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Removability of sets for sub-polyharmonic functions

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ABSTRACT. Our first aim in this paper is to generalize Bôcher's theorem for functions u whose Riesz measure $\mu = \Delta^m u$ is nonnegative in the punctured unit ball $\mathbf{B}_0$. In fact, if u satisfies a certain integral condition and $\mu = \Delta^m u \geq 0$ in $\mathbf{B}_0$, then it is shown that u can be written as the sum of a generalized potential of μ and a polyharmonic function on $\mathbf{B}$. This is nothing but the Laurent series expansion for u.

The next aim is to give a polyharmonic version of the recent results by Riihentaus [11] concerning removability of sets for subharmonic functions.

1. Introduction and statement of results

Let $\mathbf{R}^n$ be the n-dimensional Euclidean space with a point $x = (x_1, x_2, \dots, x_n)$. For a multi-index $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$, we set

$$|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n,$$

$$x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$$

and

$$D^\lambda = \left(\frac{\partial}{\partial x_1}\right)^{\lambda_1} \left(\frac{\partial}{\partial x_2}\right)^{\lambda_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\lambda_n}.$$

We denote by $B(x,r)$ the open ball centered at x with radius $r > 0$, whose boundary is written as $S(x,r) = \partial B(x,r)$. We also denote by $\mathbf{B}$ the unit ball $B(0,1)$ and by $\mathbf{B}_0$ the punctured unit ball $\mathbf{B} - \{0\}$.

A real-valued function u on an open set $G \subset \mathbf{R}^n$ is called polyharmonic of order m on G if $u \in C^{2m}(G)$ and $\Delta^m u = 0$ on G, where m is a positive integer, Δ denotes the Laplacian and $\Delta^m u = \Delta^{m-1}(\Delta u)$ (cf. [2], [10]). We denote by $H^m(G)$ the space of polyharmonic functions of order m on G. In particular, u is harmonic on G if $u \in H^1(G)$.

The fundamental solution of Δ^m is written as R_{2m}, that is,

$$R_{2m}(x) = \begin{cases} \alpha_m |x|^{2m-n} & \text{if } 2m-n \text{ is not an even nonnegative integer,} \\ \alpha_m |x|^{2m-n} \log(1/|x|) & \text{if } 2m-n \text{ is an even nonnegative integer,} \end{cases}$$

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Key words and phrases. polyharmonic functions, isolated singularities, Bôcher's theorem, Laurent series expansion, Riesz decomposition, removability of sets where the constant α_m is chosen so that $\Delta^m R_{2m}$ is the Dirac measure δ at the origin. We denote by $R_{2m,L}$ the remainder term of Taylor expansion of R_{2m}:

$$R_{2m,L}(\zeta,x) = R_{2m}(\zeta - x) - \sum_{|\lambda| \le L} \frac{\zeta^\lambda}{\lambda!}(D^\lambda R_{2m})(-x)$$

for a nonnegative integer L.

We say that a locally integrable function u on an open set $G \subset \mathbf{R}^n$ is sub-polyharmonic of order m in G if $\Delta^m u \ge 0$ in G in the weak sense, that is,

$$\int_G u(x)\Delta^m \varphi(x)dx \ge 0 \qquad \text{for all nonnegative } \varphi \in C_0^\infty(G).$$

Our first aim in this note is to establish Bôcher's theorem for sub-polyharmonic functions $u \in L^1_{loc}(2\mathbf{B}_0)$, where $2\mathbf{B}_0 = B(0,2) - \{0\}$; for polyharmonic functions, we refer the reader to the previous paper [3] as a generalization of Armitage [1].

