y)^2 = \bigoplus^{m+1} \mathscr{O}_Y(-D) \to \mathscr{O}_Y(c_1H - 2D) = \mathscr{O}_X(F - D)$, we show that β coincides with the homomorphism $\beta' \otimes \mathscr{O}_X(-D)$ where $\beta' : \oplus^{m+1}\mathscr{O}_X \to \mathscr{O}_X(F)$ is the homomorphism defined by the global sections $\{s_i|X\}$ $(1 \le i \le m+1)$ of $\mathscr{O}(F)$ to which we have associated the morphism $\varphi : X \to \mathbf{P}^m$ in (3.3.1), (3.3.2). Since the problem is local, we may assume that Y and X are defined as follows (cf. (1.1)):

$$U \times \mathbf{A}^1 \supset Y : y_i = s_{i1} + s_{i2}y = 0 \qquad (1 \le i \le m+1),$$

$$U \supset X : s_i \wedge s_1 = 0 \qquad (2 \le i \le m+1),$$

where U is an affine open subset of $\mathbf{P}^n$ such that $s_{12} \ne 0$ on U and $\mathbf{A}^1 = \mathrm{Spec}(\mathbf{C}[y])$. Let p be a point of $U, q = (p, -s_{11}(p)/s_{12}(p))$ a point of Y lying over $p, \{x_1, \ldots, x_n\}$ a regular system of parameters of $\mathbf{P}^n$ at p and let

$$w_i = dy_i = d(s_{i1} + s_{i2}y) = \sum_{j=1}^{n} \left(\frac{\partial s_{i1}}{\partial x_j} + \frac{\partial s_{i2}}{\partial x_j} y \right) dx_j + s_{i2}\, dy \qquad (1 \le i \le m+1).$$

Then we see that

$$\mathscr{I}(Y)/\mathscr{I}(Y)^2 = \textstyle\bigoplus_{i=1}^{m+1} H^0(Y, \mathscr{O}_Y)w_i, \quad \mathscr{I}(Y)/\mathscr{I}(Y)^2 \bigcap \Omega^1_{\mathbf{P}^n} \mid X = \mathscr{I}(X)/\mathscr{I}(X)^2$$

on an affine open neighborhood of q. Hence

$$\beta(w_i) = s'_{i2}\, dy \qquad (1 \le i \le m+1),$$

where s'_{i2} $(1 \leq i \leq m+1)$ is the restriction of s_{i2} to X. Since the section $s_i|X$ of $\mathcal{O}_X(F)$ on U coincides with s'_{i2}, we can check our claim.

(5.1.3) Taking the dual of $(*)$, we get the following commutative diagram of exact sequences:

$$
\begin{array}{lccccccccc}
 & & & & 0 & & 0 & & & \\
 & & & & \downarrow & & \downarrow & & & \\
 & & & & \mathcal{O}_X(2D - c_1H) & \simeq & \mathcal{O}_X(D-F) & & & \\
 & & & & \downarrow & & \downarrow{\scriptstyle \beta^*} & & & \\
(*)' & 0 & \rightarrow & T_Y \rightarrow & T_{P(E)}\,|\,Y & \rightarrow & N_{Y/P(E)} & \simeq & \bigoplus^{m+1}\mathcal{O}_X(D) & \rightarrow 0 \\
 & & & & \downarrow & & \downarrow & & & \\
 & 0 & \rightarrow & T_X \rightarrow & T_{\mathbf{P}^n}\,|\,X & \rightarrow & N_{X/\mathbf{P}^n} & \rightarrow & 0. & \\
 & & & & \downarrow & & \downarrow & & & \\
 & & & & 0 & & 0 & & &
\end{array}
$$

The injection $\beta^* : \mathcal{O}_X(D-F) \rightarrow \bigoplus^{m+1}\mathcal{O}_X(D)$ in $(*)'$ is nothing but the multiplication by the rational functions $\{f_1, \ldots, f_{m+1}\}$, where f_i $(1 \leq i \leq m+1)$ is the rational function on X defined in (3.3.3). Consequently we can prove the following.

THEOREM 10. *There exists an exact sequence*

$$0 \longrightarrow \mathcal{O}_X(D-F) \xrightarrow{\beta^*} \bigoplus^{m+1}\mathcal{O}_X(D) \longrightarrow N_{X/\mathbf{P}^n} \longrightarrow 0$$

with the injection $\beta^* : \mathcal{O}_X(D-F) \ni \gamma \rightarrow (\gamma(f_1, \ldots, \gamma f_{m+1}) \in \bigoplus^{m+1}\mathcal{O}_X(D)$, *where* f_i $(1 \leq i \leq m+1)$ *is the rational function on* X *defined in* (3.3.3).

(5.1.4) For simplicity let us denote the normal bundle $N_{X/\mathbf{P}^n}$ by N and $N \otimes \mathcal{O}_X(-D)$ by $N(-D)$. Then Theorem 10 yields the following.

COROLLARY 11. 1) *$N(-D)$ is generated by global sections and $c_i(N(-D)) = F^i$ $(0 \leq i \leq m)$.*
2) $c_i(N) = \sum_{j=0}^{i} \binom{m-j}{i-j} D^{i-j}F^j$ $(0 \leq i \leq m)$.
3) $N(-D) \simeq \varphi^*\,(T_{\mathbf{P}^m}(-1))$.

In particular, we have

$$c_1(N) = mD + F,$$

$$c_2(N) = \frac{m(m-1)}{2}D^2 + (m-1)\,DF + F^2,$$

$$c_3(N) = \frac{m(m-1)(m-2)}{6}D^3 + \frac{(m-1)(m-2)}{2}D^2F + (m-2)\,DF^2 + F^3,$$

$$\vdots \quad = \quad \vdots$$

$$c_m(N) = \sum_{j=0}^{m} D^{m-j}F^j.$$

(5.1.5) As for the infinitesimal neighborhoods of X in $\mathbf{P}^n$, we obtain the following where N^* denotes the dual bundle of N.

COROLLARY 12. 1) *If $n \geq 3$, then $H^1(S^\nu(N^*)) = 0$ for every integer $\nu \geq 1$.*
2) *If $n \geq 5$, then $\dim H^2(N^*) = 1$ and $H^2(S^\nu(N^*)) = 0$ for every integer $\nu \geq 2$. Hence we see that the first infinitesimal neighborhood is exceptional among the infinitesimal neighborhoods of X in $\mathbf{P}^n$.*

PROOF. 1). For every integer $\nu \geq 1$, there exists the following exact sequence by Theorem 10:

$$0 \to \mathcal{O}_X(D-F) \otimes S^{\nu-1}\Big(\bigoplus^{m+1}\mathcal{O}_X(D)\Big) \otimes K_X$$
$$\to S^\nu\Big(\bigoplus^{m+1}\mathcal{O}_X(D)\Big) \otimes K_X \to S^\nu(N) \otimes K_X \to 0.$$

Since $H^1(S^\nu(N^*) \simeq H^{n-m-1}(S^\nu(N) \otimes K_X)$ by Serre duality and $H^{n-m-1}(\mathcal{O}_X(\nu D) \otimes K_X) = 0$ by the Kodaira vanishing theorem and in addition $H^{n-m}(\mathcal{O}_X(\nu D - F) \otimes K_X) \simeq H^0(\mathcal{O}_X(F - \nu D)) = H^0(\mathcal{O}_X(-(\nu+1)D + c_1 H)) = 0$ by (3.4.1) 4) and (3.2.2), we obtain $H^1(S^\nu(N^*)) = 0$ from the above exact sequence. 2). Similarly we have $H^{n-m-i}(\mathcal{O}_X(\nu D) \otimes K_X) = 0$ for $1 \leq i \leq 2$ and $H^{n-m-1}(\mathcal{O}_X(\nu D - F) \otimes K_X) \simeq H^1(\mathcal{O}_X(-(\nu+1)D + c_1 H)) = 0$ for $\nu \geq 2$ (resp. $\simeq \mathbf{C}$ for $\nu = 1$) by Corollary 7. Hence it follows that $H^2(S^\nu(N^*)) = 0$ for $\nu \geq 2$ and $\dim H^2(N^*) = 1$ by an argument similar to that above.

(5.2) Tangent bundle T_X of X.

(5.2.1) Since there exist the usual exact sequence

$$0 \to T_X \to T_{\mathbf{P}^n} \,|\, X \to N_{X/\mathbf{P}^n} \to 0$$

and the exact sequence in Theorem 10

$$0 \to \mathcal{O}_X(D-F) \to \bigoplus^{m+1}\mathcal{O}_X(D) \to N_{X/\mathbf{P}^n} \to 0,$$

the following equalities on the Chern polynomials hold:

$$(1 + Ht)^{n+1} = c_t(T_X)c_t(N_{X/\mathbf{P}^n}),$$
$$(1 + Dt)^{m+1} = (1 + (D-F)t)c_t(N_{X/\mathbf{P}^n}).$$

Hence we get the following equalities on the Chern classes: for every i $(0 \leq i \leq n - m)$,

$$\sum_{j=0}^{i}\binom{m+1}{i-j}c_j(T_X)D^{i-j} = \left\{\binom{n+1}{i} - c_1\binom{n+1}{i-1}\right\}H^i + 2\binom{n+1}{i-1}H^{i-1}D.$$

We need the following to express the Chern classes $c_i(T_X)$ in terms of D and H by using the above.

LEMMA 8. *We have*

$$\sum_{k=0}^{i}(-1)^k\binom{m+1}{i-k}\binom{m+k}{k}=0 \qquad \textit{for every integer } i\ (1\le i\le m+1),$$

$$\sum_{k=0}^{i}(-1)^k\binom{m+1}{i-k}\binom{m-1+k}{k}=0 \qquad \textit{for every integer } i\ (2\le i\le m+1).$$

PROOF. It is obvious for $i=1$ and so we assume $i\ge 2$. Since

$$\binom{m+1}{i-k}\binom{m+k}{k}=(m+1)\frac{(m+k)\dots(m+k-(i-2))}{(i-k)!k!},$$

it suffices to verify

$$\sum_{i=0}^{i}(-1)^k\frac{(m+k)\dots(m+k-(i-2))}{(i-k)!k!}=0.$$

For every j $(0\le j\le i-2)$, let us put

$$h_j(x)=\sum_{k=0}^{i}(-1)^k\frac{(x+k)\dots(x+k-j)}{(i-k)!k!}.$$

We show that every $h_j(x)$ must be identically zero $(0\le j\le i-2)$ by induction on j. First it is easily seen that $h_0(x)\equiv 0$. Take the difference of $h_j(x)$:

$$h_j(x+1)-h_j(x)=(j+1)\sum_{k=0}^{i}(-1)^k\frac{(x+k)\cdots(x+k-(j-1))}{(i-k)!k!}.$$

Accordingly we see $h_j(x+1)=h_j(x)$ by the induction hypothesis. In addition,

$$h_j(0)=\sum_{k=0}^{i}(-1)^k\frac{k\dots(k-j)}{(i-k)!k!}=\sum_{k=j+1}^{i}(-1)^k\frac{k\cdots(k-j)}{(i-k)!k!}$$

$$=\sum_{k=j+1}^{i}(-1)^k\frac{1}{(i-k)!(k-(j+1))!}=0.$$

Thus $h_j(x)$ must be identically zero, from which the desired equality is obtained:

$$h_{i-2}(m)=\sum_{k=0}^{i}(-1)^k\frac{(m+k)\dots(m+k-(i-2))}{(i-k)!k!}=0.$$

Therefore we can prove the first assertion and the second one is verified similarly.

(5.2.2) As for the Chern classes $c_i(T_X)$ $(1 \leq i \leq n-m)$, the following formula shall be established where the binomial coefficients $\binom{r}{s} = 0$ for negative integers s.

