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Sharp estimates of the Green function, the Poisson kernel and the Martin kernel of cones for symmetric stable processes

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Abstract. We investigate the Green function, the Poisson kernel and the Martin kernel of circular cones in the symmetric stable case. We derive their sharp estimates. We also investigate properties of the characteristic exponent of these estimates. We prove that this exponent is a continuous function of the aperture of the cone.

1. Introduction

In recent years many results in potential theory of α-stable processes have been obtained ([2], [9], [10], [12]–[17], [19], [20], [22]–[24], [27]). In particular, the behaviour of α-harmonic functions, the Green function and the Poisson kernel in smooth domains ([19], [23]) and Lipschitz domains ([9], [10], [22]) has been investigated. In case of smooth domains the main tool is the Green function and the Poisson kernel of a ball. The estimates of the Green function and the explicit formula of the Poisson kernel are known ([8], [19], [23], [25]). In case of Lipschitz domains the situation is more complicated and it is cones that seem to be a proper tool. On the other hand, cones are Lipschitz domains themselves and they are regular enough to obtain more detailed results than those in [10]. In fact, properties of the Green function, the Poisson kernel and the exit time in bounded and unbounded cones has been studied both in the classical case ([1], [3], [18]) and α-stable case ([14], [21], [23], [26]). The latest results for $\alpha < 2$ are [2] about the so-called generalized cones and [14] about the Poisson kernel in 'smoothed' bounded cones.

The estimates presented in [23] are proved only for cones of acute aperture. Besides, they are not sharp at the vertex of the cone. The aim of this paper is to improve these estimates and to extend them to all circular cones. The basic tool is the so-called Martin kernel with pole at infinity (introduced in [2]) and its degree of homogeneity. We also use the estimates of the Green function in smooth domains ([19], [24]) and of the Poisson kernel in

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Lipschitz domains ([22]). Another important tool is a version of the Boundary Harnack Principle ([27]).

In section 2 we present basic notation and terminology. Section 3 contains the most important results. First we prove some properties of the Martin kernel with pole at infinity and its degree of homogeneity. They seem to be useful in other applications. Next we obtain estimates of the function ϕ (truncated Green function) used in [22]. With the use of this function we prove sharp estimates of the Green function and the Poisson kernels of bounded cones (Theorems 3.6 and 3.7). With some extra arguments we extend our estimates to unbounded cones (Theorem 3.11). As a consequence of Theorem 3.6 we obtain estimates of the Martin kernel in cones (Theorems 3.12 and 3.13). We also prove the existence of Martin representation in unbounded cones (cf. [9]) and we show the consistence between the Martin kernel with pole at infinity and the classical Martin kernel with the boundary point at infinity.

2. Preliminaries

We will denote by $|\cdot|$ the Euclidean norm of vectors. For $B \subset \mathbf{R}^d$, $d \geq 2$ we denote its complement by B^c and its characteristic function by $\mathbf{1}_B$. For $x \in \mathbf{R}^d$, $B(x, r)$ will denote the open ball centered at x of radius r. For a Borel set B and $r > 0$ we define $rB = \{rx : x \in B\}$ and $B + x = \{x + y : y \in B\}$. For $x \in \mathbf{R}^d$ let $\delta_x(B) = dist(x, \partial B)$.

Let D denote a bounded open set in $\mathbf{R}^d$. We say D is a Lipschitz domain if there exist constants R_0, $\lambda > 0$ such that for every $z \in \partial D$ there is a function $F : \mathbf{R}^{d-1} \to R$ and an orthonormal coordinate system $y = (y_1, \ldots, y_d)$ such that

$$D \cap B(z, R_0) = \{y : y_d > F(y_1, \ldots, y_{d-1})\} \cap B(z, R_0).$$

Moreover, F is Lipschitz with the Lipschitz constant not greater than λ. Furthermore, if F is differentiable and ∇F is Lipschitz with the Lipschitz constant not greater than λ then D is called a $C^{1,1}$ domain.

Let (X_t, P^x) be the rotation invariant('symmetric') α-stable Levy motion (i.e. homogeneous with independent increments) on $\mathbf{R}^d$ with its index $\alpha \in (0, 2)$ ([7]). For a Borel subset B of $\mathbf{R}^d$ let T_B and τ_B be the first entry time and the first exit time respectively i.e. $T_B = \inf\{t \geq 0 : X_t \in B\}$ and $\tau_B = T_{B^c}$.

In this paper constants are always positive numbers. In equations and inequalities they may change under arithmetic transformations but they will be denoted by the same symbols. The notation of the form $c = c(a, b, \ldots)$ means that the constant c depends only on $a, b, \ldots$.

A nonnegative Borel function h on $\mathbf{R}^d$ is said to be α-harmonic on D if for each bounded open set B with $\overline{B} \subset D$ and for $x \in B$ we have

$$h(x) = E^x h(X_{\tau_B}) < \infty. \tag{1}$$

If $h \equiv 0$ on D^c then it is called *singular α-harmonic* on D. If B can be replaced by D in (1) then h is called *regular α-harmonic* on D.

The following theorem ([27, Theorem 3.1]) will be one of the basic tools in our paper.

Theorem 2.1 (Boundary Harnack Principle). *Let D be an open set, $z \in \partial D$, $r > 0$, $\kappa \in (0,1)$ and $B(A, \kappa r)$ is a ball in $D \cap B(z,r)$. Then there exists some constant $C = C(d,\alpha) > 1$ such that for any two functions u, v, which are positive regular α-harmonic in $D \cap B(z,2r)$ and vanish in $D^c \cap B(z,r)$, we have*

$$C^{-1}\kappa^{d+\alpha}\frac{v(x)}{v(A)} \le \frac{u(x)}{u(A)} \le C\kappa^{-d-\alpha}\frac{v(x)}{v(A)}, \qquad x \in D \cap B(z, r/2).$$

For $x \in \mathbf{R}^d$ we define the *α-harmonic measure* of D as $\omega_D^x(B) = P^x(X_{\tau_D} \in B)$. As a function of x it is regular α-harmonic on D if B is fixed. If D is Lipschitz, its harmonic measure is concentrated on $(\bar{D})^c$ and has the density with respect to the Lebesgue measure, which is called the *Poisson kernel* (see [10]). This kernel will be denoted by $P_D(x, y)$, $x \in D$, $y \in (\bar{D})^c$. It satisfies the following *scaling property*

$$P_D(x, y) = (1/r^d)P_{(1/r)D}(x/r, y/r), \qquad r > 0. \tag{2}$$

When $D = B(0,r)$, $r > 0$, the Poisson kernel is given by the following explicit formula

$$P_r(x, y) = C_{d,\alpha}\left(\frac{r^2 - |x|^2}{|y|^2 - r^2}\right)^{\alpha/2}\frac{1}{|x-y|^d}, \qquad |x| < r,\ |y| > r,$$

where $C_{d,\alpha} = \Gamma(d/2)\pi^{-d/2-1}\sin(\pi\alpha/2)$ (see [8], [25]).

For all nonnegative Borel measurable functions f we define *the Riesz potential* of f by

$$Uf(x) = E^x\int_0^\infty f(X_t)dt = \int A_{d,\alpha}|x-y|^{\alpha-d}f(y)dy,$$

where $A_{d,\alpha} = 2^{-\alpha}\pi^{-d/2}\Gamma((d-\alpha)/2)/\Gamma(\alpha/2)$ (see [7]).

For a Borel set B we define the *Green potential* of f by

$$G_Bf(x) = E^x\int_0^{\tau_B} f(X_t)dt = \int G_B(x, y)f(y)dy,$$

where $G_B(x, y)$ is the *Green function* of B defined by

$$G_B(x, y) = A_{d,\alpha}(|x-y|^{\alpha-d} - E^x|x - X_{\tau_B}|^{\alpha-d}), \qquad x, y \in B,\ x \ne y, \tag{3}$$

$G_B(x,x)=\infty$ and $G_B(x,y)=0$ otherwise. This function is symmetric (i.e. $G_B(x,y)=G_B(y,x)$), positive in *int* B and if $B_1\subset B_2$ then $G_{B_1}\le G_{B_2}$. Furthermore, G_B satisfies the following *scaling property*

$$G_B(x,y)=(1/r^{d-\alpha})G_{(1/r)B}(x/r,y/r),\qquad r>0. \tag{4}$$

For other properties of the Green function see [24] and [19].

