Homology, Homotopy and Applications, vol. 11(2), 2009, pp.5–25

ALGEBRAIC K-THEORY AND CUBICAL DESCENT

PERE PASCUAL AND LLORENÇ RUBIÓ PONS

(*communicated by Daniel Grayson*)

Abstract

In this note we apply the Guillén-Navarro descent theorem to define a descent variant of the algebraic K-theory of varieties over a field of characteristic zero, $\mathcal{KD}(X)$, which coincides with $\mathcal{K}(X)$ for smooth varieties and to prove that there is a natural weight filtration on the groups $KD_*(X)$. After a result of Haesemeyer, we deduce that this theory is equivalent to the homotopy algebraic K-theory introduced by Weibel.

1. Introduction

In Théorème (2.1.5) of [**GN02**], F. Guillén and V. Navarro have proved a general result, which permits one to extend (in the presence of resolution of singularities) a contravariant functor compatible with smooth blow-ups on the category of smooth schemes to a functor on the category of all schemes in such a way that the extended functor is compatible with general blow-ups. In this paper we apply this result to algebraic K-theory. More specifically, we consider the algebraic K-theory functor, which to a smooth algebraic variety over a field of characteristic zero X, associates the spectrum of the cofibration category of perfect complexes, $\mathcal{K}(X)$. We apply Guillén-Navarro extension criterion to prove that this functor admits an (essentially unique) extension to all algebraic varieties, $\mathcal{KD}(X)$, which satisfies a descent property.

Moreover, by using the extension theorem in analogy with Guillén and Navarro's paper [**GN03**], we are able to prove the existence of a natural filtration on the KD-groups associated to an algebraic variety. In fact, the $\mathcal{KD}$-theory of an algebraic variety X is defined by cubical descent and therefore, if $X_\bullet$ is a cubical hyperresolution of X (see [**GNPP**, I.(2.12)]), then there is a convergent spectral sequence, see Proposition 4.3,

$$E_1^{pq} = \bigoplus_{|\alpha|=p+1} K_q(X_\alpha) \Rightarrow KD_{q-p}(X),$$

where we have written $KD_*(X) = \pi_*(\mathcal{KD}(X))$. We prove that the associated filtration on $KD_*(X)$ is independent of the chosen hyperresolution $X_\bullet$ of X.

It is well known that algebraic K-theory of schemes does not satisfy descent. C. Haesemeyer has proved in [**H**, Th. 3.5] that the homotopy algebraic K-theory,

---

Partially supported by projects DGCYT BFM2003-06001 and MT M2006-14575.
Received May 21, 2009, revised June 5, 2009; published on July 21, 2009.
2000 Mathematics Subject Classification: 19D55, 18G60, 14F.
Key words and phrases: algebraic K-theory, descent, weight filtration.
This article is available at `http://intlpress.com/HHA/v11/n2/a2`

$\mathcal{KH}$, introduced by Weibel in [**W1**], satisfies descent for varieties over a field of characteristic zero. From the uniqueness of our extension $\mathcal{KD}$ and Haesemeyer's result, it follows that, for any variety X over a field of characteristic zero, the spectra $\mathcal{KD}(X)$ and $\mathcal{KH}(X)$ are weakly equivalent. We observe that Cortiñas, Haesemeyer and Weibel have analyzed in [**CHW**] the fiber of the morphism $\mathcal{K} \longrightarrow \mathcal{KH}$ in terms of the negative cyclic homology functor.

Following [**GN02**, Théorème (2.3.6)], we also find an extension of $\mathcal{K}$ to a functor with compact support, $\mathcal{K}^c$, which once again by uniqueness is weakly equivalent to the algebraic K-theory with compact support introduced by Gillet and Soulé in [**GS**]. By our techniques we recover the weight filtration introduced in [**GS**, Theorem 7] for algebraic K-theory with compact support.

Some results of this paper have been obtained by other authors using the fibrant replacement functor for a closed model category structure on the category of presheaves on the category of schemes with a suitable topology; see the papers [**CHSW, CHW**] and [**GS**]. As remarked in [**R2**, Theorem 4.6], the two presentations are equivalent, but we think it is worthwhile to have both methods at hand. In particular, we remark that we are able to easily deduce the existence of a weight filtration on $KH_*(X)$.

Acknowledgements

We thank F. Guillén and V. Navarro for allowing us to use their preprint [**GN03**] and for many helpful discussions. We have also benefited from conversations with A. Roig.

2. The descent theorem of Guillén-Navarro

In this section we recall the main extension theorem proved by Guillén and Navarro and present some corollaries of its proof not explicitly stated in [**GN02**]. We also fix some notation.

2.1. Descent categories

The descent theorem in [**GN02**, (2.1.5)] is stated for functors from the category of smooth varieties to a cohomological descent category. This kind of category is a (higher) variation of the classical notion of triangulated category. We recall the main features of descent categories and refer to [**GN02**, (1.5.3) and (1.7)] for the precise definitions (see also the proof of Proposition 3.6).

2.1.1.

For any finite set S, the associated *cubical* set $\square_S$ is the ordered set of non-empty subsets of S and the *augmented cubical* set $\square_S^+$ is the ordered set of subsets of S, including the empty set. When $S = \{0, 1, \dots, n\}$, we simply write $\square_n$ (respectively, $\square_n^+$), which may be identified with the ordered set of $n+1$-tuples $(i_0, \dots, i_n)$, where $i_k \in \{0, 1\}$ such that there is a k with $i_k \neq 0$, and including the $(0, \dots, 0)$-tuple in the augmented case. We will write $|\alpha| = \sum_0^n i_k$.

As usual, we will denote the associated category with the same symbol. Following [**GN02**, (1.1.1)], we denote by Π the category whose objects are finite products of categories $\square_S$ and whose morphisms are the functors associated to injective maps in each component. The objects of Π will be called *cubical index categories.* Π is a symmetric monoidal category.

2.1.2.
Let $\mathcal{D}$ be a category. Given a cubical index category $\square$, a $\square$-*cubical diagram* of $\mathcal{D}$ is a functor $X\colon \square \longrightarrow \mathcal{D}$. We denote by $CoDiag_{\Pi}\mathcal{D}$ the category of cubical diagrams of $\mathcal{D}$. (According to [**GN02**, (1.2.2)], we should call these functors *cubical codiagrams*, reserving the term diagram for the contravariant functors $X\colon \square^{op} \longrightarrow \mathcal{D}$.) The objects are the pairs $(X, \square)$, where X is a $\square$-cubical diagram; a morphism from the diagram $(X, \square)$ to the diagram $(Y, \square')$ is a functor $\delta\colon \square' \longrightarrow \square$ together with a natural transformation $\delta^* X = X \circ \delta \Rightarrow Y$.

2.1.3.
A *descent category* is, essentially, a triple $(\mathcal{D}, E, \mathbf{s})$ given by a cartesian category $\mathcal{D}$ with initial object $*$, a saturated class of morphisms E of $\mathcal{D}$, called *weak equivalences*, and a functor

$$\mathbf{s}\colon CoDiag_{\Pi}\mathcal{D} \longrightarrow \mathcal{D},$$

called the *simple functor*, which satisfies the following properties:

1. *Product*: For any object X of $\mathcal{D}$, there is a natural isomorphism $\mathbf{s}_{\square_0}(X \times \square_0) \cong X$, and for any $\square \in Ob\Pi$ and any couple of $\square$-diagrams (X, Y), the morphism

   $$s_{\square}(X \times Y) \longrightarrow s_{\square}X \times s_{\square}Y$$

   is an isomorphism.
2. *Factorisation*: Let $\square, \square' \in Ob\Pi$. For any $\square \times \square'$-diagram $X = (X_{\alpha\beta})$, there is an isomorphism

   $$\mu\colon s_{\alpha\beta}X_{\alpha\beta} \longrightarrow s_{\alpha}s_{\beta}X_{\alpha\beta},$$

   natural in X.
3. *Exactness*: Let $f\colon X \longrightarrow Y$ be a morphism of $\square$-diagrams, $\square \in Ob\Pi$. If for all $\alpha \in \square$ the morphism $f_\alpha\colon X_\alpha \longrightarrow Y_\alpha$ is a weak equivalence (i.e. it is in E), then the morphism $s_{\square}f\colon s_{\square}X \longrightarrow s_{\square}Y$ is a weak equivalence.
4. *Acyclicity criterion*: Let $f\colon X_1 \longrightarrow X_0$ be a morphism of $\mathcal{D}$. Then, f is a weak equivalence if and only if the simple of the $\square_1$-diagram

   $$* \longrightarrow X_0 \xleftarrow{f} X_1$$

   is acyclic, that is, it is weakly equivalent to the final object of $\mathcal{D}$.

