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The Radon transform on an exceptional flag manifold

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Abstract. We shall give a description of certain Radon transforms between generalized flag manifolds for the simple algebraic group of type (E_7) from the view point of the D-module theory.

1. Introduction

Let G be a connected simple algebraic group over the complex number field $\mathbf{C}$, and let P and Q be parabolic subgroups of G containing the same Borel subgroup. Set $X = G/P$, $Y = G/Q$, $Z = G/(P \cap Q)$ and consider the correspondence:

$$Y \overset{q}{\leftarrow} Z \overset{p}{\rightarrow} X.$$

Assume that an invertible $\mathcal{O}_X$-module $\mathscr{L}$ and an invertible $\mathcal{O}_Y$-module $\mathscr{M}$ satisfy $q^*\mathscr{M} \otimes_{\mathcal{O}_Z} \Omega_{Z/X}^{\otimes -1} \cong p^*\mathscr{L}$, where $\Omega_{Z/X}$ denotes the sheaf of relative differential forms of maximal degree along the fibers of p. Define the sheaves of twisted differential operators $D_{X,\mathscr{L}}$ and $D_{Y,\mathscr{M}}$ on X and Y by

$$D_{X,\mathscr{L}} = \mathscr{L} \otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} \mathscr{L}^{\otimes -1}, \qquad D_{Y,\mathscr{M}} = \mathscr{M} \otimes_{\mathcal{O}_Y} D_Y \otimes_{\mathcal{O}_Y} \mathscr{M}^{\otimes -1},$$

respectively. For a $D_{Y,\mathscr{M}}$-module N set

$$R(N) = \int_p (\Omega_{Z/X}^{\otimes -1} \otimes_{\mathcal{O}_Z} q^*N).$$

It is a complex of $D_{X,\mathscr{L}}$-modules, called the Radon transform of N. This integral transform plays important roles in some aspects of the representation theory as well as in the theory of certain hypergeometric type differential equations (see Oshima [6], Tanisaki [8], and their references). In the most fundamental case where $N = D_{Y,\mathscr{M}}$ we have a canonical morphism

$$\Phi : D_{X,\mathscr{L}} \to H^0(R(D_{Y,\mathscr{M}})).$$

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This morphism Φ was investigated in Tanisaki [8] in the case where the unipotent radical of P is commutative, and it is shown there that the kernel of Φ coincides with the unique maximal G-stable left ideal of $D_{X,\mathscr{L}}$ for some choice of $Q, \mathscr{L}, \mathscr{M}$ satisfying certain technical conditions. It is also shown there in the case $G = SL_n(\mathbf{C})$ that the morphism Φ is surjective and that the cohomology groups $H^p(R(D_{Y,\mathscr{M}}))$ vanish for $p \neq 0$; however, the geometric method employed in Tanisaki [8] does not work in general for other simple groups as for the surjectivity of Φ and the vanishing of cohomology groups in non-zero degrees.

The problem was again dealt with by Marastoni-Tanisaki [4] from a more general point of view, and as one of the consequences of the representation theoretic method it was shown that the surjectivity of Φ and the vanishing of cohomology groups in non-zero degrees still hold for classical groups.

The aim of the present paper is to obtain similar results for the exceptional groups which were out of consideration in the above mentioned works. In the exceptional case there is one case where P has the commutative non-trivial unipotent radical and the necessary technical conditions on $Q, \mathscr{L}, \mathscr{M}$ are satisfied. Namely, let G be of type (E_7). Fix a Borel subgroup B of G. Let P be the unique parabolic subgroup containing B whose Levi subgroup is of type (E_6). Then its unipotent radical is commutative. Take Q to be the unique parabolic subgroup containing B with Levi subgroup of type (D_6). There exist uniquely an invertible $\mathcal{O}_X$-module $\mathscr{L}$ and an invertible $\mathcal{O}_Y$-module $\mathscr{M}$ satisfying $q^*\mathscr{M} \otimes_{\mathcal{O}_Z} \Omega_{Z/X}^{\otimes -1} \cong p^*\mathscr{L}$ (see Section 4 below for the explicit description of $\mathscr{L}$ and $\mathscr{M}$). Our main result is as follows.

