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Global existence of the Cauchy problem for a chemotactic system with predator-prey dynamics

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Abstract. In this paper we consider the Cauchy problem for the parabolic system arising in biology. By the method of the analytic semigroup developed in Osaki and Yagi [9], Yagi [14] we show existence, uniqueness and non-negativity of global solutions.

1. Introduction

Budrene and Berg [2] report that complex patterns are formed by motile cells, and this phenomenon is formulated by a parabolic system of the following type:

$$(0.1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = d_u \Delta u - \nabla \cdot \{u \nabla \chi(c)\} + f(u,v) \quad \text{in } \mathbf{R}^2 \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = d_v \Delta v + g(u,v) \quad \text{in } \mathbf{R}^2 \times (0,\infty), \\ \dfrac{\partial c}{\partial t} = d_c \Delta c + \beta u - \gamma c \quad \text{in } \mathbf{R}^2 \times (0,\infty), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ c(x,0) = c_0(x) \quad \text{in } \mathbf{R}^2 \end{cases}$$

(see, for example, Kawasaki, Mochizuki and Shigesada [6]). As one of typical systems for chemotaxis which is different from (1.1) we refer to Mimura and Tsujikawa [8]. Here $u(x,t)$, $v(x,t)$ and $c(x,t)$ are the bacterium density, the nourishment density and the concentration of the chemical substance at a place $x \in \mathbf{R}^2$ and a time $t \in [0,\infty)$, respectively. We also denote the diffusion constants of u, v, c by d_u, d_v, d_c. Furthermore β and γ are positive constants. The term $\nabla \cdot \{u \nabla \chi(c)\}$ in (0.1) represents a mobility of individuals by high concentrations of the chemical substance, which is called chemotaxis, and $\chi(c)$ is a sensitive function due to chemotaxis which is the three times continuously differentiable function on $[0,\infty)$ such that $\chi'(c) > 0$ for $c > 0$. Here we assume

$$(0.2)\qquad \sup_{0 \le c} \left| \frac{d^i \chi(c)}{dc^i} \right| < \infty, \qquad i = 1, 2, 3.$$

The smooth functions $f(u,v)$, $g(u,v)$, $u,v \in [0,\infty)$ stand for the predator-prey dynamics, which are given by

$$(0.3) \qquad \begin{aligned} f(u,v) &= (-\mu_1 - \nu_1 u + \tau_1 v)u = f_1(u,v)u, \\ g(u,v) &= (\mu_2 - \nu_2 v - \tau_2 u)v = g_1(u,v)v, \end{aligned}$$

where μ_1, μ_2, ν_1, ν_2, τ_1, τ_2 are positive constants.

Our objective of this paper is to prove the existence of a unique global nonnegative solution of (1.1) such that

$$\begin{aligned} &0 \le u \in \mathscr{C}([0,\infty); L^2(\mathbf{R}^2)) \cap \mathscr{C}^1((0,\infty); H^{-1}(\mathbf{R}^2)) \cap \mathscr{C}((0,\infty); H^1(\mathbf{R}^2)), \\ &0 \le v \in \mathscr{C}([0,\infty); H^1(\mathbf{R}^2)) \cap \mathscr{C}^1((0,\infty); L^2(\mathbf{R}^2)) \cap \mathscr{C}((0,\infty); H^2(\mathbf{R}^2)), \\ &0 \le c \in \mathscr{C}([0,\infty); H^2(\mathbf{R}^2)) \cap \mathscr{C}^1((0,\infty); H^1(\mathbf{R}^2)) \cap \mathscr{C}((0,\infty); H^3(\mathbf{R}^2)) \end{aligned}$$

for the initial functions $0 \le u_0 \in L^1(\mathbf{R}^2) \cap L^2(\mathbf{R}^2)$, $0 \le v_0 \in H^1(\mathbf{R}^2)$ and $0 \le c_0 \in H^2(\mathbf{R}^2)$.

First we show local existence, uniqueness and non-negativity of the solutions for (1.1) such that

$$\begin{aligned} &u \in \mathscr{C}([0,T_{U_0}]; L^2(\mathbf{R}^2)) \cap \mathscr{C}^1((0,T_{U_0}]; H^{-1}(\mathbf{R}^2)) \cap \mathscr{C}((0,T_{U_0}]; H^1(\mathbf{R}^2)), \\ &v \in \mathscr{C}([0,T_{U_0}]; H^1(\mathbf{R}^2)) \cap \mathscr{C}^1((0,T_{U_0}]; L^2(\mathbf{R}^2)) \cap \mathscr{C}((0,T_{U_0}]; H^2(\mathbf{R}^2)), \\ &c \in \mathscr{C}([0,T_{U_0}]; H^2(\mathbf{R}^2)) \cap \mathscr{C}^1((0,T_{U_0}]; H^1(\mathbf{R}^2)) \cap \mathscr{C}((0,T_{U_0}]; H^3(\mathbf{R}^2)) \end{aligned}$$

for initial functions $0 \le u_0 \in L^2(\mathbf{R}^2)$, $0 \le v_0 \in H^1(\mathbf{R}^2)$ and $0 \le c_0 \in H^2(\mathbf{R}^2)$. Here $T_{U_0} > 0$ is a local time depending on $\|u_0\|_{L^2} + \|v_0\|_{H^1} + \|c_0\|_{H^2}$.

Our approach to show the existence and uniqueness of the local solution is done in the framework of the abstract evolution equation with the aid of the analytic semigroup, which is studied in Osaki and Yagi [9, Theorem 3.1]. Then we show the non-negativity of solutions is applied to the truncation method used in Yagi [15]. Finally we show the global existence of the solution by deducing a priori estimate under the additional condition $u_0 \in L^1(\mathbf{R}^2)$.

We organize the paper as follows. In Section 2 we prepare the function spaces, the fundamental inequalities and the results to the Cauchy problem of the abstract evolution equation. Section 3 is devoted to proving the existence and uniqueness and the non-negativity of the local solution to the problem (1.1). Finally in Section 4 we prove the existence of the global solution to (1.1) by obtaining a priori estimates. In appendix, we describe a detailed proof of the existence of the local solution to (1.1).

2. Preliminaries

As is stated in Introduction, we use the analytic semigroup method developed in Osaki and Yagi [9, Theorem 3.1]. To do so we shall list the results in the theories of function spaces and abstract evolution equations (see Adams [1], Freidman [5], Lions and Magenes [7], Taira [10], Tanabe [11, 12], Tribel [13]).

For $-\infty < s_0 < s < s_1 < \infty$, $H^s(\mathbf{R}^2)$ is the interpolation space

$$[H^{s_0}(\mathbf{R}^2), H^{s_1}(\mathbf{R}^2)]_\theta,$$

where $s = (1-\theta)s_0 + \theta s_1$. Then we have

$$\|\cdot\|_{H^s} \le C_\theta \|\cdot\|_{H^{s_0}}^{1-\theta} \|\cdot\|_{H^{s_1}}^{\theta}.$$

When $0 \le s < 1$, $H^s(\mathbf{R}^2) \subset L^p(\mathbf{R}^2)$, $\frac{1}{p} = \frac{1-s}{2}$, with

$$\|\cdot\|_{L^p} \le C_s \|\cdot\|_{H^s}. \tag{2.1}$$

When $s = 1$, $H^1(\mathbf{R}^2) \cap L^p(\mathbf{R}^2) \subset L^q(\mathbf{R}^2)$, $1 \le p \le q < \infty$, with

$$\|\cdot\|_{L^q} \le C_{q,p} \|\cdot\|_{H^1}^{1-p/q} \|\cdot\|_{L^p}^{p/q}. \tag{2.2}$$

When $s > 1$, $H^s(\mathbf{R}^2) \subset \mathscr{B}(\mathbf{R}^2) \cap \mathscr{C}(\mathbf{R}^2)$ with

$$\|\cdot\|_{\mathscr{B}} \le C_s \|\cdot\|_{H^s}. \tag{2.3}$$

Let $1 < p < 2$ and let $0 < \varepsilon < 1$. From (2.1), (2.2) and (2.3) we have

$$\|uv\|_{L^p} \le C_p \|u\|_{L^2} \|v\|_{H^{2-2/p}}, \qquad u \in L^2(\mathbf{R}^2),\ v \in H^{2-2/p}(\mathbf{R}^2), \tag{2.4}$$

$$\|uv\|_{L^2} \le C_\varepsilon \|u\|_{L^2} \|v\|_{H^{1+\varepsilon}}, \qquad u \in L^2(\mathbf{R}^2),\ v \in H^{1+\varepsilon}(\mathbf{R}^2), \tag{2.5}$$

$$\|uv\|_{L^2} \le C_\varepsilon \|u\|_{H^\varepsilon} \|v\|_{H^{1-\varepsilon}}, \qquad u \in H^\varepsilon(\mathbf{R}^2),\ v \in H^{1-\varepsilon}(\mathbf{R}^2), \tag{2.6}$$

$$\|uv\|_{H^1} \le C_\varepsilon \|u\|_{H^1} \|v\|_{H^{1+\varepsilon}}, \qquad u \in H^1(\mathbf{R}^2),\ v \in H^{1+\varepsilon}(\mathbf{R}^2). \tag{2.7}$$

If we identify $L^2(\mathbf{R}^2)$ with its dual space, we have $H^1(\mathbf{R}^2)' = H^{-1}(\mathbf{R}^2)$. This together with (2.6) leads to

$$\|\nabla \cdot \{u \nabla \chi\}\|_{H^{-1}} \le C_\varepsilon \|u\|_{H^{1-\varepsilon}} \|\nabla \chi\|_{H^\varepsilon}, \tag{2.8}$$

for $u \in H^{1-\varepsilon}(\mathbf{R}^2)$ and $\chi \in H^{1+\varepsilon}(\mathbf{R}^2)$. We also have, from (2.7),

$$\|\nabla \cdot \{u \nabla \chi\}\|_{L^2} \le C_\varepsilon \|u\|_{H^{1+\varepsilon}} \|\nabla \chi\|_{H^1}, \tag{2.9}$$

for $u \in H^{1+\varepsilon}(\mathbf{R}^2)$ and $\chi \in H^2(\mathbf{R}^2)$.

Let $\tilde{\chi}(\cdot)$ be an extended function of $\chi(\cdot)$ in (0.2) on $(-\infty, \infty)$, which satisfies

$$\sup_{-\infty<c<\infty}\left|\frac{d^i\tilde{\chi}(c)}{dc^i}\right| < \infty, \qquad i = 1, 2, 3.$$

For any $\varepsilon \in (0, 1]$ let $c \in H^{1+\varepsilon}(\mathbf{R}^2)$. Then for any $0 \le s \le 1$, the multiplication of the $\tilde{\chi}(c)$ defines a bounded and continuous mapping of $H^s(\mathbf{R}^2)$ into itself, and further the following estimates hold:

$$\|\tilde{\chi}(c)\zeta\|_{H^\varepsilon} \le p_\varepsilon(\|c\|_{H^{1+\varepsilon}})\|\zeta\|_{H^\varepsilon}, \qquad c \in H^{1+\varepsilon}(\mathbf{R}^2),\ \zeta \in H^\varepsilon(\mathbf{R}^2), \tag{2.10}$$

$$\|\{\tilde{\chi}(c) - \tilde{\chi}(\xi)\}\zeta\|_{H^\varepsilon} \le p_\varepsilon(\|c\|_{H^{1+\varepsilon}} + \|\xi\|_{H^{1+\varepsilon}})\|c - \xi\|_{H^{1+\varepsilon}}\|\zeta\|_{H^\varepsilon}, \tag{2.11}$$
$$c, \xi \in H^{1+\varepsilon}(\mathbf{R}^2),\ \zeta \in H^\varepsilon(\mathbf{R}^2).$$

where $p_\varepsilon(\cdot)$ denotes some continuous function determined by $\tilde{\chi}(\cdot)$.

Consider the Laplace operator $\tilde{A}_0 = -a_1\Delta + a_0$ $(a_0, a_1 > 0)$ with the domain $\mathscr{D}(\tilde{A}_0) = H^1(\mathbf{R}^2)$. Then we can consider $\tilde{A}_0$ as a bounded linear operator from $\mathscr{D}(\tilde{A}_0)$ to $H^{-1}(\mathbf{R}^2)$. Then we have

$$\mathscr{D}(\tilde{A}_0^{1/2}) = L^2(\mathbf{R}^2).$$

Since

$$H^1(\mathbf{R}^2) \subset L^2(\mathbf{R}^2) \subset H^{-1}(\mathbf{R}^2),$$

the restriction $A_0 = \tilde{A}_{0|L^2(\mathbf{R}^2)}$ of $\tilde{A}_0$ to $L^2(\mathbf{R}^2)$ is defined. Then A_0 is a positive definite self-adjoint operator of $L^2(\mathbf{R}^2)$ with $\mathscr{D}(A_0) = H^2(\mathbf{R}^2)$.

For $\theta > 0$ the fractional power A_0^θ is defined in $L^2(\mathbf{R}^2)$, and is also a positive definite self-adjoint operator with the domain $\mathscr{D}(A_0^\theta) = H^{2\theta}(\mathbf{R}^2)$.

Consider the initial value problem of an abstract semilinear evolution equation

$$\begin{cases} \dfrac{dU}{dt} + AU = F(U), & 0 < t \le T, \\ U(0) = U_0 \end{cases} \tag{2.12}$$

in a Banach space X. Here A is a closed linear operator in X the spectral set of which is contained in a sectorial domain $\Sigma = \{\lambda \in \mathbf{C}; |\arg\lambda| \le \phi\}$, $0 \le \phi < \frac{\pi}{2}$, and satisfies

$$\|(\lambda - A)^{-1}\|_{\mathscr{L}(X)} \le \frac{M}{|\lambda| + 1}, \qquad \lambda \notin \Sigma. \tag{2.13}$$

Therefore, $-A$ generates an analytic semigroup $T(t) = e^{-tA}$ on X and the solution of (2.12) is represented as

$$U(t) = T(t)U_0 + \int_0^t T(t-s)F(U(s))ds, \qquad 0 < t \le T.$$

Furthermore we assume the initial function U_0 and semilinear term $F(U)$ as follows. For $0 \le \alpha \le \eta < 1$

$$U_0 \in \mathscr{D}(A^\alpha), \qquad \|A^\alpha U_0\| \le R \tag{2.14}$$

and $F : \mathscr{D}(A^\eta) \to X$ is a Lipschitz continuous function satisfying

$$\begin{aligned} \|F(U) - F(V)\|_X \le \varphi(\|A^\alpha U\|_X + \|A^\alpha V\|_X) \\ \times \{\|A^\eta(U-V)\|_X + (\|A^\eta U\|_X \\ + \|A^\eta V\|_X + 1)\|A^\alpha(U-V)\|_X\} \end{aligned} \tag{2.15}$$

for $U, V \in \mathscr{D}(A^\eta)$, where $R > 0$ is a constant and $\varphi(\cdot)$ is an increasing continuous function.