THEOREM 11. *For every i $(1 \leq i \leq n-m)$,*

$$c_i(T_X) = (-1)^i \sum\nolimits_{k=0}^{i} (-1)^k \left[\binom{n+1}{k} \left\{ \binom{m+i-k-1}{i-k} - \binom{m+i-k-1}{i-k-1} \right\} - c_1 \binom{n+1}{k-1} \binom{m+i-k}{i-k} \right] D^{i-k} H^k.$$

PROOF. We prove our formula by induction on i. Since $c_1(T_X) = -(m-1)D + (n+1-c_1)H$, it holds for $i = 1$. From (5.2.1), we obtain

$$c_i(T_X) = \left\{ \binom{n+1}{i} - c_1 \binom{n+1}{i-1} \right\} H^i + 2 \binom{n+1}{i-1} H^{i-1} D - \sum_{j=0}^{i-1} \binom{m+1}{i-j} c_j(T_X) D^{i-j}.$$

Hence by the induction hypothesis, it turns out that

$$\begin{aligned} c_i(T_X) = {} & \left\{ \binom{n+1}{i} - c_1 \binom{n+1}{i-1} \right\} H^i + 2 \binom{n+1}{i-1} H^{i-1} D \\ & + \sum\nolimits_{j=0}^{i-1} (-1)^{j+1} \binom{m+1}{i-j} \times \sum\nolimits_{k=0}^{j} (-1)^k \\ & \cdot \left[\binom{n+1}{k} \left\{ \binom{m+j-k-1}{j-k} - \binom{m+j-k-1}{j-k-1} \right\} - c_1 \binom{n+1}{k-1} \binom{m+j-k}{j-k} \right] D^{i-k} H^k. \end{aligned}$$

On the right hand side in the above, we see that the coefficient of H^i is $\binom{n+1}{i} - c_1 \binom{n+1}{i-1}$ and the coefficient of $H^{i-1}D$ is $-(m-1)\binom{n+1}{i-1} + c_1(m+1)\binom{n+1}{i-2}$.

Moreover the coefficient of $D^{i-k}H^k$ $(0 \le k \le i-2)$ coincides with

$$(-1)^{k+1}\left[\sum_{j=k}^{i-1}(-1)^j\binom{m+1}{i-j}\left\{\binom{m+j-k-1}{j-k}-\binom{m+j-k-1}{j-k-1}\right\}\right.$$

$$\left.\cdot\binom{n+1}{k}-\sum_{j=k}^{i-1}(-1)^j\binom{m+1}{i-j}\binom{m+j-k}{j-k}c_1\binom{n+1}{k-1}\right]$$

$$=(-1)^{i+k}\left[\left\{\binom{m+i-k-1}{i-k}-\binom{m+i-k-1}{i-k-1}\right\}\binom{n+1}{k}\right.$$

$$\left.-\binom{m+i-k}{i-k}c_1\binom{n+1}{k-1}\right]$$

by Lemma 7. Hence we can complete our proof of Theorem 11.

For example, we observe that

$$c_1(T_X) = -(m-1)D + (n+1-c_1)H,$$

$$c_2(T_X) = \tfrac{1}{2}(m+1)(m-2)D^2 - (m+1)(m-c_1-1)DH + \tfrac{1}{2}(n+1)(n-2c_1)H^2,$$

$$c_3(T_X) = -\tfrac{1}{6}(m+2)(m+1)(m-3)D^3 + \tfrac{1}{2}(m+1)\{(n+1)(m-2)-(m+2)c_1\}D^2H$$
$$-\tfrac{1}{2}(n+1)\{n(m-1)-2(m+1)c_1\}DH^2 + \tfrac{1}{6}(n+1)n(n-1-3c_1)H^3.$$

(5.2.3) Let $b_i := \dim_{\mathbf{C}} H^i(X,\mathbf{C})$ $(0 \le i \le 2(n-m))$ be the i-th Betti number of X. Then we find by Proposition 1 that

$$b_{2i} = b_{2(n-m-i)} = 2 \qquad \text{for } 1 \le i \le \left[\frac{n-m-1}{2}\right],$$

$$b_{2i+1} = b_{2(n-m-i)-1} = 0 \qquad \text{for } 0 \le i \le \left[\frac{n-m-2}{2}\right],$$

where $[a]$ stands for the largest integer less than or equal to a real number a.

Since $c_{n-m}(T_X) = \sum_{i=0}^{2(n-m)}(-1)^i b_i$, it turns out from Theorem 11 and Lemma 2 that the middle Betti number b_{n-m} is explicitly calculated in terms of the Chern numbers $\{c_1, c_2\}$ of E as follows.

THEOREM 12. 1) *When $n-m$ is even,*

$$\begin{aligned} b_{n-m} &= c_{n-m}(T_X) - 2(n-m) + 2 \\ &= \sum_{k=0}^{n-m} (-1)^k \left[\binom{n+1}{k} \left\{ \binom{n-k-1}{n-m-k} - \binom{n-k-1}{n-m-k-1} \right\} \right. \\ &\quad \left. - c_1 \binom{n+1}{k-1} \binom{n-k}{n-m-k} \right] p_{n-k}(c_1, c_2) + 2(n-m) + 2. \end{aligned}$$

2) *When $n-m$ is odd,*

$$\begin{aligned} b_{n-m} &= -c_{n-m}(T_X) + 2(n-m) \\ &= \sum_{k=0}^{n-m} (-1)^k \left[\binom{n+1}{k} \left\{ \binom{n-k-1}{n-m-k} - \binom{n-k-1}{n-m-k-1} \right\} \right. \\ &\quad \left. - c_1 \binom{n+1}{k-1} \binom{n-k}{n-m-k} \right] p_{n-k}(c_1, c_2) + 2(n-m). \end{aligned}$$

For example, b_{n-m} is written down explicitly for small integers n as follows.

(1) $n = 4$.

$$b_2 = 3c_1^4 - 6c_1^2 c_2 - 10c_1^3 + 15c_1 c_2 + 10c_1^2 - 10c_2 - 2$$

(2) $n = 5$.

$$b_3 = 4c_1^5 - 10c_1^3 c_2 - 18c_1^4 + 36c_1^2 c_2 + 30c_1^3 - 45c_1 c_2 - 20c_1^2 + 20c_2 + 6.$$

$\vdots$

(5.3) $\dim H^i(T_X)$ and $\dim H^i(N_{X/\mathbf{P}^n})$

In this subsection, we assume that K_X is ample and moreover $H^i(\mathbf{P}^n, E) = 0$ for $1 \leq i \leq n-m-1$ and $H^1(\mathbf{P}^4, E(c_1 - 6)) = 0$ in case $n = 4$.

Consider the following exact sequences with $N = N_{X/\mathbf{P}^n}$:

(1) $$\to H^i(T_X) \to H^i(T_{\mathbf{P}^n} \mid X) \to H^i(N) \to,$$

(2) $$\to H^i(\mathcal{O}_X) \to H^i(\bigoplus^{n+1} \mathcal{O}_X(1)) \to H^i(T_{\mathbf{P}^n} \mid X) \to,$$

(3) $$\to H^i(\mathcal{O}_X(D - F)) \to H^i(\bigoplus^{m+1} \mathcal{O}_X(D)) \to H^i(N) \to .$$

We have by Proposition 1 and Corollary 6 that $H^i(\mathcal{O}_X) = 0$ for $1 \leq i \leq n-m-1$ and $H^i(\mathcal{O}_X(1)) = 0$ for $1 \leq i \leq n-m-2$. When $n = 4$, we have also $H^1(\mathcal{O}_X(1)) = 0$ by Corollary 9 and our assumption. Hence it follows from (2) that

a) $\dim H^0(T_{\mathbf{P}^n} \mid X) = n^2 + 2n$,

b) $H^i(T_{\mathbf{P}^n} \mid X) = 0$ for $1 \leq i \leq n - m - 2$,

$\dim H^1(T_{\mathbf{P}^4} \mid X) \leq \dim H^2(\mathcal{O}_X)$.

In addition since K_X is ample, we have

c) $H^0(T_X) = 0$.

by the Nakano-Akizuki-Kodaira vanishing theorem. Therefore we find the following from (1) and a), b), c):

1) $n \geq 7$.

$\dim H^1(T_X) = \dim H^0(N) - (n^2 + 2n)$,

$\dim H^i(T_X) = \dim H^{i-1}(N)$ $(2 \leq i \leq n - m - 2)$.

2) $5 \leq n \leq 6$.

$\dim H^1(T_X) = \dim H^0(N) - (n^2 + 2n)$,

$\dim H^2(T_X) \geq \dim H^1(N)$.

3) $n = 4$.

$\dim H^1(T_X) \leq \dim H^0(N) + \chi(X) - 25$,

where $\chi(X)$ is the Euler-Poincaré characteristic of $\mathcal{O}_X$.

Moreover since $H^i(\mathcal{O}_X(D)) = 0$ $(1 \leq i \leq n - m - 1)$ by Theorem 3 and our assumption, we have

4) $\dim H^0(N) = (m+1)\dim H^0(\mathbf{P}^n, E) + \dim H^1(\mathcal{O}_X(D - F))$
$\qquad - \dim H^0(\mathcal{O}_X(D - F)) - (m+1)^2$,

$\dim H^i(N) \simeq H^{i+1}(\mathcal{O}_X(D - F))$ $(1 \leq i \leq n - m - 2)$,

$\dim H^1(N) \leq \dim H^2(\mathcal{O}_X(D - F))$ in case $n = 4$.

Summing up the above, we obtain the following because $\dim H^0(\mathcal{O}_X(D - F)) = \dim H^0(\mathcal{E}nd(E)) - 1$ and $H^i(\mathcal{O}_X(D - F)) \simeq H^i(\mathcal{E}nd(E))$ $(1 \leq i \leq n - m - 1)$ by Corollary 4.

THEOREM 13. *Assume that K_X is ample and moreover, $H^i(\mathbf{P}^n, E) = 0$ for $1 \leq i \leq n - m - 1$ and $H^1(\mathbf{P}^4, E(c_1 - 6)) = 0$ in case $n = 4$. Then we have*

1) $\dim H^0(N_{X/\mathbf{P}^n}) = (m+1)\dim H^0(\mathbf{P}^n, E) + \dim H^1(\mathbf{P}^n, \mathcal{E}nd(E))$
$\qquad - \dim H^0(\mathbf{P}^n, \mathcal{E}nd(E)) - (m^2 + 2m)$,

$H^i(N_{X/\mathbf{P}^n}) \simeq H^{i+1}(\mathbf{P}^n, \mathcal{E}nd(E))$ $(1 \leq i \leq n - m - 2)$,

$\dim H^1(N) \leq \dim H^2(\mathcal{O}_X(D - F))$ in case $n = 4$.

2) $n \geq 7$.

$\dim H^1(T_X) = (m+1)\dim H^0(\mathbf{P}^n, E) + \dim H^1(\mathcal{E}nd(E))$
$\qquad - \dim H^0(\mathbf{P}^n, \mathcal{E}nd(E)) - (m^2 + 2m) - (n^2 + 2n)$,

$\dim H^i(T_X) = \dim H^{i-1}(N_{X/\mathbf{P}^n}) = \dim H^i(\mathbf{P}^n, \mathcal{E}nd(E))$ $(2 \leq i \leq n - m - 2)$.

3) $5 \leq n \leq 6$.