The acyclicity criterion has to be also verified for higher cubical diagrams. More specifically, let X^+ be a $\square_n^+$-diagram in $\mathcal{D}$ and denote by X the cubical diagram obtained from X^+ by restriction to $\square_n$. Then the acyclicity criterion takes the following form (see Property (CD8)^{op} of Definition (1.5.3) of [**GN02**]):

4′. *Acyclicity criterion*: The augmentation morphism $\lambda_\varepsilon\colon X_0 \longrightarrow s_\square X$ is a weak equivalence if and only if the canonical morphism $* \longrightarrow s_{\square^+} X^+$ is a weak equivalence.

We remark that the transformations μ and λ of Properties 2 and 4′ are, in fact, part of the data of a descent structure.

2.1.4.
The categories of complexes give the basic examples of descent categories: if $\mathcal{A}$ is an abelian category, then the category of bounded below cochain complexes $\mathbf{C}^*(\mathcal{A})$, with the class of quasi-isomorphisms as weak equivalences and the total functor of a multicomplex as simple functor, is a descent category. See [**GN02**, (1.5.5) and (1.5.13)], for other examples.

2.1.5.
There is another technical issue in the statement of the Guillén-Navarro theorem, that of a Φ-rectified functor. If $\mathcal{D}$ is a descent category, then for each cubical index category $\square$, the simple functor induces a functor $Ho(\mathcal{D}^\square) \longrightarrow Ho\mathcal{D}$, so we can define the homotopy simple object associated to a *true* $\square$-diagram in $\mathcal{D}$. There are situations, particularly those related with resolutions, that are defined up to quasi-isomorphism, where we are interested in cubical diagrams in $Ho\mathcal{D}$; unfortunately, we have not, in general, a simple functor $(Ho\mathcal{D})^\square \longrightarrow Ho\mathcal{D}$. The notion of Φ-rectified functor corresponds, roughly speaking, to the functors $F\colon \mathcal{C} \longrightarrow Ho\mathcal{D}$ defined also, in a compatible way, on all cubical diagrams in the form $F^\square\colon \mathcal{C}^\square \longrightarrow Ho(\mathcal{D}^\square)$, so that we can take the composition $\mathcal{C}^\square \longrightarrow Ho(\mathcal{D}^\square) \xrightarrow{s} Ho\mathcal{D}$; see [**GN02**, (1.6)] for the technical details. It will be enough for us to know that any functor $\mathcal{C} \longrightarrow \mathcal{D}$ induces a Φ-rectified functor.

2.2. The Guillén-Navarro theorem

Let k be a field of characteristic zero. We denote by $\mathbf{Sch}(k)$ the category of reduced separated schemes of finite type over k, simply called *algebraic varieties*, and by $\mathbf{Sm}(k)$ the category of smooth varieties.

2.2.1.
Let

$$\begin{tikzcd} \widetilde{Y} \arrow[r, "j"] \arrow[d, "g"'] & \widetilde{X} \arrow[d, "f"] \\ Y \arrow[r, "i"] & X \end{tikzcd}$$

be a cartesian diagram of schemes, which we may consider as a $\square_1^+$-diagram. We say that it is an *acyclic square* if i is a closed immersion, f is a proper morphism and the induced morphism $\widetilde{X} \setminus \widetilde{Y} \longrightarrow X \setminus Y$ is an isomorphism.

We say that an acyclic square is an *elementary acyclic square* if all schemes in the diagram are irreducible and smooth, and f is the blow-up of X along Y.

Theorem 2.1 ([**GN02**, (2.1.5)])**.** *Let $\mathcal{D}$ be a cohomological descent category and*

$$G\colon \mathbf{Sm}(k) \longrightarrow Ho\mathcal{D}$$

a contravariant Φ-rectified functor satisfying the following conditions:

(F1) $G(\emptyset)=0$, and the canonical morphism $G(X\sqcup Y)\longrightarrow G(X)\times G(Y)$ is an isomorphism,

(F2) If $X_\bullet$ is an elementary acyclic square in $\mathbf{Sm}(k)$, then $sG(X_\bullet)$ is acyclic.

Then there is an extension of G to a Φ-rectified functor

$$GD\colon \mathbf{Sch}(k) \longrightarrow Ho\mathcal{D}$$

which satisfies the descent condition

(D) If $X_\bullet$ is an acyclic square in $\mathbf{Sch}(k)$, then $sGD(X_\bullet)$ is acyclic.

Moreover, this extension is essentially unique: if G' is another extension of G verifying the descent property (D), then there is a uniquely determined isomorphism of Φ-rectified functors $GD \Rightarrow G'$.

We will say that the functor GD has been obtained from G by cubical descent.

The proof of Guillén-Navarro's theorem gives more than stated above. In fact, if X is an algebraic variety and $X_\bullet \longrightarrow X$ is any cubical hyperresolution (see [**GNPP**, I.(2.12)], then it is proved in [**GN02**, (2.1.5)] that, under the hypothesis of the theorem,

$$GD(X)=sG(X_\bullet),$$

gives a well-defined functor from $\mathbf{Sch}(k)$ to $Ho\mathcal{D}$, independent of the chosen hyperresolution $X_\bullet$. From this explicit presentation, we easily deduce some more properties of the descent extension GD.

Proposition 2.2. *Suppose that the functor G in Theorem 2.1 is already defined for all varieties, that is, we have $G\colon \mathbf{Sch}(k)\longrightarrow Ho\mathcal{D}$, which satisfies (F1) and (F2). Then there is a natural transformation of Φ-rectified functors $G\Rightarrow GD$.*

Proof. Let X be a variety and $X_\bullet$ a cubical hyperresolution of X, indexed by a cubical set $\square$. Taking the simple of the morphism of cubical diagrams $G(X\times\square)\longrightarrow G(X_\bullet)$, we get the morphism

$$G(X)=sG(X\times\square)\longrightarrow sG(X_\bullet)=GD(X). \qquad \square$$

Looking at the construction and properties of cubical hyperresolutions, it may be proved that the extended functor GD inherits many properties of the functor G over the smooth varieties. As an example, and in view of their interest in algebraic K-theory, let us note the two properties enclosed in the following proposition.

Proposition 2.3. *Consider the hypothesis of Theorem 2.1.*

(1) Suppose that G is homotopy invariant; i.e., for any smooth variety X the projection $X\times\mathbb{A}^1\longrightarrow X$ induces an isomorphism $G(X)\cong G(X\times\mathbb{A}^1)$. Then GD is homotopy invariant: for any variety X, there is an isomorphism $GD(X)\cong GD(X\times\mathbb{A}^1)$.

(2) *Suppose that G satisfies the Mayer-Vietoris property; i.e., for any smooth variety X and any open decomposition $X = U \cup V$, the square, induced by inclusions,*

$$\begin{tikzcd} G(X) \arrow[r] \arrow[d] & G(U) \arrow[d] \\ G(V) \arrow[r] & G(U \cap V) \end{tikzcd}$$

is acyclic in $\mathcal{D}$. Then GD satisfies Mayer-Vietoris for all varieties.

Proof. Given X an algebraic variety we fix $X_\bullet$, a cubical hyperresolution of X.

(1) By the definition of GD and the homotopy invariance of G, we have a sequence of weak equivalences

$$GD(X) \cong sG(X_\bullet) \cong sG(X_\bullet \times \mathbb{A}^1) \cong GD(X \times \mathbb{A}^1),$$

so the proof follows.