Every nonnegative function which is singular α-harmonic on a bounded Lipschitz domain D has a unique representation (called the *Martin representation*)

$$f(x)=\int_{\partial D}M_D(x,z)\mu(dz), \tag{5}$$

where μ is a finite Borel measure on ∂D. The kernel function $M_D(x,z)$, called the *Martin kernel*, may be defined by

$$M_D(x,z)=\lim_{D\ni y\to z}\frac{G_D(x,y)}{G_D(x_0,y)},\qquad x\in D,\ z\in\partial D. \tag{6}$$

The existence of this limit follows from the Boundary Harnack Principle (see [9], [20]).

We will also use the following estimates for the Green function of $C^{1,1}$ domains ([19], [24]).

THEOREM 2.2. *If D is a $C^{1,1}$ domain, then there exist constants $c=c(D,\alpha)$ and $C=C(D,\alpha)$ such that for $x,y\in D$,*

$$c\min\left(\frac{A_{d,\alpha}}{|x-y|^{d-\alpha}},\frac{\delta_x^{\alpha/2}(D)\delta_y^{\alpha/2}(D)}{|x-y|^d}\right)\le G_D(x,y)$$

$$\le\min\left(\frac{A_{d,\alpha}}{|x-y|^{d-\alpha}},C\frac{\delta_x^{\alpha/2}(D)\delta_y^{\alpha/2}(D)}{|x-y|^d}\right). \tag{7}$$

From now on we will assume that $r>0$, $x\in D$, $y\in(\partial D)^c$, $z,Q\in\partial D$. $x_0\in D$ will be a fixed reference point. From now on we take R_0 so small that $\delta_{x_0}(D)\ge R_0/2$. For $r\le R_0/32$ and $Q\in\partial D$ we denote by $A_{Q,r}$ a point for which $B(A_{Q,r},\kappa r)\subset B(Q,r)\cap D$ for a certain absolute constant $\kappa=\kappa(D)=1/(2\sqrt{1+\lambda^2})$. The set of such points is nonempty and $A_{Q,r}$ is not unique. For $r>R_0/32$ we set $A_{Q,r}=x_1$, where $x_1\in D$ is another fixed point such that $|x_0-x_1|=R_0/4$. In particular, $\delta_{x_1}(D)\ge R_0/4$. Furthermore, let $\rho=\rho(x,y)=\max(\delta_x(D),\delta_y(D),|x-y|)$. For $\rho\le R_0/32$ we denote by $A=A(x,y)$ a point for which $B(A,\kappa\rho)\subset D\cap B(x,3\rho)\cap B(y,3\rho)$. The set of such points is also nonempty and if $S\in\partial D$ is such that $\delta_y=|y-S|$, we may take $A=A_{S,\delta_y}$. If $\rho>R_0/32$, we set $A(x,y)=x_1$. See [22] and [11] for details.

Let $\phi_D(x) = \min(G_D(x, x_0), C_{d,\alpha}(R_0/4)^{\alpha-d})$. We will use the following estimates for the Poisson kernel and the Green function of Lipschitz domains ([22])

THEOREM 2.3. *There are constants* $c = c(d, \alpha, \lambda, R_0, diam(D))$, $C = C(d, \alpha, \lambda, R_0, diam(D))$ *such that for* $x \in D$, $y \in (\overline{D})^c$ *we have*

$$c \frac{\phi_D(x)\phi_D(y)}{|x-y|^{d-\alpha}\phi_D^2(A(x,y))} \le G_D(x,y)$$

$$\le C \frac{\phi_D(x)\phi_D(y)}{|x-y|^{d-\alpha}\phi_D^2(A(x,y))}. \tag{8}$$

THEOREM 2.4. *There are constants* $c = c(d, \alpha, \lambda, R_0, diam(D))$, $C = C(d, \alpha, \lambda, R_0, diam(D))$ *such that for* $x \in D$, $y \in (\overline{D})^c$ *we have*

$$c \frac{\phi_D(x)\phi_D(y')}{|x-y|^{d-\alpha}\phi_D^2(A(x,y'))\delta_y^\alpha(D)(1+\delta_y(D))^\alpha} \le P_D(x,y)$$

$$\le C \frac{\phi_D(x)\phi_D(y')}{|x-y|^{d-\alpha}\phi_D^2(A(x,y'))\delta_y^\alpha(D)(1+\delta_y(D))^\alpha}, \tag{9}$$

where $y' = A_{S,\delta_y}$ *and* $S \in \partial D$ *satisfies* $\delta_y = |y - S|$.

We will also use the following version of ([10, Lemma 17]), which can be proved in an analogous way.

LEMMA 2.5. *Let* B_n *and* B *be open sets such that* $B_n \nearrow B$. *Assume that for* $x \in B$ *we have* $P^x(X_{\tau_B} \in \partial B) = 0$ *and* $P^x(\tau_B < \infty) = 1$. *Then* $\lim_{n\to\infty} P^x(\{X_{\tau_{B_n}} = X_{\tau_B}\}) = 1$. *In particular, this implies that* $\lim_{n\to\infty} G_{B_n}(x,y) = G_B(x,y)$ *for* $x, y \in B$ *and* $\lim_{n\to\infty} P_{B_n}(x,y) = P_B(x,y)$ *for* $x \in B$, $y \in int\, B^c$.

3. Main results

First we define an *unbounded cone* with vertex at $0 = (0, 0, \dots, 0)$ and symmetric with respect to the d-th axis. This is a set V defined by

$$V = \{x : \eta \cdot |(x_1, x_2, \dots, x_{d-1})| < x_d\},$$

where $\eta \in (-\infty, \infty)$. The *aperture* of V is the angle $\gamma = \arccos(\eta/\sqrt{1+\eta^2})$. For $\gamma = \pi$ we define V by $V = \mathbf{R}^d \backslash \{(0, 0, \dots, 0, x_d) : x_d \le 0\}$. Generally, an unbounded cone with vertex at 0 is a set V' isomorphic to the cone V defined above and satisfies $rV' = V'$. Finally, an unbounded cone with vertex at $Q \in \mathbf{R}^d$ is a set $V' + Q$.

For an unbounded cone V with vertex at Q we denote by $\mathbf{1}$ a point on the axis of V such that $|\mathbf{1} - Q| = 1$. Since the process X_t is homogeneous and translationally invariant, we will often assume that $Q = 0$ and $\mathbf{1} = (0, 0, \dots, 0, 1)$.

Let V be a circular cone with vertex at 0 and aperture $\gamma \in (0, \pi]$. Assume that $\mathbf{1} = (0, 0, \dots, 0, 1)$. By [2, Theorem 3.2], there exists the so-called *Martin kernel with pole at infinity*. This is a unique nonnegative function M_V on $\mathbf{R}^d$ such that $M_V(\mathbf{1}) = 1$, $M_V \equiv 0$ on V^c and M_V is regular α-harmonic on every open bounded subset of V. Moreover, M_V is locally bounded on $\mathbf{R}^d$ and homogeneous of degree $\beta \in [0, \alpha)$, that is,

$$M_V(x) = |x|^\beta M_V(x/|x|), \qquad x \in V. \tag{10}$$

Furthermore, $\beta = \beta(V, \alpha)$ is a strictly decreasing function of γ ([2, Lemma 3.3]). We will call β *the characteristics* of V.

We know the explicit formulas of M_V in several cases. If $\gamma = \pi/2$, then V is the half-space $\{(x_1, x_2, \dots, x_d) : x_d > 0\}$. In this case $M_V(x) = x_d^{\alpha/2}$, $x \in V$ and this gives $\beta = \alpha/2$. If $\gamma = \pi$ and ($d > 2$ or $\alpha \le 1$), then V^c is polar ([25]). This implies that $M_V(x) = \mathbf{1}_V(x)$ and $\beta = 0$ in this case. If $\gamma = \pi$, $d = 2$ and $\alpha > 1$, then V^c is not polar and due to [2, Theorem 3.4] we have $\beta > 0$. Actually, it is proved in [14] that $\beta = (\alpha - 1)/2$. We do not know the formula of M_V in this case.

Later on, we will prove that M_V is nothing but a constant multiple of a classical Martin kernel $M_V(x, \infty)$ defined by (6) (see Theorem 3.13).

We now present further properties of M_V. This function plays an important role in further analysis.