THEOREM 1. *Suppose that* $u \in L^1_{loc}(2\mathbf{B}_0)$ *and* $\mu = \Delta^m u$ *is a nonnegative measure on* $2\mathbf{B}_0$. *If* u *satisfies*

$$\int_{2\mathbf{B}_0} |u(x)|\,|x|^s dx < \infty \tag{1}$$

for some number $s \ge \max\{-2m, -n\}$, *then*

$$u(x) = \int_{\mathbf{B}_0} R_{2m,L}(\zeta,x)d\mu(\zeta) + h(x) + \sum_{|\lambda| \le L} C(\lambda)D^\lambda R_{2m}(x) \tag{2}$$

for a.e. $x \in \mathbf{B}_0$, *where* L *is the integer such that* $s + 2m - 1 < L \le s + 2m$, $h \in H^m(\mathbf{B})$ *and* $C(\lambda)$ *denote constants.*

The above expression is called the Laurent series expansion for u.

To prove Theorem 1, we first show that the generalized potential $\int_{\mathbf{B}_0} R_{2m,L}(\zeta,x)d\mu(\zeta)$ satisfies condition (1) for $s' > s$, and then apply Bôcher's theorem for polyharmonic functions on $\mathbf{B}_0$ given in [3].

Next we discuss removability of sets for sub-polyharmonic functions in $\mathbf{R}^n$.

We say that a continuous function h on $[0,\infty)$ is a measure function if $h(0) = 0$, h is nondecreasing and

$$h(2r) \le Mh(r) \qquad \text{for all } r > 0, \tag{3}$$

where M is a positive constant. For $\varepsilon > 0$ and $E \subset \mathbf{R}^n$, write

$$E_\varepsilon = \{x \in \mathbf{R}^n : d(x,E) < \varepsilon\},$$

where $d(x,E)$ denotes the distance of x from E, that is, $d(x,E) = \inf\{|x-y| : y \in E\}$. Then the upper Minkowski h-content of E is defined by

$$\mathscr{M}_h(E) = \limsup_{\varepsilon \to 0+} \frac{|E_\varepsilon|}{h(\varepsilon)},$$

where $|F|$ denotes the n-dimensional Lebesgue measure of a set F. If $h(r) = r^{n-\alpha}$, $0 \le \alpha < n$, then we write $\mathscr{M}_\alpha$ for $\mathscr{M}_h$.

We introduce the result by Riihentaus [11] (see also Gardiner [4]).

Theorem A (Riihentaus). *Let $\alpha \in [0, n-2]$ and let E be a closed set in Ω such that $\mathscr{M}_\alpha(E) = 0$. If f is subharmonic in $\Omega \backslash E$ and satisfies*

$$f(x) \le d(x, E)^{\alpha+2-n} \qquad \text{for all } x \in \Omega \backslash E,$$

then f has a subharmonic extension to Ω.

Now we state the following theorem.

Theorem 2. *Let h be a measure function. Suppose E is a closed set in Ω such that $\mathscr{M}_h(E) = 0$. If $u \in L^1_{loc}(\Omega \backslash E)$ is sub-polyharmonic of order m in $\Omega \backslash E$ and satisfies*

$$|u(x)| \le d(x, E)^{2m} h(d(x, E))^{-1} \qquad \text{for all } x \in \Omega \backslash E, \tag{4}$$

then u has a sub-polyharmonic extension to Ω of order m.

Let h and k be two measure functions on $[0, \infty)$ such that

$$\lim_{r \to 0} \frac{k(r)}{h(r)} = 0.$$

In Theorem 2, if $\mathscr{M}_k(E) < \infty$ and

$$|u(x)| \le d(x, E)^{2m} h(d(x, E))^{-1} \qquad \text{for all } x \in \Omega \backslash E, \tag{5}$$

then u is shown to have a sub-polyharmonic extension to Ω (see also Riihentaus [11, Theorem 2]).

2. Lemmas

Throughout this paper, let M denote various constants, not neccessarily the same on any two occurrences.

We need several lemmas to prove Theorem 1.