(2) By the definition of cubical hyperresolutions (see [**GNPP**, (I.§2)]), the restrictions of $X_\bullet$ to U, V and $U \cap V$ give hyperresolutions of these varieties. Let us denote by $U_\bullet, V_\bullet$ and $(U \cap V)_\bullet$, respectively, these restrictions. By construction, for any index α we have an open decomposition $X_\alpha = U_\alpha \cup V_\alpha$ with $U_\alpha \cap V_\alpha = (U \cap V)_\alpha$, so from the Mayer-Vietoris property for G on the category of smooth schemes, we deduce that the morphisms

$$G(X_\alpha) \longrightarrow s\left(\begin{tikzcd} & G(U_\alpha) \arrow[d] \\ G(V_\alpha) \arrow[r] & G((U \cap V)_\alpha) \end{tikzcd}\right)$$

are weak equivalences for any α. By the exactness property of descent categories, we have that

$$sG(X_\bullet) \longrightarrow s_\alpha s\left(\begin{tikzcd} & G(U_\alpha) \arrow[d] \\ G(V_\alpha) \arrow[r] & G((U \cap V)_\alpha) \end{tikzcd}\right)$$

is also a weak equivalence. But, by the factorization axiom of descent categories, the simple on the right is weak equivalent to

$$s\left(\begin{tikzcd} & sG(U_\bullet) \arrow[d] \\ sG(V_\bullet) \arrow[r] & sG((U \cap V)_\bullet) \end{tikzcd}\right).$$

So, taking into account the definition of GD, we finally deduce that the morphism $GD(X) \longrightarrow s(GD(U) \longleftarrow GD(U \cap V) \longrightarrow GD(V))$ is a weak equivalence; hence the Mayer-Vietoris property for open sets follows. □

2.3. Extension with compact support

In [**GN02**], the authors present some variations on the main theorem. In particular, they prove, in [**GN02**, (2.2.2)], that with the same hypothesis of Theorem 2.1 there is an extension G^c of G with compact support: if $\mathbf{Sch}_c(k)$ denotes the category of varieties and proper morphisms, then there is an extension of G to a Φ-rectified functor

$$G^c\colon \mathbf{Sch}_c(k) \longrightarrow Ho\mathcal{D}$$

which satisfies the descent property *(D)* and, moreover,

(D_c) If Y is a subvariety of X, then there is a natural isomorphism

$$G^c(X-Y) \cong s_{\square_0^+}(G^c(X) \longrightarrow G^c(Y)).$$

3. The descent category of spectra

In this section we prove that the category of Ω-spectra, with the homotopy limit as a simple functor, is a (cohomological) descent category in the sense of [**GN02**].

3.1. Fibrant spectra

We will work in the category of fibrant spectra of simplicial sets. Our main references will be the paper by Bousfield-Friedlander [**BF**] and Section 5 of Thomason's [**T80**].

Recall that a *prespectrum* is a sequence of pointed simplicial sets X_n, $n \geqslant 0$, together with structure maps $\Sigma X_n \longrightarrow X_{n+1}$, where for a pointed simplicial set K, $\Sigma K = S^1 \wedge K$. A prespectrum X is a *fibrant spectrum*, also called Ω-*spectrum*, if each X_n is a fibrant simplicial set and the maps $X_n \longrightarrow \Omega X_{n+1}$, obtained by adjunction of the structure maps, are weak equivalences. Morphisms between prespectra and between fibrant spectra are defined as maps in each degree that commute with the structure maps. We denote by **PreSp** and **Sp** the categories of prespectra and fibrant spectra, respectively.

The homotopy groups of a prespectrum X are defined by the direct limit

$$\pi_k(X) = \varinjlim \pi_{k+n}(X_n), \quad k \in \mathbb{Z},$$

so that if X is a fibrant spectrum, then $\pi_k(X) = \pi_{k+n}(X_n)$ for $k+n \geqslant 0$, and, more specifically, for $k \geqslant 0$, $\pi_k(X) = \pi_k(X_0)$. A map $f\colon X \longrightarrow Y$ of prespectra is a *weak equivalence* if it induces an isomorphism on homotopy groups. In this way, a map of fibrant spectra is a weak equivalence if and only if it induces weak equivalences in each degree.

3.2. Homotopy limit

Let X be a functor from an index category I to **Sp**. The homotopy limit spaces $\mathrm{holim} X_n$, $n \geqslant 0$, in the sense of Bousfield-Kan, [**BK**, Chapter XI], define a fibrant spectrum $\mathrm{holim} X$; see [**T80**, 5.6]. In fact, one can see that **PreSp** has a structure of a simplicial closed model category (see [**S**, Proposition 2.1.5]) so that we can apply the general theory of homotopy limits for these categories [**H**, Chapter 18].

The main properties we need of homotopy limits between fibrant spectra are:

(i) Functoriality and exactness on fibrant spectra: Let $f\colon X \longrightarrow Y$ be a morphism of I-diagrams spectra. Then there is a natural morphism $\mathrm{holim} f\colon \mathrm{holim} X \longrightarrow \mathrm{holim} Y$. If for each $\alpha \in I$ the morphism $f_\alpha\colon X_\alpha \longrightarrow Y_\alpha$ is a weak equivalence, then $\mathrm{holim} f$ is a weak equivalence.

(ii) Functoriality on the index category and cofinality theorem: Given a functor $\delta\colon I \longrightarrow J$ and a diagram $X\colon J \longrightarrow \mathbf{Sp}$, there is a natural map $\mathrm{holim}_J X \longrightarrow \mathrm{holim}_I \delta^* X$, where $\delta^* X = X \circ \delta$. If δ is left cofinal, then this morphism is a weak equivalence.

(iii) For any diagram $X\colon I \longrightarrow \mathbf{Sp}$, there is a natural map $\lim X \longrightarrow \mathrm{holim} X$.

3.2.1.
For a cubical diagram of spectra $X\colon \square \longrightarrow \mathbf{Sp}$, we define the *simple spectrum of* X as the homotopy limit

$$s_\square(X) = \mathrm{holim}_\square X.$$

For a fixed cubical category $\square$, $s_\square$ defines a functor $s_\square\colon \mathit{CoDiag}_\square \mathbf{Sp} \longrightarrow \mathbf{Sp}$, and by the functoriality of the homotopy limit with respect to the index category $\square$, we obtain a functor

$$\mathbf{s}\colon \mathit{CoDiag}_\Pi \mathbf{Sp} \longrightarrow \mathbf{Sp}.$$

3.2.2.
Following [**GN02**, (1.4.3)], we extend the functor s to augmented cubical diagrams by using the cone construction. For instance, if $f\colon X \longrightarrow Y$ is a $\square_0^+$-diagram of spectra, that is to say, a morphism, then it follows from *loc. cit.* that

$$s_{\square_0^+}(f) = s_{\square_1}(X \xrightarrow{f} Y \longleftarrow *),$$

which is weakly equivalent to the homotopy fiber of f.

Take an isomorphism $\square_n^+ \cong \square_0^+ \times \square_{n-1}^+$. As the cone construction is compatible with this product structure, we find

$$s_{\square_n^+} X^+ = s_{\square_{n-1}^+}(s_{\square_0^+} X^+);$$

that is, by viewing X^+ as a morphism of two $\square_{n-1}^+$-diagrams, $f\colon X_0^+ \longrightarrow X_1^+$, the simple spectrum associated to X^+ is obtained as the simple of the $\square_{n-1}^+$-cubical diagram which in each degree α has the homotopy fiber of f_α. As a consequence, the simple spectrum $s_{\square_n^+} X^+$ is isomorphic to the total fiber space of X^+ as defined by Goodwillie in [**G**, 1.1].