THEOREM 1.1. *Let G be of type (E_7), and let $P, Q, \mathscr{L}, \mathscr{M}$ be as above.*

(i) *We have $H^p(R(D_{Y,\mathscr{M}})) = 0$ for any $p \neq 0$.*

(ii) *The canonical morphism $\Phi : D_{X,\mathscr{L}} \to H^0(R(D_{Y,\mathscr{M}}))$ is surjective and its kernel is the unique maximal proper G-stable left ideal of $D_{X,\mathscr{L}}$.*

In order to check the necessary technical conditions on $Q, \mathscr{L}, \mathscr{M}$, which assures the statement on $\mathrm{Ker}(\Phi)$ by Tanisaki [8], we have used among other things a result in Morita [5] on the defining equations for the closure of P-orbits in the unipotent radical of P. The proof of the remaining statements concerning the surjectivity of Φ and the vanishing of cohomology groups for $R(D_{Y,\mathscr{M}})$ in non-zero degrees is reduced by Marastoni-Tanisaki [4] to a computation of elements in the Weyl group satisfying certain combinatorial conditions. The tedious calculation we had done by hand was also recovered on a computer by C. Marastoni.

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2. Radon transforms

For a smooth algebraic variety X over the complex number field $\mathbf{C}$ we denote by $\mathcal{O}_X, \Omega_X, D_X$ the structure sheaf, the canonical sheaf and the sheaf of differential operators on X respectively.

Let G be a connected simple algebraic group over $\mathbf{C}$. We assume that G is simply connected for the sake of simplicity. Let $\mathfrak{g}$ be the Lie algebra of G and choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We denote the root system by $\Delta \subset \mathfrak{h}^*$ and the Weyl group by W. The root subspace corresponding to $\alpha \in \Delta$ is denoted by $\mathfrak{g}_\alpha$. For $\alpha \in \Delta$ we denote the corresponding coroot by $\alpha^\vee \in \mathfrak{h}$. Fix a set of simple roots $\{\alpha_i \mid i \in I_0\}$. Let $\{\varpi_i \mid i \in I_0\} \subset \mathfrak{h}^*$ and $\{s_i \mid i \in I_0\} \subset W$ be the corresponding set of fundamental weights and simple reflections, respectively. We denote by Δ^+ the set of positive roots.

For a subset I of I_0 we set

$$\Delta_I = \Delta \cap \left(\sum_{i \in I} \mathbf{Z}\alpha_i \right), \qquad \Delta_I^+ = \Delta_I \cap \Delta^+,$$

$$W_I = \langle s_i \mid i \in I \rangle \subset W,$$

$$\mathfrak{h}_I^* = \{\lambda \in \mathfrak{h}^* \mid \lambda(\alpha_i^\vee) = 0 \ (i \in I)\},$$

$$\mathfrak{h}_{\mathbf{Z},I}^* = \{\lambda \in \mathfrak{h}_I^* \mid \lambda(\alpha_i^\vee) \in \mathbf{Z} \ (i \in I_0)\},$$

$$\mathfrak{l}_I = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_I} \mathfrak{g}_\alpha \right),$$

$$\mathfrak{n}_I = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_\alpha, \qquad \mathfrak{n}_I^- = \bigoplus_{\alpha \in \Delta^+ \setminus \Delta_I} \mathfrak{g}_{-\alpha},$$

$$\mathfrak{p}_I = \mathfrak{l}_I \oplus \mathfrak{n}_I.$$

The characters $\mathfrak{p}_I \to \mathbf{C}$ of the Lie algebra $\mathfrak{p}_I$ are in one-to-one correspondence with the elements of $\mathfrak{h}_I^*$. Denote by P_I the subgroup of G corresponding to $\mathfrak{p}_I$. Then the character $\mathfrak{p}_I \to \mathbf{C}$ corresponding to $\lambda \in \mathfrak{h}_I^*$ is integrated to a character $P_I \to \mathbf{C}^\times$ of the algebraic group P_I if and only if $\lambda \in \mathfrak{h}_{\mathbf{Z},I}^*$. In particular, the characters of P_I are in one-to-one correspondence with the elements of $\mathfrak{h}_{\mathbf{Z},I}^*$.