LEMMA 3.1. *For $n \in N$ let V_n and V be circular cones with vertices at 0, apertures γ_n and γ respectively, characteristics β_n and β respectively and Martin kernels with pole at infinity M_n and M respectively. Assume that γ_n, $\gamma \in (0, \pi]$ and $\mathbf{1} = (0, 0, \dots, 0, 1)$ is on the axis of symmetry of V and every V_n. Then $M_n(x) \to M(x)$ for all $x \in V$ if $\lim_{n\to\infty} \gamma_n = \gamma$. Furthermore, M is continuous on $\mathbf{R}^d$ iff $\gamma < \pi$ or $d = 2$, $\gamma = \pi$, $\alpha > 1$.*

PROOF. Since $\gamma_n \to \gamma$, we may assume that $\gamma_n > \gamma/2$ for all n, without loss of generality. Since $\gamma > 0$, there is a constant $\varepsilon > 0$ such that for every n, $B(\mathbf{1}, \varepsilon) \subset V_n \cap B(0, 3/2)$. We will prove that there exists $C > 0$ such that for every $|x| \le 1$, $M(x) \le C$, where C does not depend on V.

Since M is regular α-harmonic on $B(0, r) \cap V$ for $r \in [3/2, 2]$, we get

$$\begin{aligned} M(x) &= E^x M(X_{\tau_{V\cap B(0,r)}}) = E^x\{M(X_{\tau_{V\cap B(0,r)}}) : X_{\tau_{V\cap B(0,r)}} \in V\} \\ &= E^x\{M(X_{\tau_{B(0,r)}}) : X_{\tau_{V\cap B(0,r)}} \in V\} \le E^x M(X_{\tau_{B(0,r)}}) \\ &= \int P_r(x, y) M(y) dy, \end{aligned} \tag{11}$$

where $P_r(x, y) = C_{d,\alpha}\mathbf{1}_{\{|y|>r\}} \cdot \left(\frac{r^2-|x|^2}{|y|^2-r^2}\right)^{\alpha/2} \cdot \frac{1}{|x-y|^d}$. Let $P_\varepsilon(y)$ be the Poisson kernel for the ball $B(\mathbf{1}, \varepsilon)$ for the process starting from $\mathbf{1}$. We introduce a regularized version of the Poisson kernel $P_r(x, y)$ in a similar way as in [12]. Fix a nonnegative $\varphi \in C_c^\infty((3/2, 2))$ such that $\int_{3/2}^2 \varphi(r)dr = 1$ and define $\tilde{P}(x, y) = \int_{3/2}^2 \varphi(r)P_r(x, y)dr$. Then by [12, Lemma 3.11] we have

$$\tilde{P}(x, y) \le C/(1 + |y|)^{d+\alpha},$$

where C depends on d and α. Moreover, for $|y| \ge 3/2$,

$$P_\varepsilon(y) = \frac{\varepsilon^\alpha}{(|y-\mathbf{1}|^2 - \varepsilon^2)^{\alpha/2}|y-\mathbf{1}|^d} \ge \frac{C}{|y|^{d+\alpha}}$$

for some constant $C = C(d, \alpha, \varepsilon)$. Hence for $|x| \le 1$ and $|y| \ge 3/2$ we obtain $\tilde{P}(x, y) \le CP_\varepsilon(y)$. Therefore, since $\int \tilde{P}(x, y)dy = 1$, by (11) we get

$$\begin{aligned} M(x) &\le \int_{|y|>3/2} \tilde{P}(x, y)M(y)dy \le C\int_{|y|>3/2} P_\varepsilon(y)M(y)dy \\ &\le C\int_{B^c(\mathbf{1},\varepsilon)} P_\varepsilon(y)M(y)dy = CM(\mathbf{1}) = C. \end{aligned}$$

Therefore, for every n and $|x| \le 1$ we have

$$M_n(x) \le C. \tag{12}$$

This implies that for $|x| > 1$, by homogeneity of M_n we get

$$M_n(x) = |x|^{\beta_n} M_n(x/|x|) \le C|x|^{\beta_n}. \tag{13}$$

We know that $\beta_n < \alpha$ for every n and β_n increases as γ_n decreases. Therefore, if $\gamma > 0$, we may assume by (12) and (13) that there is $\beta' < \alpha$ such that for every n,

$$M_n(x) \le C \max(|x|^{\beta'}, 1). \tag{14}$$

It implies that M_n are uniformly bounded on every bounded subset $B \subset V$. Also, M_n are equicontinuous on compact subsets $F \subset V$ because of the gradient estimates (see [15]). Therefore, by the Arzeli Theorem and diagonal procedure we find a subsequence M_{n_k} almost uniformly convergent to some nonnegative function $\tilde{M}$ on V.

We prove now that $\tilde{M}$ is α-harmonic on V. Let $x \in V$ and $r > 0$ such that $B = B(x, r) \subset \overline{B(x, r)} \subset V_n$. Then we have

$$M_n(x) = \int_{B^c} M_n(y)P_r(x, y)dy. \tag{15}$$

If $R > 0$ is sufficiently large, then for $|y| > R$ we have

$$P_r(x, y) = \frac{r^\alpha}{(|y-x|^2 - r^2)^{\alpha/2}|x-y|^d} \le \frac{C}{|y|^{d+\alpha}},$$

where $C = C(d, \alpha, r, R)$. Moreover, by (14), for every n we have $M_n(y) \le C|y|^{\beta'}$, $|y| \ge 1$, and $M_n(y) \le C$, $|y| \le R$. Therefore, we get

$$M_n(y)P_r(x, y) \le \begin{cases} C|y|^{\beta'-d-\alpha}, & |y| > R, \\ CP_r(x, y), & |y| \le R. \end{cases}$$

The right hand side of the above inequality is integrable over B^c since $\beta' < \alpha$ and $\int P_r(x, y)dy = 1$. Hence, by dominated convergence, we get

$$\tilde{M}(x) = \lim_{k\to\infty} M_{n_k}(x) = \int_{B^c} \lim_{k\to\infty} M_{n_k}(y)P_r(x, y)dy = \int_{B^c} \tilde{M}(y)P_r(x, y)dy,$$

which proves that $\tilde{M}$ is α-harmonic on V. Obviously, $\tilde{M} \equiv 0$ on V^c and $\tilde{M}(\mathbf{1}) = 1$.

Now we need to prove that $\tilde{M}$ is regular α-harmonic on bounded subsets of V. Assume first that $\gamma < \pi$ or $d > 2$ or $\alpha \le 1$. Let $B = B(0, R)$ for some $R > 0$. We show that the set $\Gamma = V \cap B$ satisfies the assumptions of Lemma 2.5. This is immediate if $0 < \gamma < \pi$, since Γ are bounded Lipschitz domains in this case. If $\gamma = \pi$, then $V = \mathbf{R}^d \backslash \{(0, 0, \dots, 0, x_d) : x_d \le 0\}$. Since $d \ge 3$ or $\alpha \le 1$, the set V^c is polar ([25]) and $M \equiv \mathbf{1}_V$ is not continuous on $\mathbf{R}^d$. Let $x \in \Gamma$. Since the process does not hit ∂B when leaving B, we see that $P^x(\tau_\Gamma \in \partial\Gamma) = 0$. Obviously, $P^x(\tau_\Gamma < \infty) = 1$ as Γ is bounded. Now take open sets $B_n \subset \overline{B_n} \subset \Gamma$ such that $B_n \nearrow \Gamma$. Since $\tilde{M}$ is α-harmonic and locally bounded on V, we get $\tilde{M}(x) = E^x\tilde{M}(X_{\tau_{B_n}}) \to E^x\tilde{M}(X_{\tau_\Gamma})$ by Lemma 2.5. Hence $\tilde{M}$ is regular α-harmonic on $V \cap B$.