Lemma 1. *If u and μ are as in Theorem 1, then*

$$\int_{A(r)} d\mu(\zeta) \le M r^{-2m} \int_{C(r)} |u(\zeta)| d\zeta$$

whenever $0 < r < \frac{1}{2}$, where $A(r) = \{r \le |x| < 2r\}$ and $C(r) = \{r/2 < |x| < 4r\}$.

PROOF. Consider a function $\psi \in C_0^\infty(C(1))$ such that $\psi \geq 0$ and

$$\psi(x) = \begin{cases} 1 & \text{if } 1 \leq |x| \leq 2, \\ 0 & \text{if } |x| \leq 1/2 \text{ or } |x| \geq 4. \end{cases}$$

If we set $\psi_r(x) = \psi\left(\frac{x}{r}\right)$ for $0 < r < 1/2$, then

$$\begin{aligned} \int_{A(r)} d\mu(\zeta) &\leq \int_{C(r)} \psi_r \, d\mu(\zeta) \\ &= \int_{C(r)} (\Delta^m \psi_r) u \, d\zeta \\ &\leq \int_{C(r)} |\Delta^m \psi_r| \, |u| d\zeta \\ &\leq M r^{-2m} \int_{C(r)} |u| d\zeta. \end{aligned}$$

This proves Lemma 1.

LEMMA 2. *If u and μ are as above, then*

$$\int_{\mathbf{B}_0} |\zeta|^\ell d\mu(\zeta) < \infty \tag{6}$$

whenever $\ell \geq s + 2m$.

PROOF. Let $A_j = A(2^{-j})$ and $C_j = C(2^{-j})$; then we have by Lemma 1

$$\begin{aligned} \int_{\mathbf{B}_0} |\zeta|^\ell d\mu(\zeta) &= \sum_{j=1}^\infty \int_{A_j} |\zeta|^\ell d\mu(\zeta) \\ &\leq \sum_{j=1}^\infty 2^{\ell(-j+1)} \int_{A_j} d\mu(\zeta) \\ &\leq M \sum_{j=1}^\infty 2^{\ell(-j+1)+2mj} \int_{C_j} |u(\zeta)| d\zeta \\ &\leq M \sum_{j=1}^\infty \int_{C_j} |u(\zeta)| \, |\zeta|^s d\zeta \\ &\leq M \int_{2\mathbf{B}_0} |u(\zeta)| \, |\zeta|^s d\zeta < \infty. \end{aligned}$$

We put $I(x) = \int_{\mathbf{B}_0} |R_{2m,L}(\zeta, x)| d\mu(\zeta)$, where L is the integer such that $s + 2m - 1 < L \leq s + 2m$; note here that $L \geq 0$ and $L \geq 2m - n$ because $s \geq \max\{-2m, -n\}$. For $x \in \mathbf{B}_0$, consider the sets

$$E_1 = \left\{\zeta \in \mathbf{B}_0 : |\zeta| < \frac{|x|}{2}\right\},$$

$$E_2 = \left\{\zeta \in \mathbf{B}_0 : |\zeta - x| < \frac{|\zeta|}{2}\right\},$$

$$E_3 = \mathbf{B}_0 - (E_1 \cup E_2).$$

If $2m \geq n$, then we see from [6, Lemma 4.2] and [9, Lemmas 6, 8, 9] that

$$\begin{aligned} I(x) &\leq M \int_{E_1} |\zeta|^{L+1} |x|^{2m-n-L-1} d\mu(\zeta) \\ &\quad + M \int_{E_2} \left(|\zeta|^{2m-n} + |\zeta - x|^{2m-n} \log \frac{|\zeta|}{|\zeta - x|}\right) d\mu(\zeta) \\ &\quad + M \int_{E_3} |\zeta|^{L} |x|^{2m-n-L} \log \frac{4|\zeta|}{|x|}\, d\mu(\zeta) \\ &= M\{I_1(x) + I_2(x) + I_3(x)\}; \end{aligned}$$

if $2m < n$, then $I_2(x)$ is replaced by

$$I_2(x) = \int_{E_2} |\zeta - x|^{2m-n} d\mu(\zeta).$$

We prove the following lemma.