Then Osaki and Yagi [9, Theorem 3.1] showed the following.

Theorem 2.1 (Osaki and Yagi). *Under the conditions* (2.13), (2.14) *and* (2.15), *the initial value problem* (2.12) *admits a unique solution in the function space*

$$\begin{cases} U \in \mathscr{C}([0, T_R]; \mathscr{D}(A^\alpha)) \cap \mathscr{C}^1((0, T_R]; X) \cap \mathscr{C}((0, T_R]; \mathscr{D}(A)), \\ t^{1-\alpha} U \in \mathscr{B}((0, T_R]; \mathscr{D}(A)) \end{cases} \tag{2.16}$$

where, $T_R > 0$ is a constant determined by R.

Moreover, the estimate

$$t^{1-\alpha}\|AU(t)\| + \|A^\alpha U(t)\| \le C_R, \qquad 0 < t \le T_R$$

holds with a constant $C_R > 0$ determined by R.

3. Local solutions

In this section we consider the local solution for (1.1). To do so we extend the functions $f_1(\cdot,\cdot)$ and $g_1(\cdot,\cdot)$ on $\mathbf{R}^2$ with the property (1.3), and we write them as

$$\tilde{f}_1(\cdot,\cdot) = (-\mu_1 - \nu_1|u| + \tau_1 v)$$

and

$$\tilde{g}_1(\cdot,\cdot) = (\mu_2 - \nu_2 v - \tau_2 u).$$

Then the system (1.1) is rewritten as

$$
(\widetilde{1.1}) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = d_u \Delta u - \nabla \cdot \{u \nabla \tilde{\chi}(c)\} + \tilde{f}_1(u,v)u & \text{in } \mathbf{R}^2 \times (0,\infty), \\ \dfrac{\partial v}{\partial t} = d_v \Delta v + \tilde{g}_1(u,v)v & \text{in } \mathbf{R}^2 \times (0,\infty), \\ \dfrac{\partial c}{\partial t} = d_c \Delta c + \beta u - \gamma c & \text{in } \mathbf{R}^2 \times (0,\infty), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ c(x,0) = c_0(x) & \text{in } \mathbf{R}^2, \end{cases}
$$

where d_u, d_v, d_c, β, γ are the same positive constants in (1.1).

Theorem 3.2. *For any initial functions $u_0 \in L^2(\mathbf{R}^2)$, $v_0 \in H^1(\mathbf{R}^2)$ and $c_0 \in H^2(\mathbf{R}^2)$ the Cauchy problem $(\widetilde{1.1})$ admits a unique solution such that*

$$
u \in \mathscr{C}([0, T_{U_0}]; L^2(\mathbf{R}^2)) \cap \mathscr{C}^1((0, T_{U_0}]; H^{-1}(\mathbf{R}^2)) \cap \mathscr{C}((0, T_{U_0}]; H^1(\mathbf{R}^2)),
$$

$$
v \in \mathscr{C}([0, T_{U_0}]; H^1(\mathbf{R}^2)) \cap \mathscr{C}^1((0, T_{U_0}]; L^2(\mathbf{R}^2)) \cap \mathscr{C}((0, T_{U_0}]; H^2(\mathbf{R}^2)),
$$

$$
c \in \mathscr{C}([0, T_{U_0}]; H^2(\mathbf{R}^2)) \cap \mathscr{C}^1((0, T_{U_0}]; H^1(\mathbf{R}^2)) \cap \mathscr{C}((0, T_{U_0}]; H^3(\mathbf{R}^2)),
$$

with a constant $T_{U_0} > 0$ depending only on $\|u_0\|_{L^2} + \|v_0\|_{H^1} + \|c_0\|_{H^2}$, and

$$
(3.1) \qquad \sqrt{t}\{\|u(t)\|_{H^1} + \|v(t)\|_{H^2} + \|c(t)\|_{H^3}\} + \|u(t)\|_{L^2} + \|v(t)\|_{H^1} + \|c(t)\|_{H^2} \le C_{U_0}, \qquad 0 < t \le T_{U_0}
$$

holds, where C_{U_0} is a constant depending only on $\|u_0\|_{L^2} + \|v_0\|_{H^1} + \|c_0\|_{H^2}$.

Theorem 3.3. *Suppose $u_0 \in L^2(\mathbf{R}^2)$, $v_0 \in H^1(\mathbf{R}^2)$ and $c_0 \in H^2(\mathbf{R}^2)$ and furthermore $u_0, v_0, c_0 \ge 0$. Then the solutions u, v, c of $(\widetilde{1.1})$ are nonnegative. Therefore the functions u, v, c are solutions of* (1.1).

Proof of Theorem 3.2. By applying the Theorem 2.1, this theorem is proved. In fact, putting

$$
X = H^{-1}(\mathbf{R}^2) \times L^2(\mathbf{R}^2) \times H^1(\mathbf{R}^2),
$$

$$
U = \begin{pmatrix} u \\ v \\ c \end{pmatrix}, \qquad U_0 = \begin{pmatrix} u_0 \\ v_0 \\ c_0 \end{pmatrix}, \qquad A = \begin{pmatrix} \tilde{A}_1 & 0 & 0 \\ 0 & \tilde{A}_2 & 0 \\ -\beta & 0 & A_3 \end{pmatrix}
$$

with

$$
\tilde{A}_1 = -d_u \Delta + \mu_1, \qquad \tilde{A}_2 = -d_v \Delta + 1, \qquad A_3 = -d_c \Delta + \gamma
$$

and

$$(3.2) \qquad F(U) = \begin{pmatrix} -\nabla \cdot \{u\nabla\tilde{\chi}(c)\} + \{\mu_1 + \tilde{f}_1(u,v)\}u \\ \{1 + \tilde{g}_1(u,v)\}v \\ 0 \end{pmatrix},$$

we can rewrite $(\widetilde{1.1})$ as in (2.12). Moreover we see A is the closed operator with the domain

$$\mathscr{D}(A) = H^1(\mathbf{R}^2) \times H^2(\mathbf{R}^2) \times H^3(\mathbf{R}^2)$$

the spectral set of which is contained in a sectorial domain, and satisfies (2.13). By taking initial functions U_0 such that

$$U_0 \in \mathscr{D}(A^{1/2}) = L^2(\mathbf{R}^2) \times H^1(\mathbf{R}^2) \times H^2(\mathbf{R}^2)$$

the condition (2.14) is satisfied with $\alpha = \frac{1}{2}$. For $0 < \varepsilon_0 < \frac{1}{2}$ put $\eta = 1 - \frac{\varepsilon_0}{2}$, from which $\alpha < \eta < 1$ holds. The nonlinear term (3.2)

$$F(U) : \mathscr{D}(A^\eta) = H^{1-\varepsilon_0}(\mathbf{R}^2) \times H^{2-\varepsilon_0}(\mathbf{R}^2) \times H^{3-\varepsilon_0}(\mathbf{R}^2) \to X$$

satisfies

$$\begin{aligned} (3.3) \qquad \|F(U) - F(W)\|_X \le \varphi(&\|A^{1/2}U\|_X + \|A^{1/2}W\|_X) \\ &\times \{\|A^{1-\varepsilon_0/2}(U - W)\|_X + (\|A^{1-\varepsilon_0/2}U\|_X \\ &\quad + \|A^{1-\varepsilon_0/2}W\|_X + 1)\|A^{1/2}(U - W)\|_X\} \end{aligned}$$

for any

$$U = \begin{pmatrix} u \\ v \\ c \end{pmatrix}, \qquad W = \begin{pmatrix} w \\ \rho \\ \xi \end{pmatrix}, \qquad U, W \in \mathscr{D}(A^\eta)$$

which corresponds to the condition (2.15) and is proved in Appendix. Thus from Theorem 2.1 we have Theorem 3.2.

Proof of Theorem 3.3. We show non-negativity of the solutions, which is proved by the same method in Yagi [15, Theorem 2.1]. First, we verify that u, v and c are real valued. Denote the complex conjugate of u, v and c by $\bar{u}$, $\bar{v}$ and $\bar{c}$, respectively. Then $\bar{u}$, $\bar{v}$ and $\bar{c}$ is also a solution to $(\widetilde{1.1})$. Therefore, by uniqueness of the solution, we have $u = \bar{u}$, $v = \bar{v}$ and $c = \bar{c}$.

Next, let $0 \le u_0 \in L^2(\mathbf{R}^2)$, $0 \le v_0 \in H^1(\mathbf{R}^2)$ and $0 \le c_0 \in H^2(\mathbf{R}^2)$. Consider C^3 class function $H(u)$ in $u \in (-\infty, \infty)$ which satisfies

$$\begin{cases} H(u) = 0 & \text{for } u \ge 0, \\ H(u) > 0 & \text{for } u < 0 \end{cases}$$

and further

$$\begin{cases} 0 \le H(u) \le Cu^2, \\ 0 \le H'(u)u \le CH(u), \\ 0 \le H''(u)u^2 \le CH(u) \end{cases}$$

with a some positive constant C. Such the function is, for example,

$$\begin{cases} H(u) = \frac{1}{2}u^2 + \frac{2}{3}u + \frac{1}{4} \quad \text{if } \ u \le -1, \\ H(u) = \frac{1}{12}u^4 \quad \text{if } \ -1 \le u \le 0, \\ H(u) = 0 \quad \text{if } \ u \ge 0. \end{cases}$$

Then we have $H'(u) \in H^1(\mathbf{R}^2)$ for $u \in H^1(\mathbf{R}^2)$. In fact this follows form

$$\begin{aligned} \|H'(u)\|_{L^2} &\le \begin{cases} C(\|u\|_{L^2} + 1), & u \le -1 \\ C\|u^3\|_{L^2}, & u \ge -1 \end{cases} \\ &\le C'(1 + \|u\|_{L^2} + \|u\|_{L^6}^3) \le C''(1 + \|u\|_{L^2} + \|u\|_{H^1}^2 \|u\|_{L^2}) \end{aligned}$$

and

$$\|\nabla H'(u)\|_{L^2} = \|H''(u)\nabla u\|_{L^2} \le \sup_{u \in \mathbf{R}} |H''(u)| \, \|\nabla u\|_{L^2} \le C\|\nabla u\|_{L^2}.$$

Put

$$\varphi(t) = \int_{\mathbf{R}^2} H(u(x,t))dx, \qquad 0 \le t \le T_{U_0}.$$

Then, $\varphi(t) \ge 0$, $0 \le t \le T_{U_0}$. Since $u \in \mathscr{C}([0, T_{U_0}]; L^2(\mathbf{R}^2))$, $\varphi(t)$ is continuous in $t \in [0, T_{U_0}]$. It is easy to see $\varphi(t)$ is also C^1 class in $t \in [0, T_{U_0}]$. Since for any $t > 0$, $u \in H^1(\mathbf{R}^2)$ and $H(u) \in H^1(\mathbf{R}^2)$, the first equality in $(\widetilde{1.1})$ leads to

$$\begin{aligned} \varphi'(t) &= \int_{\mathbf{R}^2} H'(u(x,t)) \frac{\partial u}{\partial t} \, dx \\ &= \int_{\mathbf{R}^2} H'(u)[d_u \Delta u - \nabla \cdot \{u \nabla \tilde{\chi}(c)\} + \tilde{f}_1(u,v)u]dx \\ &= -d_u \int_{\mathbf{R}^2} \nabla H'(u) \cdot \nabla u \, dx + \int_{\mathbf{R}^2} u \nabla H'(u) \cdot \nabla \tilde{\chi}(c) dx + \int_{\mathbf{R}^2} H'(u) \tilde{f}_1(u,v)u \, dx \\ &= -d_u \int_{\mathbf{R}^2} H''(u)|\nabla u|^2 dx + \int_{\mathbf{R}^2} uH''(u) \nabla u \cdot \nabla \tilde{\chi}(c) dx + \int_{\mathbf{R}^2} H'(u) \tilde{f}_1(u,v)u \, dx \\ &= -d_u \Phi + \Psi + \Omega, \qquad 0 \le t \le T_{U_0}. \end{aligned}$$

Here $\Phi \geq 0$. From the Cauchy inequality, (0.2) and (2.5) it follows that for any $0 < \varepsilon_0 < 1$ and $\varepsilon > 0$

$$\begin{aligned}
\Psi &\leq \left(\int_{\mathbf{R}^2} |uH''(u)^{1/2}\nabla\tilde{\chi}(c)|^2 dx\right)^{1/2} \left(\int_{\mathbf{R}^2} |H''(u)^{1/2}\nabla u|^2 dx\right)^{1/2} \\
&= \|uH''(u)^{1/2}\nabla\tilde{\chi}(c)\|_{L^2}\|H''(u)^{1/2}\nabla u\|_{L^2} \\
&\leq \frac{1}{4\varepsilon}\|uH''(u)^{1/2}\nabla\tilde{\chi}(c)\|_{L^2}^2 + \varepsilon\|H''(u)^{1/2}\nabla u\|_{L^2}^2 \\
&= \frac{1}{4\varepsilon}\|uH''(u)^{1/2}\tilde{\chi}'(c)\nabla c\|_{L^2}^2 + \varepsilon\Phi \\
&\leq \frac{C}{4\varepsilon}\|uH''(u)^{1/2}\nabla c\|_{L^2}^2 + \varepsilon\Phi \\
&\leq \frac{C}{4\varepsilon}C_{\varepsilon_0}\|uH''(u)^{1/2}\|_{L^2}^2\|\nabla c\|_{H^{1+\varepsilon_0}}^2 + \varepsilon\Phi \\
&\leq C'\int_{\mathbf{R}^2} H(u)dx\|c\|_{H^{2+\varepsilon_0}}^2 + \varepsilon\Phi \\
&= C'\|c\|_{H^{2+\varepsilon_0}}^2\varphi(t) + \varepsilon\Phi, \qquad 0 < t \leq T_{U_0}.
\end{aligned}$$