Now let $\gamma = \pi$, $d = 2$ and $\alpha > 1$. This case requires different arguments since V^c is not polar. Let $x \in V$. Fix $z \in \partial V$ and take $R > 2$ such that $z \in B(0, R/4)$. First we prove that there exists a constant $C = C(d, \alpha, R)$ such that for $x \in V \cap B(0, R/4)$,

$$\tilde{M}(x) \le CM(x). \tag{16}$$

To show it we define $f_n(x) = P^x(X_{\tau_{\Gamma_n}} \in B^c)$ and $f(x) = P^x(X_{\tau_\Gamma} \in B^c)$ where $\Gamma_n = V_n \cap B$, $\Gamma = V \cap B$. As $\Gamma_n \nearrow \Gamma$, we may assume that $x \in \Gamma_n$ for all n for a given x. Furthermore, $B(\mathbf{1}, 1/2) \subset \Gamma_n$. Next we observe that f_n and M_n are regular α-harmonic on Γ_n and vanish on $V_n^c \cap B$. Similarly, f and M are regular α-harmonic on Γ and vanish on $V^c \cap B$. Therefore, by the Boundary Harnack Principle we obtain

$$c\frac{f_n(\mathbf{1})}{M_n(\mathbf{1})} \le \frac{f_n(x)}{M_n(x)} \le C\frac{f_n(\mathbf{1})}{M_n(\mathbf{1})},$$

$$c\frac{f(\mathbf{1})}{M(\mathbf{1})} \le \frac{f(x)}{M(x)} \le C\frac{f(\mathbf{1})}{M(\mathbf{1})}, \tag{17}$$

where c, C depend on d, α, R. Since $f_n(x) \le f(x)$ and $1 \ge f(\mathbf{1}) \ge f_n(\mathbf{1}) \ge P^{\mathbf{1}}(X_{\tau_{B(\mathbf{1},1/2)}} \in B^c) = C > 0$, (17) gives $M_n(x) \le C'M(x)$ and letting $n \to \infty$ we obtain (16).

Next we show that M is continuous on $\mathbf{R}^d$. We only need to show that

$$\lim_{x \to z} M(x) = 0. \tag{18}$$

First assume that $z = 0$. M is locally bounded and $M(x) = |x|^{\beta} M(x/|x|)$ with $\beta > 0$ so (18) is immediate. Now let $z \ne 0$. From (17) we get $f(x) \le CM(x) \to 0$ as $x \to 0$. Let $y = x - z$. Observe that $\Gamma - z \subset V \cap B(-z, R)$. Hence

$$f(x) = P^x(X_{\tau_\Gamma} \in B^c) = P^y(X_{\tau_{\Gamma - z}} \in B^c(-z, R))$$
$$\le P^y(X_{\tau_{V \cap B(-z,R)}} \in B^c(-z, R)) = g(y).$$

Take $r > 0$ such that $B(0, r) \subset B(-z, R)$. Then g is regular α-harmonic on $V \cap B(0, r)$ and vanishes on $V^c \cap B(0, r)$. Hence for y sufficiently close to 0 we obtain $g(y) \le CM(y)$ by similar arguments as for (17). Therefore, $\lim_{y \to 0} g(y) = 0$, which implies $\lim_{x \to z} f(x) = 0$. Now (18) follows from (17). Combining this with (16) we see that $\tilde{M}$ is continuous on $\mathbf{R}^d$. But we know that if a function is α-harmonic on a bounded set D and is continuous on $\bar{D}$, then is regular α-harmonic on D ([27]). Hence, as in the previous cases, $\tilde{M}$ is regular α-harmonic on every open bounded subset of V. Therefore, by the uniqueness of M ([2, Theorem 3.2]), $\tilde{M} \equiv M$ and $\lim_{n \to \infty} M_n = M$ on V. Finally, continuity of M for $\gamma < \pi$ will follow from Lemma 3.3 stated below. This completes the proof. □

The next theorem provides some information about β.

Theorem 3.2. *Let V_n and V be as in Lemma* 3.1. *Let $\gamma \in (0, \pi]$. Then*
(i) *If $\lim_{n \to \infty} \gamma_n = \gamma$, then $\lim_{n \to \infty} \beta_n = \beta$.*
(ii) *If $\lim_{n \to \infty} \gamma_n = 0$, then $\lim_{n \to \infty} \beta_n = \alpha$.*
(iii) *If $\gamma = \pi$ then $\beta = (\alpha - 1)/2$ in case $d = 2$ and $\alpha > 1$, otherwise $\beta = 0$.*
(iv) *If $\gamma = \pi/2$, then $\beta = \alpha/2$.*

Proof. From (10) we have $M_n((1/2) \cdot \mathbf{1}) = (1/2)^{\beta_n}$ and $M((1/2) \cdot \mathbf{1}) = (1/2)^{\beta}$. To prove (i) we use Lemma 3.1. We have $M_n((1/2) \cdot \mathbf{1}) \to M((1/2) \cdot$

1) and it gives $\lim_{n\to\infty}\beta_n=\beta$. It remains to prove (ii) as (iii) and (iv) were discussed before. By [23, Theorem 4.6], for $x\in V_n\cap B(0,2)$ we have

$$c\delta_x^{\alpha/2}(V_n)|x|^{\alpha/2-\varepsilon'}\le G_{V_n}(x,8\cdot\mathbf{1})\le C\delta_x^{\alpha/2}(V_n)|x|^{\alpha/2-\varepsilon},\tag{19}$$

where c, C depend on d, α, n and $0<\varepsilon'<\varepsilon$ satisfy $\varepsilon,\varepsilon'\to 0$ if $\gamma_n\to 0$ [23, Lemma 3.7]. Since M_n and $G_{V_n}(\cdot,8\cdot\mathbf{1})$ are regular α-harmonic on $V_n\cap B(0,6)$ and vanish on $V_n^c\cap B(0,6)$, we obtain by the Boundary Harnack Principle for all n and $x=r\cdot\mathbf{1}$, $r\le 1$

$$\frac{M_n(r\cdot\mathbf{1})}{M_n(\mathbf{1})}=\frac{G_{V_n}(r\cdot\mathbf{1},8\cdot\mathbf{1})}{G_{V_n}(\mathbf{1},8\cdot\mathbf{1})}.$$

Combining this with (19) we see that $cr^{\alpha-\varepsilon'}\le M_n(r\cdot\mathbf{1})\le Cr^{\alpha-\varepsilon}$. From (10) we have $M_n(r\cdot\mathbf{1})=r^{\beta_n}M_n(\mathbf{1})=r^{\beta_n}$ and this implies that $\alpha-\varepsilon\le\beta_n\le\alpha-\varepsilon'$. Since $\varepsilon,\varepsilon'\to 0$ in case $n\to\infty$, we get $\beta_n\to\alpha$. This completes the proof. □

The rest of the section is devoted to study the behaviour of the Green function, Poisson kernels and Martin kernels for cones of apertures less than π. For any cone V with its vertex at 0 and aperture $\gamma<\pi$ we define its *inner smooth set* B_V. This will be a fixed $C^{1,1}$ domain such that

$$(B(0,3/2)\backslash B(0,1/16))\cap V\subset B_V\subset(B(0,2)\backslash B(0,1/32))\cap V.$$

We notice that B_V can be chosen to be dependent only on d and γ. Next we define a *bounded* ('smooth') cone of length 2 and vertex at 0 by

$$V_2=B_V\cup(V\cap B(0,1/16)).$$

Similarly we define a bounded ('smooth') cone of length $R>0$ and vertex at 0 by $V_R=(R/2)V_2$. Finally, a bounded ('smooth') cone of length R and vertex at Q is a set of the form $Q+V_R$.

From now on V and V_R will denote, respectively, unbounded and bounded cones with vertices at a fixed Q. We also set $R_V=(1/8)\sin\gamma\le 1/8$. This implies that $V\cap B(z,4R_V)\subset B_V$ if $z\in\partial V\cap\partial B(Q,3/4)$. Furthermore, let $x'=(1/16)\cdot\mathbf{1}=(0,0,\dots,0,1/16)$.

The next lemma provides sharp estimates for M_V.

