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WAVELET CHARACTERIZATION OF THE POINTWISE MULTIPLIER SPACE $\dot{X}_r$

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Abstract: In the present note we characterize the function space $\dot{X}_r$, which is the set of pointwise multipliers which map L^2 into $\dot{H}^{-r}$. To this end, we use wavelets and capacity.

Keywords: pointwise multiplier, wavelet decomposition

1. Introduction

The aim of the present paper is to characterize the function space $\dot{X}_r$ in terms of wavelet expansion, where the space $\dot{X}_r$ is the set of pointwise multipliers which map L^2 into $\dot{H}^{-r}$, which is defined as follows:

Definition 1.1. *For $0 \leqslant r < \frac{d}{2}$, the space $\dot{X}_r$ is defined as the space of functions $f \in L^2_{\mathrm{loc}}\left(\mathbb{R}^d\right)$ that satisfy the following inequality:*

$$\|f\|_{\dot{X}_r} = \sup_{\|g\|_{\dot{H}^r} \leqslant 1} \|fg\|_{L^2} < \infty,$$

where $\dot{H}^r\left(\mathbb{R}^d\right)$ stands for the completion of the space $\mathcal{D}\left(\mathbb{R}^d\right)$ with respect to the norm $\|u\|_{\dot{H}^r} = \left\|(-\Delta)^{\frac{r}{2}} u\right\|_{L^2}$.

We refer to [2] for the reference of this field which contains a vast amount of researches of the multiplier spaces. Here and below we place ourselves in the setting of $\mathbb{R}^d$ with $d \geqslant 3$.

We shall characterize this norm in terms of the $\dot{H}^r$ capacity and wavelets. In the present paper we use the compactly supported wavelet functions with r-regularity ($r \geqslant 1$) proposed by I. Daubechies [3]. For $j \in \mathbb{Z}$ and $\gamma \in \mathbb{Z}^d$, we write $Q_{j,\gamma} = \left\{x \in \mathbb{R}^d : 2^j x - \gamma \in [0,1)^d\right\}$. Let $\mathcal{Q}$ be the set of all dyadic cubes in $\mathbb{R}^d$, i.e., $\mathcal{Q} = \left\{Q = Q_{j,\gamma} : j \in \mathbb{Z}, \gamma = (\gamma_1, \gamma_2, ..., \gamma_d) \in \mathbb{Z}^d\right\}$. Suppose φ and ψ are r-regular

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compactly supported functions obtained by multiresolution approximations. Let $\psi_0 = \varphi$ and $\psi_1 = \psi$. For any $\varepsilon \in E := \{0, 1, \cdots, 2^d - 1\}$, we use binary expansion to write

$$\varepsilon = \sum_{j=1}^{d} 2^{j-1}\varepsilon_j, \qquad \varepsilon_j \in \{0, 1\}.$$

For $Q_{j,\gamma} \in \mathcal{Q}$ and $\varepsilon = 1, 2, \cdots, 2^d - 1$, we let

$$\psi_{\varepsilon,j,\gamma}(x) = \psi_{\varepsilon,Q_{j,\gamma}} = 2^{j\frac{d}{2}}\psi_\varepsilon(2^j x_1 - \gamma_1)\cdots\psi_{\varepsilon_d}(2^j x_d - \gamma_d).$$

It is known that the $\psi_{\varepsilon,j,\gamma}$'s enjoy the following properties :

(a) The system $\{\psi_{\varepsilon,j,\gamma}\}_{Q_{j,\gamma}\in\mathcal{Q},\varepsilon\in E}$ forms an orthonormal basis of $L^2\left(\mathbb{R}^d\right)$;
(b) $\operatorname{supp}(\psi_{\varepsilon,j,\gamma}) \subset M\ Q_{j,\gamma}$, $M \geqslant 1$, where MQ is the cube concentric with Q but with the side lenght M times that of Q (i.e., the M-times expansion);
(c) $\left\|\frac{\partial^\alpha}{\partial x^\alpha}\psi_{\varepsilon,j,\gamma}\right\|_\infty \leqslant C2^{\frac{jd}{2}+|\alpha|j}$, $|\alpha| \leqslant r$;
(d) $\int x^\alpha \psi_{\varepsilon,j,\gamma}(x)dx = 0$, $|\alpha| \leqslant r$.

Here we present the definition of capacity (see [1], [2]).