Since $0 \leq H'(u)u \leq CH(u)$ and $v \in \mathscr{C}((0, T_{U_0}]; H^2(\mathbf{R}^2))$, it follows from (2.3) that for $0 < \varepsilon_0 < 1$

$$\begin{aligned}
\Omega &= \int_{\mathbf{R}^2} H'(u)u(-\mu_1 - \nu_1|u|)dx + \tau_1\int_{\mathbf{R}^2} H'(u)uv\,dx \\
&\leq \mu_1\int_{\mathbf{R}^2} H'(u)u\,dx + \tau_1\int_{\mathbf{R}^2} H'(u)u|v|dx \\
&\leq C\int_{\mathbf{R}^2} H(u)dx + C'\sup_{x\in\mathbf{R}^2}|v|\int_{\mathbf{R}^2} H(u)dx \\
&\leq C_{\varepsilon_0}(1 + \|v\|_{H^{1+\varepsilon_0}})\varphi(t), \qquad 0 < t \leq T_{U_0}.
\end{aligned}$$

Summing up these, for sufficiently small $\varepsilon > 0$ we have

$$\begin{aligned}
\varphi'(t) &\leq (-d_u + \varepsilon)\Phi + C(\|c\|_{H^{2+\varepsilon_0}}^2 + \|v\|_{H^{1+\varepsilon_0}} + 1)\varphi(t) \\
&\leq C(\|c\|_{H^{2+\varepsilon_0}}^2 + \|v\|_{H^{1+\varepsilon_0}} + 1)\varphi(t), \qquad 0 < t \leq T_{U_0}.
\end{aligned}$$

By the Gronwall inequality we have

$$\varphi(t) \leq \varphi(0)e^{C\int_0^t(\|c(s)\|_{H^{2+\varepsilon_0}}^2 + \|v(s)\|_{H^{1+\varepsilon_0}} + 1)ds}, \qquad 0 < t \leq T_{U_0}. \tag{3.4}$$

From (3.1) we have

$$\|v(t)\|_{H^{1+\varepsilon_0}} \le \|v(t)\|_{H^2} \le \frac{C_{U_0}}{\sqrt{t}}, \qquad 0 < t \le T_{U_0}$$

and so

$$\int_0^t \|v(s)\|_{H^{1+\varepsilon_0}} ds \le \int_0^t \frac{C_{U_0}}{\sqrt{s}} ds < \infty, \qquad 0 < t \le T_{U_0}.$$

Thus for $0 < \varepsilon_0 < 1$ we have $v \in L^1(0, T_{U_0}; H^{1+\varepsilon_0}(\mathbf{R}^2))$. From (3.1)

$$\sqrt{t}\|c(t)\|_{H^3} + \|c(t)\|_{H^2} \le C_{U_0}, \qquad 0 < t \le T_{U_0}$$

holds. By applying interpolation inequality it follows that for $0 < \varepsilon_0 < 1$

$$\|c(t)\|_{H^{2+\varepsilon_0}} \le C_{\varepsilon_0}\|c(t)\|_{H^2}^{1-\varepsilon_0}\|c(t)\|_{H^3}^{\varepsilon_0} \qquad 0 < t \le T_{U_0}.$$

Combining these, we have

$$\|c(t)\|_{H^{2+\varepsilon_0}} \le C_{\varepsilon_0} C_{U_0}^{1-\varepsilon_0}\left(\frac{C_{U_0}}{\sqrt{t}}\right)^{\varepsilon_0} \le Ct^{-\varepsilon_0/2} \qquad 0 < t \le T_{U_0}.$$

Thus for $0 < \varepsilon_0 < 1$

$$\int_0^t \|c(s)\|_{H^{2+\varepsilon_0}}^2 ds \le C^2 \int_0^t s^{-\varepsilon_0} ds < \infty, \qquad 0 < t \le T_{U_0}$$

holds, which means $c \in L^2(0, T_{U_0}; H^{2+\varepsilon_0}(\mathbf{R}^2))$ for $0 < \varepsilon_0 < 1$. Since $u_0 \ge 0$, we have $H(u_0) = 0$, which leads to $\varphi(0) = 0$. By (3.4) for $t \in [0, T_{U_0}]$ we have $\varphi(t) = 0$, that is, $u(t) \ge 0$. Similarly we have $v(t), c(t) \ge 0$, $0 \le t \le T_{U_0}$. The proof is complete.

4. Global solutions

In this section we prove the global existence of (0.1).

Proposition 4.1. *Let* $0 \le u_0 \in L^1(\mathbf{R}^2) \cap L^2(\mathbf{R}^2)$, $0 \le v_0 \in H^1(\mathbf{R}^2)$ *and* $0 \le c_0 \in H^2(\mathbf{R}^2)$. *Let* u, v, c *be local solutions of* (0.1) *on* $[0, T_U)$ *such that*

$$0 \le u \in \mathscr{C}([0, T_U); L^2(\mathbf{R}^2)) \cap \mathscr{C}^1((0, T_U); H^{-1}(\mathbf{R}^2)) \cap \mathscr{C}((0, T_U); H^1(\mathbf{R}^2)),$$

$$0 \le v \in \mathscr{C}([0, T_U); H^1(\mathbf{R}^2)) \cap \mathscr{C}^1((0, T_U); L^2(\mathbf{R}^2)) \cap \mathscr{C}((0, T_{U_0}); H^2(\mathbf{R}^2)),$$

$$0 \le c \in \mathscr{C}([0, T_U); H^2(\mathbf{R}^2)) \cap \mathscr{C}^1((0, T_U); H^1(\mathbf{R}^2)) \cap \mathscr{C}((0, T_{U_0}); H^3(\mathbf{R}^2)).$$

Then $u \in \mathscr{C}([0, T_U); L^1(\mathbf{R}^2))$. *Furthermore, for some continuous increasing function* $p(\cdot)$ *independent of* u, v, c *we have*

$$\|u(t)\|_{L^1} + \|u(t)\|_{L^2} + \|v(t)\|_{H^1} + \|c(t)\|_{H^2} \le p(t + \|u_0\|_{L^1} + \|u_0\|_{L^2} + \|v_0\|_{H^1} + \|c_0\|_{H^2}), \qquad 0 \le t < T_U. \tag{4.1}$$

Proof. First we show $u \in \mathscr{C}([0, T_U); L^1(\mathbf{R}^2))$. We consider the first equation of (1.1) in $H^{-1}(\mathbf{R}^2)$ and rewrite as

$$\begin{cases} \dfrac{du}{dt} + Lu = G(t), & 0 < t < T_U, \\ u(0) = u_0, \end{cases}$$

where

$$L = -d_u \varDelta + \mu_1$$

is a linear operator of $H^1(\mathbf{R}^2)$ to $H^{-1}(\mathbf{R}^2)$ and

$$G(t) = -\nabla \cdot \{u \nabla \chi(c)\} + \{\mu_1 + f_1(u, v)\} u$$

is a function on $\mathscr{C}((0, T_U); H^{-1}(\mathbf{R}^2))$. Let $u_0 \in L^1(\mathbf{R}^2) \cap L^2(\mathbf{R}^2)$. Then u is rewritten as

$$u(t) = e^{-tL} u_0 + \int_0^t e^{-(t-s)L} G(s) ds, \qquad 0 < t < T_U.$$

From (3.1) we have

$$\begin{aligned}
\|G(t)\|_{L^1} &\le \|\nabla \cdot \{u \nabla \chi(c)\}\|_{L^1} + \|\mu_1 + f_1(u, v) u\|_{L^1} \\
&\le \|\nabla u \cdot \nabla \chi(c)\|_{L^1} + \|u \varDelta \chi(c)\|_{L^1} + \nu_1 \|u^2\|_{L^1} + \tau_1 \|uv\|_{L^1} \\
&\le C(\|\nabla u \cdot \nabla c\|_{L^1} + \|u |\nabla c|^2\|_{L^1} + \|u \varDelta c\|_{L^1} + \|u\|_{L^2}^2 + \|u\|_{L^2} \|v\|_{L^2}) \\
&\le C(\|\nabla u\|_{L^2} \|\nabla c\|_{L^2} + \|u\|_{L^2} \|\nabla c\|_{L^4}^2 + \|u\|_{L^2} \|\varDelta c\|_{L^2} + \|u\|_{L^2}^2 \\
&\qquad + \|u\|_{L^2} \|v\|_{L^2}) \\
&\le C' \{\|u\|_{H^1} \|c\|_{H^1} + \|u\|_{L^2} (\|\nabla c\|_{H^1}^2 + \|c\|_{H^2}) + \|u\|_{L^2}^2 + \|u\|_{L^2} \|v\|_{L^2}\} \\
&\le C'' \{\|u\|_{H^1} \|c\|_{H^1} + \|u\|_{L^2} \|c\|_{H^2} (\|c\|_{H^2} + 1) + \|u\|_{L^2}^2 + \|u\|_{L^2} \|v\|_{L^2}\} \\
&\le C_U \left(\frac{1}{\sqrt{t}} + 1 \right), \qquad 0 < t < T_U.
\end{aligned}$$

Therefore we see

$$\int_0^t \|G(s)\|_{L^1} ds < \infty, \qquad 0 < t < T_U. \tag{4.2}$$

By this together with the fact that $-L$ generates an analytic semigroup on $L^1(\mathbf{R}^2)$ (Tanabe, [12, Sec. 5.4]), we have $u \in \mathscr{C}([0, T_U); L^1(\mathbf{R}^2))$.

In fact, let

$$T(t) = e^{-tL},$$

and put

$$W(t) = T(t)u_0, \qquad V(t) = \int_0^t T(t-s)G(s)ds, \qquad 0 < t < T_U.$$

Then

$$u(t) = W(t) + V(t), \qquad 0 < t < T_U.$$

We consider the case $t \neq 0$. Since

$$\|W(t)\|_{L^1} = \|T(t)u_0\|_{L^1} \le C\|u_0\|_{L^1} < \infty,$$

we have $W(t) \in L^1(\mathbf{R}^2)$. Since, for $0 < t + \varepsilon < T_U$,

$$\begin{aligned}\|W(t+\varepsilon) - W(t)\|_{L^1} &= \|T(t+\varepsilon)u_0 - T(t)u_0\|_{L^1}\\ &= \|(T(\varepsilon) - I)T(t)u_0\|_{L^1}\\ &\to 0 \qquad (\varepsilon \to 0),\end{aligned}$$

we have $W(t) \in \mathscr{C}((0, T_U); L^1(\mathbf{R}^2))$. Next we consider $V(t)$. For small $\varepsilon > 0$, put

$$V_\varepsilon(t) = \int_0^{t-\varepsilon} T(t-s)G(s)ds, \qquad \varepsilon < t < T_U.$$

Then it follows from (4.2) that

$$\|V_\varepsilon(t) - V(t)\|_{L^1} \le C\int_{t-\varepsilon}^t \|G(s)\|_{L^1} ds \to 0, \qquad (\varepsilon \to 0),$$

which means V_ε converges to $V(t)$ uniformly in $t \in (0, T_U)$. Since, for $0 < t + h < T_U$,

$$\begin{aligned}&V_\varepsilon(t+h) - V_\varepsilon(t)\\ &\quad= \int_0^{t+h-\varepsilon} T(t+h-s)G(s)ds - \int_0^{t-\varepsilon} T(t-s)G(s)ds\\ &\quad= \int_{t-\varepsilon}^{t+h-\varepsilon} T(t+h-s)G(s)ds + \int_0^{t-\varepsilon} \{T(t+h-s) - T(t-s)\}G(s)ds\\ &\quad= \int_{t-\varepsilon}^{t+h-\varepsilon} T(t+h-s)G(s)ds + \{T(h) - I\}\int_0^{t-\varepsilon} T(t-s)G(s)ds,\end{aligned}$$

according to (4.2) we have

$$\|V_\varepsilon(t+h) - V_\varepsilon(t)\|_{L^1} \le C \int_{t-\varepsilon}^{t+h-\varepsilon} \|G(s)\|_{L^1} ds + \left\| \{T(h) - I\} \int_0^{t-\varepsilon} T(t-s)G(s)ds \right\|_{L^1}$$

$$\to 0 \qquad (h \to 0),$$

which leads us to $V_\varepsilon(t) \in \mathscr{C}((0, T_U); L^1(\mathbf{R}^2))$. Since V_ε converges to $V(t)$ uniformly in t, we have

$$V(t) \in \mathscr{C}((0, T_U); L^1(\mathbf{R}^2)).$$

Now we proceed to the proof of (4.1). To do so we remember the lemma in Osaki and Yagi [9, Lemma 5.2]. Let $\varphi_R(r)$ be a smooth function in $r \in [0, \infty)$ which satisfies the following

$$\begin{cases} \varphi_R = 1 \quad \text{for } 0 \le r \le R, \\ 0 \le \varphi_R \le 1 \quad \text{for } R < r < R+1, \\ \varphi_R = 0 \quad \text{for } R+1 \le r. \end{cases}$$

and

$$\sup_{0 \le r < \infty} |\varphi_R^{(i)}(r)| \le C \qquad (i = 1, 2, 3).$$

Lemma 4.1. *For $u \in L^1(\mathbf{R}^2)$, $U \in L^2(\mathbf{R}^2)$ and $V \in H^1(\mathbf{R}^2)$*

$$\lim_{R \to \infty} \int_{\mathbf{R}^2} \Delta u \varphi_R(|x|) dx = 0, \tag{4.3}$$

$$\lim_{R \to \infty} \int_{\mathbf{R}^2} U \nabla V \cdot \nabla \varphi_R(|x|) dx = 0 \tag{4.4}$$

hold in distribution sense.

Proof of inequality (4.1). We shall prove (4.1) in series of steps.

Step 1 (L^p estimate of v ($p \ge 2$)) From the assumption of Proposition 4.1 we see $v \in H^1(\mathbf{R}^2)$ for each $t \in [0, T_U)$. Put $U = v^{p-1}$, $V = v$. Since

$$\|v(t)^{p-1}\|_{L^2} = \|v(t)\|_{L^{2(p-1)}}^{p-1} \le \|v(t)\|_{H^1}^{p-1} < \infty,$$

it follows $U \in L^2(\mathbf{R}^2)$, $V \in H^1(\mathbf{R}^2)$. Thus we apply Lemma 4.1.