LEMMA 3.3. *There exist constants* $c=c(d,\alpha,\gamma)$, $C=C(d,\alpha,\gamma)$ *such that for every* $x\in V$ *we have*

$$c|x-Q|^{\beta-\alpha/2}\delta_x^{\alpha/2}(V)\le M_V(x)\le C|x-Q|^{\beta-\alpha/2}\delta_x^{\alpha/2}(V).\tag{20}$$

PROOF. We may assume that $Q=0$ and **1** lies on the axis of V. Let $z\in\partial V\cap\partial B(0,1)$. Since M_V is homogeneous, we obtain

$$M_V(x)=M_V(x/|x|)|x|^{\beta}.\tag{21}$$

By definition of B_V and R_V we have $x' \in B_V$ and $|z - x'| > 4R_V$. So the functions $M_V(\cdot)$ and $G_{B_V}(\cdot, x')$ are regular α-harmonic on $V \cap B(z, 4R_V)$ and vanish on $V^c \cap B(z, 4R_V)$. Therefore, if $x/|x| \in B(z, R_V)$, then by the Boundary Harnack Principle we obtain

$$c\frac{G_{B_V}(x/|x|, x')}{G_{B_V}(A_{z,R_V}, x')} \le \frac{M_V(x/|x|)}{M_V(A_{z,R_V})} \le C\frac{G_{B_V}(x/|x|, x')}{G_{B_V}(A_{z,R_V}, x')}, \tag{22}$$

where c, C depend only on d, α, γ, and the definition of the points A_{z,R_V} refers to the set B_V. Since $\delta_{A_{z,R_V}}(V) = \delta_{A_{z,R_V}}(B_V) \ge \kappa R_V$, all the points A_{z,R_V} are in a compact subset of V. Since $M_V(\cdot)$ is continuous, positive on V and $G_{B_V}(\cdot, x')$ is continuous, positive on $B_V \backslash \{x'\}$, we get

$$\begin{aligned} c &\le G_{B_V}(A_{z,R_V}, x') \le C, \\ c &\le M_V(A_{z,R_V}) \le C, \end{aligned} \tag{23}$$

and, again, c, C depend only on d, α, γ. By the definition of B_V we have $\delta_{x/|x|}(B_V) = \delta_{x/|x|}(V)$ if $x/|x| \in B(z, R_V)$. Therefore, from Theorem 2.2 we have

$$\begin{aligned} c'\delta_{x/|x|}^{\alpha/2}(V) &\le c\frac{\delta_{x/|x|}^{\alpha/2}(B_V)\delta_{x'}^{\alpha/2}(B_V)}{|x/|x| - x'|^d} \le G_{B_V}(x/|x|, x') \\ &\le C\frac{\delta_{x/|x|}^{\alpha/2}(B_V)\delta_{x',B_V}^{\alpha/2}}{|x/|x| - x'|^d} \le C'\delta_{x/|x|}^{\alpha/2}(V) \end{aligned} \tag{24}$$

for x such that $x/|x| \in B(z, R_V)$ and for any $z \in \partial V \cap \partial B(0,1)$. Now (24) extends easily to all $x/|x| \in V \cap \partial B(0,1)$ since $\delta_x(V)$ and $G_{B_V}(x, x')$ are positive and continuous functions of x on $V \cap \partial B(0,1)$. Since $\delta_{x/|x|}(V) = \delta_x(V)/|x|$, we obtain (20) by combining (21), (22), (23) and (24). The proof is completed. □

A bounded cone is a Lipschitz domain with a localization radius R_0 and a Lipschitz constant $\lambda \ge \eta$. Recall that $\phi_{V_R}(x) = \min(G_{V_R}(x, x_0), C_{d,\alpha}R_0^{\alpha-d})$. We know that if D is a Lipschitz domain with its localization radius r_0, then rD is a Lipschitz domain with its localization radius rr_0. This implies that ϕ_{V_R} has the same scaling property as G_{V_R}.

With this remark we are ready to present estimates for ϕ_{V_R}.

Lemma 3.4. *There exist constants $c = c(d, \alpha, \gamma)$, $C = C(d, \alpha, \gamma)$ such that for every $x, y \in V_R$ we have*

$$cR^{\alpha-d-\beta}\delta_x^{\alpha/2}(V_R)|x - Q|^{\beta-\alpha/2} \le \phi_{V_R}(x) \le CR^{\alpha-d-\beta}\delta_x^{\alpha/2}(V_R)|x - Q|^{\beta-\alpha/2}.$$

Proof. The steps taken are analogous to those in the proof of Lemma 3.3. We may assume that $Q = 0$. Let $R = 2$. We know that $\phi_{V_2}(x) = G_{V_2}(x, x_0)$ for x sufficiently close to ∂V_2. Let $z \in \partial V_2$ and $x \in B(z, R_V)$. If $|z| \leq 1$, then applying the Boundary Harnack Principle to the functions $G_{V_2}(\cdot, x_0)$, M_V and using Lemma 3.3, we obtain $\delta_x^{\alpha/2}(V)|x|^{\beta-\alpha/2} \leq G_{V_2}(x, x_0) \leq C\delta_x^{\alpha/2}(V)|x|^{\beta-\alpha/2}$. Since $(1/2)\delta_x(V) \leq \delta_x(V_2) \leq \delta_x(V)$, this leads to the desired result. If $|z| \geq 1$, then we have $c \leq |x|^{\beta-\alpha/2} \leq C$. In this case we apply the Boundary Harnack Principle to the functions $G_{V_2}(\cdot, x_0)$ and $G_{B_V}(\cdot, x_0)$. We obtain $cG_{B_V}(x, x_0) \leq G_{V_2}(x, x_0) \leq CG_{B_V}(x, x_0)$. Now Theorem 2.2 completes the proof for ϕ_{V_2}. The scaling property of ϕ_{V_R} gives the estimates for all V_R. □

We also need the following technical lemma.

Lemma 3.5. *Let $Q = 0$. Assume that $x, y \in V_2$. Then there exist constants $c = c(d, \alpha, \gamma)$, $C = C(d, \alpha, \gamma)$ such that for $A = A(x, y)$ we have*

$$c \max(|x|, |y|) \leq |A| \leq C \max(|x|, |y|). \tag{25}$$

Proof. We note that for V_2, R_0 can be chosen to be dependent only on d and γ. Recall that $\rho = \max(\delta_x(V_2), \delta_y(V_2), |x - y|)$. This gives

$$\rho \leq \max(|x|, |y|, |x| + |y|) \leq 2 \max(|x|, |y|). \tag{26}$$

Assume first that $\rho \leq R_0/32$. Then, by definition, $A \in B(x, 3\rho) \cap B(y, 3\rho)$ and $\kappa\rho \leq \delta_A(V_2) \leq \rho$. This, combined with (26), leads to $|A| \leq |x| + 3\rho \leq 7 \max(|x|, |y|)$. Furthermore, $|A| \geq |x| - 3\rho$, $|A| \geq |y| - 3\rho$. This implies that

$$|A| \geq \max(|x|, |y|) - 3\rho \tag{27}$$

and

$$|A| \geq \delta_A(V_2) \geq \kappa\rho. \tag{28}$$

If $\rho \leq \max(|x|, |y|)/(\kappa + 3)$, then from (27), $|A| \geq \max(|x|, |y|)\kappa/(\kappa + 3)$. If $\rho \geq \max(|x|, |y|)/(\kappa + 3)$, then $|A| \geq \max(|x|, |y|)\kappa/(\kappa + 3)$ from (28). Hence we proved that $7 \max(|x|, |y|) \geq |A| \geq \max(|x|, |y|)\kappa/(\kappa + 3)$.

If $\rho > R_0/32$, then we have $A = x_1$ and by (26), $2 \geq \max(|x|, |y|) \geq R_0/64$. Hence (25) holds immediately. The proof is completed. □

Now, with the use of Theorem 2.3 and the estimates for ϕ_{V_R} we obtain sharp estimates for the Green functions of bounded cones. This theorem improves [23, Theorem 4.7].

Theorem 3.6. *Let $R > 0$. There exist constants $c = c(d, \alpha, \gamma)$, $C = C(d, \alpha, \gamma)$ such that for every $x, y \in V_R$ we have*

$$
\begin{aligned}
& c \min\left(\frac{A_{d,\alpha}}{|x-y|^{d-\alpha}}, \frac{\delta_x^{\alpha/2}(V_R)\delta_y^{\alpha/2}(V_R)}{|x-y|^d}\left(\frac{\min(|x-Q|,|y-Q|)}{\max(|x-Q|,|y-Q|)}\right)^{\beta-\alpha/2}\right) \\
& \quad \le G_{V_R}(x,y) \\
& \quad \le \min\left(\frac{A_{d,\alpha}}{|x-y|^{d-\alpha}}, C\frac{\delta_x^{\alpha/2}(V_R)\delta_y^{\alpha/2}(V_R)}{|x-y|^d}\left(\frac{\min(|x-Q|,|y-Q|)}{\max(|x-Q|,|y-Q|)}\right)^{\beta-\alpha/2}\right).
\end{aligned}
$$

Proof. Let $R = 2$ and $Q = 0$. We note that in this case the choice of R_0 depends only on d and γ. Assume that $x, y \in V_2$ and $A = A(x,y)$ is as in Theorem 2.3. From Lemma 3.4 we have

$$
\begin{aligned}
c\delta_x^{\alpha/2}(V_2)|x|^{\beta-\alpha/2} &\le \phi_{V_2}(x) \le C\delta_x^{\alpha/2}(V_2)|x|^{\beta-\alpha/2}, \\
c\delta_y^{\alpha/2}(V_2)|y|^{\beta-\alpha/2} &\le \phi_{V_2}(y) \le C\delta_y^{\alpha/2}(V_2)|y|^{\beta-\alpha/2}, \qquad (29) \\
c\delta_A^{\alpha/2}(V_2)|A|^{\beta-\alpha/2} &\le \phi_{V_2}(A) \le C\delta_A^{\alpha/2}(V_2)|A|^{\beta-\alpha/2}.
\end{aligned}
$$

Recall that $\rho = \max(\delta_x(V_2), \delta_y(V_2), |x-y|)$. We consider two cases.