$\dim H^1(T_X) = \dim H^0(N) - (n^2 + 2n)$,

$\dim H^2(T_X) \geq \dim H^1(N)$

4) $n = 4$.

$$\begin{aligned}\dim H^1(T_X) \le 3\dim H^0(\mathbf{P}^4, E) + \dim H^1(\mathbf{P}^4, \mathscr{E}nd(E)) \\ - \dim H^0(\mathbf{P}^4, \mathscr{E}nd(E)) + \chi(X) - 33,\end{aligned}$$

where $\chi(X)$ is the Euler-Poincaré characteristic of $\mathcal{O}_X$.

(5.4) Hilbert schemes

Let $\mathscr{H}ilb$ be the Hilbert scheme of $\mathbf{P}^n$. In this subsection, we shall study some geometric structures of $\mathscr{H}ilb$ at determinantal subvarieties using Lemma 7, Theorem 10 and Theorem 13 and as a by-product, we shall give a splitting theorem which states that the geometry of $\mathscr{H}ilb$ is related to the splitting of rank two bundles on $\mathbf{P}^n$ $(n \ge 4)$.

(5.4.1) Let X be a determinantal subvariety associated to E. Then the following are well-known:

(a) The Zariski tangent space of $\mathscr{H}ilb$ at X is isomorphic to $H^0(N_{X/\mathbf{P}^n})$.

(b) The dimension of every irreducible component of $\mathscr{H}ilb$ at X is at least $\dim H^0(N_{X/\mathbf{P}^n}) - \dim H^1(N_{X/\mathbf{P}^n})$.

In this subsection, we assume that K_X is ample and moreover, $H^i(\mathbf{P}^n, E) = 0$ for $1 \le i \le n - m - 1$ and $H^1(\mathbf{P}^4, E(c_1 - 6)) = 0$ in case $n = 4$.

By Theorem 13, we observe that

$$\begin{aligned}\dim H^0(N) &= (m+1)\dim H^0(E) - (m+1)^2 + \dim H^1(\mathscr{E}nd(E)) \\ &\quad - \dim H^0(\mathscr{E}nd(E)) + 1 \\ &= \dim U' + \dim H^1(\mathscr{E}nd(E)) - \dim H^0(\mathscr{E}nd(E)) + 1,\end{aligned}$$

where U' is the open subscheme of $G = \mathrm{Grass}(H^0(E), m+1)$ in Theorem 1 which parametrizes the determinantal subvarieties associated to E and

$$H^i(N_{X/\mathbf{P}^n}) \simeq H^{i+1}(\mathscr{E}nd(E)) \qquad \text{for } 1 \le i \le n - m - 2,$$

$$\dim H^1(N_{X/\mathbf{P}^n}) \le \dim H^2(\mathcal{O}_X(D - F)) \qquad \text{in case } n = 4.$$

(5.4.2) Let $\mathrm{Aut}(E)$ be the automorphism group of E. Then $\mathrm{Aut}(E)$ is a connected linear algebraic group of dimension $\dim H^0(\mathscr{E}nd(E))$ and is considered as a closed subgroup of $GL(H^0(\mathbf{P}^n, E))$ canonically because E is generated by global sections. For example, if E is a simple vector bundle (resp. $E \simeq \bigoplus^2 \mathcal{O}(a)$ or $E \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$ $(a > b)$), then $\mathrm{Aut}(E)$ is the multiplicative group $\mathbf{C}^*$ (resp. $GL(2, \mathbf{C})$ for $E \simeq \bigoplus^2 \mathcal{O}(a)$ or a semi-direct product of a vector group $H^0(\mathbf{P}^n, \mathcal{O}(a - b))$ and a two-dimensional torus group $(\mathbf{C}^*)^2$ for $E \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$).

For every element $g \in \mathrm{Aut}(E)$ and $s = \langle s_1, \ldots, s_{m+1} \rangle \in G$, we define

$$g \cdot s = \langle g(s_1), \ldots, g(s_{m+1}) \rangle,$$

where $g(s_i)$ is the composite of s_i with g. Then it defines an action of $\mathrm{Aut}(E)$ on G and we have

$$g \cdot s_i \wedge g \cdot s_j = \det g \cdot s_i \wedge s_j \qquad (1 \leq i, j \leq m+1),$$

where $\det : \mathrm{Aut}(E) \ni g \to \det(g) \in \mathbf{C}^*$ is the determinant character. Hence if we denote by X_s the determinantal subvariety defined by $s \in U'$ as in (1.3.3), then we find that X_s coincides with $X_{g \cdot s}$. Therefore $\mathrm{Aut}(E)$ acts on U'.

(5.4.3) As for the stabilizer $\mathrm{Stab}(s)$ of $s \in U'$, we have the following.

Lemma 9. $\mathrm{Stab}(s)$ *is equal to the multiplicative group* $\mathbf{C}^*$.

Proof. First we check that $\{s_i \wedge s_j\}$ $(1 \leq i < j \leq m+1)$ consists of linearly independent elements of $H^0(\mathbf{P}^n, \mathcal{O}(c_1))$. Suppose that there is a non-trivial relation

$$(*) \qquad s_1 \wedge s_2 = \sum_{k<l,\,(k,l) \neq (1,2)} a_{kl} s_k \wedge s_l \qquad (a_{kl} \in \mathbf{C})$$

and let U be an affine open subset of $\mathbf{P}^n$ such that $U \cap \bigcap_{i \geq 3} W(s_i) \neq \varnothing$ and $s_{12} \neq 0$ on U, where $E \mid U \simeq \bigoplus^2 \mathcal{O}_U$ and $s_i \mid U = (s_{i1}, s_{i2})$ $(s_{i1}, s_{i2} \in H^0(U, \mathcal{O}_U)$, $1 \leq i \leq m+1)$. As we have shown in (1.1), X_s is defined by the equations $s_1 \wedge s_i = 0$ $(2 \leq i \leq m+1)$ on U. Let T be the closed subscheme defined by the equations $s_1 \wedge s_i = 0$ $(3 \leq i \leq m+1)$ on U. Then we obtain the following equality on T:

$$s_1 \wedge s_2 = \sum_{i \geq 3} \left(-\frac{a_{2i} s_{i2}}{s_{12}} \right) s_1 \wedge s_2$$

by restricting the relation $(*)$ to T because $s_{i1} = (s_{11}/s_{12}) s_{i2}$ $(3 \leq i \leq m+1)$. Since T is unmixed and $s_1 \wedge s_2$ is a prime element of height 1 in the affine coordinate ring $A(T)$ of T, it yields that $1 + \sum_{i \geq 3} (a_{2i}/s_{12}) s_{i2}$ is a nilpotent element of $A(T)$, which is a contradiction.

Next we show that $\mathrm{Stab}(s)$ is equal to the multiplicative group $\mathbf{C}^*$. Assuming that g is contained is $\mathrm{Stab}(s)$, let us put

$$g \cdot s_i = \sum_{k=1}^{m+1} a_{ki} s_k \qquad (a_{ki} \in \mathbf{C},\ 1 \leq i \leq m+1).$$

Then it follows that for any i and j $(1 \leq i < j \leq m+1)$,

$$g \cdot s_i \wedge g \cdot s_j = \sum_{k,l} a_{ki} a_{lj} s_k \wedge s_l = \sum_{k<l} (a_{ki} a_{lj} - a_{li} a_{kj}) s_k \wedge s_l,$$

from which are derived

$$a_{ki} a_{lj} = a_{li} a_{kj} \qquad \text{for } (k, j) \neq (i, j), \qquad a_{ii} a_{jj} - a_{ji} a_{ij} = \det g.$$

Hence the matrix $(a_{ij}) \in GL(m+1, \mathbf{C})$ is a scalar matrix, i.e., $g \cdot s_i = \alpha s_i$ for some nonzero element $\alpha \in \mathbf{C}$ $(1 \le i \le m+1)$. Let U be an affine open subset of $\mathbf{P}^n$ such that $E \mid U \simeq \bigoplus^2 \mathcal{O}_U$, $s_i \mid U = (s_{i1}, s_{i2})$ $(1 \le i \le m+1)$ and let

$$g \mid U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, H^0(U, \mathcal{O}_U)).$$

Then we see that for $1 \le i \le m+1$,

$$(a-\alpha)s_{i1} + bs_{i2} = 0, \qquad cs_{i1} + (d-\alpha)s_{i2} = 0.$$

Hence $a = d = \alpha$ and $b = c = 0$. Since this holds on every affine open subset of $\mathbf{P}^n$ where E is trivial, we find that g is the multiplication by $\alpha \in \mathbf{C}$.

As a trivial corollary of Lemma 9, we get the following.

COROLLARY 13. *Every orbit is isomorphic to the quotient group* $\mathrm{Aut}(E)/\mathbf{C}^*$. *Hence the action of* $\mathrm{Aut}(E)$ *on* U' *is closed, i.e., every orbit is closed in* U'.

(5.4.4) let $q : U' \ni s \to X_s \in \mathscr{H}ilb$ be the induced morphism (cf. Theorem 1). Then we see that q is an orbit morphism, i.e., q is constant on any orbit $O(s) = \{g \cdot s \mid g \in \mathrm{Aut}(E)\}$.

(5.4.4.1) First we begin with proving that $q^{-1}(q(s))$ $(s \in U')$ consists of finitely many orbits.

1) Let $(dq)_s : T_{s,U'} \to T_{X_s, \mathscr{H}ilb}$ be the differential of q at $s = \langle s_1, \dots, s_{m+1} \rangle$, where $T_{s,U'}$ (resp. $T_{X_s,\mathscr{H}ilb}$) is the Zariski tangent space of U' at s (resp. the one of $\mathscr{H}ilb$ at X_s). Then it is known that

a) $T_{s,U'} \simeq \mathrm{Hom}(\langle s_1, \dots, s_{m+1} \rangle, H^0(E)/\langle s_1, \dots, s_{m+1} \rangle)$
$\simeq H^0(\bigoplus^{m+1} \mathcal{O}_{X_s}(D)) \simeq H^0(N_{Y_s/P(E)})$ by Theorem 3
$= \{$embedded deformations of Y_s in $P(E)\}$,

where Y_s is the smooth complete intersection subvariey of $P(E)$ defined by s in (1.3).

b) $T_{X_s,\mathscr{H}ilb} \simeq H^0(N_{X_s/\mathbf{P}^n}) = \{$embedded deformations of X_s in $\mathbf{P}^n\}$.

Let $H^0(E) = \langle s_1, \dots, s_{m+1} \rangle \oplus V$ be a direct sum decomposition of $H^0(E)$ by vector subspaces. Let $t = \{t_1, \dots, t_{m+1} \mid t_i \in V\}$ be a set of $m+1$ elements of V and consider the following embedded deformation $\mathscr{Y} \subset P(E) \times \mathrm{Spec}(\mathbf{C}[\varepsilon])$ ($\mathbf{C}[\varepsilon]$ being the ring of dual numbers) of Y_s in $P(E)$: Locally on $\pi^{-1}(U) \times \mathrm{Spec}(\mathbf{C}[\varepsilon])$, $\mathscr{Y}$ is defined by the equations

$$\mathscr{Y} : (s_{i1}X_1 + s_{i2}X_2) + \varepsilon(t_{i1}X_1 + t_{i2}X_2) = 0 \qquad (1 \le i \le m+1),$$

where U is an affine open subset such that $E \mid U \simeq \bigoplus^2 \mathcal{O}_U$, $s_i \mid U = (s_{i1}, s_{i2})$ and $t_i \mid U = (t_{i1}, t_{i2})(s_{ij}, t_{ij} \in H^0(U, \mathcal{O}_U))$.