Multiply the second equation of (0.1) by v^{p-1}, $p \ge 2$ and $\varphi_R(|x|) \ge 0$ and integrate this on $\mathbf{R}^2$. Then we have

$$\begin{aligned}
&\frac{1}{p}\int_{\mathbf{R}^2}\varphi_R(|x|)\frac{\partial}{\partial t}v^p\,dx\\
&\qquad = d_v\int_{\mathbf{R}^2}v^{p-1}\Delta v\varphi_R(|x|)dx+\int_{\mathbf{R}^2}v^{p-1}g(u,v)\varphi_R(|x|)dx\\
&\qquad = -d_v\int_{\mathbf{R}^2}\{(p-1)v^{p-2}|\nabla v|^2\varphi_R(|x|)+v^{p-1}\nabla v\cdot\nabla\varphi_R(|x|)\}dx\\
&\qquad\quad +\int_{\mathbf{R}^2}v^{p-1}g(u,v)\varphi_R(|x|)dx\\
&\qquad = -d_v\int_{\mathbf{R}^2}(p-1)v^{p-2}|\nabla v|^2\varphi_R(|x|)dx-d_v\int_{\mathbf{R}^2}U\nabla V\cdot\nabla\varphi_R(|x|)dx\\
&\qquad\quad +\int_{\mathbf{R}^2}v^{p-1}g(u,v)\varphi_R(|x|)dx.
\end{aligned}$$

Letting $R\to\infty$, we have

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbf{R}^2}v^p\,dx=-d_v\int_{\mathbf{R}^2}(p-1)v^{p-2}|\nabla v|^2dx+\int_{\mathbf{R}^2}v^{p-1}g(u,v)dx,$$

because $\lim_{R\to\infty}\varphi_R(|x|)=1$. From

$$v^{p-1}g(u,v)=v^{p-1}(\mu_2-v_2v-\tau_2u)v\le\mu_2v^p$$

it follows that

$$\frac{1}{p}\frac{d}{dt}\int_{\mathbf{R}^2}v^p\,dx\le\mu_2\int_{\mathbf{R}^2}v^p\,dx.$$

Multiply this by $e^{-p\mu_2s}\ge 0$ and integrate in $s\in(0,t)$. Then we have

$$\frac{1}{p}\int_0^t e^{-p\mu_2s}\frac{d}{ds}\int_{\mathbf{R}^2}v^p\,dxds\le\int_0^t\mu_2e^{-p\mu_2s}\int_{\mathbf{R}^2}v^p\,dxds$$

and therefore

$$\begin{aligned}
&\frac{1}{p}(e^{-p\mu_2t}\|v(t)\|_{L^p}^p-\|v_0\|_{L^p}^p)-\frac{1}{p}\int_0^t-p\mu_2e^{-p\mu_2s}\int_{\mathbf{R}^2}v^p\,dxds\\
&\qquad\le\int_0^t\mu_2e^{-p\mu_2s}\int_{\mathbf{R}^2}v^p\,dxds.
\end{aligned}$$

Consequently we have

$$\|v(t)\|_{L^p}^p\le e^{p\mu_2t}\|v_0\|_{L^p}^p\le e^{p\mu_2t}\|v_0\|_{H^1}^p,\qquad 0\le t<T_U. \tag{4.5}$$

Step 2 (L^1 estimate of $u(t,\cdot)$ and $L^2(0,t)$-estimate of $\|u(s)\|_{L^2}$) First we shall get an L^1 estimate of u. Now we note that the assumptions of Proposition 4.1 lead to

$$u \in \mathscr{C}([0,T_U); L^1(\mathbf{R}^2)).$$

For $t \in [0,T_U)$ put $U = u\chi'(c)$, $V = c$. Since $u \in L^2(\mathbf{R}^2)$, $c \in H^2(\mathbf{R}^2)$, we have

$$\|u\chi'(c)\|_{L^2} \le \sup_{0\le c} |\chi'(c)| \, \|u\|_{L^2} \le C\|u\|_{L^2} < \infty,$$

and so $U \in L^2(\mathbf{R}^2)$, $V \in H^1(\mathbf{R}^2)$. Thus we can apply the Lemma 4.1. Multiply the first equation of (1.1) by $\varphi_R(|x|)$, and integrate this on $\mathbf{R}^2$. Then we have

$$\begin{aligned}
&\int_{\mathbf{R}^2} \varphi_R(|x|)\frac{\partial}{\partial t} u \, dx \\
&\quad = d_u \int_{\mathbf{R}^2} \Delta u \varphi_R(|x|)dx - \int_{\mathbf{R}^2} \nabla \cdot \{u\nabla\chi(c)\}\varphi_R(|x|)dx + \int_{\mathbf{R}^2} f(u,v)\varphi_R(|x|)dx \\
&\quad = d_u \int_{\mathbf{R}^2} \Delta u \varphi_R(|x|)dx + \int_{\mathbf{R}^2} U\nabla V \cdot \nabla\varphi_R(|x|)dx + \int_{\mathbf{R}^2} f(u,v)\varphi_R(|x|)dx.
\end{aligned}$$

Letting $R \to \infty$, we have

$$\frac{d}{dt}\int_{\mathbf{R}^2} u \, dx = \int_{\mathbf{R}^2} f(u,v)dx.$$

Integrating this in $s \in (0,t)$, we get

$$\|u(t)\|_{L^1} = \|u_0\|_{L^1} + \int_0^t \int_{\mathbf{R}^2} f(u,v)dxds, \qquad 0 \le t < T_U. \tag{4.6}$$

From the Cauchy inequality it follows that

$$\begin{aligned}
f(u,v) &= -\mu_1 u - v_1 u^2 + \tau_1 uv \\
&\le -\mu_1 u - v_1 u^2 + \tau_1\left(\varepsilon u^2 + \frac{1}{4\varepsilon}v^2\right) \\
&\le -(v_1 - \tau_1\varepsilon)u^2 + \frac{\tau_1}{4\varepsilon}v^2.
\end{aligned} \tag{4.7}$$

Take $\varepsilon = \dfrac{v_1}{\tau_1}$. Then from (4.7) we have

$$\begin{aligned}
\int_0^t \int_{\mathbf{R}^2} f(u,v)dxds &\le \frac{\tau_1^2}{4v_1}\int_0^t \|v(s)\|_{L^2}^2 ds \le \frac{\tau_1^2}{4v_1}\|v_0\|_{L^2}^2 \int_0^t e^{2\mu_2 s}\, ds \\
&\le \frac{\tau_1^2}{4v_1}\|v_0\|_{L^2}^2 \frac{1}{2\mu_2}(e^{2\mu_2 t} - 1) \le Ce^{2\mu_2 t}\|v_0\|_{L^2}^2, \qquad 0 \le t < T_U.
\end{aligned}$$

Therefore we have

$$\|u(t)\|_{L^1} \le \|u_0\|_{L^1} + Ce^{2\mu_2 t}\|v_0\|_{L^2}^2, \qquad 0 \le t < T_U. \tag{4.8}$$

Furthermore from (4.6) it follows that

$$-\int_0^t \int_{\mathbf{R}^2} f(u,v)\,dxds \le \|u_0\|_{L^1}.$$

Taking $\varepsilon = \dfrac{\nu_1}{2\tau_1}$ in (4.7), we have

$$u^2 \le -\frac{2}{\nu_1} f(u,v) + \left(\frac{\tau_1}{\nu_1}\right)^2 v^2.$$

By integration in $x \in \mathbf{R}^2$ and $s \in (0,t)$,

$$\begin{aligned}
\int_0^t \|u(s)\|_{L^2}^2 ds &\le -\frac{2}{\nu_1}\int_0^t \int_{\mathbf{R}^2} f(u,v)\,dxds + \left(\frac{\tau_1}{\nu_1}\right)^2 \int_0^t \|v(s)\|_{L^2}^2 ds \\
&\le \frac{2}{\nu_1}\|u_0\|_{L^1} + \left(\frac{\tau_1}{\nu_1}\right)^2 \int_0^t \|v(s)\|_{L^2}^2 ds \\
&\le \frac{2}{\nu_1}\|u_0\|_{L^1} + \left(\frac{\tau_1}{\nu_1}\right)^2 \|v_0\|_{L^2}^2 \int_0^t e^{2\mu_2 s}\, ds \\
&\le \frac{2}{\nu_1}\|u_0\|_{L^1} + Ce^{2\mu_2 t}\|v_0\|_{L^2}^2 \\
&\le C'(\|u_0\|_{L^1} + e^{2\mu_2 t}\|v_0\|_{L^2}^2), \qquad 0 \le t < T_U.
\end{aligned} \tag{4.9}$$

Step 3 (H^1-estimate of c) First note that we see from Proposition 4.1 that for $t \in (0, T_U)$, $c \in H^3(\mathbf{R}^2)$ and $c_t \in H^1(\mathbf{R}^2)$. Then putting $U = c$, $U' = c_t$, $V = c$, we have $U, U' \in L^2(\mathbf{R}^2)$, $V \in H^1(\mathbf{R}^2)$. Now we apply Lemma 4.1. Multiply the third equation of (1.1) by $c \ge 0$ and $\varphi_R(|x|) \ge 0$, and integrate this on $\mathbf{R}^2$. Then we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbf{R}^2} \varphi_R(|x|)\frac{\partial}{\partial t}c^2\, dx \\
&\quad = d_c \int_{\mathbf{R}^2} c\Delta c \varphi_R(|x|)dx + \beta \int_{\mathbf{R}^2} cu\varphi_R(|x|)dx - \gamma \int_{\mathbf{R}^2} c^2 \varphi_R(|x|)dx \\
&\quad = -d_c \int_{\mathbf{R}^2} |\nabla c|^2 \varphi_R(|x|)dx - d_c \int_{\mathbf{R}^2} U\nabla V \cdot \nabla \varphi_R(|x|)dx \\
&\qquad + \beta \int_{\mathbf{R}^2} cu\varphi_R(|x|)dx - \gamma \int_{\mathbf{R}^2} c^2 \varphi_R(|x|)dx.
\end{aligned}$$

Letting $R \to \infty$, we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^2} c^2\,dx = -d_c\int_{\mathbf{R}^2}|\nabla c|^2 dx + \beta\int_{\mathbf{R}^2} cu\,dx - \gamma\int_{\mathbf{R}^2} c^2\,dx.$$

From Cauchy's inequality it follows that

$$\begin{aligned}\beta\int_{\mathbf{R}^2} cu\,dx &\le \beta\int_{\mathbf{R}^2}\left(\frac{\gamma}{2\beta}c^2 + \frac{\beta}{2\gamma}u^2\right)dx\\ &= \frac{\gamma}{2}\int_{\mathbf{R}^2} c^2\,dx + \frac{\beta^2}{2\gamma}\int_{\mathbf{R}^2} u^2\,dx,\end{aligned}$$

which leads to

$$\frac{d}{dt}\int_{\mathbf{R}^2} c^2\,dx + 2d_c\int_{\mathbf{R}^2}|\nabla c|^2 dx + \gamma\int_{\mathbf{R}^2} c^2\,dx \le \frac{\beta^2}{\gamma}\|u(t)\|_{L^2}^2.$$

Integrating this in t, we have, by (4.9)

$$\begin{aligned}\|c(t)\|_{L^2}^2 &\le \|c_0\|_{L^2}^2 + \frac{\beta^2}{\gamma}C(\|u_0\|_{L^1} + e^{2\mu_2 t}\|v_0\|_{L^2}^2)\\ &\le C'(\|u_0\|_{L^1} + e^{2\mu_2 t}\|v_0\|_{L^2}^2 + \|c_0\|_{L^2}^2), \qquad 0 \le t < T_U.\end{aligned} \tag{4.10}$$

Next we shall obtain the L^2-estimate of ∇c. Multiply the third equation of (1.1) by Δc and $\varphi_R(|x|) \ge 0$ and integrate this on $\mathbf{R}^2$. Then we have

$$\begin{aligned}&\int_{\mathbf{R}^2}\varphi_R(|x|)\Delta c\frac{\partial}{\partial t}c\,dx\\ &\quad= d_c\int_{\mathbf{R}^2}|\Delta c|^2\varphi_R(|x|)dx + \beta\int_{\mathbf{R}^2} u\Delta c\varphi_R(|x|)dx - \gamma\int_{\mathbf{R}^2} c\Delta c\varphi_R(|x|)dx.\end{aligned}$$

As for the left hand side we have

$$\int_{\mathbf{R}^2}\varphi_R(|x|)\Delta c\frac{\partial}{\partial t}c\,dx = -\frac{1}{2}\int_{\mathbf{R}^2}\varphi_R(|x|)\frac{\partial}{\partial t}|\nabla c|^2 dx - \int_{\mathbf{R}^2} U'\nabla V\cdot\nabla\varphi_R(|x|)dx.$$

On the other hand, as for the third term of the right hand side we have

$$-\gamma\int_{\mathbf{R}^2} c\Delta c\varphi_R(|x|)dx = \gamma\int_{\mathbf{R}^2}|\nabla c|^2\varphi_R(|x|)dx + \gamma\int_{\mathbf{R}^2} U\nabla V\cdot\nabla\varphi_R(|x)dx.$$

Therefore letting $R \to \infty$, we have

$$-\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^2}|\nabla c|^2 dx = d_c\int_{\mathbf{R}^2}|\Delta c|^2 dx + \beta\int_{\mathbf{R}^2} u\Delta c\,dx + \gamma\int_{\mathbf{R}^2}|\nabla c|^2 dx,$$

from which it follows that

$$\frac{d}{dt}\int_{\mathbf{R}^2}|\nabla c|^2dx+2d_c\int_{\mathbf{R}^2}|\Delta c|^2dx+2\gamma\int_{\mathbf{R}^2}|\nabla c|^2dx\le 2\beta\int_{\mathbf{R}^2}|u\Delta c|dx.$$

Since Cauchy's inequality yields

$$2\beta\int_{\mathbf{R}^2}|u\Delta c|dx\le 2\beta\int_{\mathbf{R}^2}\left(\frac{\beta}{2d_c}u^2+\frac{d_c}{2\beta}|\Delta c|^2\right)dx=\frac{\beta^2}{d_c}\int_{\mathbf{R}^2}u^2\,dx+d_c\int_{\mathbf{R}^2}|\Delta c|^2dx,$$

we have

(4.11)
$$\frac{d}{dt}\int_{\mathbf{R}^2}|\nabla c|^2dx+d_c\int_{\mathbf{R}^2}|\Delta c|^2dx+2\gamma\int_{\mathbf{R}^2}|\nabla c|^2dx\le\frac{\beta^2}{d_c}\|u(t)\|_{L^2}^2,\qquad 0\le t<T_U.$$

Integrate this in t. Then it follows from (4.9) that

$$\begin{aligned}\|\nabla c(t)\|_{L^2}^2&\le\|\nabla c_0\|_{L^2}^2+\frac{\beta^2}{d_c}C(\|u_0\|_{L^1}+e^{2\mu_2t}\|v_0\|_{L^2}^2)\\&\le C'(\|u_0\|_{L^1}+e^{2\mu_2t}\|v_0\|_{L^2}^2+\|\nabla c_0\|_{L^2}^2),\qquad 0\le t<T_U.\end{aligned}$$