Lemma 3. *If μ is a nonnegative measure on $\mathbf{B}_0$ satisfying* (6) *and $s' > s \geq \max\{-2m, -n\}$, then*

$$\int_{\mathbf{B}} I(x)|x|^{s'} dx < \infty.$$

Proof. We have only to treat s' satisfying $s' > s$ and

$$s' - 1 < L - 2m < s'.$$

First, since $(2m - n - L - 1 + s') + n = s' - (L - 2m + 1) < 0$, we have

$$\begin{aligned} \int_{\mathbf{B}} I_1(x)|x|^{s'} dx &= \int_{\mathbf{B}} \left(\int_{E_1} |\zeta|^{L+1} |x|^{2m-n-L-1} d\mu(\zeta)\right) |x|^{s'} dx \\ &\leq \int_{\mathbf{B}_0} |\zeta|^{L+1} \left(\int_{\{x:|x| \geq 2|\zeta|\}} |x|^{2m-n-L-1+s'} dx\right) d\mu(\zeta) \\ &= M \int_{\mathbf{B}_0} |\zeta|^{2m+s'} d\mu(\zeta) < \infty \end{aligned}$$

with the aid of (6).

Next, noting that $|\zeta|/2 < |x| < 2|\zeta|$ when $\zeta \in E_2$, we have

$$\begin{aligned}
\int_{\mathbf{B}} I_2(x)|x|^{s'}dx &= \int_{\mathbf{B}}\left\{\int_{E_2}\left(|\zeta|^{2m-n} + |\zeta - x|^{2m-n}\log\frac{|\zeta|}{|\zeta - x|}\right)d\mu(\zeta)\right\}|x|^{s'}dx \\
&\le \int_{\mathbf{B}_0}|\zeta|^{2m-n}\left(\int_{\{x:|\zeta|/2<|x|<2|\zeta|\}}|x|^{s'}dx\right)d\mu(\zeta) \\
&\quad + \int_{\mathbf{B}_0}\left(\int_{\{x:|\zeta-x|\le|\zeta|/2\}}|\zeta - x|^{2m-n}\log\frac{|\zeta|}{|\zeta - x|}|x|^{s'}dx\right)d\mu(\zeta) \\
&\le M\int_{\mathbf{B}_0}|\zeta|^{2m+s'}d\mu(\zeta) \\
&\quad + M\int_{\mathbf{B}_0}|\zeta|^{s'}\left(\int_{\{x:|\zeta-x|\le|\zeta|/2\}}|\zeta - x|^{2m-n}\log\frac{|\zeta|}{|\zeta - x|}\,dx\right)d\mu(\zeta) \\
&\le M\int_{\mathbf{B}_0}|\zeta|^{2m+s'}d\mu(\zeta) < \infty.
\end{aligned}$$

Finally, since $(2m - n - L + s') + n = s' - (L - 2m) > 0$, we establish

$$\begin{aligned}
\int_{\mathbf{B}} I_3(x)|x|^{s'}dx &= \int_{\mathbf{B}}\left(\int_{E_3}|\zeta|^{L}|x|^{2m-n-L}\log\frac{4|\zeta|}{|x|}\,d\mu(\zeta)\right)|x|^{s'}dx \\
&\le \int_{\mathbf{B}_0}|\zeta|^{L}\left(\int_{\{x:|x|\le 2|\zeta|\}}|x|^{2m-n-L+s'}\log\frac{4|\zeta|}{|x|}\,dx\right)d\mu(\zeta) \\
&\le M\int_{\mathbf{B}_0}|\zeta|^{2m+s'}d\mu(\zeta) < \infty.
\end{aligned}$$

Thus we have obtained

$$\int_{\mathbf{B}} I(x)|x|^{s'}dx < \infty,$$

as required.