For $\lambda \in \mathfrak{h}_I^*$ we define a $U(\mathfrak{g})$-module $M_I(\lambda)$ by

$$M_I(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_I)} \mathbf{C}_\lambda,$$

where $\mathbf{C}_\lambda$ denotes the one-dimensional $\mathfrak{p}_I$-module corresponding to λ. It is a highest weight module with highest weight λ, called a generalized Verma module. We denote by $K_I(\lambda)$ the unique maximal proper submodule of $M_I(\lambda)$.

Set

$$X_I = G/P_I.$$

For $\lambda \in \mathfrak{h}^*_{\mathbf{Z},I}$ we denote by $\mathscr{O}_{X_I}(\lambda)$ the G-equivariant invertible $\mathscr{O}_{X_I}$-module such that the action of P_I on the fiber at eP_I is given by λ. We define a sheaf of differential operators $D_{X_I,\lambda}$ by

$$D_{X_I,\lambda} = \mathscr{O}_{X_I}(\lambda) \otimes_{\mathscr{O}_{X_I}} D_{X_I} \otimes_{\mathscr{O}_{X_I}} \mathscr{O}_{X_I}(-\lambda).$$

The fiber $\mathbf{C} \otimes_{\mathscr{O}_{X_I,eP_I}} (D_{X_I,\lambda})_{eP_I}$ of $D_{X_I,\lambda}$ at the origin eP_I is a $U(\mathfrak{g})$-module isomorphic to $M_I(-\lambda)$.

By Tanisaki [8] (see also Kashiwara [3]) we have the following.

PROPOSITION 2.1. *G-stable left ideals $\mathscr{K}$ of $D_{X_I,\lambda}$ are in one-to-one correspondence with $U(\mathfrak{g})$-submodules K of $M_I(-\lambda)$ via $\mathscr{K} \mapsto K = \mathbf{C} \otimes_{\mathscr{O}_{X_I,eP_I}} \mathscr{K}_{eP_I}$.*

In particular $D_{X_I,\lambda}$ contains a unique maximal proper G-stable left ideal $\mathscr{K}_{X_I,\lambda}$.

We note that the opposite correspondence $K \mapsto \mathscr{K}$ is also described in Tanisaki [8].

In the rest of this paper we fix subsets I and J of I_0, and consider the correspondence

$$X_J \overset{p_2}{\leftarrow} X_{I\cap J} \overset{p_1}{\rightarrow} X_I.$$

Set $\Omega_{p_1} = \Omega_{X_{I\cap J}} \otimes_{\mathscr{O}_{X_{I\cap J}}} p_1^* \Omega_{X_I}^{\otimes -1}$. It is isomorphic as a G-equivariant $\mathscr{O}_{X_{I\cap J}}$-module to $\mathscr{O}_{X_{I\cap J}}(\gamma)$, where $\gamma = \sum_{\alpha \in \Delta_I^+ \setminus \Delta_J} \alpha$. Hereafter, we fix $\lambda \in \mathfrak{h}^*_{\mathbf{Z},I}$ and $\mu \in \mathfrak{h}^*_{\mathbf{Z},J}$ satisfying

$$\lambda = \mu - \gamma \tag{2.1}$$

(such λ and μ uniquely exist if $I \cup J = I_0$, see Marastoni-Tanisaki [4]). Then we have $p_2^* \mathscr{O}_{X_J}(\mu) \otimes_{\mathscr{O}_{X_{I\cap J}}} \Omega_{p_1}^{\otimes -1} \cong p_1^* \mathscr{O}_{X_I}(\lambda)$, and hence we can define, for a $D_{X_J,\mu}$-module N, a complex of $D_{X_I,\lambda}$-modules

$$R(N) = \int_{p_1} (\Omega_{p_1}^{\otimes -1} \otimes_{\mathscr{O}_{X_{I\cap J}}} p_2^* N).$$

We shall consider the most fundamental case where $N = D_{X_J,\mu}$. In this case we have a canonical morphism

$$\Phi : D_{X_I,\lambda} \to H^0(R(D_{X_J,\mu}))$$

by the definition of R.