This together with (4.10) leads to

$$\|c(t)\|_{H^1}^2\le C'(\|u_0\|_{L^1}+e^{2\mu_2t}\|v_0\|_{L^2}^2+\|c_0\|_{H^1}^2),\qquad 0\le t<T_U. \tag{4.12}$$

Step 4 (L^1-estimate of $(u+1)\log(u+1)$) Since it follows from Proposition 4.1 that $u\in H^1(\mathbf{R}^2)$ and $c\in H^2(\mathbf{R}^2)$ for $t\in(0,T_U)$,

$$\begin{aligned}&\|\log(u+1)\|_{L^2}\le\|u\|_{L^2},\\&\|\log(u+1)u\chi'(c)\|_{L^2}\le\sup_{0\le c}|\chi'(c)|\,\|u^2\|_{L^2}\le C\|u\|_{H^1}^2,\\&\|(\log(u+1)-u)\chi'(c)\|_{L^2}\le\sup_{0\le c}|\chi'(c)|2\|u\|_{L^2}\le C\|u\|_{L^2}.\end{aligned}$$

Therefore putting

$$U=\log(u+1),\qquad U_1=\log(u+1)u\chi'(c),\qquad U_2=(\log(u+1)-u)\chi'(c),$$
$$V=u,\qquad V_1=c,$$

we have

$$U,U_1,U_2\in L^2(\mathbf{R}^2),\qquad V,V_1\in H^1(\mathbf{R}^2).$$

Therefore we can apply Lemma 4.1. Multiply the first equation of (1.1) by $\log(u+1)$ and $\varphi_R(|x|)$, and integrate this on $\mathbf{R}^2$. Then we have

$$\begin{aligned}(4.13)\qquad &\int_{\mathbf{R}^2} \varphi_R(|x|)\log(u+1)\frac{\partial}{\partial t}u\,dx \\ &= d_u\int_{\mathbf{R}^2}\log(u+1)\Delta u\varphi_R(|x|)dx \\ &\quad -\int_{\mathbf{R}^2}\log(u+1)\nabla\cdot\{u\nabla\chi(c)\}\varphi_R(|x|)dx \\ &\quad +\int_{\mathbf{R}^2}\log(u+1)f(u,v)\varphi_R(|x|)dx \\ &= d_u\Phi-\Psi+\int_{\mathbf{R}^2}\log(u+1)f(u,v)\varphi_R(|x|)dx,\end{aligned}$$

where

$$\Phi=\int_{\mathbf{R}^2}\log(u+1)\Delta u\varphi_R(|x|)dx \quad\text{and}\quad \Psi=\int_{\mathbf{R}^2}\log(u+1)\nabla\cdot\{u\nabla\chi(c)\}\varphi_R(|x|)dx.$$

By

$$\log(u+1)\frac{\partial}{\partial t}u=\frac{\partial}{\partial t}\{(u+1)\log(u+1)-u\},$$

we have

$$(4.14)\qquad \int_{\mathbf{R}^2}\varphi_R(|x|)\log(u+1)\frac{\partial}{\partial t}u\,dx=\int_{\mathbf{R}^2}\varphi_R(|x|)\frac{\partial}{\partial t}\{(u+1)\log(u+1)-u\}dx.$$

On the other hand since

$$\Phi=-\int_{\mathbf{R}^2}\frac{1}{u+1}|\nabla u|^2\varphi_R(|x|)dx-\int_{\mathbf{R}^2}U\nabla V\cdot\nabla\varphi_R(|x|)dx,$$

by $R\to\infty$ we have

$$(4.15)\qquad \Phi\to-\int_{\mathbf{R}^2}\frac{1}{u+1}|\nabla u|^2dx.$$

Similarly we have

$$\begin{aligned}\Psi&=-\int_{\mathbf{R}^2}\frac{u}{u+1}\nabla u\cdot\nabla\chi(c)\varphi_R(|x|)dx-\int_{\mathbf{R}^2}U_1\nabla V_1\cdot\nabla\varphi_R(|x|)dx \\ &=-\Omega-\int_{\mathbf{R}^2}U_1\nabla V_1\cdot\nabla\varphi_R(|x|)dx,\end{aligned}$$

where

$$\Omega = \int_{\mathbf{R}^2} \frac{u}{u+1} \nabla u \cdot \nabla \chi(c) \varphi_R(|x|) dx.$$

From

$$\begin{aligned}
&\int_{\mathbf{R}^2} \{\log(u+1) - u\} \Delta \chi(c) \varphi_R(|x|) dx \\
&\quad = -\int_{\mathbf{R}^2} \frac{u}{u+1} \nabla u \cdot \nabla \chi(c) \varphi_R(|x|) dx - \int_{\mathbf{R}^2} U_2 \nabla V_1 \cdot \nabla \varphi_R(|x|) dx \\
&\quad = -\Omega - \int_{\mathbf{R}^2} U_2 \nabla V_1 \cdot \nabla \varphi_R(|x|) dx,
\end{aligned}$$

we have

$$\begin{aligned}
\Psi = &\int_{\mathbf{R}^2} \{\log(u+1) - u\} \Delta \chi(c) \varphi_R(|x|) dx \\
&+ \int_{\mathbf{R}^2} U_2 \nabla V_1 \cdot \nabla \varphi_R(|x|) dx - \int_{\mathbf{R}^2} U_1 \nabla V_1 \cdot \nabla \varphi_R(|x|) dx.
\end{aligned}$$

By $R \to \infty$ we have

$$\Psi \to \int_{\mathbf{R}^2} \{\log(u+1) - u\} \Delta \chi(c) dx.$$

Therefore by (4.13), (4.14) and (4.15) we have

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbf{R}^2} \{(u+1) \log(u+1) - u\} dx \\
&\quad = -d_u \int_{\mathbf{R}^2} \frac{1}{u+1} |\nabla u|^2 dx - \int_{\mathbf{R}^2} \{\log(u+1) - u\} \Delta \chi(c) dx \\
&\qquad + \int_{\mathbf{R}^2} \log(u+1) f(u, v) dx.
\end{aligned}$$

From

$$\Delta \chi(c) = \chi''(c) |\nabla c|^2 + \chi'(c) \Delta c,$$

it follows that

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbf{R}^2} \{(u+1) \log(u+1) - u\} dx + d_u \int_{\mathbf{R}^2} \frac{1}{u+1} |\nabla u|^2 dx \\
&\quad \le \int_{\mathbf{R}^2} |\{\log(u+1) - u\} \Delta \chi(c)| dx + \int_{\mathbf{R}^2} \log(u+1) f(u, v) dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\mathbf{R}^2} |\{\log(u+1)-u\}\chi''(c)|\nabla c|^2|dx + \int_{\mathbf{R}^2} |\{\log(u+1)-u\}\chi'(c)\Delta c|dx \\
&\quad + \int_{\mathbf{R}^2} \log(u+1) f(u,v)dx \\
&= \phi + \psi + \omega,
\end{aligned}
$$

where

$$
\begin{aligned}
\phi &= \int_{\mathbf{R}^2} |\{\log(u+1)-u\}\chi''(c)|\nabla c|^2|dx, \\
\psi &= \int_{\mathbf{R}^2} |\{\log(u+1)-u\}\chi'(c)\Delta c|dx, \\
\omega &= \int_{\mathbf{R}^2} \log(u+1)(-\mu_1 u - v_1 u^2 + \tau_1 uv)dx.
\end{aligned}
$$

From (2.2) we have

$$
\begin{aligned}
&\phi \leq C\|u\|_{L^2}\| \, |\nabla c|^2\|_{L^2} = C\|u\|_{L^2}\|\nabla c\|_{L^4}^2 \leq C\|u\|_{L^2}(C_{4,2}\|\nabla c\|_{H^1}^{1-2/4}\|\nabla c\|_{L^2}^{2/4})^2 \\
&\quad \leq C'\|u\|_{L^2}\|c\|_{H^2}\|c\|_{H^1} \leq C'\left\{\frac{d_c}{4C'}\|c\|_{H^2}^2 + \frac{1}{4\frac{d_c}{4C'}}(\|u\|_{L^2}\|c\|_{H^1})^2\right\} \\
&\quad = \frac{d_c}{4}\|\Delta c\|_{L^2}^2 + \frac{d_c}{4}\|c\|_{H^1}^2 + \frac{(C')^2}{d_c}\|u\|_{L^2}^2\|c\|_{H^1}^2 \leq \frac{d_c}{4}\|\Delta c\|_{L^2}^2 + C''\|c\|_{H^1}^2(\|u\|_{L^2}^2 + 1).
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
\psi &= \int_{\mathbf{R}^2} |\{\log(u+1)-u\}\chi'(c)\Delta c|dx \\
&\leq C\|u\|_{L^2}\|\Delta c\|_{L^2} \leq \frac{d_c}{4}\|\Delta c\|_{L^2}^2 + C'\|u\|_{L^2}^2.
\end{aligned}
$$

From Young's inequality it follows that for small $\varepsilon > 0$

$$
\begin{aligned}
\omega &= \int_{\mathbf{R}^2} \log(u+1)(-\mu_1 u - v_1 u^2 + \tau_1 uv)dx \\
&\leq -v_1 \int_{\mathbf{R}^2} \log(u+1)u^2 \, dx + \tau_1 \int_{\mathbf{R}^2} \log(u+1)uv \, dx \\
&\leq -v_1 \|\log(u+1)\|_{L^3}^3 + \tau_1 \|\log(u+1)uv\|_{L^1} \\
&\leq -v_1 \|\log(u+1)\|_{L^3}^3 + \tau_1 \|\log(u+1)\|_{L^3}\|u\|_{L^2}\|v\|_{L^6}
\end{aligned}
$$

$$\leq -\nu_1 \|\log(u+1)\|_{L^3}^3 + \tau_1 \left\{ \varepsilon \|\log(u+1)\|_{L^3}^3 + C\left(\frac{1}{\varepsilon}\right) (\|u\|_{L^2} \|v\|_{L^6})^{3/2} \right\}$$

$$\leq \tau_1 C\left(\frac{1}{\varepsilon}\right) \left\{ \frac{3}{4} \|u\|_{L^2}^{(3/2)\cdot(4/3)} + \frac{1}{4} \|v\|_{L^6}^{(3/2)\cdot 4} \right\}$$

$$\leq C(\|u\|_{L^2}^2 + \|v\|_{L^6}^6).$$

Consequently we have

$$\frac{d}{dt}\int_{\mathbf{R}^2} \{(u+1)\log(u+1) - u\} dx$$
$$\leq \frac{d_c}{2} \|\Delta c\|_{L^2}^2 + C\|c\|_{H^1}^2 (\|u\|_{L^2}^2 + 1) + C' \|u\|_{L^2}^2 + C'' \|v\|_{L^6}^6,$$

from which together with (4.11) it follows that

$$\frac{d}{dt}\int_{\mathbf{R}^2} [\{(u+1)\log(u+1) - u\} + |\nabla c|^2] dx$$
$$\leq \frac{d_c}{2} \|\Delta c\|_{L^2}^2 + C\|c\|_{H^1}^2 (\|u\|_{L^2}^2 + 1)$$
$$+ C' \|u\|_{L^2}^2 + C'' \|v\|_{L^6}^6 - d_c \|\Delta c\|_{L^2}^2 + \frac{\beta^2}{d_c} \|u\|_{L^2}^2$$
$$\leq C''' \{ \|c\|_{H^1}^2 \|u\|_{L^2}^2 + \|c\|_{H^1}^2 + \|u\|_{L^2}^2 + \|v\|_{L^6}^6 \}$$
$$\leq C'''' \{ (\|c\|_{H^1}^2 + 1)(\|u\|_{L^2}^2 + 1) + \|v\|_{L^6}^6 \}, \qquad 0 < t < T_U.$$

Integrating this in t, we have

$$\begin{aligned} \|(u(t)+1)\log(u(t)+1)\|_{L^1} &- \|(u_0+1)\log(u_0+1)\|_{L^1} \\ &- (\|u(t)\|_{L^1} - \|u_0\|_{L^1}) + \|\nabla c(t)\|_{L^2}^2 - \|\nabla c_0\|_{L^2}^2 \\ &\leq C\int_0^t \{ (\|c(s)\|_{H^1}^2 + 1)(\|u(s)\|_{L^2}^2 + 1) + \|v(s)\|_{L^6}^6 \} ds. \end{aligned} \tag{4.16}$$

Now put

$$N_{\log}^1(u) = \|(u+1)\log(u+1)\|_{L^1}, \qquad u \in L^1(\mathbf{R}^2) \cap L^2(\mathbf{R}^2),\ u \geq 0.$$

Make use of the inequalities (4.5), (4.9) and (4.12). Then from (4.16) we have

$$
\begin{aligned}
N_{\log}^1(u(t)) &\le N_{\log}^1(u_0) + \|u(t)\|_{L^1} + \|\nabla c_0\|_{L^2}^2 \\
&\quad + C(\|u_0\|_{L^1} + e^{2\mu_2 t}\|v_0\|_{L^2}^2 + \|c_0\|_{H^1}^2)\int_0^t (\|u(s)\|_{L^2}^2 + 1)ds \\
&\quad + C\int_0^t e^{6\mu_2 s}\|v_0\|_{L^6}^6 ds \\
&\le N_{\log}^1(u_0) + \|u_0\|_{L^1} + Ce^{2\mu_2 t}\|v_0\|_{L^2}^2 + \|\nabla c_0\|_{L^2}^2 \\
&\quad + C(\|u_0\|_{L^1} + e^{2\mu_2 t}\|v_0\|_{L^2}^2 + \|c_0\|_{H^1}^2 + 1)(\|u_0\|_{L^1} + e^{2\mu_2 t}\|v_0\|_{L^2}^2 + t) \\
&\quad + Ce^{6\mu_2 t}\|v_0\|_{L^6}^6 \\
&\le N_{\log}^1(u_0) + Ce^{6\mu_2 t}(1 + t + \|u_0\|_{L^1} + \|v_0\|_{H^1} + \|c_0\|_{H^1})^6, \quad 0 < t < T_U.
\end{aligned}
$$

In the following Step 5, we need the following lemma which is shown by Biler, Hebisch and Nadzieja [4, p. 1199] and is used by Osaki and Yagi [9, Lemma 5.3].

Lemma 4.2. *For any* $\zeta > 0$

$$\|u\|_{L^3}^3 \le \zeta\|u\|_{H^1}^2 N_{\log}^1(u) + q(\zeta^{-1})\|u\|_{L^1}, \qquad u \in L^1(\mathbf{R}^2) \cap H^1(\mathbf{R}^2),\ u \ge 0$$

holds, where $q(\cdot)$ *is an increasing continuous function.*

Step 5 (Final stage) Now we are in a position to prove our estimate. So we begin to get an L^2-estimate of u, then H^1-estimate of v and finally H^2-estimate of c.