3.2.3.
If X^+ is a $\square_n^+$-diagram and X denotes its restriction to $\square_n$, then it follows from the general properties of homotopy limits outlined above that there is a natural map $X_0 \longrightarrow \mathrm{holim} X$. As a consequence of [**G**, 1.1b] (compare also with [**P**, Proposition (3.3)], for a similar situation), we obtain:

Proposition 3.1. *Let* $X^+\colon \square_n^+ \longrightarrow \mathbf{Sp}$ *be an augmented cubical diagram of spectra and* X *be the cubical diagram obtained by restriction to* $\square_n$*. The simple* $s_{\square_n^+} X^+$ *is isomorphic to the homotopy fiber of the morphism* $X_0 \longrightarrow s_\square X = \mathrm{holim} X$*.*

Denote by $*$ the initial object of $\mathbf{Sp}$. The following corollary relates the simple of a cubical diagram with the simple of an augmented diagram.

Corollary 3.2. *Let* $X\colon \square_n \longrightarrow \mathbf{Sp}$ *be a cubical diagram of spectra and let* $\widetilde{X}$ *the augmented cubical diagram obtained from* X *by adding* $X_0 = *$. *Then,*

$$s_{\square_n^+}\widetilde{X} = \Omega s_{\square_n} X.$$

We also deduce the following result, which will be used later:

Corollary 3.3. *Let* $X_\bullet$ *be a* $\square_n$*-diagram of spectra. Then there is a convergent spectral sequence*

$$E_1^{pq} = \bigoplus_{|\alpha|=p+1} \pi_q(X_\alpha) \Longrightarrow \pi_{q-p}(s_{\square_n} X_\bullet).$$

Proof. Consider the cubical diagrams $F^p X_\bullet$ defined by

$$(F^p X_\bullet)_\alpha = \begin{cases} X_\alpha, & \text{if } |\alpha| \leqslant p+1, \\ *, & \text{if } |\alpha| > p+1. \end{cases}$$

Observe that $F^{-1}X_\bullet$ is the constant diagram defined by $*$ and that $F^n X_\bullet = X_\bullet$. We obtain a sequence of cubical diagrams

$$F^n X_\bullet \longrightarrow F^{n-1}X_\bullet \longrightarrow \cdots \longrightarrow F^0 X_\bullet \longrightarrow *,$$

which is a degreewise sequence of fibrations of spectra. Hence, taking homotopy limits there is a sequence of fibrations

$$s_\square(F^n X_\bullet) \longrightarrow s_\square(F^{n-1}X_\bullet) \longrightarrow \cdots \longrightarrow s_\square(F^0 X_\bullet) \longrightarrow *.$$

The Bousfield-Kan spectral sequence associated to the tower of fibrations obtained by adjoining identities from the left converges to the homotopy of $s_\square(F^n X_\bullet) = s_\square X_\bullet$. The E_1-terms are $E_1^{pq} = \pi_{q-p}(sGr^p X_\bullet)$, where $Gr^p X_\bullet$ is the $\square$-diagram obtained degreewise as the fibers of the morphism $s_\square(F^p X_\bullet) \longrightarrow s_\square(F^{p-1}X_\bullet)$. But, reasoning as in the proof of Proposition (3.3) of [**P**], for these diagrams we have

$$s_{\square_n} Gr^p X_\bullet = \prod_{|\alpha|=p+1} \Omega^p X_\alpha;$$

hence it follows that

$$E_1^{pq} = \pi_{q-p}\left(\prod_{|\alpha|=p+1} \Omega^p X_\alpha\right) = \bigoplus_{|\alpha|=p+1} \pi_q(X_\alpha).$$

Convergence is a consequence of Lemma 5.48 of [**T80**]. ☐

3.2.4.
We say that an augmented cubical diagram of spectra X^+ is *acyclic* if the canonical morphism $* \longrightarrow s_{\square^+} X^+$ is a weak equivalence. The acyclic diagrams are also called *homotopy cartesian* diagrams (see [**G**, §1] and [**W1**]). From Proposition 3.1, the corollary below follows immediately (see also [**W1**, Proposition 1.1]):

Corollary 3.4. *Let* $X^+\colon \square_n^+ \longrightarrow \mathbf{Sp}$ *be an augmented cubical diagram of spectra and* X *the cubical diagram obtained by restriction to* $\square_n$. *Then* X^+ *is acyclic if and only if the natural morphism* $X_0 \longrightarrow \operatorname{holim} X$ *is a weak equivalence.*

Remark 3.5. Observe that for $n = 2$ this result reduces to the well-known fact that a square of fibrant spectra

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d, "h"] \\
X' \arrow[r, "k"] & Y'
\end{tikzcd}$$

is acyclic (or homotopy cartesian) if and only if the natural map from X to the homotopy limit of $X' \xrightarrow{k} Y' \xleftarrow{h} Y$ is a weak equivalence.

3.2.5.
After the remarks above, we have on **Sp** a class of weak equivalences and a simple functor $\mathbf{s}\colon CoDiag_{\Pi}\mathbf{Sp} \longrightarrow \mathbf{Sp}$. According to [**GN02**, Définition (1.7.1)] to have a (cohomological) descent category on **Sp** we also need the following data:

(i) a natural transformation $\mu\colon s_\square \circ s_{\square'} \Rightarrow s_{\square\times\square'}$,

(ii) a natural transformation $\lambda_\square\colon id_{\mathbf{Sp}} \Rightarrow s_\square \circ i_\square$,

in such a way that $(s, \mu, \lambda_0)\colon \Pi \longrightarrow CoReal_{\Pi}\mathbf{Sp}$ defines a comonoidal quasi-strict functor; see *loc. cit.* As the homotopy limit is the end of a functor, by the Fubini theorem (see [**M**, §IX.8]) there is a natural transformation

$$\mu\colon s_\square \circ s_{\square'} \Rightarrow s_{\square\times\square'},$$

such that for any diagram X, μ_X is an isomorphism. As for λ, recall that $s_\square(X \times \square)$ is the function space from the classifying space of the index category $\square$ to X, so one defines

$$\lambda_\square(X)\colon X \longrightarrow s_\square(X \times \square)$$

by constant functions.

Proposition 3.6. *The category of fibrant spectra* **Sp** *with weak homotopy equivalences as weak equivalences and the homotopy limit* holim *as a simple functor for cubic diagrams, and the natural transformations μ, λ defined above, is a cohomological descent category.*

Proof. The actual definition of a cohomological descent category consists of eight axioms, which are dual to the axioms (CD1)–(CD8) of [**GN02**, Definition (1.5.3)] (see also their (1.7)). Most of them are immediate from the definitions and the properties of homotopy limits, so we comment on the four axioms summarized in Subsection 2.1.3, (see also [**R**, Theorem 5.1] for an extension of this result to stable simplicial model categories).

It is clear from the definitions that **Sp** is a cartesian category with initial object $*$.

1. *Product*: Since the homotopy limit is an end, it is compatible with products, so for any $\square$-diagrams X, Y of **Sp**, there is a natural isomorphism

$$s_\square(X \times Y) \cong s_\square(X) \times s_\square(Y).$$

2. *Factorisation*: In addition, because of the Fubini theorem for ends, if X is a $\square \times \square'$-diagram, then there are natural isomorphisms

$$s_\square s_{\square'} X_{\alpha\beta} \cong s_{\square\times\square'} X_{\alpha\beta} \cong s_{\square'} s_\square X_{\alpha\beta};$$

   see [**T80**, Lemma 5.7].

3. *Exactness*: If $f\colon X \longrightarrow Y$ is a morphism of $\square$-diagrams in **Sp** such that for any $\alpha \in \square$ the morphism f_α is a weak equivalence, then $s_\square f\colon s_\square X \longrightarrow s_\square Y$is a weak equivalence, since the homotopy limit preserves weak equivalences between fibrant spectra; see [**T80**, 5.5]. Observe that this property is not true for prespectra.

4′. *Acyclicity criterion*: This is exactly the result of Corollary 3.4. ☐

4. Descent algebraic K-theory

4.1.