Let us explain a part of the results in Marastoni-Tanisaki [4] which gives some information on the morphism Φ. Set

$$\Gamma = \{x \in W_J \mid x^{-1}\Delta^+_{I\cap J} \subset \Delta^+_J, (x(\mu-\rho))(\alpha^\vee) \neq 0 \ (\alpha \in \Delta_I)\}.$$

For $x \in \Gamma$ denote by y_x the unique element of W_I satisfying $(y_x x(\mu-\rho))(\alpha^\vee) < 0$ for any $\alpha \in \Delta^+_I$. Define functions $\ell : \Gamma \to \mathbf{Z}_{\geqq 0}$ and $m : \Gamma \to \mathbf{Z}_{\geqq 0}$ by

$$\ell(x) = \sharp\{\alpha \in \Delta^+_J \backslash \Delta_I \mid (x(\mu-\rho))(\alpha^\vee) > 0\},$$
$$m(x) = \sharp\{\alpha \in \Delta^+_I \backslash \Delta_J \mid (x(\mu-\rho))(\alpha^\vee) < 0\}.$$

Then we have the following.

Proposition 2.2 (Marastoni-Tanisaki [4]). (i) *We have*

$$\sum_p (-1)^p [H^p(R(D_{X_J,\mu}))]$$
$$= \sum_{x\in\Gamma} (-1)^{\ell(x)-m(x)} [D_{X_I,\lambda} \otimes_{\mathcal{O}_{X_I}} \mathcal{O}_{X_I}(\lambda-\rho-y_x x(\mu-\rho))]$$

in the Grothendieck group of the category of quasi-G-equivariant $D_{X_I,\lambda}$-*modules.*

(ii) *We have* $H^p(R(D_{X_J,\mu})) = 0$ *for any* $p \neq 0$ *if and only if* $\ell(x) \geqq m(x)$ *for any* $x \in \Gamma$.

(iii) *Assume that* $H^p(R(D_{X_J,\mu})) = 0$ *for any* $p \neq 0$. *Then* Φ *is an epimorphism if and only if* $\ell(x) > m(x)$ *for any* $x \in \Gamma \backslash \{e\}$.

(iv) *Assume that* $H^p(R(D_{X_J,\mu})) = 0$ *for any* $p \neq 0$. *Then* Φ *is an isomorphism if and only if* $\Gamma = \{e\}$.

In fact, in Marastoni-Tanisaki [4], a result more precise than the statement (i) is formulated in the derived category, and the remaining statements (ii), (iii), (iv) are deduced from it.

3. The commutative parabolic case

In the rest of this paper we assume that

$$\mathfrak{n}_I \text{ is commutative and non-zero.} \tag{3.2}$$

In this case, we have $I = I_0 \backslash \{i_0\}$ for some $i_0 \in I_0$. Moreover, there exist only finitely many L_I-orbits on $\mathfrak{n}_I$. Let $\mathscr{A}_0$ denote the zeroes of Gyoja's b-function of the prehomogeneous vector space $(L_I, \mathfrak{n}_I)$, and set $\mathscr{A} = \{-a-1 \mid a \in \mathscr{A}_0\}$ (see Gyoja [2] for the definition of the b-function and its explicit description in each case). Then $\mathscr{A}$ is a finite set consisting of non-negative rational numbers. Let $\mathscr{C}$ denote the set of non-open L_I-orbits on $\mathfrak{n}_I$. Then $\mathscr{C}$ is a totally ordered set with respect to the closure relation. Moreover, we have $|\mathscr{C}| = |\mathscr{A}|$. Hence there exists a unique bijection $\mathscr{A} \to \mathscr{C}$ $(a \mapsto C_a)$ satisfying $C_a \subset \bar{C}_b$ for $a, b \in \mathscr{A}$ such that $a \leqq b$.