Step 5.1 (L^2-estimate of u) Put $U = V = u$, $U_1 = u^2\chi'(c)$, $V_1 = c$. Then by proved part of Proposition 4.1 we have $U, U_1 \in L^2(\mathbf{R}^2)$, $V, V_1 \in H^1(\mathbf{R}^2)$. So we apply Lemma 4.2. Multiplying the first equation of (1.1) by $u \ge 0$ and $\varphi_R(|x|) \ge 0$, and integrating this on $\mathbf{R}^2$, we have

$$
\begin{aligned}
(4.17)\qquad \frac{1}{2}\int_{\mathbf{R}^2} \varphi_R(|x|)\frac{\partial}{\partial t}u^2\,dx
&= d_u\int_{\mathbf{R}^2} u\Delta u\varphi_R(|x|)dx - \int_{\mathbf{R}^2} u\nabla\cdot\{u\nabla\chi(c)\}\varphi_R(|x|)dx \\
&\quad + \int_{\mathbf{R}^2} uf(u,v)\varphi_R(|x|)dx \\
&= d_u\Phi - \Psi + \int_{\mathbf{R}^2} uf(u,v)\varphi_R(|x|)dx.
\end{aligned}
$$

Since

$$\Phi = \int_{\mathbf{R}^2} u \Delta u \varphi_R(|x|) dx$$
$$= -\int_{\mathbf{R}^2} |\nabla u|^2 \varphi_R(|x|) dx - \int_{\mathbf{R}^2} U \nabla V \cdot \nabla \varphi_R(|x|) dx,$$

by $R \to \infty$ we have

$$\Phi \to -\int_{\mathbf{R}^2} |\nabla u|^2 dx.$$

Similarly, since

$$\Psi = \int_{\mathbf{R}^2} u \nabla \cdot \{u \nabla \chi(c)\} \varphi_R(|x|) dx$$
$$= \frac{1}{2} \int_{\mathbf{R}^2} u^2 \Delta \chi(c) \varphi_R(|x|) dx - \frac{1}{2} \int_{\mathbf{R}^2} U_1 \nabla V_1 \cdot \nabla \varphi_R(|x|) dx,$$

by $R \to \infty$ we have

$$\Psi \to \frac{1}{2} \int_{\mathbf{R}^2} u^2 \Delta \chi(c) dx.$$

Consequently, form (4.17) we have

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^2} u^2 \, dx = -d_u \int_{\mathbf{R}^2} |\nabla u|^2 dx - \frac{1}{2} \int_{\mathbf{R}^2} u^2 \Delta \chi(c) dx + \int_{\mathbf{R}^2} u f(u, v) dx. \tag{4.18}$$

As for the second term of this equation we have

$$\Delta \chi(c) = \chi'(c) \Delta c + \chi''(c) |\nabla c|^2,$$

so we get

$$-\frac{1}{2} \int_{\mathbf{R}^2} u^2 \Delta \chi(c) dx = -\frac{1}{2} \int_{\mathbf{R}^2} \chi'(c) u^2 \Delta c \, dx - \frac{1}{2} \int_{\mathbf{R}^2} \chi''(c) u^2 |\nabla c|^2 dx$$
$$= \phi + \psi.$$

From (2.2) and interpolation inequalities it follows that

$$\phi = -\frac{1}{2} \int_{\mathbf{R}^2} \chi'(c) u^2 \Delta c \, dx \le C \int_{\mathbf{R}^2} |u^2 \Delta c| dx$$
$$\le C \|u^2\|_{L^{3/2}} \|\Delta c\|_{L^3} = C \|u\|_{L^3}^2 \|\Delta c\|_{L^3}$$
$$\le C \|u\|_{L^3}^2 (C_{3,2} \|\Delta c\|_{H^1}^{1-2/3} \|\Delta c\|_{L^2}^{2/3}) \le C' \|u\|_{L^3}^2 \|c\|_{H^3}^{1/3} \|c\|_{H^2}^{2/3}$$
$$\le C' \|u\|_{L^3}^2 \|c\|_{H^3}^{1/3} (C_{1/2} \|c\|_{H^1}^{1-1/2} \|c\|_{H^3}^{1/2})^{2/3} \le C'' \|u\|_{L^3}^2 \|c\|_{H^3}^{2/3} \|c\|_{H^1}^{1/3}.$$

From Lemma 4.2 for any $\zeta > 0$ we have

$$\begin{aligned}
\phi &\le C(\zeta\|u\|_{H^1}^2 N_{\log}^1(u) + q(\zeta^{-1})\|u\|_{L^1})^{2/3}\|c\|_{H^3}^{2/3}\|c\|_{H^1}^{1/3} \\
&\le C\{(\zeta\|u\|_{H^1}^2 N_{\log}^1(u))^{2/3} + (q(\zeta^{-1})\|u\|_{L^1})^{2/3}\}\|c\|_{H^3}^{2/3}\|c\|_{H^1}^{1/3} \\
&\le C\zeta^{2/3}\|u\|_{H^1}^{4/3}\|c\|_{H^3}^{2/3}N_{\log}^1(u)^{2/3}\|c\|_{H^1}^{1/3} + q(\zeta^{-1})\|u\|_{L^1}^{2/3}\|c\|_{H^3}^{2/3}\|c\|_{H^1}^{1/3} \\
&\le C\zeta^{2/3}\left(\frac{2}{3}\|u\|_{H^1}^{(4/3)\cdot(3/2)} + \frac{1}{3}\|c\|_{H^3}^{(2/3)\cdot 3}\right)N_{\log}^1(u)^{2/3}\|c\|_{H^1}^{1/3} \\
&\quad + q(\zeta^{-1})\left\{\varepsilon\|c\|_{H^3}^{(2/3)\cdot 3} + \frac{C}{\varepsilon}(\|u\|_{L^1}^{2/3}\|c\|_{H^1}^{1/3})^{3/2}\right\} \\
&\le C'\zeta^{2/3}(\|u\|_{H^1}^2 + \|c\|_{H^3}^2)N_{\log}^1(u)^{2/3}\|c\|_{H^1}^{1/3} \\
&\quad + \varepsilon q(\zeta^{-1})\|c\|_{H^3}^2 + \frac{C}{\varepsilon}q(\zeta^{-1})\|u\|_{L^1}\|c\|_{H^1}^{1/2}.
\end{aligned}$$

Here putting $\varepsilon = \zeta^{2/3}/q(\zeta^{-1})$ and $\zeta = \zeta_1^{3/2}/(C'' N_{\log}^1(u)\|c\|_{H^1}^{1/2})$ with arbitrary constant $\zeta_1 \ge 0$, we have

$$\begin{aligned}
\phi &\le C''\zeta^{2/3}(\|u\|_{H^1}^2 + \|c\|_{H^3}^2)(N_{\log}^1(u)^{2/3}\|c\|_{H^1}^{1/3} + 1) + C\left(\frac{q(\zeta^{-1})}{\zeta^{2/3}}\right)q(\zeta^{-1})\|u\|_{L^1}\|c\|_{H^1}^{1/2} \\
&\le \zeta_1(\|u\|_{H^1}^2 + \|c\|_{H^3}^2) + p(N_{\log}^1(u) + \|c\|_{H^1} + \zeta_1^{-1}),
\end{aligned}$$

where $p(\cdot)$ and $q(\cdot)$ are certain increasing continuous functions stated in Proposition 4.1 and Lemma 4.2, respectively. Similarly, we have

$$\begin{aligned}
\psi &= -\frac{1}{2}\int_{\mathbf{R}^2}\chi''(c)u^2|\nabla c|^2 dx \le C\|u^2\|_{L^{3/2}}\|\,|\nabla c|^2\|_{L^3} = C\|u\|_{L^3}^2\|\nabla c\|_{L^6}^2 \\
&\le C\|u\|_{L^3}^2(C_{6,2}\|\nabla c\|_{H^1}^{1-2/6}\|\nabla c\|_{L^2}^{2/6})^2 \le C'\|u\|_{L^3}^2\|c\|_{H^2}^{4/3}\|c\|_{H^1}^{2/3} \\
&\le C'\|u\|_{L^3}^2(C_{1/2}\|c\|_{H^1}^{1-1/2}\|c\|_{H^3}^{1/2})^{4/3}\|c\|_{H^1}^{2/3} \le C''\|u\|_{L^3}^2\|c\|_{H^3}^{2/3}\|c\|_{H^1}^{4/3} \\
&\le C'''\zeta^{2/3}(\|u\|_{H^1}^2 + \|c\|_{H^3}^2)(N_{\log}^1(u)^{2/3}\|c\|_{H^1}^{4/3} + 1) + C\left(\frac{q(\zeta^{-1})}{\zeta^{2/3}}\right)q(\zeta^{-1})\|u\|_{L^1}\|c\|_{H^1}^2 \\
&\le \zeta_1(\|u\|_{H^1}^2 + \|c\|_{H^3}^2) + p(N_{\log}^1(u) + \|c\|_{H^1} + \zeta_1^{-1}).
\end{aligned}$$

Then it follows from (4.18) that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^2} u^2\,dx + d_u\int_{\mathbf{R}^2}|\nabla u|^2 dx \tag{4.19}$$
$$\le \zeta(\|u\|_{H^1}^2 + \|c\|_{H^3}^2) + p(N_{\log}^1(u) + \|c\|_{H^1} + \zeta^{-1}) + \int_{\mathbf{R}^2} uf(u,v)dx.$$

Since

$$\begin{aligned}\|\Delta c\|_{L^2}^2 &\le (C\|\nabla\Delta c\|_{L^2}^{2/3}\|c\|_{L^2}^{1/3})^2 = C^2\|\nabla\Delta c\|_{L^2}^{4/3}\|c\|_{L^2}^{2/3}\\ &\le C^2\left(\frac{2}{3}\|\nabla\Delta c\|_{L^2}^2 + \frac{1}{3}\|c\|_{L^2}^2\right)\\ &\le C'(\|\nabla\Delta c\|_{L^2}^2 + \|c\|_{L^2}^2),\end{aligned}$$

we have

$$\|c\|_{H^3}^2 \le C(\|c\|_{H^1}^2 + \|\nabla\Delta c\|_{L^2}^2).$$

By Young's inequality we have

$$\begin{aligned}uf(u,v) &= -\mu_1 u^2 - \nu_1 u^3 + \tau_1 u^2 v\\ &\le -\mu_1 u^2 - \nu_1 u^3 + \tau_1\left(\varepsilon u^3 + C\left(\frac{1}{\varepsilon}\right)v^3\right)\\ &= -\mu_1 u^2 + (-\nu_1 + \varepsilon\tau_1)u^3 + \tau_1 C\left(\frac{1}{\varepsilon}\right)v^3.\end{aligned}$$

Therefore by taking $\varepsilon = \dfrac{\nu_1}{4\tau_1} > 0$ it follows that

$$\int_{\mathbf{R}^2} uf(u,v)dx \le -\mu_1\|u\|_{L^2}^2 - \frac{3}{4}\nu_1\|u\|_{L^3}^3 + C\|v\|_{L^3}^3,$$

from which the inequality (4.19) is rewritten as

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^2} u^2\,dx + d_u\int_{\mathbf{R}^2}|\nabla u|^2 dx\\ &\quad\le \zeta(\|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|c\|_{H^1}^2 + \|\nabla\Delta c\|_{L^2}^2) + p(N_{\log}^1(u) + \|c\|_{H^1} + \zeta^{-1})\\ &\qquad - \mu_1\|u\|_{L^2}^2 - \frac{3}{4}\nu_1\|u\|_{L^3}^3 + C\|v\|_{L^3}^3\\ &\quad\le (\zeta - \mu_1)\|u\|_{L^2}^2 + \zeta(\|\nabla u\|_{L^2}^2 + \|\nabla\Delta c\|_{L^2}^2) - \frac{3}{4}\nu_1\|u\|_{L^3}^3\\ &\qquad + p(N_{\log}^1(u) + \|v\|_{L^3} + \|c\|_{H^1} + \zeta^{-1}).\end{aligned}$$

Consequently we have

$$(4.20)\quad \frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^2} u^2\,dx + (d_u - \zeta)\int_{\mathbf{R}^2} |\nabla u|^2 dx - \zeta\|\nabla \Delta c\|_{L^2}^2$$
$$\leq (\zeta - \mu_1)\|u\|_{L^2}^2 - \frac{3}{4}v_1\|u\|_{L^3}^3 + p(N_{\log}^1(u) + \|v\|_{L^3} + \|c\|_{H^1} + \zeta^{-1}),$$
$$0 < t < T_U.$$

Step 5.2 (L^2-estimate of ∇v) Put $U = v_t$, $U_1 = v^2$, $V = v$. Then from the proved part in Proposition 4.1 we see that $U, U_1 \in L^2(\mathbf{R}^2)$, $V \in H^1(\mathbf{R}^2)$. Multiply the second equation of (1.1) by Δv and $\varphi_R(|x|) \geq 0$, and integrate this in $\mathbf{R}^2$. Then we have

$$(4.21)\quad \int_{\mathbf{R}^2} \varphi_R(|x|)\Delta v \frac{\partial}{\partial t} v\,dx = d_v \int_{\mathbf{R}^2} |\Delta v|^2 \varphi_R(|x|)dx + \int_{\mathbf{R}^2} g(u,v)\Delta v \varphi_R(|x|)dx.$$

As for the left hand side of (4.21) we have

$$\int_{\mathbf{R}^2} \varphi_R(|x|)\Delta v \frac{\partial}{\partial t} v\,dx = -\frac{1}{2}\int_{\mathbf{R}^2} \varphi_R(|x|)\frac{\partial}{\partial t}|\nabla v|^2 dx - \int_{\mathbf{R}^2} U\nabla V \cdot \nabla \varphi_R(|x|)dx.$$

Therefore by $R \to \infty$ we have

$$\int_{\mathbf{R}^2} \Delta v \frac{\partial}{\partial t} v\,dx = -\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^2} |\nabla v|^2 dx.$$