Case 1. Let $|x-y| \ge (1/10)\max(\delta_x(V_2), \delta_y(V_2))$. We have $|x-y| \le \rho \le 10|x-y|$. If $\rho \le R_0/32$, then by definition, $\kappa\rho \le \delta_A(V_2) \le \rho$. If $\rho > R_0/32$, then $\delta_A(V_2) = \delta_{x_1}(V_2)$ and $4 \ge |x-y| \ge R_0/320$. This implies that

$$
c|x-y| \le \delta_A(V_2) \le C|x-y|. \qquad (30)
$$

Now, combining (25), (29), (30) and (8) we obtain

$$
\begin{aligned}
G_{V_2}(x,y) &\ge c\frac{\delta_x^{\alpha/2}(V_2)\delta_y^{\alpha/2}(V_2)}{|x-y|^d}\left(\frac{\min(|x|,|y|)}{\max(|x|,|y|)}\right)^{\beta-\alpha/2} \\
&\ge \min\left(\frac{A_{d,\alpha}}{|x-y|^{d-\alpha}}, c\frac{\delta_x^{\alpha/2}(V_2)\delta_y^{\alpha/2}(V_2)}{|x-y|^d}\left(\frac{\min(|x|,|y|)}{\max(|x|,|y|)}\right)^{\beta-\alpha/2}\right),
\end{aligned}
$$

and

$$
G_{V_2}(x,y) \le C\frac{\delta_x^{\alpha/2}(V_2)\delta_y^{\alpha/2}(V_2)}{|x-y|^d}\left(\frac{\min(|x|,|y|)}{\max(|x|,|y|)}\right)^{\beta-\alpha/2}.
$$

Since $G_{V_2}(x,y) \le A_{d,\alpha}|x-y|^{\alpha-d}$ by definition, this completes the proof in Case 1.

Case 2. Let $|x-y| \le (1/10)\max(\delta_x(V_2), \delta_y(V_2))$. We may assume that $\delta_x(V_2) \ge \delta_y(V_2)$. This means that $|x-y| \le (1/10)\delta_x(V_2)$. Let $S \in \partial V_2$ be a point for which $\delta_y(V_2) = |y-S|$. Then we obtain $\delta_x(V_2) \le |x-S| \le |x-y| + |y-S| \le (1/10)\delta_x(V_2) + \delta_y(V_2)$. This implies that

$$
\delta_y(V_2) \le \delta_x(V_2) \le (10/9)\delta_y(V_2), \qquad (31)
$$

which leads to $|x-y| \le (1/9)\min(\delta_x(V_2), \delta_y(V_2))$. Hence, by [22, Lemma 11],

$$c|x-y|^{\alpha-d} \le G_{V_2}(x,y) \le A_{d,\alpha}|x-y|^{\alpha-d},$$

where c depends on d, α. Hence, immediately,

$$G_{V_2}(x,y) \ge c\min\left(\frac{A_{d,\alpha}}{|x-y|^{d-\alpha}}, \frac{\delta_x^{\alpha/2}(V_2)\delta_y^{\alpha/2}(V_2)}{|x-y|^d}\left(\frac{\min(|x|,|y|)}{\max(|x|,|y|)}\right)^{\beta-\alpha/2}\right).$$

Furthermore, using (31) we have $|x| \le |x-y| + |y| \le (1/10)\delta_x(V_2) + |y| \le (1/9)\delta_y(V_2) + |y| \le (10/9)|y|$ and, similarly, $|y| \le |x-y| + |x| \le (1/10)\delta_x(V_2) + |x| \le (11/10)|x|$. This implies that

$$(9/10)|x| \le |y| \le (11/10)|x|.$$

This means that

$$c \le \left(\frac{\min(|x|,|y|)}{\max(|x|,|y|)}\right)^{\beta-\alpha/2} \le C.$$

As we have $|x-y|^\alpha \le ((1/9)\min(\delta_x(V_2), \delta_y(V_2)))^\alpha \le (1/9)^\alpha \delta_x^{\alpha/2}(V_2)\delta_y^{\alpha/2}(V_2)$,

$$G_{V_2}(x,y) \le \frac{A_{d,\alpha}}{|x-y|^{d-\alpha}} \le C\frac{\delta_x^{\alpha/2}(V_2)\delta_y^{\alpha/2}(V_2)}{|x-y|^d}\left(\frac{\min(|x|,|y|)}{\max(|x|,|y|)}\right)^{\beta-\alpha/2}.$$

This completes the proof for G_{V_2}. The scaling property (4) extends the estimates to all G_{V_R}. □

In an analogous way we can obtain estimates for P_{V_R} as it is stated in the next theorem. We present the proof for the convenience of the reader. The estimates for $P_{V_R}(x,y)$ when $\delta_y(V_R) \le R_0/32$ are the same as in [14, Theorem 3.2] and they have been proved independently and simultaneously.

THEOREM 3.7. *Let $R > 0$. Then there exist constants $c = c(d,\alpha,\gamma)$, $C = C(d,\alpha,\gamma)$ such that for $x \in V_R$, $y \in int\, V_R^c$ we have*

$$c\frac{\delta_x^{\alpha/2}(V_R)}{\delta_y^{\alpha/2}(V_R)|x-y|^d}\left(\frac{\min(|x-Q|,|y-Q|)}{\max(|x-Q|,|y-Q|)}\right)^{\beta-\alpha/2} \le P_{V_R}(x,y)$$
$$\le C\frac{\delta_x^{\alpha/2}(V_R)}{\delta_y^{\alpha/2}(V_R)|x-y|^d}\left(\frac{\min(|x-Q|,|y-Q|)}{\max(|x-Q|,|y-Q|)}\right)^{\beta-\alpha/2}$$

if $\delta_y(V_R) \le R_0/32$, and

$$c\frac{\delta_x^{\alpha/2}(V_R)|x-Q|^{\beta-\alpha/2}R^{\alpha-\beta}}{|y|^{d+\alpha}} \le P_{V_R}(x,y) \le C\frac{\delta_x^{\alpha/2}(V_R)|x-Q|^{\beta-\alpha/2}R^{\alpha-\beta}}{|y|^{d+\alpha}}$$

if $\delta_y(V_R) > R_0/32$.

Proof. Let $R = 2$ and $Q = 0$. We note that in this case the choice of R_0 depends only on d and γ. Assume first that $x \in V_2$, $y \in int\, V_R^c$ and $\delta_y(V_2) \le R_0/32$. Let y' and $A = A(x, y')$ be as in Theorem 2.4. From Lemma 3.4 we have

$$\begin{aligned} c\delta_x^{\alpha/2}(V_2)|x|^{\beta-\alpha/2} &\le \phi_{V_2}(x) \le C\delta_x^{\alpha/2}(V_2)|x|^{\beta-\alpha/2}, \\ c\delta_{y'}^{\alpha/2}(V_2)|y'|^{\beta-\alpha/2} &\le \phi_{V_2}(y') \le C\delta_{y'}^{\alpha/2}(V_2)|y'|^{\beta-\alpha/2}, \\ c\delta_A^{\alpha/2}(V_2)|A|^{\beta-\alpha/2} &\le \phi_{V_2}(A) \le C\delta_A^{\alpha/2}(V_2)|A|^{\beta-\alpha/2}. \end{aligned} \tag{32}$$

First we prove that

$$c\delta_y^{\alpha/2}(V_2)|y|^{\beta-\alpha/2} \le \phi_{V_2}(y') \le C\delta_y^{\alpha/2}(V_2)|y|^{\beta-\alpha/2}, \tag{33}$$

where c, C depend only on d, α, γ. We observe that, by definition we have $\kappa\delta_y(V_2) \le \delta_{y'}(V_2) \le \delta_y(V_2)$. Next, we see that if $\delta_y(V_2) = |y - S|$, then $|y - y'| \le |y - S| + |y' - S| \le 2\delta_y(V_2)$. This implies that $|y| \le |y - y'| + |y'| \le 2\delta_y(V_2) + |y'| \le (2/\kappa)\delta_{y'}(V_2) + |y'| \le (3/\kappa)|y'|$ and similarly $|y'| \le |y - y'| + |y| \le 2\delta_y(V_2) + |y| \le 3|y|$. This gives

$$(\kappa/3)|y| \le |y'| \le 3|y|. \tag{34}$$

Hence, (33) follows from (32).