LEMMA 4. *If u and μ are as above, then*

$$v(x) \equiv u(x) - \int_{\mathbf{B}_0} R_{2m,L}(\zeta, x)d\mu(\zeta) \in H^m(\mathbf{B}_0)$$

with L as before.

PROOF. It is sufficient to show that $\Delta^m v = 0$ in $\mathbf{B}_0$ in the weak sense.

Let $\varphi \in C_0^\infty(\mathbf{B}_0)$. In view of Lemma 3, we can apply Fubini's theorem to obtain

$$\begin{aligned}
\langle u - v, \Delta^m \varphi \rangle &= \left\langle \int_{\mathbf{B}_0} \left(R_{2m}(\zeta - x) - \sum_{|\lambda| \le L} \frac{\zeta^\lambda}{\lambda!} (D^\lambda R_{2m})(-x) \right) d\mu(\zeta), \Delta^m \varphi \right\rangle \\
&= \int_{\mathbf{B}_0} \left\{ \int_{\mathbf{B}_0} \left(R_{2m}(\zeta - x) - \sum_{|\lambda| \le L} \frac{\zeta^\lambda}{\lambda!} (D^\lambda R_{2m})(-x) \right) \Delta^m \varphi(x) dx \right\} d\mu(\zeta) \\
&= \int_{\mathbf{B}_0} \left(\varphi(\zeta) - \sum_{|\lambda| \le L} \frac{\zeta^\lambda}{\lambda!} D^\lambda \varphi(0) \right) d\mu(\zeta) \\
&= \int_{\mathbf{B}_0} \varphi(\zeta) d\mu(\zeta) \\
&= \langle u, \Delta^m \varphi \rangle,
\end{aligned}$$

since φ vanishes in a neighborhood of the origin. This proves

$$\langle v, \Delta^m \varphi \rangle = 0,$$

as required.

3. Proof of Theorem 1

From Lemmas 3 and 4, we see that $v \in H^m(\mathbf{B}_0)$ and

$$\int_{\mathbf{B}_0} |v(x)| \, |x|^{s'} dx < \infty$$

for all $s' > s$. In view of [3], we can find $h \in H^m(\mathbf{B})$ and constants $C(\lambda)$ for which

$$v(x) = h(x) + \sum_{|\lambda| \le L} C(\lambda) D^\lambda R_{2m}(x)$$

holds a.e. on $\mathbf{B}_0$, where L is the integer such that $s + 2m - 1 < L \le s + 2m$. This implies that u is of the form

$$u(x) = \int_{\mathbf{B}_0} R_{2m,L}(\zeta, x) d\mu(\zeta) + h(x) + \sum_{|\lambda| \le L} C(\lambda) D^\lambda R_{2m}(x)$$

for a.e. $x \in \mathbf{B}_0$, as required.

In case $m = 1$, our theorem gives the following simple result.

Corollary. *If u is a subharmonic function on $2\mathbf{B}_0$ satisfying*

$$\int_{2\mathbf{B}_0} u^+(x)|x|^{-2}dx < \infty, \tag{7}$$

then u can be extended to a subharmonic function on $\mathbf{B}$, where $u^+(x) = \max\{u(x), 0\}$.

Proof. Since u^+ is subharmonic on $2\mathbf{B}_0$ and satisfies (1) with $s = -2$, we can take $L = 0$ in Theorem 1, and show that u^+ is of the form

$$u^+(x) = \int_{\mathbf{B}_0} R_2(\zeta - x)d\mu(\zeta) + h(x) + CR_2(x)$$

for $x \in \mathbf{B}_0$, where $\mu = \Delta u^+ \geq 0$, $\mu(\mathbf{B}_0) < \infty$, h is harmonic in $\mathbf{B}$ and C is a constant. In view of (7),