As for the second term of the right hand side of (4.21) it follows that

$$\int_{\mathbf{R}^2} g(u,v)\Delta v \varphi_R(|x|)dx$$
$$= \mu_2 \int_{\mathbf{R}^2} v\Delta v \varphi_R(|x|)dx - v_2 \int_{\mathbf{R}^2} v^2 \Delta v \varphi_R(|x|)dx - \tau_2 \int_{\mathbf{R}^2} uv\Delta v \varphi_R(|x|)dx$$
$$= \mu_2 \int_{\mathbf{R}^2} v\Delta v \varphi_R(|x|)dx - v_2\Omega - \tau_2 \int_{\mathbf{R}^2} uv\Delta v \varphi_R(|x|)dx,$$

where

$$\Omega = \int_{\mathbf{R}^2} v^2 \Delta v \varphi_R(|x|)dx = -2\int_{\mathbf{R}^2} v|\nabla v|^2 \varphi_R(|x|)dx - \int_{\mathbf{R}^2} U_1 \nabla V \cdot \nabla \varphi_R(|x|)dx.$$

By $R \to \infty$

$$\Omega \to -2\int_{\mathbf{R}^2} v|\nabla v|^2 dx.$$

Consequently it follows from (4.21) that

$$-\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^2}|\nabla v|^2dx = d_v\int_{\mathbf{R}^2}|\Delta v|^2dx + \mu_2\int_{\mathbf{R}^2}v\Delta v\,dx$$
$$+ 2\nu_2\int_{\mathbf{R}^2}v|\nabla v|^2dx - \tau_2\int_{\mathbf{R}^2}uv\Delta v\,dx,$$

which leads to

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^2}|\nabla v|^2dx + d_v\int_{\mathbf{R}^2}|\Delta v|^2dx + 2\nu_2\int_{\mathbf{R}^2}v|\nabla v|^2dx$$
$$\leq \mu_2\int_{\mathbf{R}^2}|v\Delta v|dx + \tau_2\int_{\mathbf{R}^2}|uv\Delta v|dx$$
$$= \phi + \psi.$$

It follows from Cauchy's inequality that for any $\varepsilon > 0$

$$\phi = \mu_2\|v\Delta v\|_{L^1} \leq \mu_2\|v\|_{L^2}\|\Delta v\|_{L^2} \leq \mu_2\left(\varepsilon\|\Delta v\|_{L^2}^2 + \frac{1}{4\varepsilon}\|v\|_{L^2}^2\right).$$

By taking $\varepsilon = \dfrac{d_v}{4\mu_2} > 0$ in this inequality we have

$$\phi \leq \frac{d_v}{4}\|\Delta v\|_{L^2}^2 + C\|v\|_{L^2}^2.$$

From Young's inequality it follows that for any $\varepsilon, \varepsilon' > 0$

$$\psi = \tau_2\|uv\Delta v\|_{L^1} \leq \tau_2\|u\|_{L^3}\|v\|_{L^6}\|\Delta v\|_{L^2}$$
$$\leq \tau_2\left(\varepsilon\|\Delta v\|_{L^2}^2 + \frac{1}{4\varepsilon}\|u\|_{L^3}^2\|v\|_{L^6}^2\right)$$
$$\leq \varepsilon\tau_2\|\Delta v\|_{L^2}^2 + \frac{\tau_2}{4\varepsilon}\left(\varepsilon'\|u\|_{L^3}^3 + C\left(\frac{1}{\varepsilon'}\right)\|v\|_{L^6}^6\right).$$

By taking $\varepsilon = \dfrac{d_v}{4\tau_2} > 0$ and $\varepsilon' = \dfrac{d_v\nu_1}{4(\tau_2)^2} > 0$ we have

$$\psi \leq \frac{d_v}{4}\|\Delta v\|_{L^2}^2 + \frac{\nu_1}{4}\|u\|_{L^3}^3 + C\|v\|_{L^6}^6.$$

Therefore we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^2}|\nabla v|^2dx + \frac{d_v}{2}\int_{\mathbf{R}^2}|\Delta v|^2dx \tag{4.22}$$
$$\leq \frac{\nu_1}{4}\|u\|_{L^3}^3 + C(\|v\|_{L^2}^2 + \|v\|_{L^6}^6), \qquad 0 < t < T_U.$$

Step 5.3 (L^2-estimate of Δc) In this step we put $U = \Delta c$, $V = c_t$, $V_1 = c$. Then we see that $U \in L^2(\mathbf{R}^2)$, $V, V_1 \in H^1(\mathbf{R}^2)$. Apply the third equation of (1.1) by ∇, and take the inner product between this and $\nabla \Delta c$, and multiply this by $\varphi_R(|x|) \geq 0$. Integrating this on $\mathbf{R}^2$, we have

$$\begin{aligned}(4.23) \qquad & \int_{\mathbf{R}^2} \varphi_R(|x|) \nabla \Delta c \cdot \nabla \left(\frac{\partial}{\partial t} c \right) dx \\ &= d_c \int_{\mathbf{R}^2} |\nabla \Delta c|^2 \varphi_R(|x|) dx + \beta \int_{\mathbf{R}^2} \nabla \Delta c \cdot \nabla u \varphi_R(|x|) dx \\ &\quad - \gamma \int_{\mathbf{R}^2} \nabla \Delta c \cdot \nabla c \varphi_R(|x|) dx.\end{aligned}$$

As for the left hand side of (4.23) we have

$$\begin{aligned}\int_{\mathbf{R}^2} \varphi_R(|x|) \nabla \Delta c \cdot \nabla \left(\frac{\partial}{\partial t} c \right) dx &= - \int_{\mathbf{R}^2} \varphi_R(|x|) \Delta c \cdot \left(\frac{\partial}{\partial t} \Delta c \right) dx \\ &\quad - \int_{\mathbf{R}^2} U \nabla V \cdot \nabla \varphi_R(|x|) dx.\end{aligned}$$

By $R \to \infty$ we have

$$\int_{\mathbf{R}^2} \nabla \Delta c \cdot \nabla \left(\frac{\partial}{\partial t} c \right) dx = -\frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^2} |\Delta c|^2 dx.$$

As for the third term of the right hand side of (4.23) we have

$$-\gamma \int_{\mathbf{R}^2} \nabla \Delta c \cdot \nabla c \varphi_R(|x|) dx = \gamma \int_{\mathbf{R}^2} |\Delta c|^2 \varphi_R(|x|) dx + \gamma \int_{\mathbf{R}^2} U \nabla V_1 \cdot \nabla \varphi_R(|x|) dx$$

By $R \to \infty$ we have

$$-\gamma \int_{\mathbf{R}^2} \nabla \Delta c \cdot \nabla c \, dx = \gamma \int_{\mathbf{R}^2} |\Delta c|^2 dx.$$

Therefore from (4.23) it follows that

$$-\frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^2} |\Delta c|^2 dx = d_c \int_{\mathbf{R}^2} |\nabla \Delta c|^2 dx + \beta \int_{\mathbf{R}^2} \nabla \Delta c \cdot \nabla u \, dx + \gamma \int_{\mathbf{R}^2} |\Delta c|^2 dx,$$

from which for any $\varepsilon > 0$ we have

$$\begin{aligned}\frac{1}{2} \frac{d}{dt} \int_{\mathbf{R}^2} |\Delta c|^2 dx + d_c \int_{\mathbf{R}^2} |\nabla \Delta c|^2 dx + \gamma \int_{\mathbf{R}^2} |\Delta c|^2 dx &\leq \beta \int_{\mathbf{R}^2} |\nabla \Delta c \cdot \nabla u| dx \\ &\leq \beta \left\{ \varepsilon \|\nabla \Delta c\|_{L^2}^2 + \frac{1}{4\varepsilon} \|\nabla u\|_{L^2}^2 \right\}.\end{aligned}$$

Taking $\varepsilon = \dfrac{d_c}{2\beta}$, we have

(4.24)
$$\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^2}|\Delta c|^2dx + \frac{d_c}{2}\int_{\mathbf{R}^2}|\nabla\Delta c|^2dx + \gamma\int_{\mathbf{R}^2}|\Delta c|^2dx \le \frac{\beta^2}{2d_c}\|\nabla u\|_{L^2}^2, \qquad 0 < t < T_U.$$

Step 5.4 (Proof of (4.1)) Multiply (4.24) by $\dfrac{d_u d_c}{\beta^2}$, and add (4.20) and (4.22) to this. Then we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^2}\left(u^2 + |\nabla v|^2 + \frac{d_u d_c}{\beta^2}|\Delta c|^2\right)dx + \left(\frac{d_u}{2} - \zeta\right)\int_{\mathbf{R}^2}|\nabla u|^2dx \\
&\qquad + \frac{d_v}{2}\int_{\mathbf{R}^2}|\Delta v|^2dx + \left(\frac{d_u d_c^2}{2\beta^2} - \zeta\right)\int_{\mathbf{R}^2}|\nabla\Delta c|^2dx + \frac{d_u d_c\gamma}{\beta^2}\int_{\mathbf{R}^2}|\Delta c|^2dx \\
&\quad \le (\zeta - \mu_1)\|u\|_{L^2}^2 - \frac{\nu_1}{2}\|u\|_{L^3}^3 + p(N_{\log}^1(u) + \|v\|_{L^3} + \|c\|_{H^1} + \zeta^{-1}) \\
&\qquad + C(\|v\|_{L^2}^2 + \|v\|_{L^6}^6) \\
&\quad \le (\zeta - \mu_1)\|u\|_{L^2}^2 + p(N_{\log}^1(u) + \|v\|_{L^2} + \|v\|_{L^3} + \|v\|_{L^6} + \|c\|_{H^1} + \zeta^{-1}).
\end{aligned}$$

For small $\zeta > 0$ we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^2}\left(u^2 + |\nabla v|^2 + \frac{d_u d_c}{\beta^2}|\Delta c|^2\right)dx + \frac{d_u}{4}\int_{\mathbf{R}^2}|\nabla u|^2dx \\
&\qquad + \frac{d_v}{2}\int_{\mathbf{R}^2}|\Delta v|^2dx + \frac{d_u d_c^2}{4\beta^2}\int_{\mathbf{R}^2}|\nabla\Delta c|^2dx + \frac{d_u d_c\gamma}{\beta^2}\int_{\mathbf{R}^2}|\Delta c|^2dx \\
&\quad \le p(N_{\log}^1(u) + \|v\|_{L^p} + \|c\|_{H^1} + 1), \qquad 0 < t < T_U,
\end{aligned}$$

that is,

$$\frac{d}{dt}\int_{\mathbf{R}^2}(u^2 + |\nabla v|^2 + |\Delta c|^2)dx \le p(N_{\log}^1(u) + \|v\|_{L^p} + \|c\|_{H^1} + 1), \qquad 0 < t < T_U.$$

Integrating this in t, we have

$$\begin{aligned}
&\|u(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 + \|\Delta c(t)\|_{L^2}^2 \\
&\quad \le \|u_0\|_{L^2}^2 + \|\nabla v_0\|_{L^2}^2 + \|\Delta c_0\|_{L^2}^2 \\
&\qquad + \int_0^t p(N_{\log}^1(u(s)) + \|v(s)\|_{L^p} + \|c(s)\|_{H^1} + 1)ds
\end{aligned}$$

$$\begin{aligned}
&\le \|u_0\|_{L^2}^2 + \|\nabla v_0\|_{L^2}^2 + \|\Delta c_0\|_{L^2}^2 + \int_0^t p(N_{\log}^1(u_0) \\
&\quad + Ce^{6\nu_2 s}(1 + s + \|u_0\|_{L^1} + \|v_0\|_{H^1} + \|c_0\|_{H^1})^6 \\
&\quad + e^{\mu_2 s}\|v_0\|_{L^p} + \|u_0\|_{L^1} + e^{\mu_2 s}\|v_0\|_{L^2} + \|c_0\|_{H^1} + 1)ds \\
&\le \|u_0\|_{L^2}^2 + \|\nabla v_0\|_{L^2}^2 + \|\Delta c_0\|_{L^2}^2 + p(N_{\log}^1(u_0) \\
&\quad + Ce^{6\nu_2 t}(1 + t + \|u_0\|_{L^1} + \|v_0\|_{H^1} + \|c_0\|_{H^1})^6 \\
&\quad + e^{\mu_2 t}\|v_0\|_{L^p} + \|u_0\|_{L^1} + e^{\mu_2 t}\|v_0\|_{L^2} + \|c_0\|_{H^1} + 1)\int_0^t ds,
\end{aligned}$$

which together with (4.5) and (4.12) leads to

$$\begin{aligned}
&\|u(t)\|_{L^2}^2 + \|v(t)\|_{H^1}^2 + \|c(t)\|_{H^2}^2 \\
&\quad \le C(\|u_0\|_{L^1} + \|u_0\|_{L^2}^2 + \|v_0\|_{H^1}^2 + \|c_0\|_{H^2}^2) \\
&\qquad + p(t + \|u_0\|_{L^1} + N_{\log}^1(u_0) + \|v_0\|_{H^1} + \|c_0\|_{H^1}), \qquad 0 < t < T_U.
\end{aligned}$$

From this and (4.8) we have (4.1).