Let X be a noetherian separated scheme. We denote by $\mathcal{K}(X)$ the K-spectrum associated to the category of perfect complexes on X; see [**TT**, Definition 3.1]. It defines a contravariant functor from the category of noetherian separated schemes to the category of spectra **Sp** ([**TT**, 3.14]), so it defines a rectified functor; see 2.1.5. Moreover, it is a covariant functor for perfect projective maps and for proper flat morphisms ([**TT**, 3.16]).

Theorem 4.1. *Let k be a field of characteristic zero. The rectified functor*

$$\mathcal{K}\colon \mathbf{Sm}(k) \longrightarrow Ho\mathbf{Sp}$$

admits a unique extension, up to a unique isomorphism of Φ-rectified functors, to a functor

$$\mathcal{KD}\colon \mathbf{Sch}(k) \longrightarrow Ho\mathbf{Sp}$$

such that it satisfies the descent property (D):

(D) if $X_\bullet$ is an acyclic square in $\mathbf{Sch}(k)$*, then $s\mathcal{KD}(X_\bullet)$ is acyclic.*

Proof. By Proposition 3.6, we know that **Sp** is a descent category. So, in order to apply the Guillén-Navarro descent theorem 2.1, we have to verify properties *(F1)* and *(F2)*. The first one is immediate, while *(F2)* follows from Thomason's calculation in [**T93**, Théorème], of the algebraic K-theory of a blow up along a regularly immersed subscheme, as has been observed by many authors (see, for example, [**H**, Theorem 3.6], [**GS**, Theorem 5], or [**CHSW**, Remark 1.6]).

In the context of cubical spectra we propose the following presentation of property *(F2)*. Consider an elementary acyclic square as in 2.2.1 and the square of spectra obtained by application of the algebraic K functor

$$\begin{tikzcd}
\mathcal{K}(X) \arrow[r, "i^*"] \arrow[d, "f^*"'] & \mathcal{K}(Y) \arrow[d, "g^*"] \\
\mathcal{K}(\widetilde{X}) \arrow[r, "j^*"] & \mathcal{K}(\widetilde{Y}).
\end{tikzcd}$$

We have to prove that this square is an acyclic square of spectra. If N is the conormal bundle of Y in X, then $\widetilde{Y} = \mathbb{P}(N)$, so the morphism $\Psi\colon \prod^d \mathcal{K}(Y) \longrightarrow \mathcal{K}(\widetilde{Y})$ induced by the functor which is defined on a sequence of perfect complexes by

$$(E_0, \dots, E_{d-1}) \mapsto \bigoplus_{i=0}^{d-1} \mathcal{O}_{\mathbb{P}(N)}(-i) \otimes Lg^* E_i$$

is a weak equivalence; see [**TT**, Theorem 4.1] and also [**T91**].

For the blown up variety $\widetilde{X}$, it has been proved by Thomason (see [**T93**, Théorème 2.1]) that the morphism $\Phi\colon \mathcal{K}(X) \times \prod^{d-1} \mathcal{K}(Y) \longrightarrow \mathcal{K}(\widetilde{X})$, which is induced by the functor on perfect complexes given by

$$(F, E_1, \dots, E_{d-1}) \mapsto f^*F \oplus \bigoplus_{i=1}^{d-1} j_*(\mathcal{O}_{\mathbb{P}(N)}(-i) \otimes Lg^* E_i)$$

is also a weak equivalence.

Define

$$j'\colon \mathcal{K}(X) \times \prod^{d-1} \mathcal{K}(Y) \longrightarrow \prod^{d} \mathcal{K}(Y)$$

componentwise by g^*i^* on the first component and the morphism given by multiplication by $\lambda_{-1}(N)$ in the Y-components. After the self-intersection formula [**T93**, (3.1.4)], the diagram

$$\begin{tikzcd}
\mathcal{K}(X) \times \prod^{d-1} \mathcal{K}(Y) \arrow[r, "\Phi"] \arrow[d, "j'"'] & \mathcal{K}(\widetilde{X}) \arrow[d, "j^*"] \\
\prod^{d} \mathcal{K}(Y) \arrow[r, "\Psi"] & \mathcal{K}(\widetilde{Y})
\end{tikzcd}$$

is commutative. Since Φ, Ψ are weak equivalences, it is an acyclic diagram.

Consider now the augmented commutative cubical diagram

$$\begin{tikzcd}
& \mathcal{K}(X) \arrow[rr] \arrow[dl, equal] \arrow[dd] & & \mathcal{K}(X) \times \prod^{d-1} \mathcal{K}(Y) \arrow[dl, "\Phi"'] \arrow[dd, "j'"] \\
\mathcal{K}(X) \arrow[rr, "f^*"] \arrow[dd, "i^*"'] & & \mathcal{K}(\widetilde{X}) \arrow[dd, "j^*"] & \\
& \mathcal{K}(Y) \arrow[rr] \arrow[dl, equal] & & \prod^{d} \mathcal{K}(Y) \arrow[dl, "\Psi"'] \\
\mathcal{K}(Y) \arrow[rr, "g^*"] & & \mathcal{K}(\widetilde{Y}), &
\end{tikzcd}$$

where the horizontal back arrows are the inclusion on the first factor.

As the right and left side squares are acyclic, it follows from the definition in 3.2.2 that it is an acyclic cubical diagram. But the back square is acyclic because the two horizontal morphisms have the same cofiber, so the front square must be acyclic, which is what has to be proved. □

For a k-variety X, we will denote by $KD_*(X)$ the homotopy groups of $\mathcal{KD}(X)$,

$$KD_*(X) := \pi_*(\mathcal{KD}(X)).$$

The descent property *(D)* gives rise to exact sequences:

Corollary 4.2. *Let $X_\bullet$ an acyclic square in* $\mathbf{Sch}(k)$*. Then there is an exact sequence*

$$\cdots \to KD_n(X) \xrightarrow{f^*-i^*} KD_n(\widetilde{X}) \oplus KD_n(Y) \xrightarrow{j^*+g^*} KD_n(\widetilde{Y}) \xrightarrow{\delta} KD_{n-1}(X) \to \cdots .$$

More generally, if X is a k-variety, then $\mathcal{KD}(X)$ is defined as the simple of the cubical diagram of spectra $\mathcal{K}(X_\bullet)$, where $X_\bullet$ is a cubical hyperresolution, so from Proposition 3.3 we deduce:

Proposition 4.3. *Let k be a field of characteristic zero and X be an algebraic k-variety. Let $X_\bullet$ be a cubical hyperresolution of X. Then there is a convergent spectral sequence*

$$E_1^{pq} = \bigoplus_{|\alpha|=p+1} K_q(X_\alpha) \Longrightarrow KD_{q-p}(X).$$

If X is of dimension d, then we can take cubical hyperresolutions of size $\leqslant d$ (see [**GNPP**, I.2.15]), so it follows:

Corollary 4.4. *Let k be a field of characteristic zero and X be an algebraic k-variety of dimension d. Then,*

$$KD_n(X) = 0, \qquad n < -d.$$

4.2. Some properties of $\mathcal{KD}$

As explained in Section 1, $\mathcal{KD}$ inherits many properties of the algebraic K-theory of smooth schemes. For example, from Proposition 2.3 and the properties of homotopy invariance and Mayer-Vietoris for the K-theory of smooth schemes (see [**Q**, 7.1 and 7.3.5]), we deduce immediately:

Proposition 4.5. *The descent $\mathcal{KD}$-theory satisfies:*

(1) $\mathcal{KD}$ is homotopy invariant; that is, for any variety X, the projection $X \times \mathbb{A}^1 \longrightarrow X$ induces a weak equivalence $\mathcal{KD}(X) \cong \mathcal{KD}(X \times \mathbb{A}^1)$.

(2) $\mathcal{KD}$ has the Mayer-Vietoris property; that is, if $X = U \cup V$, with U, V open sets, then the square

$$\begin{tikzcd} \mathcal{K}D(X) \arrow[r] \arrow[d] & \mathcal{K}D(U) \arrow[d] \\ \mathcal{K}D(V) \arrow[r] & \mathcal{K}D(U \cap V) \end{tikzcd}$$

is homotopy cartesian.