Recall that $\rho = \max(\delta_x(V_2), \delta_{y'}(V_2), |x - y'|)$. We show that

$$(\kappa/(\kappa + 2))|x - y| \le \rho \le 3|x - y|. \tag{35}$$

First, notice that $\max(\delta_x(V_2), \delta_y(V_2)) \le |x - y|$ for all $x \in V_2$, $y \notin V_2$. Hence $|x - y'| \le |x - y| + |y - y'| \le |x - y| + 2\delta_y(V_2) \le 3|x - y|$ and $\delta_{y'}(V_2) \le \delta_y(V_2) \le |x - y|$. This gives the upper bound in (35). Next, $|x - y| \le |x - y'| + |y - y'| \le |x - y'| + 2\delta_y(V_2) \le \rho + (2/\kappa)\delta_{y'}(V_2) \le (1 + 2/\kappa)\rho$. Hence, (35) is proved.

From (35) by the same arguments as for (30) we obtain

$$c|x - y| \le \delta_A(V_2) \le C|x - y|. \tag{36}$$

Finally, $1 \le 1 + \delta_y(V_2) \le 1 + R_0/32$. Now using (9) and combining this with (25) and (32)–(36), we obtain the desired estimates for $R = 2$.

Now let $\delta_y(V_2) > R_0/32$. Then $y' = x_1$ and $\rho \ge \delta_{x_1}(V_2) \ge R_0/4$ so $A = x_1$, too. Next, $R_0/32 \le \delta_y(V_2) \le |y| \le |y - S| + |S| \le 2 + \delta_y(V_2) \le (1 + 64/R_0)\delta_y(V_2)$ and $\delta_y(V_2) \le |x - y| \le |x| + |y| \le 2 + |y| \le (1 + 64/R_0)|y|$. Combining this with (32) and (9), we complete the proof for $R = 2$. The scaling property (2) for the Poisson kernels completes the proof for every $R > 0$. □

As a simple application we estimate the probability that the process starting 'near' the vertex of a cone V_R exits the cone 'far' from the vertex. We will use this result in further analysis.

LEMMA 3.8. *Let $R > 1$ and $x \in V_R \cap B(Q, R/2)$. Then there are constants $c = c(d, \alpha, \gamma)$, $C = C(d, \alpha, \gamma)$ such that*

$$c\delta_x^{\alpha/2}(V_R)|x - Q|^{\beta-\alpha/2}/R^\beta \le P^x(X_{\tau_{V_R}} \in B^c(Q, 3R/4))$$
$$\le C\delta_x^{\alpha/2}(V_R)|x - Q|^{\beta-\alpha/2}/R^\beta.$$

PROOF. Take $Q = 0$. From the scaling property (2) we see that

$$P^x(X_{\tau_{V_R}} \in B^c(0, 3R/4)) = P^{x/R}(X_{\tau_{V_1}} \in B^c(0, 3/4))$$
$$= \int_{|y| \ge 3/4} P_{V_1}(x/R, y)dy. \tag{37}$$

For $|x/R| < 1/2$ and $|y| \ge 3/4$ we have $(1/3)|y| \le |x/R - y| \le (5/3)|y|$. Since $R > 1$, $(1/2)\delta_x(V_R)/R \le \delta_{x/R}(V_1) \le 2\delta_x(V_R)/R$. Let $x' \in V_{1/2}$. Combining this with Theorem 3.7, we obtain

$$\frac{P_{V_1}(x/R, y)}{P_{V_1}(x', y)} \le C\delta_{x/R}^{\alpha/2}(V_1)(|x|/R)^{\beta-\alpha/2} \le \frac{C2^{\alpha/2}\delta_x^{\alpha/2}(V_R)|x|^{\beta-\alpha/2}}{R^\beta}.$$

Hence, by (37),

$$P^x(X_{\tau_{V_R}} \in B^c(0, 3R/4)) \le CP^{x'}(X_{\tau_{V_1}} \in B^c(0, 3/4))\frac{\delta_x^{\alpha/2}(V_R)|x|^{\beta-\alpha/2}}{R^\beta}.$$

The proof is completed. □

Now we consider the case of unbounded cones. We have the following

LEMMA 3.9. *For all $x \in V$ we have $P^x(\tau_V < \infty) = 1$ and $P^x(X_{\tau_V} \in \partial V) = 0$.*

PROOF. Since $\beta > 0$, from [2, Theorem 4.1] we have $E^x\tau_V^t < \infty$ in case $t \in (0, \beta/\alpha)$. This implies that $P^x(\tau_V < \infty) = 1$.

Now take $x \in V \cap B(Q, R/2)$. Let $R > 1$. Since the Lebesgue measure of ∂V is 0 and $P^x(X_{\tau_{V_R}} \in \partial V_R) = 0$, we obtain $P^x(X_{\tau_{V_R}} \in \partial V) = 0$. Hence we have

$$P^x(X_{\tau_V} \in \partial V) = P^x(X_{\tau_V} \in \partial V, X_{\tau_{V_R}} \in V^c) + P^x(X_{\tau_V} \in \partial V, X_{\tau_{V_R}} \in V \setminus V_R)$$
$$= P^x(X_{\tau_{V_R}} \in \partial V) + P^x(X_{\tau_V} \in \partial V, X_{\tau_{V_R}} \in V \setminus V_R)$$
$$\le P^x(X_{\tau_{V_R}} \in B^c(Q, 3R/4))$$

and, by Lemma 3.8, the last term tends to 0 as $R \to \infty$. This completes the proof. □

With this lemma we are able to describe the behaviour of the Green function and the Poisson kernel of V_R when $R \to \infty$. As an immediate consequence of Lemma 2.5 we obtain

Lemma 3.10. *For* $x, y \in \mathbf{R}^d$ *we have* $\lim_{R\to\infty} G_{V_R}(x, y) = G_V(x, y)$ *and* $\lim_{R\to\infty} P_{V_R}(x, y) = P_V(x, y)$.

By this result and Theorem 3.7 we easily obtain the following estimates for G_V and P_V.

Theorem 3.11. *There exist constants* $c = c(d, \alpha, \gamma)$, $C = C(d, \alpha, \gamma)$ *such that*
(i) *for* $x, y \in V$ *we have*

$$c \min\left(\frac{A_{d,\alpha}}{|x-y|^{d-\alpha}}, \frac{\delta_x^{\alpha/2}(V)\delta_y^{\alpha/2}(V)}{|x-y|^d}\left(\frac{\min(|x-Q|,|y-Q|)}{\max(|x-Q|,|y-Q|)}\right)^{\beta-\alpha/2}\right)$$

$$\le G_V(x, y)$$

$$\le \min\left(\frac{A_{d,\alpha}}{|x-y|^{d-\alpha}}, C\frac{\delta_x^{\alpha/2}(V)\delta_y^{\alpha/2}(V)}{|x-y|^d}\left(\frac{\min(|x-Q|,|y-Q|)}{\max(|x-Q|,|y-Q|)}\right)^{\beta-\alpha/2}\right),$$

(ii) *for* $x \in V$, $y \in$ *int* V^c *we have*

$$c\frac{\delta_x^{\alpha/2}(V)}{\delta_y^{\alpha/2}(V)|x-y|^d}\left(\frac{\min(|x-Q|,|y-Q|)}{\max(|x-Q|,|y-Q|)}\right)^{\beta-\alpha/2} \le P_V(x, y)$$

$$\le C\frac{\delta_x^{\alpha/2}(V)}{\delta_y^{\alpha/2}(V)|x-y|^d}\left(\frac{\min(|x-Q|,|y-Q|)}{\max(|x-Q|,|y-Q|)}\right)^{\beta-\alpha/2}.$$

Remark 1. If $V = \{(0, 0, \dots, 0, x_d) : x_d > 0\}$, we know the explicit formula of P_V. This is equal to $P_V(x, y) = C_{d,\alpha} x_d^{\alpha/2} |y|^{-\alpha/2} |x-y|^{-d}$ (see [9, Example 2]). This confirms the estimates from Theorem 3.11 as $\beta = \alpha/2$ in this case. Note that this formula can be found by choosing a sequence of balls $B_n \nearrow V$, using the explicit formulas of the Poisson kernels for B_n and applying Lemma 2.5.