THEOREM 4.4. *For any initial functions* $0 \le u_0 \in L^1(\mathbf{R}^2) \cap L^2(\mathbf{R}^2)$, $0 \le v_0 \in H^1(\mathbf{R}^2)$ *and* $0 \le c_0 \in H^2(\mathbf{R}^2)$ *the system* (1.1) *admits a unique global solution such that*

$$\begin{aligned}
&0 \le u \in \mathscr{C}([0,\infty); L^2(\mathbf{R}^2)) \cap \mathscr{C}^1((0,\infty); H^{-1}(\mathbf{R}^2)) \cap \mathscr{C}((0,\infty); H^1(\mathbf{R}^2)), \\
&0 \le v \in \mathscr{C}([0,\infty); H^1(\mathbf{R}^2)) \cap \mathscr{C}^1((0,\infty); L^2(\mathbf{R}^2)) \cap \mathscr{C}((0,\infty); H^2(\mathbf{R}^2)), \\
&0 \le c \in \mathscr{C}([0,\infty); H^2(\mathbf{R}^2)) \cap \mathscr{C}^1((0,\infty); H^1(\mathbf{R}^2)) \cap \mathscr{C}((0,\infty); H^3(\mathbf{R}^2)).
\end{aligned}$$

PROOF. Let $0 \le u_0 \in L^2(\mathbf{R}^2)$, $0 \le v_0 \in H^1(\mathbf{R}^2)$ and $0 \le c_0 \in H^2(\mathbf{R}^2)$. Then from Theorems 3.2 and 3.3 the system (1.1) has a unique local solution $u(\cdot) \in L^2(\mathbf{R}^2)$, $v(\cdot) \in H^1(\mathbf{R}^2)$ and $c(\cdot) \in H^2(\mathbf{R}^2)$ on $[0, T_{U_0})$. If, furthermore, we assume $u_0 \in L^1(\mathbf{R}^2)$, then $u(\cdot) \in L^1(\mathbf{R}^2)$. Now for any $s \in (0, T_{U_0})$ we consider $u(s)$, $v(s)$, $c(s)$ as initial functions of (1.1) then from Theorems 3.1 and 3.2 it follows there exists a unique solution of (1.1) such that

$$\begin{aligned}
&0 \le \bar{u} \in \mathscr{C}([s, s + T_{Us}); L^2(\mathbf{R}^2)) \cap \mathscr{C}^1((s, s + T_{Us}); \\
&\qquad H^{-1}(\mathbf{R}^2)) \cap \mathscr{C}((s, s + T_{Us}); H^1(\mathbf{R}^2)), \\
&0 \le \bar{v} \in \mathscr{C}([s, s + T_{Us}); H^1(\mathbf{R}^2)) \cap \mathscr{C}^1((s, s + T_{Us}); \\
&\qquad L^2(\mathbf{R}^2)) \cap \mathscr{C}((s, s + T_{Us}); H^2(\mathbf{R}^2)),
\end{aligned}$$

$$0 \le \bar{c} \in \mathscr{C}([s, s+T_{Us}); H^2(\mathbf{R}^2)) \cap \mathscr{C}^1((s, s+T_{Us});$$
$$H^1(\mathbf{R}^2)) \cap \mathscr{C}((s, s+T_{Us}); H^3(\mathbf{R}^2)),$$

where $T_{Us} > 0$ is a local time which is determined by $\|u(s)\|_{L^2} + \|v(s)\|_{H^1} + \|c(s)\|_{H^2}$. Moreover by the uniqueness of solutions we have

$$\bar{u} = u, \qquad \bar{v} = v, \qquad \bar{c} = c.$$

Here we put

$$r = p(T_{U_0} + \|u_0\|_{L^1} + \|u_0\|_{L^2} + \|v_0\|_{H^1} + \|c_0\|_{H^2}).$$

From Proposition 4.1

$$\begin{aligned}
&\|u(s)\|_{L^1} + \|u(s)\|_{L^2} + \|v(s)\|_{H^1} + \|c(s)\|_{H^2} \\
&\qquad \le p(s + \|u_0\|_{L^1} + \|u_0\|_{L^2} + \|v_0\|_{H^1} + \|c_0\|_{H^2}) \\
&\qquad \le p(T_{U_0} + \|u_0\|_{L^1} + \|u_0\|_{L^2} + \|v_0\|_{H^1} + \|c_0\|_{H^2}) \\
&\qquad = r
\end{aligned}$$

holds, from which we see the norm $\|u(s)\|_{L^2} + \|v(s)\|_{H^1} + \|c(s)\|_{H^2}$ depends only on r. Consequently the $T_{Us} > 0$ depend only on r, not on s. Thus the local time $T_{Us} > 0$ is determined by r, and independent on s, that is $T_{Us} = T_r > 0$. Since s is arbitrary in $[0, T_{U_0})$, the time interval $[0, T_{U_0})$ is extended by $T_r > 0$. Repeating this procedure we can construct the global solution.

5. Appendix

Here we continue verifying the inequality (3.3) holds. In fact from (3.2) we have

$$\begin{aligned}
&\|F(U) - F(W)\|_X \\
&\quad = \left\| \begin{pmatrix} -\nabla \cdot \{u\nabla\tilde{\chi}(c)\} + \{\mu_1 + \tilde{f}_1(u,v)\}u \\ +\nabla \cdot \{w\nabla\tilde{\chi}(\xi)\} - \{\mu_1 + \tilde{f}_1(w,\rho)\}w \\ \{1 + \tilde{g}_1(u,v)\}v - \{1 + \tilde{g}_1(w,\rho)\}\rho \\ 0 \end{pmatrix} \right\|_X \\
&\quad = \left\| \begin{pmatrix} -\nabla \cdot \{(u-w)\nabla\tilde{\chi}(c)\} - \nabla \cdot \{w\nabla[\tilde{\chi}(c) - \tilde{\chi}(\xi)]\} \\ +\{\tilde{f}_1(u,v) - \tilde{f}_1(w,\rho)\}u + \{\mu_1 + \tilde{f}_1(w,\rho)\}(u-w) \\ v - \rho + \{\tilde{g}_1(u,v) - \tilde{g}_1(w,\rho)\}v + \tilde{g}_1(w,\rho)(v-\rho) \\ 0 \end{pmatrix} \right\|_X
\end{aligned}$$

$$\leq \|\nabla \cdot \{(u-w)\nabla\tilde{\chi}(c)\}\|_{H^{-1}} + \|\nabla \cdot \{w\nabla[\tilde{\chi}(c) - \tilde{\chi}(\xi)]\}\|_{H^{-1}}$$
$$+ \|\{\tilde{f}_1(u,v) - \tilde{f}_1(w,\rho)\}u\|_{H^{-1}} + \|\{\mu_1 + \tilde{f}_1(w,\rho)\}(u-w)\|_{H^{-1}}$$
$$+ \|v-\rho\|_{L^2} + \|\{\tilde{g}_1(u,v) - \tilde{g}_1(w,\rho)\}v\|_{L^2}$$
$$+ \|\tilde{g}_1(w,\rho)(v-\rho)\|_{L^2}.$$

Owing to (2.8) and (2.10) we obtain

$$\begin{aligned}
\|\nabla \cdot \{(u-w)\nabla\tilde{\chi}(c)\}\|_{H^{-1}} &\leq C_{\varepsilon_0}\|u-w\|_{H^{1-\varepsilon_0}}\|\nabla\tilde{\chi}(c)\|_{H^{\varepsilon_0}} \\
&= C_{\varepsilon_0}\|u-w\|_{H^{1-\varepsilon_0}}\|\tilde{\chi}'(c)\nabla c\|_{H^{\varepsilon_0}} \\
&\leq C_{\varepsilon_0}\|u-w\|_{H^{1-\varepsilon_0}} p_{\varepsilon_0}(\|c\|_{H^{1+\varepsilon_0}})\|\nabla c\|_{H^{\varepsilon_0}} \\
&\leq p'_{\varepsilon_0}(\|c\|_{H^{1+\varepsilon_0}})\|u-w\|_{H^{1-\varepsilon_0}}.
\end{aligned}$$

Form (2.8), (2.10) and (2.11) we have

$$\begin{aligned}
&\|\nabla \cdot \{w\nabla[\tilde{\chi}(c) - \tilde{\chi}(\xi)]\}\|_{H^{-1}} \\
&\quad \leq C_{\varepsilon_0}\|w\|_{H^{1-\varepsilon_0}}\|\nabla[\tilde{\chi}(c) - \tilde{\chi}(\xi)]\|_{H^{\varepsilon_0}} \\
&\quad = C_{\varepsilon_0}\|w\|_{H^{1-\varepsilon_0}}\|\tilde{\chi}'(c)\nabla c - \tilde{\chi}'(\xi)\nabla\xi\|_{H^{\varepsilon_0}} \\
&\quad = C_{\varepsilon_0}\|w\|_{H^{1-\varepsilon_0}}\|\tilde{\chi}'(c)\nabla c - \tilde{\chi}'(\xi)\nabla c + \tilde{\chi}'(\xi)\nabla c - \tilde{\chi}'(\xi)\nabla\xi\|_{H^{\varepsilon_0}} \\
&\quad \leq C_{\varepsilon_0}\|w\|_{H^{1-\varepsilon_0}}\{\|[\tilde{\chi}'(c) - \tilde{\chi}'(\xi)]\nabla c\|_{H^{\varepsilon_0}} + \|\tilde{\chi}'(\xi)\nabla(c-\xi)\|_{H^{\varepsilon_0}}\} \\
&\quad \leq C_{\varepsilon_0}\|w\|_{H^{1-\varepsilon_0}}\{p_{\varepsilon_0}(\|c\|_{H^{1+\varepsilon_0}} + \|\xi\|_{H^{1+\varepsilon_0}})\|c-\xi\|_{H^{1+\varepsilon_0}}\|\nabla c\|_{H^{\varepsilon_0}} \\
&\qquad + p'_{\varepsilon_0}(\|\xi\|_{H^{1+\varepsilon_0}})\|\nabla(c-\xi)\|_{H^{\varepsilon_0}}\} \\
&\quad \leq p''_{\varepsilon_0}(\|c\|_{H^{1+\varepsilon_0}} + \|\xi\|_{H^{1+\varepsilon_0}})\|w\|_{H^{1-\varepsilon_0}}\|c-\xi\|_{H^{1+\varepsilon_0}}.
\end{aligned}$$

From Hölder's inequality and (2.4) and (2.5) it follows that

$$\begin{aligned}
&\|\{\tilde{f}_1(u,v) - \tilde{f}_1(w,\rho)\}u\|_{H^{-1}} \\
&\quad = \|\{-\mu_1 - v_1 u + \tau_1 v - (-\mu_1 - v_1 w + \tau_1 \rho)\}u\|_{H^{-1}} \\
&\quad \leq v_1\|(u-w)u\|_{H^{-1}} + \tau_1\|(v-\rho)u\|_{H^{-1}} \\
&\quad \leq C\left\{\sup_{\|\phi\|_{H^1}\leq 1}\int_{\mathbf{R}^2}|(u-w)u\phi|\,dx + \sup_{\|\phi\|_{H^1}\leq 1}\int_{\mathbf{R}^2}|(v-\rho)u\phi|\,dx\right\} \\
&\quad \leq C\left\{\sup_{\|\phi\|_{H^1}\leq 1}\|(u-w)u\|_{L^{2/(1+\varepsilon_0)}}\|\phi\|_{L^{2/(1-\varepsilon_0)}} + \sup_{\|\phi\|_{H^1}\leq 1}\|(v-\rho)u\|_{L^2}\|\phi\|_{L^2}\right\}
\end{aligned}$$

$$\le C\{\|(u-w)u\|_{L^{2/(1+\varepsilon_0)}} + \|(v-\rho)u\|_{L^2}\}$$
$$\le C\{C_{2/(1+\varepsilon_0)}\|u\|_{L^2}\|u-w\|_{H^{2-(1+\varepsilon_0)}} + C_{\varepsilon_0}\|u\|_{L^2}\|v-\rho\|_{H^{1+\varepsilon_0}}\}$$
$$\le C'\|u\|_{L^2}\{\|u-w\|_{H^{1-\varepsilon_0}} + \|v-\rho\|_{H^{2-\varepsilon_0}}\}.$$

Similarly from Hölder's inequality (2.4) and (2.6) we have

$$\|\{\mu_1 + \tilde{f}_1(w,\rho)\}(u-w)\|_{H^{-1}}$$
$$= \|(\mu_1 - \mu_1 - \nu_1 w + \tau_1\rho)(u-w)\|_{H^{-1}}$$
$$\le \nu_1\|w(u-w)\|_{H^{-1}} + \tau_1\|\rho(u-w)\|_{H^{-1}}$$
$$\le C\left\{\sup_{\|\phi\|_{H^1}\le 1}\int_{\mathbf{R}^2}|w(u-w)\phi|dx + \sup_{\|\phi\|_{H^1}\le 1}\int_{\mathbf{R}^2}|\rho(u-w)\phi|dx\right\}$$
$$\le C\{C_{2/(1+\varepsilon_0)}\|w\|_{L^2}\|u-w\|_{H^{1-\varepsilon_0}} + \|\rho(u-w)\|_{L^2}\}$$
$$\le C\{C_{2/(1+\varepsilon_0)}\|w\|_{L^2}\|u-w\|_{H^{1-\varepsilon_0}} + C_{\varepsilon_0}\|\rho\|_{H^{\varepsilon_0}}\|u-w\|_{H^{1-\varepsilon_0}}\}$$
$$\le C'(\|w\|_{L^2} + \|\rho\|_{H^{\varepsilon_0}})\|u-w\|_{H^{1-\varepsilon_0}}.$$

From (2.6) we have

$$\|\{\tilde{g}_1(u,v) - \tilde{g}_1(w,\rho)\}v\|_{L^2}$$
$$= \|\{\mu_2 - \nu_2 v - \tau_2 u - (\mu_2 - \nu_2\rho - \tau_2 w)\}v\|_{L^2}$$
$$\le \nu_2\|(v-\rho)v\|_{L^2} + \tau_2\|(u-w)v\|_{L^2}$$
$$\le C(C_{\varepsilon_0}\|v\|_{H^{\varepsilon_0}}\|v-\rho\|_{H^{1-\varepsilon_0}} + C_{\varepsilon_0}\|v\|_{H^{\varepsilon_0}}\|u-w\|_{H^{1-\varepsilon_0}})$$
$$\le C'\|v\|_{H^{\varepsilon_0}}(\|v-\rho\|_{H^{1-\varepsilon_0}} + \|u-w\|_{H^{1-\varepsilon_0}})$$

and

$$\|\tilde{g}_1(w,\rho)(v-\rho)\|_{L^2}$$
$$= \|(\mu_2 - \nu_2\rho - \tau_2 w)(v-\rho)\|_{L^2}$$
$$\le C(\|v-\rho\|_{L^2} + \|\rho(v-\rho)\|_{L^2} + \|w(v-\rho)\|_{L^2})$$
$$\le C(\|v-\rho\|_{H^{1-\varepsilon_0}} + C_{\varepsilon_0}\|\rho\|_{H^{\varepsilon_0}}\|v-\rho\|_{H^{1-\varepsilon_0}} + C_{\varepsilon_0}\|w\|_{H^{\varepsilon_0}}\|v-\rho\|_{H^{1-\varepsilon_0}})$$
$$\le C'(1 + \|\rho\|_{H^{\varepsilon_0}} + \|w\|_{H^{\varepsilon_0}})\|v-\rho\|_{H^{1-\varepsilon_0}}$$
$$\le C'(\|w\|_{H^{1-\varepsilon_0}} + \|\rho\|_{H^{\varepsilon_0}} + 1)\|v-\rho\|_{H^1}.$$

Summing up these we have, with a continuous function $p(\cdot)$

$$\|F(U) - F(W)\|_X \le p(\|u\|_{L^2} + \|v\|_{H^{\varepsilon_0}} + \|c\|_{H^{1+\varepsilon_0}} + \|w\|_{L^2} + \|\rho\|_{H^{\varepsilon_0}} + \|\xi\|_{H^{1+\varepsilon_0}})$$
$$\times \{\|u - w\|_{H^{1-\varepsilon_0}} + \|v - \rho\|_{H^{2-\varepsilon_0}}$$
$$+ (\|w\|_{H^{1-\varepsilon_0}} + \|\rho\|_{H^{\varepsilon_0}} + 1)(\|v - \rho\|_{H^1} + \|c - \xi\|_{H^{1+\varepsilon_0}})\}$$

which is the desired estimate.

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