$$\liminf_{r\to 0} r^{-1} \int_{S(0,r)} u^+(x)dS(x) = 0.$$

Moreover, by [8, Theorem 4.3.1] we see easily that

$$\lim_{r\to 0}[r\kappa(r)]^{-1} \int_{S(0,r)} \left(\int_{\mathbf{B}_0} R_2(\zeta - x)d\mu(\zeta)\right) dS(x) = 0,$$

where $\kappa(r) = 1$ for $n \geq 3$ and $\kappa(r) = \log(1/r)$ for $n = 2$, which shows that $C = 0$. Thus u^+ is extended to a subharmonic function on $\mathbf{B}$. Since $u \leq u^+$, u is bounded above near the origin, so that u is extended to a subharmonic function on $\mathbf{B}$ by [6, Theorem 5.18].

4. Removability of sets

To prove Theorem 2, we need the following lemma, which is a version of partition of unity (cf. [7]).

Lemma 5. *Let $\{B_i : i = 1, \dots, N\}$, $B_i = B(x_i, r_i)$, be a finite collection of balls such that $\{5^{-1}B_i\}$ is mutually disjoint. Then there is a family of nonnegative functions $\varphi_i \in C_0^\infty(\mathbf{R}^n)$ with support $\operatorname{supp} \varphi_i \subset 2B_i$ such that $\sum_{i=1}^N \varphi_i(x) = 1$ for $x \in \bigcup_{i=1}^N B_i$. Furthermore, for each multi-index λ, there is a constant C_λ such that*

$$|D^\lambda \varphi_i(x)| \leq C_\lambda r_i^{-|\lambda|} \qquad \text{for all } x \in \mathbf{R}^n \text{ and } i = 1, \dots, N. \tag{8}$$

Proof of Theorem 2. By our assumption that $\mathscr{M}_h(E) = 0$, for $\varepsilon > 0$, there is r_0, $0 < r_0 < 1$, such that

$$|E_r| \leq \varepsilon h(r) \qquad \text{whenever } 0 \leq r \leq r_0. \tag{9}$$

We first show that

$$\int_{E_r \setminus E} d(x,E)^{2m} h(d(x,E))^{-1} dx \le Mr^{2m}\varepsilon. \tag{10}$$

If we put $K_j = \{x \in \mathbf{R}^n \mid d(x,E) < r2^{-j}\}$, then

$$E_r \setminus E = \bigcup_{j=0}^{\infty} (K_j \setminus K_{j+1}).$$

Hence we have by (9)

$$\begin{aligned}
\int_{E_r \setminus E} d(x,E)^{2m} h(d(x,E))^{-1} dx &= \sum_{j=0}^{\infty} \int_{K_j \setminus K_{j+1}} d(x,E)^{2m} h(d(x,E))^{-1} dx \\
&\le \sum_{j=0}^{\infty} (r2^{-j})^{2m} h(r2^{-(j+1)})^{-1} |K_j| \\
&\le Mr^{2m}\varepsilon \sum_{j=0}^{\infty} 2^{-2mj} \\
&= Mr^{2m}\varepsilon. \qquad (11)
\end{aligned}$$

From (4) and (11) it follows that

$$\int_{E_r \setminus E} |u| dx \le Mr^{2m}\varepsilon. \tag{12}$$

If we set $u = 0$ on E, then we see that $u \in L^1_{loc}(\Omega)$.

Next we show that

$$\int_{\Omega} u(x) \Delta^m \varphi(x) dx \ge 0 \tag{13}$$

for nonnegative $\varphi \in C_0^{\infty}(\Omega)$. We may assume that $0 \le \varphi \le 1$ and $|D^{\lambda}\varphi| \le 1$ for every multi-index $|\lambda| \le 2m$. We put $K = \operatorname{supp} \varphi$ and take $r_0 > 0$ such that $K_{r_0} \subset \Omega$.