On the other hand, let us denote by $\mathscr{X}$ the closed subscheme of $\mathbf{P}^n \times \mathrm{Spec}(\mathbf{C}[\varepsilon])$ defined by the equations

$$s_i \wedge s_j + \varepsilon(s_i \wedge t_j - s_j \wedge t_i) = 0 \qquad (1 \leq i < j \leq m+1).$$

Then we find by the same argument as in Lemma 1 that if we restrict the morphism $\pi \times \mathrm{id} : P(E) \times \mathrm{Spec}(\mathbf{C}[\varepsilon]) \to \mathbf{P}^n \times \mathrm{Spec}(\mathbf{C}[\varepsilon])$ to $\mathscr{Y}$, then it gives an isomorphism between $\mathscr{Y}$ and $\mathscr{X}$. Consequently $\mathscr{X}$ is the embedded deformation of X_s in $\mathbf{P}^n$ which corresponds to $\mathscr{Y}$ under $\pi \times \mathrm{id}$.

2) Here we recall the exact sequence in Theorem 10

$$0 \to \mathcal{O}_{X_s}(D-F) \to \bigoplus^{m+1} \mathcal{O}_{X_s}(D) \simeq N_{Y_s/P(E)} \to N_{X_s/\mathbf{P}^n} \to 0,$$

which gives rise to the exact sequence

$$0 \to H^0(\mathcal{O}_{X_s}(D-F)) \to \bigoplus^{m+1} H^0(\mathcal{O}_{X_s}(D)) \to H^0(N_{X_s/\mathbf{P}^n}) \to .$$

Then by the results in 1), the homomorphism $\bigoplus^{m+1} H^0(\mathcal{O}_{X_s}(D)) \to H^0(N_{X_s/\mathbf{P}^n})$ coinsides with $(dq)_s$. Hence we find that $\mathrm{Ker}(dq)_s \simeq H^0(\mathcal{O}_{X_s}(D-F))$. Meanwhile, since q is an orbit morphism, it is obvious that the Zariski tangent space $T_{s,O(s)}$ of $O(s)$ at s is contained in $\mathrm{Ker}(dq)_s$. In addition, we find that $\mathrm{Ker}(dq)_s = T_{s,O(s)}$ because $\dim T_{s,O(s)} = \dim H^0(\mathscr{E}nd(E)) - 1 = \dim H^0(\mathcal{O}_{X_s}(D-F))$ by Corollary 4 and Corollary 12.

3) Let $q^{-1}(q(s))_{\mathrm{red}} = Q_1 \cup \cdots \cup Q_r$ be the irreducible decomposition. Then every Q_i is an $\mathrm{Aut}(E)$-invariant and irreducible closed subvariety of U'. Suppose that Q_i is not a single orbit. Let x be a smooth point of Q_i and consider an irreducible curve Γ of Q_i which is smooth at x and $T_{x,\Gamma}$ is not contained in $T_{x,O(x)}$. Then it follows from 2) that $(dq)_x(T_{x.\Gamma}) \neq 0$. On the other hand, we have $(dq)_x(T_{x.\Gamma}) = 0$, which is a contradiction. Hence every Q_i is a single orbit.

(5.4.4.2) Secondly we shall prove that $q^{-1}(q(s))_{\mathrm{red}}$ $(s \in U')$ consists of a single orbit when $n=4$ or 5. Indeed, assume $X_s = X_{s'}$ for $s, s' \in U'$. Then there exists a non-singular matrix $A = (a_{ijkl}) \in GL(3, \mathbf{C})$ such that

$$s_i \wedge s_j = \sum_{1 \leq k < l \leq 3} a_{klij} s'_k \wedge s'_l,$$

because $\{s_i \wedge s_j\}$ (resp. $\{s'_i \wedge s'_j\}$) is a set of linearly independent elements of $H^0(\mathbf{P}^n, \mathcal{O}(c_1))$.

Consider the following exact sequences corresponding to s and s' respectively in Lemma 7:

$$(*) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & E^* & \xrightarrow{\beta} & \bigoplus^3 \mathcal{O}_{\mathbf{P}^n} & \xrightarrow{\alpha} & I_{X_s} \otimes \mathcal{O}(c_1) & \longrightarrow & 0, \\ & & & & \downarrow f & & \| & & \\ 0 & \longrightarrow & E^* & \xrightarrow{\beta'} & \bigoplus^3 \mathcal{O}_{\mathbf{P}^n} & \xrightarrow{\alpha'} & I_{X_{s'}} \otimes \mathcal{O}(c_1) & \longrightarrow & 0, \end{array}$$

and let $f : \bigoplus^3 \mathcal{O}_{\mathbf{P}^n} \to \bigoplus^3 \mathcal{O}_{\mathbf{P}^n}$ be the induced automorphism defined by A. Then since $\alpha = \alpha' \circ f$, it turns out from the diagram $(*)$ that there is an automorphism $g \in \mathrm{Aut}(E)$ such that $f \circ \beta = \beta' \circ g^*$, where g^* is the dual automorphism of g. We prove that $\langle s_1, s_2, s_3 \rangle = \langle g(s_1'), g(s_2'), g(s_3') \rangle$. Let U be an affine open subset of $\mathbf{P}^n$ with a trivialization $E \mid U \simeq \bigoplus^2 \mathcal{O}_U$ and let ℓ be an element of $H^0(U, E^*)$. Then since we have

$$\begin{aligned}(f \circ \beta)(\ell) &= f(\ell(s_3)e_1 \wedge e_2 - \ell(s_2)e_1 \wedge e_3 + \ell(s_1)e_2 \wedge e_3)\\ &= \ell(s_3))f(e_1 \wedge e_2) - \ell(s_2)f(e_1 \wedge e_3) + \ell(s_1)f(e_2 \wedge e_3)\end{aligned}$$

and

$$\begin{aligned}(\beta' \circ g^*)(\ell) = \beta'(g^*(\ell)) &= g^*(\ell)(s_3')e_1 \wedge e_2 - g^*(\ell)(s_2')e_1 \wedge e_2 + g^*(\ell)(s_1')e_2 \wedge e_3\\ &= \ell(g(s_3'))e_1 \wedge e_2 - \ell(g(s_2'))e_1 \wedge e_3 + \ell(g(s_1'))e_2 \wedge e_3,\end{aligned}$$

it follows that there is a non-singular matrix $B = (b_{ij}) \in GL(3, \mathbf{C})$ such that

$$\ell(g(s_i')) = \sum_{j=1}^{3} b_{ji}\ell(s_j) \qquad (1 \le i \le 3),$$

from which we obtain the desired equality $\langle s_1, s_2, s_3 \rangle = \langle g(s_1'), g(s_2'), g(s_3') \rangle$.

Let $P(n) = \chi(\mathcal{O}_{X_s}(n))$ $(n \in \mathbf{Z})$ be the Hilbert polynomial of X_s, $\mathscr{H}ilb_P = \{$closed subschemes X of $\mathbf{P}^n$ with $\chi(\mathcal{O}_X(n)) = P(n)\}$ the Hilbert scheme with respect to $P = P(n)$ and let $\mathscr{H}ilb_P^0$ be an irreducible component of $\mathscr{H}ilb_P$ containing the determinantal subvarieties associated to E, i.e., that of $\mathscr{H}ilb_P$ which contains $\overline{q(U')}$ where $\overline{q(U')}$ is the closure of $q(U')$ in $\mathscr{H}ilb_P$. Then summing up the above, we obtain the following:

THEOREM 14. *For any point $s \in U'$, $q^{-1}(q(s))_{\mathrm{red}}$ consists of finitely many orbits. Moreover, if $n = 4$ or 5, then $q^{-1}(q(s))_{\mathrm{red}}$ is a single orbit. Hence we have* $\dim \mathscr{H}ilb_P^0 \ge \dim \overline{q(U')} = \dim U' - \dim \mathrm{Aut}(E)/\mathbf{C}^*$.

(5.4.4.3) Let $PGL(n+1, \mathbf{C})$ be the automorphism group of $\mathbf{P}^n$ and let $T_\sigma : \mathbf{P}^n \ni x \to \sigma x \in \mathbf{P}^n$ be the transformation of $\mathbf{P}^n$ defined by $\sigma \in PGL(n+1, \mathbf{C})$. Then we have:

1) $PGL(n+1, \mathbf{C})$ acts on $\mathscr{H}ilb_P$. Moreover since $PGL(n+1, \mathbf{C})$ is connected, it acts on $\mathscr{H}ilb_P^0$.

2) The pull back $T_\sigma^*(E)$ of E by T_σ is a very ample vector bundle on $\mathbf{P}^n$.

3) Fir every point $s = \langle s_1, \ldots, s_{m+1} \rangle \in U'$, we find that $\sigma(X_s) = X_{\sigma(s)}$, where $X_{\sigma(s)}$ is the determinantal subvariety associated to $T_\sigma^*(E)$ which is defined by $\sigma(s) = \langle T_\sigma^*(s_1), \ldots, T_\sigma^*(s_{m+1}) \rangle$.

Hence if we denote by U_σ' the open subscheme of $\mathrm{Grass}(H^0(T_\sigma^*(E)), m+1)$ in Theorem 1 and by q_σ the morphism from U_σ' to $\mathscr{H}ilb_P^0$ for $T_\sigma^*(E)$, we get the following commutative diagram:

$$\begin{array}{ccccc} U' & \ni & s & \xrightarrow{q} & X_s \in \mathscr{H}ilb_P^0 \\ \downarrow & & & & \downarrow \\ U'_\sigma & \ni & \sigma(s) & \xrightarrow{q\sigma} & X_{\sigma(s)} \in \mathscr{H}ilb_P^0. \end{array}$$

(5.4.5) If $H^1(\mathscr{E}nd(E)) = 0$, say E is a direct sum of line bundles, then we observe from (5.4.1)

$$\dim T_{X_s, \mathscr{H}ilb_P} = \dim U' - \dim \mathrm{Aut}(E)/\mathbf{C}^* = \dim \overline{q(U')}.$$

Hence Theorem 14 implies that $\mathscr{H}ilb_P^0 = \overline{q(U')}$ and $\mathscr{H}ilb_P^0$ is smooth at the determinantal subvarieties. Therefore, the following is obtained.

COROLLARY 14. *If $H^1(\mathbf{P}^n, \mathscr{E}nd(E)) = 0$, then*

1) *$\mathscr{H}ilb_P^0$ coincides with $\overline{q(U')}$. In particular, $\mathscr{H}ilb_P^0$ is unirational and the set of the determinantal subvarieties associated to E is Zariski dense in $\mathscr{H}ilb_P^0$.*

2) *$\mathscr{H}ilb_P^0$ is smooth at the determinantal subvarieties associated to E.*

(5.4.6) G. Kempf [34] gave the following splitting theorem.

Theorem (G. Kempf) *Let E be a holomorphic vector bundle on $\mathbf{P}^n (n \geq 3)$ and let P be a projective hyperplane in $\mathbf{P}^n$. Then E splits if and only if*

$$H^1(P, \mathscr{E}nd(E)(-l)) = 0 \qquad \text{for any } l > 0.$$

As an application of our previous observations, we shall establish another similar splitting theorem for rank two vector bundles on $\mathbf{P}^n$ $(n \geq 4)$. Let P be a 4- or 5-dimensional projective linear subspace of $\mathbf{P}^n$ and let $\bar{E} = E|P$ be the restriction of E to P.