One may prove in a similar manner that $\mathcal{KD}$ satisfies the fundamental Bass theorem. Also, following [**T80**, Proposition 2.3], or [**W1**, Corollary 5.2], one can prove the existence of a Brown-Gersten type spectral sequence:

$$E_2^{pq} = H^p(X, \widetilde{KD}_{-q}) \Rightarrow KD_{-p-q}(X),$$

where $\widetilde{KD}_*$ stands for the sheaf in the Zariski topology associated to the presheaf KD_*.

4.3. Equivalence with homotopy algebraic K-theory

In [**H**, Theorem 3.5], Haesemeyer has proved that the homotopy algebraic K-theory $\mathcal{KH}$ of an algebraic variety X defined by Weibel in [**W1**] (see also [**TT**, Exercise 9.11]), satisfies the descent axiom *(D)*. As the KH-theory coincides with K-theory for smooth varieties, we can apply the uniqueness property of the extension theorem 2.1 to obtain:

Corollary 4.6. *Let X be an algebraic variety over a field of characteristic zero. There is a natural morphism $\mathcal{KD}(X) \longrightarrow \mathcal{KH}(X)$, in $Ho\mathbf{Sp}$, which is a weak equivalence.*

This may also be stated as a uniqueness result for KH-theory:

Corollary 4.7. *Let k be a field of characteristic zero. The homotopy algebraic K-theory $\mathcal{KH}$ is the unique (Φ-rectifiable) functor $\mathbf{Sch}(k) \longrightarrow Ho\mathbf{Sp}$, up to equivalence, which satisfies the descent property (D) and is equivalent to the algebraic K-functor $\mathcal{K}$ over smooth algebraic varieties.*

4.4. Algebraic K-theory with compact support

We can apply the same arguments of the proof of Theorem 4.1 jointly with the compact support extension theorem in [**GN02**, (2.3.6)], to extend the algebraic K-theory of smooth projective varieties over a field of characteristic zero to a theory with compact support:

Theorem 4.8. *Let k be a field of characteristic zero and $\mathbf{V}(k)$ be the category of smooth projective k-varieties. The rectified contravariant functor*

$$\mathcal{K}\colon \mathbf{V}(k) \longrightarrow Ho\mathbf{Sp}$$

admits a unique extension, up to unique isomorphism of Φ-rectified functors, to a functor

$$\mathcal{K}^c\colon \mathbf{Sch}_c(k) \longrightarrow Ho\mathbf{Sp}$$

such that it satisfies the descent property (D) and the compact support descent property:

(D_c) If Y is a subvariety of X, then there is a natural isomorphism

$$\mathcal{K}^c(X \setminus Y) \cong \operatorname{holim}(\mathcal{K}^c(X) \longrightarrow \mathcal{K}^c(Y) \longleftarrow *).$$

In other words, property (D_c) says that the sequence

$$\mathcal{K}^c(X \setminus Y) \longrightarrow \mathcal{K}^c(X) \longrightarrow \mathcal{K}^c(Y)$$

is a fibration sequence in $Ho\mathbf{Sp}$, so that taking homotopy groups and writing $K^c_*(X) = \pi_*(\mathcal{K}^c(X))$, it gives rise to a long exact sequence

$$\cdots \longrightarrow K^c_n(X \setminus Y) \longrightarrow K^c_n(X) \longrightarrow K^c_n(Y) \longrightarrow K^c_{n-1}(X \setminus Y) \longrightarrow \cdots .$$

In [**GS**, Theorem 7], Gillet-Soulé defined a $\mathcal{K}$-theory with compact support satisfying (D_c), so by the uniqueness of the compact support extension we find:

Corollary 4.9. *Let X be an algebraic variety over a field of characteristic zero. Then $\mathcal{K}^c(X)$ is naturally isomorphic in $Ho\mathbf{Sp}$ to the algebraic K-theory with compact support introduced by Gillet and Soulé in [**GS**, Theorem 7].*

5. Weight filtration

In this section we prove that there are well-defined filtrations on the groups $KD_*(X)$, or equivalently on $KH_*(X)$, and on the groups $K^c_*(X)$, which are trivial for X smooth. In the compact support case we recover the weight filtration obtained by Gillet-Soulé [**GS**].

We fix a field k of characteristic zero.

5.1.

Let X be an algebraic variety. The spectral sequence in Proposition 4.3 associated to a cubical hyperresolution $X_\bullet$ of X induces a filtration on the groups $KD_n(X)$. Our next goal is to prove that this filtration on $KD_n(X)$ is independent of the cubical hyperresolution $X_\bullet$. We will follow Section 3 of [**GN03**] closely, where the authors analyze the weight filtration in an abelian setting.

5.2. Towers of fibrant spectra

First, we introduce a cohomological descent structure on the category of towers of fibrations $\mathbf{tow}(\mathbf{Sp})$.

5.2.1.

A tower of fibrations $X(-)$ is a sequence of fibrations of spectra

$$\cdots \longrightarrow X(n) \longrightarrow X(n-1) \longrightarrow \cdots \longrightarrow X(1) \longrightarrow X(0) \longrightarrow *.$$

A morphism of towers of fibrations is a morphism of diagrams. We denote by $\mathbf{tow}(\mathbf{Sp})$ the category of towers of fibrations.

Defining weak equivalences of towers of fibrations and simple functors for cubical diagrams degreewise, it is immediate to prove the following result:

Proposition 5.1. *The category of towers of fibrations $\mathbf{tow}(\mathbf{Sp})$ together with weak equivalences and simple functors for cubical diagrams defined degreewise is a descent category.*

5.2.2.
We now introduce a second descent structure on $\mathbf{tow}(\mathbf{Sp})$. Recall that if $X(-)$ is a tower of fibrations, then there is a functorial spectral sequence

$$E_1^{pq} = \pi_{q-p}(F(p)) \Longrightarrow \pi_{q-p}(X),$$

where $F(p)$ is the fiber of the morphism $X(p) \longrightarrow X(p-1)$ and $X = \lim X(p)$; see [**T80**, 5.43] (where convergence is understood in the sense of Bousfield-Kan).

Definition 5.2. We say that a morphism of towers $f\colon X(-) \longrightarrow Y(-)$ is an E_2-*weak equivalence* if the morphism $E_2^{**}(f)$ induced on the E_2-terms of the corresponding spectral sequences is an isomorphism.

Observe that if $f_p\colon X(p) \longrightarrow Y(p)$ is a weak equivalence, for all $p \geqslant 0$, then f induces an isomorphism in the E_1-terms of the spectral sequence and hence it is also a E_2-weak equivalence.

5.2.3.
Now we define a simple construction, for cubical diagrams of type $\square$,

$$s_2\colon (\square, \mathbf{tow}(\mathbf{Sp})) \longrightarrow \mathbf{tow}(\mathbf{Sp}),$$

compatible with the E_2-weak equivalences: given a tower of fibrations $X(-)$ and a positive integer $n \geqslant 0$, we denote by $X[n](-)$ the tower of fibrations defined by

$$X[n](p) := \begin{cases} *, & 0 \leqslant p < n, \\ X(p-n), & p \geqslant n, \end{cases}$$

with the evident morphisms, so that the new tower is obtained by translating n places to the left the tower $X(-)$.