Let $\xi \in \mathfrak{h}_I^*$. By the Poincaré-Birkhoff-Witt theorem we have a canonical isomorphism $U(\mathfrak{n}_I^-) \to M_I(\xi)$ of $U(\mathfrak{n}_I^-)$-modules sending 1 to the highest weight vector of $M_I(\xi)$. Since $\mathfrak{n}_I^-$ is commutative, the enveloping algebra $U(\mathfrak{n}_I^-)$ is isomorphic to the symmetric algebra $S(\mathfrak{n}_I^-)$ which is naturally identified with the algebra $\mathbf{C}[(\mathfrak{n}_I^-)^*]$ consisting of polynomial functions on $(\mathfrak{n}_I^-)^*$. Identifying $(\mathfrak{n}_I^-)^*$ with $\mathfrak{n}_I$ via the Killing form we obtain a linear isomorphism

$$F_\xi : \mathbf{C}[\mathfrak{n}_I] \to M_I(\xi).$$

Recall that $K_I(\xi)$ denote the unique maximal proper submodule of $M_I(\xi)$.

PROPOSITION 3.1. *For $a \in \mathscr{A}$ we have $K_I(-a\varpi_{i_0}) = F_{-a\varpi_{i_0}}(I(\bar{C}_a))$, where $I(\bar{C}_a)$ denotes the defining ideal of $\bar{C}_a$.*

It was noticed by several people that for certain ξ, which are sometimes called weights in the Wallach set, $K_I(\xi)$ corresponds to the defining ideal of the closure of an L_I-orbit under the bijection F_ξ (see Enright-Joseph [1] and its references). The above description of the parameter ξ in terms of Gyoja's b-function was pointed out by Tanisaki [7], and its unified proof was given by Wachi [9] subsequently.

Let us return to the Radon transforms. Hereafter, we consider the particular case where

$$\lambda = -a\varpi_{i_0}, \tag{3.3}$$

$$\overline{\mathrm{Ad}(L_I)(\mathfrak{n}_I \cap \mathfrak{n}_J)} = \bar{C}_a \tag{3.4}$$

for some $a \in \mathscr{A}$ in the notation of Section 2. In this case the maximal proper G-stable left ideal $\mathscr{K}_{X_I,\lambda}$ of $D_{X_I,\lambda}$ admits an explicit description in view of Proposition 2.1 and Proposition 3.1. Then we have the following.

PROPOSITION 3.2 (Tanisaki [8]). *We have* $\mathrm{Ker}\,\Phi = \mathscr{K}_{X_I,\lambda}$.

Indeed, it was proved in Tanisaki [8] that $\mathrm{Ker}\,\Phi \supset \mathscr{K}_{X_I,\lambda}$ using (3.4) and Proposition 3.1. Since $\mathrm{Ker}\,\Phi$ is G-stable, $\mathrm{Ker}\,\Phi$ coincides with $\mathscr{K}_{X_I,\lambda}$ or $D_{X_I,\lambda}$ by the maximality of $\mathscr{K}_{X_I,\lambda}$. We see easily that Φ is non-trivial, and hence the possibility $\mathrm{Ker}\,\Phi = D_{X_I,\lambda}$ is excluded.

4. The exceptional case

Let us call the quadruple (I, J, λ, μ) consisting of $I, J \subset I_0, \lambda \in \mathfrak{h}_{\mathbf{Z},I}^*, \mu \in \mathfrak{h}_{\mathbf{Z},J}^*$ an admissible quadruple when they satisfy the conditions (2.1), (3.2), (3.3) and

(3.4). In Tanisaki [8] admissible quadruples are listed up in the cases where G is of classical type. It is also proved there that we have

$$H^p(R(D_{X_J,\mu})) = 0 \qquad (p \neq 0), \tag{4.5}$$

$$\Phi \text{ is an epimorphism} \tag{4.6}$$

for any admissible quadruple for $G = SL_n(\mathbf{C})$. As for the other classical groups it is proved in Marastoni-Tanisaki [4] that both (4.5) and (4.6) hold. Hence we have a complete description of $R(D_{X_J,\mu})$ in these cases.

In the present paper we shall deal with the remaining exceptional cases. We have one case.