THEOREM 15. *E splits into line bundles if and only if $H^1(P, \mathscr{E}nd(\bar{E})) = 0$.*

PROOF. Since it is well-known that E splits if and only if $\bar{E}$ splits, we may assume that E is a rank two vector bundle on $\mathbf{P}^n$ (n being either 4 or 5). In addition after multiplying E by a suitable line bundle, we may assume that E is a very ample vector bundle enjoying the assumptions in Theorem 13. Suppose that $H^1(\mathscr{E}nd(E)) = 0$. Then Corollary 13 states that $\mathscr{H}ilb_P^0 = \overline{q(U')}$ and $\mathscr{H}ilb_P^0$ is smooth at the determinantal subvarieties associated to E. On the other hand, we see by (5.4.4.3) that $\mathscr{H}ilb_P^0$ is invariant under the action of $PGL(n+1, \mathbf{C})$. Hence it follows that $\overline{q(U')} = \overline{\sigma q(U')}$ for every element $\sigma \in PGL(n+1, \mathbf{C})$. Since $q(U')$ is a constructible set, there exist two elements $s, s' \in U'$ satisfying $X_s = \sigma(X_s) = X_{\sigma(s)}$. Consider the following exact sequences in Lemma 7 corresponding to s' and $\sigma(s)$, respectively:

$$0 \to E^* \otimes \mathcal{O}(-c_1) \to \bigoplus^3 \mathcal{O}(-c_1) \to I_{X_{s'}} \to 0.$$

$$0 \to T_\sigma^*(E^*) \otimes \mathcal{O}(-c_1) \to \bigoplus^3 \mathcal{O}(-c_1) \to I_{X_{\sigma(s)}} \to 0.$$

Then we observe by an argument similar to that in (5.4.4.2) that $T_\sigma^*(E)$ is isomorphic to E, which implies that E is a homogeneous bundle. Since every homogeneous bundle on $\mathbf{P}^n$ of rank $r < n$ is a direct sum of line bundles (cf. [27]), we can complete our proof.

(5.4.7) Remark 2. The proof of Theorem 15 tells us that the following inverse problem is a key to solving the splitting problem for a rank two vector bundle on $\mathbf{P}^4$ or $\mathbf{P}^5$.

The inverse problem. Let E be a rank two vector bundle on $\mathbf{P}^4$ (resp. $\mathbf{P}^5$), P the Hilbert polynomial of a determinantal surface (resp. a determinantal threefold) associated to E and let X be a general point of $\mathscr{H}ilb_P^0$. Then is X a determinantal surface (resp. determinantal threefold) associated to E?

Indeed, suppose that the answer to the above inverse problem is affirmative for E. Then $\mathscr{H}ilb_P^0$ coincides with $\overline{q(U')}$ and so our proof of Theorem 15 insures that E is a direct sum of line bundles.

See (6.1) for examples and their geometric structures of determinantal surfaces associated to rank two bundles on $\mathbf{P}^4$.

6. Geography of determinantal surfaces

The geography of algebraic surfaces and threefolds of general type has been extensively studied by many mathematicians (e.g., F. Catanese [5], [6], [7], B. Hunt [17], U. Persson [28], G. Xiao [42], [43], ...). As was shown in the previous sections, our determinantal surfaces and threefolds are simply connected and of general type in the general cases where they are not complete intersections. In this section, we shall describe how the geography of those surfaces look like and consequently we shall single out some new species in certain botanical gardens. Unfortunately we do not know any indecomposable rank two vector bundles on $\mathbf{P}^n$ $(n \geq 5)$. Hence we shall postpone describing the geography of determinantal threefolds to the future, although we can calculate the important invariants $c_1^3(T_X)$, $c_1(T_X)c_2(T_X)$, $c_3(T_X)$ for the geography of determinantal threefolds in terms of the Chern numbers $\{c_1, c_2\}$ of E.

(6.1) Determinantal surfaces.

Before describing the geography of determinantal surfaces, we shall explain algebro-geometric structures of determinantal surfaces by using the results in the previous sections.

(6.1.1) Ampleness of the divisor F on X (cf. (3.3)). Assume that X is a determinantal surface with $\rho(X) = 2$ (cf. Lemma 2). Suppose that F is not ample, i.e., $\varphi : X \to \mathbf{P}^2$ is not a finite morphism.

(1). Let $\Sigma = \{y \in \mathbf{P}^2 \mid \dim \varphi^{-1}(y) = 1\}$. Then the cardinality of Σ is equal to 1. Indeed, let W_1, W_2 be irreducible curves such that $y_i = \varphi(W_i)$ $(i = 1, 2)$ and $y_1 \neq y_2$. Since $F \cdot W_i = 0$, it follows that $W_i^2 < 0$ by the Hodge index theorem and so the intersection matrix

$$\begin{bmatrix} F^2 & 0 & 0 \\ 0 & W_1^2 & 0 \\ 0 & 0 & W_2^2 \end{bmatrix}$$

is non-singular. Thus $\{F, W_1, W_2\}$ are linearly independent in $\mathrm{Pic}(X) \otimes \mathbf{R}$. This contradicts $\rho(X) = 2$.

(2) Let y be the unique point of Σ and W an irreducible curve contained in $\varphi^{-1}(y)$. Then by an argument similar to that in (1), W is a unique irreducible curve in $\varphi^{-1}(y)$. Hence W is determined uniquely.

(3) Since $\{F, H\}$ is a basis of $\mathrm{Pic}(X)$, let us put $W = xF + yH$ $(x, y \in \mathbf{Z})$. Then we have $0 = FW = xF^2 + yFH = xc_2^2 + yc_1c_2$ and hence $y = -(c_2/c_1)x$. On the other hand, we see that $0 < WH = x(FH - (c_2/c_1)H^2) = x(c_1c_2 - c_2(c_1^2 - c_2)/c_1) = (c_2^2/c_1)x$ and so $W = q(c_1F - c_2H)$ for a positive rational number q.

(4) Conversely if there exists an effective member of the complete linear system $|n(c_1F - c_2H)|$ for some positive integer n, then it is easily seen that F is not ample. Therefore we obtain the following.

PROPOSITION 4. *Under the above assumption, F is ample if and only if the complete linear system $|n(c_1F - c_2H)|$ is empty for all positive integers n.*

(6.1.2) Remark 3. Let us put $W = c_1F - c_2H = -c_1D + (c_1^2 - c_2)H$ which may not be an effective curve in general. If F is not ample, then it follows from Proposition 4 that $|nW| \neq \varnothing$ for a positive integer n and so the extremal ray $\ell_2 = \mathbf{R}_{\geq 0}[W]$ (cf. (3.4.1)) is realized by an effective curve and $\theta_2 = (c_1^2 - c_2)/c_1$.

(6.1.3) Let E be a very ample rank two bundle on $\mathbf{P}^4$ and X a determinantal surface associated to E. The Horrocks-Mumford bundle E_{HM} [16] is essentialy the only known indecomposable rank two bundle on $\mathbf{P}^4$. Hence according to the following cases, we shall show algebro-geometric structures of the determinatal surfaces X.

1) $E = \mathcal{O}(a) \oplus \mathcal{O}(a)$ $(a \geq 1)$,
2) $E = \mathcal{O}(b) \oplus \mathcal{O}(a)$ $(1 \leq b < a)$,
3) $E = E_{\mathrm{HM}}(a)$ $(a \geq 2)$.

In each case, it is easily seen that a is the largest integer such that $H^0(\mathbf{P}^4, E(-a)) \neq 0$.

(6.1.4) $E = \mathcal{O}(a) \oplus \mathcal{O}(a)$ $(a \geq 1)$.

1) In this case, $P(E)$ is isomorphic to $\mathbf{P}^4 \times \mathbf{P}^1$. Let $\{s_i = (s_{i1}, s_{i2}) \mid s_{ik} \in H^0(\mathbf{P}^4, \mathcal{O}(a)),\ 1 \leq i \leq 3, 1 \leq k \leq 2\}$ be a set of sufficiently general three sections of E which satisfies the condition $(*)$ in (1.1). Then Y is defined by the following equations:

$$Y : s_{i1}X_1 + s_{i2}X_2 = 0 \qquad (1 \leq i \leq 3),$$

where $\{X_1, X_2\}$ is a system of parameters of homogeneous coordinates of $\mathbf{P}^1$. Since $c_1 = 2a$ and $E(-a) \simeq \bigoplus^2 \mathcal{O}_{\mathbf{P}^4}$, we observe that the second projection $P(E) \to \mathbf{P}^1$ coincides with the morphism defined by the complete linear system $|D - a\pi^*(H)| = |D - (c_1 - a)\pi^*(H)|$.

2) Since Z (resp. Z^*) is the restriction of $D - a\pi^*(H)$ (resp. $D - (c_1 - a)\pi^*(H)$) to X, the complete linear system $|Z| = |Z^*|$ satisfies the following.

a) $|Z|$ is base points free, $Z^2 = 0$ and $\dim |Z| = 1$.

If we denote by $\psi : X \to \mathbf{P}^1$ the morphism defined by $|Z|$, then ψ is the restriction of the above second projection to X. Hence every member of $|Z|$ is a fiber of ψ and so it has the following equations:

$$s_{i1}\eta_1 + s_{i2}\eta_2 = 0 \qquad (1 \leq i \leq 3),$$

where (η_1, η_2) is a point of $\mathbf{P}^1$.

b) Since $K_X = D + (2a - 5)H$, general members of $|Z|$ are smooth curves with genus $g(Z) = 1 + (3a - 5)a^3/2$.

c) Every member of $|Z|$ is irreducible because $\theta_1 = a$ by Corollary 5. Thus $\psi : X \to \mathbf{P}^1$ determines an irreducible fiber space structure of X over $\mathbf{P}^1$.

$$\begin{array}{ccc} X & \xrightarrow{\psi} & \mathbf{P}^1 \\ \downarrow{\scriptstyle \varphi} & & \\ \mathbf{P}^2 & & \end{array}$$

3) If $a = 1$, then $\psi : X \to \mathbf{P}^1$ is a $\mathbf{P}^1$-bundle over $\mathbf{P}^1$ and moreover X is isomorphic to the rational ruled surface $F_1 = P(\mathcal{O} \oplus \mathcal{O}(1))$. In addition it follows that the (-1)-curve W coincides with $-2D + 3H = 2F - H$ and the birational morphism $\varphi : X \to \mathbf{P}^2$ is the blowing-up of $\mathbf{P}^2$ with one point as the center. Hence F is not ample and $\overline{NE}_1(X) = NE_1(X)$ has the following extremal rays:

$$\ell_1 = \mathbf{R}_{\geq 0}[Z], \qquad \ell_2 = \mathbf{R}_{\geq 0}[W].$$

4) If $a \geq 2$, then $K_X = D + (2a - 5)H$ is very ample. Hence $\psi : X \to \mathbf{P}^1$

is an irreducible fiber space with smooth non-hyperelliptic curves Z of genus $g(Z) = 1 + (3a-5)a^3/2$ as general fibers. In this case, we prove that F is ample. Assuming that there exists a unique irreducible curve W that is mapped to the point y in $\mathbf{P}^2$, we derive a contradiction. Since $c_1F - c_2H = a(-2D + 3aH)$, we find $W = q(-2D + 3aH)$ with a positive integer q when a is odd (resp. a half integer q when a is even). Hence we consider the following cases.

(1) a is odd. Take a line ℓ passing through the point y. Then $\varphi^*(\ell) = Z' + \alpha W$ ($a \in \mathbf{N}$) where Z' is an effective curve, which implies that $F - \alpha W = (2\alpha q - 1)D - (3\alpha q - 2)aH$ is effective. Hence we get $\alpha = q = 1$ because $(3\alpha q - 2/(2\alpha q - 1) \le 1$ by Corollary 5 and so we obtain $W = -2D + 3aH = F - Z$, i.e., $Z + W \in |F|$. Since every member of $|F|$ is a pull-back of a line under φ by Corollary 3, $\varphi(Z)$ is a line for every fiber Z of ψ. On the other hand, the morphism $\varphi : X \to \mathbf{P}^2$ is defined as follows:

$$\varphi : X \ni x \to (s_{11}(x) : s_{21}(x) : s_{31}(x)) = (s_{12}(x) : s_{22}(x) : s_{32}(x)) \in \mathbf{P}^2.$$

Therefore it follows from the Hilbert Nullstellensatz that the six sections $\{s_{ij} \in H^0(\mathbf{P}^4, \mathcal{O}(a)) \mid 1 \le i \le 3, 1 \le j \le 2\}$ are linearly dependent over k. However when $a \ge 2$, they are linearly independent by the choice of those sections because of $\dim H^0(\mathbf{P}^4, \mathcal{O}(a)) \ge 15$, which is a contradiction.