Definition 5.3. Let $\square$ be a cubical category and $X_\bullet(-)$ be a $\square$-diagram of towers of fibrations. Denote by $dX(-)$ the $\square$-diagram of towers of fibrations given by

$$(dX)_\alpha(-) = X_\alpha[|\alpha| - 1](-),$$

with morphisms induced by $X_\bullet$. We define the s_2 *simple* of $X_\bullet(-)$ as the tower of fibrations obtained by applying homotopy limits in each cubical degree of $dX_\bullet(-)$; that is,

$$s_2(X_\bullet)(p) := s(dX_\bullet(p)) = \mathrm{holim}_\alpha X_\alpha(p - |\alpha| + 1).$$

For example, given a $\square_1$-diagram $X_\bullet(-)$ of towers of fibrations

$$\begin{tikzcd}
\cdots \arrow[r] & X(1) \arrow[r] \arrow[d] & X(0) \arrow[r] \arrow[d] & * \\
\cdots \arrow[r] & Y(1) \arrow[r] & Y(0) \arrow[r] & * \\
\cdots \arrow[r] & Z(1) \arrow[r] \arrow[u] & Z(0) \arrow[r] \arrow[u] & *
\end{tikzcd}$$

the new diagram $dX_\bullet(-)$ is the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & X(1) \arrow[r] \arrow[d] & X(0) \arrow[r] \arrow[d] & * \\
\cdots \arrow[r] & Y(0) \arrow[r] & * \arrow[r] & * \\
\cdots \arrow[r] & Z(1) \arrow[r] \arrow[u] & Z(0) \arrow[r] \arrow[u] & *
\end{tikzcd}$$

and it follows that its s_2 simple in degree p corresponds to the spectrum

$$\operatorname{holim}(X(p) \longrightarrow Y(p-1) \longleftarrow Z(p)).$$

Lemma 5.4. *For any cubical diagram of towers of fibrations $X_\bullet(-)$ there is a canonical quasi-isomorphism of complexes of abelian groups*

$$E_1^{*q}(s_2X_\bullet(-)) \longrightarrow s(\alpha \mapsto E_1^{*q}(X_\alpha(-))).$$

Proof. The notation $s(\alpha \mapsto E_1^{*q}(X_\alpha(-)))$ refers to the ordinary simple functor for complexes of abelian groups, also called the total complex associated to a cubical complex. The group in degree p of this complex is

$$s(\alpha \mapsto E_1^{*q}(X_\alpha(-)))^p = s(\alpha \mapsto \pi_{q-r}F_\alpha(r)) = \bigoplus_{|\alpha|+r=p+1} \pi_{q-r}(F_\alpha(r)),$$

while the differential is induced by the differentials of the Bousfield-Kan spectral sequence of the tower $X_\alpha(-)$.

On the other hand, by definition, for each p, $s_2X_\bullet(p)$ is the ordinary simple of the cubical diagram of spectra $dX_\bullet(p)$, so the complex $E_1^{*q}(s_2X_\bullet(-))$ is the E_1-term of the Bousfield-Kan spectral sequence associated to the tower of fibrations

$$\cdots \longrightarrow sdX_\bullet(p) \longrightarrow \cdots \longrightarrow sdX_\bullet(1) \longrightarrow sdX_\bullet(0) \longrightarrow * .$$

Denote by $F_\alpha(p)$ the fiber of the fibration $X_\alpha(p) \longrightarrow X_\alpha(p-1)$. Since homotopy limits commute, the fiber of the fibration

$$sdX_\bullet(p) \longrightarrow sdX_\bullet(p-1)$$

is isomorphic to the simple spectrum associated to the cubical diagram $dF_\bullet(p)$. But, in this diagram all morphisms are constant, so

$$sdF_\bullet(p) = \prod_\alpha \Omega^{|\alpha|-1}F_\alpha(p-|\alpha|+1) = \prod_{|\alpha|+r=p+1} \Omega^{p-r}F_\alpha(r);$$

hence its homotopy groups are given by

$$E_1^{pq} = \pi_{q-p}(sdF_\bullet(p)) = \bigoplus_{|\alpha|+r=p+1} \pi_{q-r}(F_\alpha(r)).$$

The differential is also induced by the differentials of the Bousfield-Kan spectral sequence of the tower $X_\alpha(-)$. □

Proposition 5.5. *The simple s_2 and the E_2-weak equivalences define a cohomological descent category structure on* **tow(Sp)**.

Proof. Observe that a morphism between towers of fibrations f is a E_2-weak equivalence if and only if the morphism $E_1(f)$ of the corresponding spectral sequence is a quasi-isomorphism of complexes. If $Gr\mathbf{C}^*(\mathbb{Z})$ denotes the category of graded complexes of abelian groups, then the functor

$$\begin{aligned} E_1\colon \mathbf{tow}(\mathbf{Sp}) &\longrightarrow Gr\mathbf{C}^*(\mathbb{Z}) \\ X(-) &\longmapsto E_1^* \end{aligned}$$

commutes with direct sums and, by the previous result, it commutes with the simple s_2 functor, so the result follows from [**GN02**, (1.5.12)]. □

5.3. An extension criterion for towers

In the next result we write $Ho_2(\mathbf{tow}(\mathbf{Sp}))$ for the homotopy category obtained from $\mathbf{tow}(\mathbf{Sp})$ by inverting E_2-weak equivalences.

The following result, remarked by V. Navarro several years ago in the abelian context, is the key point in order to extend some functors on $\mathbf{Sm}(k)$ with values in the category of spectra to functors defined for all varieties and taking values in $Ho_2(\mathbf{tow}(\mathbf{Sp}))$.

Proposition 5.6 (Compare with [**GN03**, Proposition (3.10)])**.** *Let $G\colon \mathbf{Sm}(k) \longrightarrow Ho\mathbf{Sp}$ be a Φ-rectifiable functor and denote also by*

$$G\colon \mathbf{Sm}(k) \longrightarrow Ho_2(\mathbf{tow}(\mathbf{Sp}))$$

the associated constant functor. Then, G satisfies property (F2) if and only if for every elementary acyclic square the sequence

$$0 \longrightarrow \pi_n G(X) \overset{f^*-i^*}{\longrightarrow} \pi_n G(\widetilde{X}) \oplus \pi_n G(Y) \overset{j^*+g^*}{\longrightarrow} \pi_n G(\widetilde{Y}) \longrightarrow 0$$

is exact.

Proof. By the acyclicity criterion on descent categories, the property *(F2)* of Theorem 2.1 for the extended functor G says that the morphism

$$E_1^{*,q}(G(X)) \longrightarrow E_1^{*,q} s_2 G(X_\bullet)$$

is a quasi-isomorphism. Observe that we have

$$E_1^{*,q}(G(X)) = \begin{cases} \pi_q(G(X)), & p = 0, \\ 0, & p \neq 0. \end{cases}$$

On the other hand, the E_1-page of the spectral sequence of $s_2G(X_\bullet)$ reduces to the exact sequence

$$\pi_n G(\widetilde{X}) \oplus \pi_n G(Y) \overset{j^*+g^*}{\longrightarrow} \pi_n G(\widetilde{Y}),$$

so the *(F2)* property is equivalent to the fact that the morphism of complexes of abelian groups

$$\pi_n G(X) \overset{f^*-i^*}{\longrightarrow} (\pi_n G(\widetilde{X}) \oplus \pi_n G(Y) \overset{j^*+g^*}{\longrightarrow} \pi_n G(\widetilde{Y}))$$

is a quasi-isomorphism, which is precisely the condition stated in the proposition. □

5.3.1.
We return now to the applications to algebraic K-theory. The following proposition has also been proved by Gillet-Soulé directly from Thomason's calculations; see [**GS**, Theorem 5]:

Proposition 5.7. *For any elementary acyclic square of* $\mathbf{Sm}(k)$ *and any* $n \geqslant 0$*, the sequence*

$$0 \longrightarrow K_n(X) \xrightarrow{f^*-i^*} K_n(\widetilde{X}) \oplus K_n(Y) \xrightarrow{j^*+g^*} K_n(\widetilde{Y}) \longrightarrow 0$$

is exact.