Now let us focus on Martin kernels of cones. Since a bounded cone is a bounded Lipschitz domain, its Martin kernel may be defined as in (6). Hence, as an immediate consequence of Theorem 3.6 we obtain the following estimates.

Theorem 3.12. *Let* $R > 0$. *Every singular* α*-harmonic function on* V_R *admits the Martin representation in the sense of* (5) *with the Martin kernel* M_{V_R}

defined as in (6). *There are constants* $c = c(d, \alpha, \gamma)$, $C = C(d, \alpha, \gamma)$ *such that for* $x \in V_R$, $z \in \partial V_R$ *we have*

$$c \frac{|x_0 - z|^d}{|x - z|^d} \frac{\delta_x^{\alpha/2}(V_R)}{\delta_{x_0}^{\alpha/2}(V_R)} \left(\frac{\min(|x - Q|, |z - Q|) \cdot \max(|x_0 - Q|, |z - Q|)}{\max(|x - Q|, |z - Q|) \cdot \min(|x_0 - Q|, |z - Q|)} \right)^{\beta - \alpha/2}$$

$$\le M_{V_R}(x, z)$$

$$\le C \frac{|x_0 - z|^d}{|x - z|^d} \frac{\delta_x^{\alpha/2}(V_R)}{\delta_{x_0}^{\alpha/2}(V_R)} \left(\frac{\min(|x - Q|, |z - Q|) \cdot \max(|x_0 - Q|, |z - Q|)}{\max(|x - Q|, |z - Q|) \cdot \min(|x_0 - Q|, |z - Q|)} \right)^{\beta - \alpha/2}$$

if $z \neq Q$ *and*

$$c \frac{|x_0 - Q|^{d + \beta - \alpha/2}}{|x - Q|^{d + \beta - \alpha/2}} \frac{\delta_x^{\alpha/2}(V_R)}{\delta_{x_0}^{\alpha/2}(V_R)} \le M_{V_R}(x, Q) \le C \frac{|x_0 - Q|^{d + \beta - \alpha/2}}{|x - Q|^{d + \beta - \alpha/2}} \frac{\delta_x^{\alpha/2}(V_R)}{\delta_{x_0}^{\alpha/2}(V_R)}.$$

The case of unbounded cones is more complicated. However, the results are similar, with a natural extension $\{\infty\} \in \partial V$.

Theorem 3.13. *Every singular* α*-harmonic function on* V *admits Martin the representation in the sense of* (5) *with the Martin kernel* M_V *defined as in* (6). *There are constants* $c = c(d, \alpha, \gamma)$, $C = C(d, \alpha, \gamma)$ *such that for* $x \in V$ *we have*

$$c \frac{|x_0 - z|^d}{|x - z|^d} \frac{\delta_x^{\alpha/2}(V)}{\delta_{x_0}^{\alpha/2}(V)} \left(\frac{\min(|x - Q|, |z - Q|) \cdot \max(|x_0 - Q|, |z - Q|)}{\max(|x - Q|, |z - Q|) \cdot \min(|x_0 - Q|, |z - Q|)} \right)^{\beta - \alpha/2}$$

$$\le M_V(x, z)$$

$$\le C \frac{|x_0 - z|^d}{|x - z|^d} \frac{\delta_x^{\alpha/2}(V)}{\delta_{x_0}^{\alpha/2}(V)} \left(\frac{\min(|x - Q|, |z - Q|) \cdot \max(|x_0 - Q|, |z - Q|)}{\max(|x - Q|, |z - Q|) \cdot \min(|x_0 - Q|, |z - Q|)} \right)^{\beta - \alpha/2}$$

if $z \in \partial V \setminus \{Q \cup \infty\}$ *and*

$$c \frac{|x_0 - Q|^{d + \beta - \alpha/2}}{|x - Q|^{d + \beta - \alpha/2}} \frac{\delta_x^{\alpha/2}(V)}{\delta_{x_0}^{\alpha/2}(V)} \le M_V(x, Q) \le C \frac{|x_0 - Q|^{d + \beta - \alpha/2}}{|x - Q|^{d + \beta - \alpha/2}} \frac{\delta_x^{\alpha/2}(V)}{\delta_{x_0}^{\alpha/2}(V)},$$

$$c \frac{|x - Q|^{\beta - \alpha/2}}{|x_0 - Q|^{\beta - \alpha/2}} \frac{\delta_x^{\alpha/2}(V)}{\delta_{x_0}^{\alpha/2}(V)} \le M_V(x, \infty) \le C \frac{|x - Q|^{\beta - \alpha/2}}{|x_0 - Q|^{\beta - \alpha/2}} \frac{\delta_x^{\alpha/2}(V)}{\delta_{x_0}^{\alpha/2}(V)}.$$

Furthermore, we have

$$M_V(x, \infty) = M_V(x) \cdot M_V(\mathbf{1}, \infty).$$

Proof. The existence of the Martin representation is not immediate since V is unbounded. To prove it, we introduce *the Kelvin transform* T (with pole at 0) as

$$Tx = x/|x|^2, \qquad x \neq 0. \tag{38}$$

We have $T^{-1} = T$ and $|Tx| = 1/|x|$. Next we introduce *the Kelvin transform* of a function as

$$Tu(x) = |x|^{\alpha-d}u(Tx). \tag{39}$$

See [17] for further details.

We may assume that $Q = (0, 0, \dots, 0, 1)$ and $\mathbf{1} = (0, 0, \dots, 0, 2)$. V is a circular domain and its profile consists of two half-lines starting from Q. By symmetry, the image of V under T, TV is also circular and its profile is the image of the profile of V. It is easy to check that the image of a line under T is a circle provided the line does not pass through 0. Hence the profile of TV consists of two arcs of the same radius and these arcs have two common points Q and 0. It implies that TV is a bounded Lipchitz domain so, according to [9], there exists the Martin representation on TV with classical Martin kernels $M_{TV}(x', z')$, $x' \in TV$, $z' \in \partial TV$. Then, by [9, Lemma 8], the classes of singular α-harmonic functions on V and TV are isomorphic and there exists Martin representation on V with its kernel function $TM_{TV}(\cdot, z')$. Since $0 \notin TV$ and $T^{-1} = T$, we have from [17, Proposition 2.6]

$$|x|^{\alpha-d}|y|^{\alpha-d}G_{TV}(Tx, Ty) = G_V(x, y), \qquad x, y \in V. \tag{40}$$

Let T_{x_0} be the reference point for the kernel $M_{TV}(\cdot, \cdot)$. From (40) we obtain

$$\frac{G_V(x, y)}{G_V(x_0, y)} = \frac{|x|^{\alpha-d}}{|x_0|^{\alpha-d}} \frac{G_{TV}(Tx, Ty)}{G_{TV}(Tx_0, Ty)}. \tag{41}$$

If $y \to z$, then $Ty \to Tz$ with a natural extension $T0 = \infty$, $T\infty = 0$. Hence, letting $y \to z$ in (41) and using (6), we obtain

$$M_V(x, z) = \frac{|x|^{\alpha-d}M_{TV}(Tx, Tz)}{|x_0|^{\alpha-d}}.$$

By (39) this implies that $M_V(x, z) = |x_0|^{d-\alpha}TM_{TV}(x, Tz)$. It means that classical Martin kernels $M_V(\cdot, \cdot)$ exist and the kernels $TM_{TV}(\cdot, z')$ are their constant multiples. Hence there exists the classical Martin representation on V with the kernel functions $M_V(\cdot, \cdot)$. The estimates of $M_V(\cdot, \cdot)$ are an immediate consequence of (6) and Theorem 3.6.

It remains to prove the last part of the theorem. We see that the function $f(x) = M_V(x, \infty)/M_V(\mathbf{1}, \infty)$ is nonnegative singular α-harmonic in V and $f(\mathbf{1}) = 1$. Moreover, the estimates of $M_V(\cdot, \infty)$ imply that f is locally bounded. Hence, by [10, Lemma 17], f is regular α-harmonic on every open bounded subset of V. From the uniqueness of $M_V(\cdot)$ ([2, Theorem 3.2]) we obtain $f \equiv M_V(\cdot)$, which completes the proof. $\square$

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