(2) a is even. Let $a = 2d$ ($d \in \mathbf{N}$). Then we find $W = q(-D + 3dH)$ ($q \in \mathbf{N}$). By an argument similar to that in (1), it follows that $q = 1$ or 2 and that we have either $F - W = dH$ or $F - 2W = Z$ for $q = 1$ and $F - W = Z$ for $q = 2$. In (1), it was proved that $F - Z$ is not effective. If $F = dH + W$, then we have $\dim H^0(\mathcal{O}_X(F)) \ge \dim H^0(\mathcal{O}_X(H))$, which is a contradiction because of $3 = \dim H^0(\mathcal{O}_X(F))$ by Corollary 3 and $\dim H^0(\mathcal{O}_X(H)) = 5$ by Corollary 6.

(6.1.5) $E = \mathcal{O}(b) \oplus \mathcal{O}(a)$ $(1 \le b < a)$.

1) Since $E(-a) = \mathcal{O}(b-a) \oplus \mathcal{O}$, it follows that $\dim|D - a\pi^*(H)| = 0$ and the unique effective member of $|D - a\pi^*(H)|$ which we denote by Γ is the section of $\pi : P(E) \to \mathbf{P}^4$ corresponding to the quotient line bundle $\mathcal{O}(b)$. In addition since $E(-b) = \mathcal{O} \oplus \mathcal{O}(a-b)$ is generated by global sections and $H^0(E(-b))$ has a basis $\{(1,0), (0, X_0^{\alpha_0} \cdots X_4^{\alpha_4}) \mid \sum_0^4 \alpha_i = a - b, \alpha_i \ge 0\}$ where $\{X_0, X_1, \dots, X_4\}$ is a system of homogeneous coordinates of $\mathbf{P}^4$, $|D - b\pi^*(H)|$ determines a morphism $\psi : P(E) \to \mathbf{P}^N$ such that:

(1) $N = \begin{pmatrix} a-b+4 \\ a-b \end{pmatrix}.$

(2) Let $U_0 = \{X_0 \ne 0\}$ be an affine open subset of $\mathbf{P}^4$ and $\{\xi_1, \xi_2\}$ a system of homogeneous coordinates of $\mathbf{P}^1$. Then ψ is expressed on $\pi^{-1}(U_0) \simeq U_0 \times \mathbf{P}^1$ as follows:

$$U_0 \times \mathbf{P}^1 \ni \left(\left(\frac{X_1}{X_0}, \ldots, \frac{X_4}{X_0}\right), (\xi_1 : \xi_2)\right) \to \left(\xi_1 : \xi_2 : \frac{X_1}{X_0}\xi_2 : \cdots : \frac{X_4^{a-b}}{X_0^{a-b}}\xi_2\right) \in \mathbf{P}^N$$

and Γ is defined by $\xi_2 = 0$. Hence it follows that $\psi(\Gamma) = (1 : 0 : \cdots : 0)$ and ψ is an immersion outside of Γ. Let Ξ be the divisor corresponding to the quotient line bundle $\mathcal{O}(a)$ of E. Then Ξ is a member of $|D - b\pi^*(H)|$ and Ξ is defined by $\xi_1 = 0$ on $\pi^{-1}(U_0)$ and hence $\psi : \Xi \to \mathbf{P}^{N-1}$ is nothing but the Veronese embedding of $\mathbf{P}^4$ of degree $(a-b)^4$. Thus $\psi(P(E))$ is the cone having $O = (1 : 0 : \cdots : 0)$ as the vertex and $\psi(P(E))$ as the base variety. Moreover since $\mathcal{O}_\Gamma(D) \simeq \mathcal{O}_\Gamma(bH)$, it is easily seen that $R^i\psi_*(\mathcal{O}_{P(E)})_O = 0$ $(i \geq 1)$, i.e., O is a rational singular point with multiplicity $(a-b)^4$.

2) We have $H^0(\mathcal{O}_{(P(E))}(D - b\pi^*(H)) \simeq H^0(\mathcal{O}_X(D - bH))$ by Theorem 3 and Z^* is the intersection of Ξ with X. Thus:

a) $|Z^*|$ is free from base points, $(Z^*)^2 = a^3(a-b)$ and the morphism defined by $|Z^*|$ is the restriction of ψ to X which we denote also by ψ.

b) Since Z is the intersection of Γ with three generic very ample divisors, Z is a smooth curve with $Z \cdot Z^* = 0$, $Z^2 = -(a-b)b^3$ and genus $g(Z) = 1 + (3b-5)b^3/2$. Z is mapped to the point O under ψ and ψ is an isomorphism outside of Z. Hence ψ is a contraction morphism of X along Z to a normal surface $\psi(X)$.

c) We have $R^1\psi_*(\mathcal{O}_X)_O = \varprojlim H^1(\mathcal{O}_{nZ})$. Consider the exact sequence

$$0 \to \mathcal{O}_Z(-nZ) \to \mathcal{O}_{(n+1)Z} \to \mathcal{O}_{nZ} \to 0.$$

This gives rise to an exact sequence

$$\to H^0(\mathcal{O}_{(n+1)Z}) \to H^0(\mathcal{O}_{nZ}) \to H^1(\mathcal{O}_Z(-nZ)) \to H^1(\mathcal{O}_{(n+1)Z}) \to H^1(\mathcal{O}_{nZ}) \to 0.$$

Hence it follows that $\dim R^1\psi_*(\mathcal{O}_X)_O = g(Z)$ if and only if $H^1(\mathcal{O}_Z(-Z)) = 0$. Since $\mathcal{O}_Z(D) \simeq \mathcal{O}_Z(bH)$, we get that $H^1(\mathcal{O}_Z(-Z)) \simeq H^0(\mathcal{O}_Z(2Z + K_X)) = H^0(\mathcal{O}_Z((4b - a - 5H)) = 0$ is equivalent to $b < (a+5)/4$. Summing up the above, we see that O is a rational singular point if and only if $b = 1$. In addition when $a = 2$ and $b = 1$, Z is a (-1)-curve and ψ is the blowing-up of a smooth K3 surface $\psi(X)$ with center O that is a complete intersection of three quadrics in $\mathbf{P}^5$. Since $K_X = D + (a+b-5)H$, X is of general type except for the case $a = 2$, $b = 1$.

3) We prove that F is also ample in these cases. As in the proof in (6.1.5) 4), let W be the irreducible curve satisfying $\varphi(W) = y$. Let $a = ea'$ and $b = eb'$ where e is the greatest common divisor of a and b. Then we see $W = q\{-(a'+b')D + e(a'^2 + a'b' + b'^2)H\}$ with a positive integer q, provided $a' + b'$ and e are relatively prime. By an argument similar to that in (6.1.4) 4), it follows that $q = b' = 1$ and $a'Z + W \in |F|$. Since there exists an exact

sequence

$$0 \to H^0(\mathcal{O}_X(a'Z)) \to H^0(\mathcal{O}_X(F)) \to H^0(\mathcal{O}_W) \to,$$

we get $3 = \dim H^0(\mathcal{O}_X(F)) \le \dim H^0(\mathcal{O}(a'Z)) + 1$, which is a contradiction because of $\dim H^0(\mathcal{O}_X(rZ)) = 1$ for all positive integers r. In the remaining case, we can also derive a contradiction similarly.

(6.1.6) $E = E_{\mathrm{HM}}(a)$ $(a \ge 2)$.

(6.1.6.1) 1) Let $E = E_{\mathrm{HM}}(a)$ $(a \in \mathbf{Z})$ where $E = E_{\mathrm{HM}}$ is the Horrocks-Mumford bundle [16] that is a stable rank two vector bundle of $\mathbf{P}^4$ with $c_1(E_{\mathrm{HM}}) = 5$, $c_2(E_{\mathrm{HM}}) = 10$. First we shall show that E is very ample if and only if $a \ge 2$. To this end, we shall recall the following on the Horrocks-Mumford bundle (cf. [3], [4], [38]).

(1) We can construct E_{HM} geometrically by elementary transformations of algebraic vector bundles as follows. Consider the following quintic hypersurface X in $\mathbf{P}^4$:

$$X : \sum_{i=0}^{4} X_i^5 - 5\prod_{i=0}^{4} X_i = 0,$$

where $\{X_0, \dots, X_4\}$ is a system of homogeneous coordinates of $\mathbf{P}^4$. Then we observe that

$$\mathrm{Sing}(X) = \left\{ (\xi^{a_0} : \cdots : \xi^{a_4}) \in \mathbf{P}^4 \;\middle|\; \sum_{i=0}^{4} a_i \equiv 0 \pmod 5 \right\},$$

where $\mathrm{Sing}(X)$ is the singular locus of X and $\xi = \exp(2\pi i/5)$ is a primitive fifth root of unit and $|\mathrm{Sing}(X)| = 125$.

Consider the diagonal subgroup $B = \{(\xi^{a_0}, \dots, \xi^{a_4}) \mid \sum_{i=0}^{4} a_i \equiv 0 \pmod 5\}$ of $\mathrm{PGL}(5, \mathbf{C})$. Then B is isomorphic to $\bigoplus^3 \mathbf{Z}/(5)$ and is normalized by the symmetric group S_5 of degree 5. If we define $G = B \cdot S_5$ (semi-direct product) $(|G| = 15{,}000)$, then we see that G leaves X invariant and it acts on $\mathrm{Sing}(X)$ transitively. We represent elements σ, τ, μ of $\mathrm{SL}(5, \mathbf{C})$ by matrices:

$$\sigma = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 & 0 \\ 0 & 0 & \xi^2 & 0 & 0 \\ 0 & 0 & 0 & \xi^3 & 0 \\ 0 & 0 & 0 & 0 & \xi^4 \end{pmatrix}, \quad \mu = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then one can easily verify the following relations:

$$\sigma^{-1}\tau\sigma = \xi\tau, \qquad \mu^{-1}\tau\mu = \tau^2, \qquad \tau^{-1}\sigma\tau = \xi^4\sigma, \qquad \mu^{-1}\sigma\mu = \sigma^3.$$

Let σ_i be the i-th elementary symmetric function with respect to $\{X_0, \dots, X_4\}$ $(0 \le i \le 4)$. Since the defining equation of X is expressed as

$$\sum_{i=0}^{4} X_i^5 - 5\prod_{i=0}^{4} X_i = \sigma_1 \sum X_i^4 - \sigma_2 \sum X_i^3 + \sigma_3 \sum X_i^2 - \sigma_4 \sum X_i,$$

X contains a smooth quadric surface $S : \sigma_1 = \sigma_2 = 0$.

$W := \sum_{i=0}^{4} \tau^i(S)$ is an effective Weil divisor of X of degree 10. Then the complete linear system $|W|$ of W satisfies the following properties:

(a) $\dim |W| = 1$.

(b) The base locus $\mathrm{Bs}(|W|)$ is contained in $\mathrm{Sing}(X)$ $(|\mathrm{Bs}(|W|)| = 100)$.

(c) General members of $|W|$ are abelian surfaces.