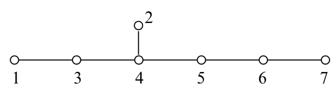

Fig. 1. Dynkin diagram of type (E_7)

Proposition 4.1. *Let G be of type* (E_7), *and use the labelling of I_0 given in Figure* 1. *Set $I = I_0 \setminus \{7\}$, $J = I_0 \setminus \{1\}$, $\lambda = 8\varpi_7$, $\mu = 12\varpi_1$. Then (I, J, λ, μ) is an admissible quadruple.*

Proof. In this case $\mathfrak{n}_I$ is commutative and we have $\mathscr{A} = \{0, 4, 8\}$. Hence $\mathfrak{n}_I$ consists of four L_I-orbits C_0, C_4, C_8, and the open orbit. They satisfy

$$\{0\} = C_0 \subset \bar{C}_4 \subset \bar{C}_8 \subset \mathfrak{n}_I.$$

$\bar{C}_4$ is the zeroes of 27-polynomials with degree 2, and $\bar{C}_8$ is the zeroes of a single polynomial with degree 3 (relative invariant). The condition (2.1) is easily checked using the table of the roots for type (E_6). Hence the only non-trivial part is to check the condition $\overline{\mathrm{Ad}(L_I)(\mathfrak{n}_I \cap \mathfrak{n}_J)} = \bar{C}_8$. We have to show $\mathfrak{n}_I \cap \mathfrak{n}_J \subset \bar{C}_8$, $\mathfrak{n}_I \cap \mathfrak{n}_J \not\subset \bar{C}_4$. They can be checked using the explicit description of the relative invariant and the 27-polynomials with degree 2 defining $\bar{C}_4$ given in Morita [5]. Details are omitted. □

Let G be of type (E_7), and let (I, J, λ, μ) be the admissible quadruple given in Proposition 4.1. By Proposition 3.2 the kernel of the canonical homomorphism

$$\Phi : D_{X_I,\lambda} \to H^0(R(D_{X_J,\mu}))$$

is the maximal proper G-stable submodule of $D_{X_I,\lambda}$ which is locally generated by a single section corresponding to the relative invariant.

Let us also consider the conclusion of Proposition 2.2.

PROPOSITION 4.2. *Let G be of type* (E_7), *and let* (I, J, λ, μ) *be the admissible quadruple given in Proposition* 4.1. *Then the set Γ consists of two elements e and $x = s_5 s_6 s_7$, and we have $\ell(e) = m(e) = 0$, $\ell(x) = 3$, $m(x) = 2$. Moreover, we have $\lambda - \rho - y_x x(\mu - \rho) = -2\varpi_7$ (see Section 2 for the definition of Γ).*

We omit the details of the calculation. The fact was also checked by C. Marastoni on a computer.

It follows that (4.5) and (4.6) also hold in our case. Moreover, by using a result in Marastoni-Tanisaki [4] which is stronger than Proposition 2.2(i) we see that there exists an exact sequence

$$0 \to D_{X_I,\lambda} \otimes_{\mathcal{O}_{X_I}} \mathcal{O}_{X_I}(-2\varpi_7) \to D_{X_I,\lambda} \to H^0(R(D_{X_J,\mu})) \to 0.$$

In conclusion we have obtained the following result.

THEOREM 4.3. *Let G be of type* (E_7), *and set $I = I_0 \backslash \{7\}$, $J = I_0 \backslash \{1\}$ in the labelling of I_0 given in Figure* 1. *Then the quadruple $(I, J, 8\varpi_7, 12\varpi_1)$ is an admissible quadruple, and we have the following.*

(i) $H^p(R(D_{X_J,12\varpi_1})) = 0$ *for any* $p \neq 0$.

(ii) *The canonical morphism $\Phi : D_{X_I,8\varpi_7} \to H^0(R(D_{X_J,12\varpi_1}))$ is an epimorphism.*

(iii) *The kernel of Φ is the maximal proper G-stable left ideal of $D_{X_I,8\varpi_7}$, which is locally generated by a section which can be described using the relative invariant of the prehomogeneous vector space $(L_I, \mathfrak{n}_I)$. Moreover, we have* $\mathrm{Ker}(\Phi) \cong D_{X_I,8\varpi_7} \otimes_{\mathcal{O}_{X_I}} \mathcal{O}_{X_I}(-2\varpi_7)$.

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