Proof. As we have recalled in the proof of Theorem 4.1, an elementary acyclic square gives rise to a homotopy cartesian square of algebraic K-theory spectra, so we have an exact sequence

$$\cdots \longrightarrow K_n(X) \xrightarrow{f^*-i^*} K_n(\widetilde{X}) \oplus K_n(Y) \xrightarrow{j^*+g^*} K_n(\widetilde{Y}) \xrightarrow{\delta} K_{n-1}(X) \longrightarrow \cdots .$$

But, by Thomason's calculation of the algebraic K-theory of a blow up ([**T93**, Théorème 2.1]), there are isomorphisms

$$\varphi\colon K_n(X) \bigoplus_{i=1}^{d-1} K_n(Y) \longrightarrow K_n(\widetilde{X}),$$
$$\psi\colon \bigoplus_{i=0}^{d-1} K_n(Y) \longrightarrow K_n(\widetilde{Y}),$$

given, respectively, by

$$\varphi(x, y_1, \ldots, y_{d-1}) = f^*(x) + \bigoplus_{i=1}^{d-1} j_*(\ell^{-i} \cup g^*(y_i)),$$
$$\psi(y_0, y_1, \ldots, y_{d-1}) = \bigoplus_{i=0}^{d-1} j_*(\ell^{-i} \cup g^*(y_i)).$$

With this identifications the morphism f^* corresponds to the inclusion of $K_n(X)$ on the first factor of $K_n(\widetilde{X})$, and so the morphism $f^* - i^*$ is injective. This splits the exact sequence above into the required short exact sequences. □

Now, by Propositions 5.6 and 5.7 we can apply the extension criterion of Theorem 2.1, so we find:

Corollary 5.8. *Let* k *be a field of characteristic zero. The constant algebraic* K*-theory functor* $\mathcal{K}\colon \mathbf{Sm}(k) \longrightarrow Ho_2(\mathbf{tow}(\mathbf{Sp}))$ *admits an essentially unique extension* $\mathcal{KD}(-)\colon \mathbf{Sch}(k) \longrightarrow Ho_2(\mathbf{tow}(\mathbf{Sp}))$ *which satisfies the descent property (D). Moreover, for any variety* X*, the tower of fibrations* $\mathcal{KD}(-)(X)$ *satisfies* $\mathcal{KD}(n)(X) = \mathcal{KD}(X)$ *for* $n \gg 0$*.*

Proof. We have only to justify the last sentence. Take an algebraic variety X and an hyperresolution $X_\bullet$, whose type $\square$ is of length ℓ. By the definition of the descent functor $\mathcal{KD}(-)$, the tower $\mathcal{KD}(-)(X)$ is the s_2-simple tower associated to the diagram of constant towers $\mathcal{K}(X_\bullet)$; that is, it is the tower whose spectra are the homotopy limits of the diagram $d\mathcal{K}(X_\bullet)(n)$ for each n (see Definition 5.3). Observe that this diagram is constant for $n \geqslant \ell$ and, moreover, it is precisely the cubical diagram $\mathcal{K}(X_\bullet)$, so the result follows. ☐

Since the spectral sequence of a tower of fibrations is functorial in the category $Ho_2(\mathbf{tow}(\mathbf{Sp}))$ from the E_2-term on, we deduce from the corollary above:

Corollary 5.9. *There is a well-defined and functorial finite-increasing filtration* F^p *on* $KD_n(X)$ *which is trivial for smooth varieties.*

Remark 5.10. Equivalently, by 4.6, for any variety X the last corollary defines a functorial finite filtration on the homotopy algebraic K-theory groups $KH_n(X)$.

5.4.

Finally, we observe that the same procedure may be applied to the algebraic K-theory with compact support. In this case, from the uniqueness property of descent extensions and [**GS**, Theorem 7], we deduce:

Corollary 5.11. *There is a well-defined and functorial finite-increasing filtration* $W^pK_n^c(X)$ *which is trivial for complete smooth varieties. This filtration coincides with the weight filtration defined by Gillet-Soulé in [**GS**, Theorem 7].*

References

[**BF**] A. Bousfield and E. Friedlander, Homotopy theory of Γ-spaces, spectra, and bisimplicial sets, in *Geometric applications of homotopy theory* (Proc. Conf., Evanston, IL, 1977), Lecture Notes in Math. **658**, 80–130, Springer-Verlag, New York (1978).

[**BK**] A. Bousfield and D. Kan, *Homotopy limits, completions, and localizations*, Lecture Notes in Math. **304**, Springer-Verlag, New York (1972).

[**CHSW**] G. Cortiñas, C. Haesemeyer, M. Schlichting, and C. Weibel, Cyclic homology, cdh-cohomology and negative K-theory, *Annals of Math.* **167** (2008), no. 2, 549–573.

[**CHW**] G. Cortiñas, C. Haesemeyer, and C. Weibel, K-regularity, *cdh*-fibrant Hochschild homology, and a conjecture of Vorst, *J. Amer. Math. Soc.* **21** (2008), no. 2, 547–561.

[**GN02**] F. Guillén and V. Navarro Aznar, Un critère d'extension des foncteurs définis sur les schémas lisses, *Publ. Math. I.H.E.S.* **95** (2002), 1–91.

[**GN03**] F. Guillén and V. Navarro Aznar, Cohomological descent and weight filtration, Communication at the joint meeting of the AMS and the Spanish Math. Soc., Sevilla, 2003.

[**GNPP**] F. Guillén, V. Navarro, P. Pascual, and F. Puerta, *Hyperrésolutions cubiques et descente cohomologique*, Lecture Notes in Math. **1335**, Springer-Verlag, New York, 1988.

[**GS**] H. Gillet and C. Soulé, Descent, motives and K-theory, *J. Reine Angew. Math.* **478** (1996), 127–176.

[G] T. Goodwillie, Calculus II: Analytic functors, *K-Theory* **5** (1992), no. 4, 295–332.

[H] C. Haesemeyer, Descent properties for homotopy K-theory, *Duke Math. J.* **125** (2004), no. 3, 589–620.

[M] S. MacLane, *Categories for the working mathematician*, Grad. Texts in Math. **5**, Springer-Verlag, New York, 1971.

[P] P. Pascual, On the simple object associated to a diagram in a closed model category, *Math. Proc. Cambridge Philos. Soc.* **100** (1986), no. 3, 459–474.

[Q] D. Quillen, Higher algebraic K-theory. I, in *Algebraic K-theory, II. Higher K-theories*, Lecture Notes in Math. **341**, 85–147, Springer-Verlag, New York (1973).

[R] Ll. Rubió Pons, Model categories and cubical descent, *Bol. Soc. Mat. Mexicana (3)* **13** (2007), no. 2, 293–305.

[R2] Ll. Rubió Pons, Fibrant presheaves of spectra and Guillén-Navarro extension, *Rend. Sem. Mat. Univ. Padova*, 2009, to appear.

[S] S. Schwede, Spectra in model categories and applications to the algebraic cotangent complex, *J. Pure Appl. Algebra* **120** (1997), no. 1, 77–104.

[T80] R. Thomason, Algebraic K-theory and étale cohomology, *Annals Sci. École Normale Sup.* **13** (1980), 437–552.

[T91] R. Thomason, Les K-groupes d'un fibré projectif, in (J. Jardine et al., eds.), in *Algebraic K-theory and algebraic topology* (Lake Louise, AB, 1991), Kluwer Acad. Publ., Dordrecht, 1993, 243–248.

[T93] R. Thomason, Les K-groupes d'un schéma éclaté et une formule d'intersection excédentaire, *Invent. Math.* **112** (1993), no. 1, 195–215.

[TT] R. Thomason and T. Trobaugh, Higher Algebraic K-theory of schemes and of derived categories, *The Grothendieck Festschrift*, Vol. III, Progress in Math. **88**, 247–435, Birkhäuser Boston, Boston, MA, 1990.

[W1] C. Weibel, Homotopy algebraic K-theory, in *Algebraic K-theory and number theory* (Honolulu, HI, 1987), Contemp. Math. **83**, 461–488, Amer. Math. Soc., Providence, RI, 1989.

[W2] C. Weibel, A Quillen-type spectral sequence for the K-theory of varieties with isolated singularities, *K-Theory* **3** (1989), no. 3, 261–270.

Pere Pascual `pere.pascual@upc.edu`

Departament de Matemàtica Aplicada 1, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain.

Llorenç Rubió Pons `llorenc.rubio@upc.edu`

Departament de Matemàtica Aplicada 1, Universitat Politècnica de Catalunya, Diagonal 647, 08028 Barcelona, Spain.