Hence the above observations insure that the triple $(\mathbf{P}^4, X, |W|)$ satisfies the conditions for the elementary transformatios of algebraic vector bundles [39] and so there exist a rank two stable bundle which is isomorphic to the Horrocks-Mumford bundle E_{HM} with G as symmetries and two sections $\{s_1, s_2\}$ $(s_i \in H^0(\mathbf{P}^4, E_{\mathrm{HM}}),\ i = 1, 2)$ such that

$$X = Z(s_1 \wedge s_2) : \text{the zero locus of } s_1 \wedge s_2,$$

$$W = Z(s_1) : \text{the zero locus of } s_1.$$

Accordingly we obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{P}^4} \xrightarrow{s_1} E_{\mathrm{HM}} \xrightarrow{s_1 \wedge} I(W) \otimes \mathcal{O}(5) \longrightarrow 0,$$

where $I(W)$ is the defining ideal sheaf of W. In addition, $Z(s) \in |W|$ is an abelian surface for a general section $s \in H^0(\mathbf{P}^4, E_{\mathrm{HM}})$.

(2) Let $\{Y_i\}$ $(0 \le i \le 4)$ be the following new system of homogeneous coordinates of $\mathbf{P}^4$:

$$Y_i = \sum_{j=0}^{4} \xi^{-ij} X_j \qquad (0 \le i \le 4).$$

Then it follows that each $S_i = \tau^i(S)$ is defined by equations

$$Y_i = Y_{i+1} Y_{i-1} + Y_{i+2} Y_{i-2} = 0$$

and $H^0(I(W)(5))$ is a 3-dimensional vector space with a basis $\{f_0, f_1, f_2\}$, where

$$f_0 = Y_0 Y_1 Y_2 Y_3 Y_4, \qquad f_1 = \sum_{i=0}^{4} \tau^i(Y_1^2 Y_3 (Y_1 Y_4 + Y_2 Y_3)),$$

$$f_2 = \sum_{i=0}^{4} \tau^i(Y_1 Y_2^2 (Y_1 Y_4 + Y_2 Y_3)).$$

If we put Y to be the closed subscheme of $\mathbf{P}^4$ defined by

$$f_0 = f_1 = f_2 = 0,$$

then we observe that Y has the following 30 irreducible components:

(a) S_i $(0 \leq i \leq 4)$,

(b) $\ell_{ij} : \tau^j(Y_0) = 0,\ \ \tau^j(Y_1) = \varepsilon_i^3\tau^j(Y_4),\ \ \tau^j(Y_2) = \varepsilon_i\tau(Y_3)$ $(0 \leq i \leq j \leq 4)$,
where $\varepsilon_i^5 = -1$ $(0 \leq i \leq 4)$.

The last 25 lines are called the HM—*lines* (Horrocks-Mumford lines).

Consider the following commutative diagram of exact sequences:

$$(*) \qquad \begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & & \downarrow & & & & \\ & & & & \mathcal{O}_{\mathbf{P}^4} & & & & \\ & & & & \downarrow & & & & \\ 0 & \to & F & \to & E_{\mathrm{HM}} & \to & E_{\mathrm{HM}}/F & \to & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J(W)(5) & \to & I(W)(5) & \to & I(W)/J(W)(5) & \to & 0 \\ & & & & \downarrow & & & & \\ & & & & 0 & & & & \end{array}$$

where F is the subsheaf of E_{HM} generated by global sections of E_{HM} (resp. $J(W)(5)$ is the subsheaf of $I(W)(5)$ generated by global sections of $I(W)(5)$). Then since the diagram $(*)$ yields an isomorphism $E_{\mathrm{HM}}/F \simeq I(W)/J(W)(5)$, it turns out that $\mathrm{Supp}(E_{\mathrm{HM}}/F)$ consists of the above HM-lines. Consequently we find that E_{HM} is generated by global sections outside of those 25 HM-lines (cf. [16]).

(3) There exists the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \mathcal{O}(1) & \to & G & \to & J(W)(6) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \mathcal{O}(1) & \to & E_{\mathrm{HM}}(1) & \to & I(W)(6) & \to & 0, \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & E_{\mathrm{HM}}(1)/G & \simeq & I(W)(6)/J(W)(6) & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

where G is the subsheaf of $E_{\mathrm{HM}}(1)$ generated by global sections of $E_{\mathrm{HM}}(1)$ (resp. $J(W)(6)$ is the subsheaf of $I(W)(6)$ generated by global sections). Hence $\mathrm{Supp}\,(E_{\mathrm{HM}}(1)/G) = \mathrm{Supp}(I(W)(6)/J(W)(6))$. Since $E_{\mathrm{HM}}(1)$ is generated by global sections outside of the 25 HM-lines, $\mathrm{Supp}(I(W)(6)/J(W)(6))$, which is invariant under the action of τ, is contained in $\bigcup \ell_{ij}$.

It is known that $H^0(I(W)(6))$ is a 30-dimensional vector space with a basis $\{\tau^i(g_j)\}$ $(0 \le i \le 4,\ 0 \le j \le 5)$, where

$$g_j = Y_0 f_j \qquad (0 \le j \le 2),$$

$$g_3 = Y_0 Y_1 Y_4 (Y_0 Y_1 Y_4 + Y_1^2 Y_3 + Y_2 Y_4^2),$$

$$g_4 = Y_0 Y_2 Y_3 (Y_0 Y_2 Y_3 + Y_1 Y_2^2 + Y_3^2 Y_4),$$

$$g_5 = Y_1 Y_2 Y_3 Y_4 (Y_1 Y_4 + Y_2 Y_3).$$

Hence by an argument similar to that in (1), $\mathrm{Supp}(I(W)(6)/J(W)(6)) = \emptyset$, i.e., $E_{\mathrm{HM}}(1)$ is generated by global sections.

(4) For a line ℓ in $\mathbf{P}^4$, let us put

$$E_{\mathrm{HM}}|\ell = \mathcal{O}(2-k) \oplus \mathcal{O}(3+k) \qquad (k \ge 0).$$

Then it is known that $k = 0$ for a generic line ℓ by the Grauert-Mühlich theorem and that $k \ge 3$, i.e., ℓ is a triple jumping line if and only if ℓ is one of the HM-lines by [3]. Hence we observe that $0 \le k \le 3$ for every line ℓ and $k = 3$ if and only if ℓ is an HM-line because $E_{\mathrm{HM}}(1)$ is generated by global sections. In addition, we see that $E_{\mathrm{HM}}(1)$ is 1-ample in the sense of A. Sommese and $E = E_{\mathrm{HM}}(a)$ is very ample if and only if $a \ge 2$.

Summing up the above, we obtain the following.

PROPOSITION 5. *$E_{\mathrm{HM}}(1)$ is generated by global sections and it is 1-ample in the sense of A. Sommese. Hence $E = E_{\mathrm{HM}}(a)$ is very ample if and only if $a \ge 2$.*

(6.1.6.2) We shall investigate the geometric structures of the determinantal surfaces associated to $E = E_{\mathrm{HM}}(a)$ $(a \ge 2)$.

1) The following are known (cf. [16]):

a) $\dim H^0(\mathbf{P}^4, E_{\mathrm{HM}}) = 4$, $\dim H^0(\mathbf{P}^4, E_{\mathrm{HM}}(-1)) = 0$ and the zero locus $W(s)$ is an abelian surface for a general section $s \in H^0(\mathbf{P}^4, E_{\mathrm{HM}})$.

b) E_{HM} is generated by global sections outside of 25 HM-lines $\{\ell_{ij}\}$ and $E_{\mathrm{HM}} \mid \ell_{ij} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(6)$.

Let $\pi : P(E_{\mathrm{HM}}) \to \mathbf{P}^4$ be the structure morphism, D_{HM} the tautological divisor of $P(E_{\mathrm{HM}})$ and let ℓ'_{ij} be the section of $\pi_{ij} : P(E_{\mathrm{HM}}|\ell_{ij}) \to \ell_{ij}$ corresponding to the quotient line bundle: $E_{\mathrm{HM}}|\ell_{ij} \to \mathcal{O}(-1)$. Then we find that generic members of the complete linear system $|D_{\mathrm{HM}}|$ are irreducible and smooth and the base point locus $\mathrm{B_s}(|D_{\mathrm{HM}}|)$ of $|D_{\mathrm{HM}}|$ coincides with $\bigcup \ell'_{ij}$. Hence if we denote by ψ the dominant rational map $P(E) \to \mathbf{P}^3$ associated to $|D_{\mathrm{HM}}|$, then ψ is regular on $P(E_{\mathrm{HM}}) \setminus \bigcup \ell'_{ij}$.

2) Let us put $D = D_{\mathrm{HM}} + a\pi^*(H)$ $(a \geq 2)$ which is a very ample divisor by Proposition 5. If we denote by X a determinantal surface associated to $E = E_{\mathrm{HM}}(a)$, then:

(1) $K_X = D + 2aH$ is very ample.

(2) $\dim|Z| = 3$, $Z^2 = 5(a^3 + 9a^2 + 15a - 5) > 0$ and a generic member of $|Z|$ is an irreducible smooth curve of genus $1 + (3a^4 + 55a^3 + 225a^2 + 225a - 50)/2$ because Z is the restriction of D_{HM} to X and $c_1 = 2a + 5$ and $c_2 = a^2 + 5a + 10$.

(3) The base point locus $\mathrm{B_s}(|Z|)$ of $|Z|$ consists of at most finitely many points. Hence $|Z|$ defines a rational map:

$$\psi_{|Z|} : X \to \mathbf{P}^3,$$

which is the restriction of ψ to X. Indeed, suppose that some ℓ'_{ij} is contained in X. Then we find that $Z \cdot \ell'_{ij} = -1$, which contradicts $Z^2 > 0$. Hence it follows from 1) that $\mathrm{B_s}(|Z|)$ is at most a finite set. If we take a sufficiently general X, then $\psi_{|Z|}$ is a regular morphism because $\mathrm{B_s}(|Z|) = \varnothing$.

(6.1.6.3) We shall describe the geometric structure of $\mathscr{H}ilb_P^0$ for $E = E_{\mathrm{HM}}(a)$ $(a \geq 2)$. It is known (cf. [8]) that

$$\dim H^1(\mathscr{E}nd(E)) = 24, \qquad \dim H^2(\mathscr{E}nd(E)) = 2.$$

Hence we have

$$\dim H^0(N) = \dim q(U') + 24, \qquad \dim H^1(N) \leq 2.$$

Consider the canonical morphism

$$\xi : PGL(5, \mathbf{C}) \times q(U') \ni (\sigma, X_s) \to \sigma(X_s) \in \mathscr{H}ilb_P^0.$$

Then ξ is a quasi-finite morphism and so $\dim \xi(PGL(5, \mathbf{C}) \times q(U')) = \dim q(U') + 24$ because $\{\sigma \in PGL(5, \mathbf{C}) \mid T_\sigma^*(E_{\mathrm{HM}}) \simeq E_{\mathrm{HM}}\}$ is a finite group. Thus we find that $\mathscr{H}ilb_P^0$ coincides with $\overline{\xi(PGL(5, \mathbf{C}) \times q(U')}$ and it is smooth at all determinantal surfaces associated to $T_\sigma^*(E)$ for any $\sigma \in PGL(5, \mathbf{C})$.

(6.2) The geography of determinantal surfaces.

Let $c_1(X) = c_1(T_X)$ be the anti-canonical divisor and $c_2(X) = c_2(T_X)$ the Euler-characteristic of a smooth projective surface X. Then $c_1^2(X)$ and $\chi(X) = (c_1^2(X) + c_2(X)/12$ are two fundamental invariants for the geography of surfaces of general type.