Let $0 < 4r < r_0$. By a covering lemma, we can find a finite collection of balls $B_i = B(x_i, r)$ such that $\{5^{-1}B_i\}$ is mutually disjoint and

$$\bigcup_{i=1}^{N} B_i \supset K.$$

By re-indexing if necessary, we can find N^* such that

$$\begin{cases} 2B_i \cap E \neq \varnothing & \text{for } i = 1, \dots, N^*; \\ 2B_i \cap E = \varnothing & \text{for } i = N^* + 1, \dots, N. \end{cases}$$

Let φ_i be as in Lemma 5. Since u is sub-polyharmonic of order m in $\Omega \backslash E$, we see that

$$\int_{2B_i} u\Delta^m(\varphi\varphi_i)dx \geq 0$$

for $i = N^* + 1, \dots, N$, so that

$$\begin{aligned} \int_\Omega u\Delta^m \varphi \, dx &= \int_\Omega u\Delta^m \left\{ \varphi \left(\sum_{i=1}^N \varphi_i \right) \right\} dx \\ &= \sum_{i=1}^N \int_{2B_i} u\Delta^m(\varphi\varphi_i)dx \\ &\geq \sum_{i=1}^{N^*} \int_{2B_i} u\Delta^m(\varphi\varphi_i)dx \\ &\geq -\frac{M}{r^{2m}} \sum_{i=1}^{N^*} \int_{2B_i} |u|dx \end{aligned}$$

with the aid of (8). Thus by (12) we have

$$\int u\Delta^m \varphi \, dx \geq -M\varepsilon,$$

which gives (13). Consequently, u is sub-polyharmonic of order m in Ω.

For a measure function h and $f \in L^1_{loc}(\Omega)$, define

$$A_{f,h}(x) = \sup_B r^{-2m} h(r)^{-1} \inf_v \int_B |f(y) - v(y)|dy,$$

where the supremum is taken over all balls $B = B(x, r) \subset \Omega$ and the infimum is taken over all $v \in L^1_{loc}(\Omega)$ such that $\Delta^m v \geq 0$ on B. Further consider the set S_f of all $x \in \Omega$ such that

$$\limsup_{r \to 0} r^{-n-2m} \int_{B(x,r)} |f(y) - v(y)|dy > 0$$

for all functions $v \in L^1_{loc}(\Omega)$ satisfying $\Delta^m v \geq 0$ on a neighborhood of x.

As in [7] we can prove the following theorem, which gives an extension of a theorem in Gardiner [4].

Theorem 3. *If* $A_{u,h} \in L^{\infty}(\Omega)$ *and* $H_h(S_u) = 0$, *then* u *has a sub-polyharmonic extension to* Ω, *where* H_h *denotes the Hausdorff measure with a measure function* h.

5. Remarks on Theorem 1

Suppose that $u \in L^1_{loc}(2\mathbf{B}_0)$ and $\mu = \Delta^m u$ is a nonnegative measure on $2\mathbf{B}_0$. Then, as in the book of Hayman-Kennedy [6], u can be represented as

$$u(x) = \int_{\mathbf{B}_0} R_{2m, L(|\zeta|)}(\zeta, x) d\mu(\zeta) + h(x) + \sum_{\lambda} C(\lambda) D^{\lambda} R_{2m}(x) \tag{14}$$

for a.e. $x \in \mathbf{B}_0$, where $L(r)$ is a nonincreasing positive function on $(0, 1]$, $h \in H^m(\mathbf{B})$ and $C(\lambda)$ denote constants. To prove this, we use the estimate

$$|R_{2m,\ell}(\zeta, x)| \le M^{\ell} |\zeta|^{\ell+1} |x|^{2m-n-\ell-1}$$

whenever $2|\zeta| \le |x|$ and $2m - n < \ell + 1$, where M is a positive constant depending only on m and n.

Thus our theorem gives a condition which assures that L is bounded and the above sum contains only finite terms.

Remark. We do not know whether u has a similar Laurent expansion or not, if we replace condition (1) by

$$\int_{2\mathbf{B}_0} u^+(x)|x|^s dx < \infty.$$

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