Definition 1.2. *The quantity* $\operatorname{cap}(e, \dot{H}^r)$ *stands for the* $\dot{H}^r$*-capacity of a compact set* $e \subset \mathbb{R}^d$*, which is defined by*

$$\operatorname{cap}\left(e, \dot{H}^r\right) = \inf\left\{\|u\|^2_{\dot{H}^r(\mathbb{R}^d)} : u \in \mathcal{D}\left(\mathbb{R}^d\right),\ u \geqslant 1 \text{ on } e\right\}.$$

Having clarified the definition of capacity, let us now formulate our main result.

Theorem 1.1. *Let* $0 \leqslant r < \frac{d}{2}$. *Then the following statements are equivalent:*

(i) $f \in \dot{X}_r\left(\mathbb{R}^d\right)$.
(ii) *The function* f *can be expanded as follows:*

$$f = \sum_{\varepsilon=1}^{2^d-1} \sum_{(j,\gamma)\in\mathbb{Z}\times\mathbb{Z}^d} \lambda_{\varepsilon,j,\gamma}\psi_{\varepsilon,j,\gamma}(x),$$

where $\{\lambda_{\varepsilon,j,\gamma}\}_{\varepsilon=1,2,\cdots,2^d-1,\,(j,\gamma)\in\mathbb{Z}\times\mathbb{Z}^d}$ *satisfies*

$$\sum_{\varepsilon=1}^{2^d-1} \sum_{(j,\gamma)\in\mathbb{Z}\times\mathbb{Z}^d} |\lambda_{\varepsilon,j,\gamma}|^2 \int_e |\psi_{\varepsilon,j,\gamma}(x)|^2\,dx \leqslant C\,cap\left(e, \dot{H}^r\right)$$

for any compact set e *of* $\mathbb{R}^d$.

Finally let us make a remark on the usage of the constant C; C denotes a constant independent of f. However, it varies at each occurrence.

2. Proof of Theorem 1.1.

Denote by M the centered Hardy-Littlewood maximal operator.

$$Mf(x) = \sup_{c(Q)=x} \frac{1}{|Q|} \int_Q |f(x)|\, dx,$$

where Q runs over all compact cubes in $\mathbb{R}^d$ and $c(Q)$ denotes the center of the cube Q.

Lemma 2.1. *Let e be a compact set. If we set $E_\kappa = \{x \in \mathbb{R}^d : M\chi_e(x) > \kappa\}$, then we have*

$$\text{cap}\left(\overline{E_\kappa}, \dot{H}^r\right) \leqslant c\,\kappa^{-2}\text{cap}\left(e, \dot{H}^r\right)$$

Proof. Choose $u \in \mathcal{D}\left(\mathbb{R}^d\right)$ so that

$$u \geqslant 1 \quad \textit{on } e \qquad \textit{and} \qquad \|u\|_{\dot{H}^r} \leqslant 2\text{cap}\left(e, \dot{H}^r\right).$$

Pick a function $\psi \in \mathcal{D}\left(\mathbb{R}^d\right)$ so that $\chi_{Q(1)} \leqslant \psi \leqslant \chi_{Q(2)}$, where, if $R > 0$, we wrote $Q(R)$ for the cube given by

$$Q(R) = \{x = (x_1, x_2, \cdots, x_d) : \max(|x_1|, |x_2|, \cdots, |x_d|) \leqslant R\}.$$

Let $x \in \overline{E_\kappa}(\subset E_{\kappa/2})$. Then, by the definition of the centered Hardy-Littlewood maximal operator, there exists a cube Q centered at x such that $|Q \cap e| \geqslant \dfrac{\kappa}{2}|Q|$. Let us write $\ell(Q) = \dfrac{|Q|^{\frac{1}{d}}}{2}$ and $\psi_{\ell(Q)}(x) = \dfrac{1}{|Q|}\psi\left(\dfrac{x}{\ell(Q)}\right)$. Therefore, we have

$$\psi_{\ell(Q)} * u(x) \geqslant \frac{1}{|Q|}\int_e \psi_{\ell(Q)}(x-y)\, dy \geqslant \frac{|e \cap Q|}{|Q|} \geqslant \frac{\kappa}{2}.$$

Hence it follows that

$$\text{cap}\left(\overline{E_\kappa}, \dot{H}^r\right) \leqslant c\,\kappa^{-2}\|\psi_{\ell(Q)} * u\|_{\dot{H}^r} \leqslant c\,\kappa^{-2}\|u\|_{\dot{H}^r} \leqslant c\,\kappa^{-2}\text{cap}\left(e, \dot{H}^r\right). \qquad \blacksquare$$

Corollary 2.1. *Let $0 \leqslant r < \frac{d}{2}$. The following statements are equivalent.*