(6.2.1) We have shown in Theorem 11 that

$$c_1(X) = -D - (c_1 - 5)H, \qquad c_2(X) = -(5 - 3c_1)DH + (10 - 5c_1)H^2.$$

Hence $c_1^2(X)$, $c_2(X)$ and $\chi(X)$ are described in terms of the Chern numbers $\{c_i\}$ $(i=1,2)$ of E as follows.

LEMMA 10. 1) $c_1^2(X)=4c_1^4-20c_1^3-8c_1^2c_2+25c_1^2+30c_1c_2+c_2^2-25c_2$.
2) $c_2(X)=3c_1^4-10c_1^3-6c_1^2c_2+10c_1^2+15c_1c_2-10c_2$.
3) $\chi(X)=(7c_1^4-30c_1^3-14c_1^2c_2+35c_1^2+45c_1c_2+c_2^2-35c_2)/12$.
In particular we observe that $c_1^2(X)$, $c_2(X)$ and $\chi(X)$ depend only on the Chern numbers of E.

(6.2.2) REMARK 4. According to the cases 1) $E=\mathcal{O}(a)\oplus\mathcal{O}(b)$ $(a,b\in\mathbf{N})$ and 2) $E=E_{\mathrm{HM}}(a)$ $(a\geq 2)$, the above invariants in Lemma 9 can be calculated explicitly as the following.

1) $E=\mathcal{O}(a)\oplus\mathcal{O}(b)$.

$$c_1^2(X)=4a^4+8a^3b+9a^2b^2+8ab^3+4b^4-20a^3-30a^2b-30ab^2 -20b^3+25a^2+25ab+25b^2,$$

$$c_2(X)=3a^4+6a^3b+6a^2b^2+6ab^3+3b^4-10a^3-15a^2b-15ab^2 -10b^3+10a^2+10ab+10b^2,$$

$$\chi(X)=(7a^4+14a^3b+15a^2b^2+14ab^3+7b^4-30a^3-45a^2b-45ab^2-30b^3 +35a^2+35ab+35b^2)/12.$$

2) $E=E_{\mathrm{HM}}(a)$.

$$c_1^2(X)=33a^4+230a^3+450a^2+225a-25,$$

$$c_2(X)=24a^4+190a^3+465a^2+375a+25,$$

$$\chi(X)=(57a^4+420a^3+915a^2+600a)/12.$$

(6.2.3) Let $\mathbf{R}^2$ be the Euclidean plane with the coordinates $(\chi(X),c_1^2(X))$. Then it is known that surfaces of general type enjoy the following inequalities:

a) $c_1^2(X)>0$, $c_2(X)>0$, $\chi(X)>0$,

b) $c_1^2(X)\geq 2\chi(X)-6$. The line $c_1^2=2\chi(X)-6$ is called the Noether line.

c) $c_1^2(X)\leq 9\chi(X)$. The line $c_1^2(X)=9\chi(X)$ is called the Yau-Miyaoka line.

d) If X is a complete intersection, then $c_1^2(X)\leq 8\chi(X)$.

e) If X admits a genus two fibration, then $c_1^2(X)\leq 8\chi(X)$.

We shall study the geography of invariants of determinantal surfaces of general type according to whether 1) $c_1^2-4c_2\geq 0$ or 2) $c_1^2-4c_2<0$, where c_i $(i=1,2)$ is the i-th Chern number of E.

1) $c_1^2 - 4c_2 \geq 0$.

a) By Lemma 10, we find that

$$19c_1^2(X) - 132\chi(X) = 8c_2^2 + (2c_1^2 + 75c_1 - 90)c_2 - (c_1^4 + 50c_1^3 - 90c_1^2).$$

Hence let us put

$$\begin{aligned} f(c_2) &= 8c_2^2 + (2c_1^2 + 75c_1 - 90)c_2 - (c_1^4 + 50c_1^3 - 90c_1^2) \\ &= 8\{c_2 + (2c_1^2 + 75c_1 - 90)/16\}^2 \\ &\quad - (36c_1^4 + 1900c_1^3 + 2385c_1^2 - 13500c_1 + 8100)/32. \end{aligned}$$

Then we observe

$$f\left(\frac{c_1^2}{4}\right) = -\frac{125}{4}c_1^2\left(c_1 - \frac{54}{25}\right) < 0 \qquad \text{for } c_1 \geq 3.$$

Hence it follows that $f(c_2) < 0$ for $c_1 \geq 3$, which implies that

$$c_1^2(X) < \frac{132}{19}\chi(X) \qquad \text{when } c_1 \geq 3.$$

b) In addition for a natural number n, consider the determinantal surfaces X' associated to $E' = E(n)$. Then since we have

$$c_1' = c_1 + 2n, \qquad c_2' = c_2 + c_1 n + n^2,$$

where c_i' $(i = 1, 2)$ is the i-th Chern number of E', they yield that

$$c_1^2(X') = 33n^4 + (\text{lower order term}), \qquad c_2(X') = 24n^4 + (\text{lower order term}),$$

$$\chi(X') = \frac{57}{12}n^4 + (\text{lower order term}),$$

which tell us that $\lim_{n\to\infty} c_1^2(X')/\chi(X') = 132/19$. Therefore we find that the line $c_1^2(X) = 132/19\chi(X)$ is an asymptote of the geography of determinantal surfaces.

c) It is easily checked that

$$c_1^2(X) - 4\chi(X) = \tfrac{1}{3}\{2c_2^2 - (10c_1^2 - 45c_1 + 40)c_2 + (5c_1^4 - 30c_1^3 + 40c_1^2)\}.$$

Hence if we put

$$g(c_2) = 2c_2^2 - (10c_1^2 - 45c_1 + 40)c_2 + (5c_1^4 - 30c_1^3 + 40c_1^2),$$

then

$$g\left(\frac{c_1^2}{4}\right) = \frac{3}{8}c_1^2(7c_1^2 - 50c_1 + 80) > 0 \qquad \text{for } c_1 \geq 5,$$

which implies that

$$4 < c_1^2(X)/\chi(X) \qquad \text{for } c_1 \geq 5.$$

In the cases $3 \leq c_1 \leq 4$, it follows from Corollary 10 that $c_1^2(X)/\chi(X) = 54/17 = 3.1764\ldots$ when $E = \mathcal{O}(1) \oplus \mathcal{O}(3)$ and $c_1^2(X)/\chi(X) = 28/11 = 2.5454\ldots$ when $E = \mathcal{O}(2) \oplus \mathcal{O}(2)$.

Summing up the above, we obtain the following.

THEOREM 16. *If $c_1^2 - 4c_2 \geq 0$, then we have $4\chi(X) < c_1^2(X) < (132/19)\chi(X)$ except for the cases $E = \mathcal{O}(1) \oplus \mathcal{O}(3)$ and $E = \mathcal{O}(2) \oplus \mathcal{O}(2)$. The line $c_1^2(X) = (132/19)\chi(X)$ is an asymptote of the geography of determinantal surfaces.*

2) $c_1^2 - 4c_2 < 0$.

a) Let us put

$$\mathscr{E} = \{E | E \text{ is a rank two very ample bundle on } \mathbf{P}^4 \text{ with } c_1^2 - 4c_2 < 0\}/\text{isom}.$$

and define a real number θ_3 as follows:

$$\theta_3 = \sup_{E \in \mathscr{E}} \left\{ \frac{12(4c_1^4 - 8c_1^2 c_2 + c_2^2)}{7c_1^4 - 14c_1^2 c_2 + c_2^2} \right\}.$$

First we prove the inequality $132/19 < \theta_3 \leq 8$. To prove $\theta_3 \leq 8$, it is enough to check that for every member $E \in \mathscr{E}$, the following inequality holds

$$\frac{12(4c_1^4 - 8c_1^2 c_2 + c_2^2)}{7c_1^4 - 14c_1^2 c_2 + c_2^2} < 8,$$

which is equivalent to $2c_1^4 - 4c_1^2 c_2 - c_2^2 > 0$. Since $c_i > 0$ $(i = 1, 2)$, $c_1^2 - 2c_2 > 0$ and $c_1^4 - 3c_1^2 c_2 + c_2^2 > 0$ by (3.2.2), we observe that

$$2c_1^4 - 4c_1^2 c_2 - c_2^2 = 2(c_1^4 - 3c_1^2 c_2 + c_2^2) + 2c_2(c_1^2 - 2c_2) + c_2^2 > 0.$$

Moreover since $c_1^2 - 4c_2 < 0$, it is easily seen that $8c_2^2 + 2c_1^2 c_2 - c_1^4 = -(c_1^2 + 2c_2)(c_1^2 - 4c_2) > 0$, which is equivalent to

$$\frac{12(4c_1^4 - 8c_1^2 c_2 + c_2^2)}{7c_1^4 - 14c_1^2 c_2 + c_2^2} > \frac{132}{19}$$

and hence we can obtain the inequality $\theta_3 > 132/19$.

b) Next we prove that $c_1^2(X)/\chi(X) < \theta_3$ holds. To this end, it suffices to verify that

$$\frac{c_1^2(X)}{\chi(X)} < \frac{12(4c_1^4 - 8c_1^2 c_2 + c_2^2)}{7c_1^4 - 14c_1^2 c_2 + c_2^2},$$

that is equivalent to

$$20c_1^7-70c_1^5c_2+50c_1^3c_2^2-35c_1^6+105c_1^4c_2-60c_1^2c_2^2+15c_1c_2^3-10c_2^3>0.$$

As mentioned in a), we find that $c_1^2/4<c_2<\xi c_1^2$, where $\xi=(3-\sqrt{5})/2$. Let us put

$$f(x)=(3c_1-2)x^3+2c_1^2(5c_1-6)x^2-7c_1^4(2c_1-3)x+c_1^6(4c_1-7).$$

Then we find that

$$f(\xi c_1^2)=5(9-4\sqrt{5})c_1-\frac{(71-31\sqrt{5})}{2}>0,$$

from which we obtain by a standard argument that $f(c_2)>0$ for $c_1^2/4<c_2<\xi c_1^2$. Hence we can establish the desired positivity.

c) Let $f:\mathbf{P}^4\to\mathbf{P}^4$ be a finite morphism of degree n^4 and $E'=f^*(E)$ the pull-back of E by f. Then we see that $c_1'=nc_1$, $c_2'=n^2c_2$, where $\{c_i'\}$ $(i=1,2)$ is the i-th Chern number of E'. If X' denotes a determinantal surface associated to E', then

$$c_1^2(X')=(4c_1^4-8c_1^2c_2+c_2^2)n^4-10(2c_1^3-3c_1c_2)n^3+25(c_1^2-c_2)n^2,$$

$$\chi(X')=\frac{(7c_1^4-14c_1^2c_2+c_2^2)n^4-15(2c_1^3-3c_1c_2)n^3+35(c_1^2-c_2)n^2}{12},$$

$$\lim_{n\to\infty}\frac{c_1^2(X')}{\chi(X')}=\frac{12(4c_1^4-8c_1^2c_2+c_2^2)}{7c_1^4-14c_1^2c_2+c_2^2}.$$

Therefore the line $c_1^2(X)=\theta_3\chi(X)$ is an asymptote of the geography of determinantal surfaces. Thus we obtain the following.

THEOREM 17. *If $c_1^2-4c_2<0$, then it follows that $c_1^2(X)<\theta_3\chi(X)$, where $\theta_3=\sup_{E\in\mathscr{E}}\{12(4c_1^4-8c_1^2c_2+c_2^2)/(7c_1^4-14c_1^2c_2+c_2^2)\}$ and $\mathscr{E}=\{E|E$ is a rank two very ample bundle on $\mathbf{P}^4$ with $c_1^2-4c_2<0\}/$isomorphism. Then line $c_1^2(X)=\theta_3\chi(X)$ is another asymptote of the geography of determinantal surfaces.*

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