(1) *For any compact set e of $\mathbb{R}^d$,*

$$\sum_{\varepsilon=1}^{2^d-1} \sum_{(j,\gamma)\in\mathbb{Z}\times\mathbb{Z}^d} |\lambda_{\varepsilon,j,\gamma}|^2 \int_e |\psi_{\varepsilon,j,\gamma}(x)|^2\, dx \leqslant C\, cap\left(e, \dot{H}^r\right). \tag{1}$$

(2) *For any compact set e of $\mathbb{R}^d$,*

$$\sum_{\varepsilon=1}^{2^d-1} \sum_{(j,\gamma)\in\mathbb{Z}\times\mathbb{Z}^d} |\lambda_{\varepsilon,j,\gamma}|^2 \int_{\mathbb{R}^d} |\psi_{\varepsilon,j,\gamma}(x)|^2\, M[\chi_e](x)^{\frac{4}{5}}\, dx \leqslant C\, cap\left(e, \dot{H}^r\right). \tag{2}$$

Needless to say, significance of this corollary is that (1) implies (2).

Proof. We may assume that $|e| > 0$. Otherwise, the right-hand sides of (1) and (2) are zero and there is nothing to prove. Also, we freeze $\varepsilon = 1, 2, \cdots, 2^d - 1$; the estimates will be independent of ε. We write $E_\kappa = \{M\chi_e > \kappa\}$ as before. For all $x \in \mathbb{R}^d$, there exists a large cube Q, which is centered at x, that engulfs the compact set e. Hence it follows that $M\chi_e(x) \geqslant \dfrac{|e|}{|Q|} > 2^{-l}$ for some $l \in \mathbb{Z}$. Consequently we have $\mathbb{R}^d = \bigcup\limits_{k=1}^{\infty} E_{2^{-k}}$. We decompose $\mathbb{R}^d$ by using this collection $\{E_{2^{-k}}\}$. The result is

$$
\begin{aligned}
&\sum_{(j,\gamma)\in\mathbb{Z}\times\mathbb{Z}^d} |\lambda_{\varepsilon,j,\gamma}|^2 \int_{\mathbb{R}^d} |\psi_{\varepsilon,j,\gamma}(x)|^2\, M[\chi_e](x)^{\frac{4}{5}}\, dx \\
&\qquad \leqslant \sum_{(j,\gamma)\in\mathbb{Z}\times\mathbb{Z}^d} |\lambda_{\varepsilon,j,\gamma}|^2 \int_e |\psi_{\varepsilon,j,\gamma}(x)|^2\, dx \\
&\qquad\quad + \sum_{k=1}^{\infty} \sum_{(j,\gamma)\in\mathbb{Z}\times\mathbb{Z}^d} |\lambda_{\varepsilon,j,\gamma}|^2 \int_{E_{2^{-k}}\setminus E_{2^{-k+1}}} |\psi_{\varepsilon,j,\gamma}(x)|^2\, M[\chi_e](x)^{\frac{4}{5}}\, dx \\
&\qquad \leqslant \sum_{(j,\gamma)\in\mathbb{Z}\times\mathbb{Z}^d} |\lambda_{\varepsilon,j,\gamma}|^2 \int_e |\psi_{\varepsilon,j,\gamma}(x)|^2\, dx \\
&\qquad\quad + \sum_{k=1}^{\infty} \sum_{(j,\gamma)\in\mathbb{Z}\times\mathbb{Z}^d} 2^{-\frac{4d}{5}(k-1)}\, |\lambda_{\varepsilon,j,\gamma}|^2 \int_{E_{2^{-k}}\setminus E_{2^{-k+1}}} |\psi_{\varepsilon,j,\gamma}(x)|^2\, dx.
\end{aligned}
$$

From the assumption (1) we deduce

$$
\begin{aligned}
\sum_{(j,\gamma)\in\mathbb{Z}\times\mathbb{Z}^d} |\lambda_{\varepsilon,j,\gamma}|^2 \int_{E_{2^{-k}}\setminus E_{2^{-k+1}}} |\psi_{\varepsilon,j,\gamma}(x)|^2\, dx &\leqslant C\,\mathrm{cap}(E_{2^{-k}} \setminus E_{2^{-k+1}}, \dot{H}^r) \\
&\leqslant C\,\mathrm{cap}(E_{2^{-k}}, \dot{H}^r).
\end{aligned}
$$

If we invoke Lemma 2.1 with $\kappa = 2^{-k}$, then we have

$$
\sum_{(j,\gamma)\in\mathbb{Z}\times\mathbb{Z}^d} |\lambda_{\varepsilon,j,\gamma}|^2 \int_{E_{2^{-k}}\setminus E_{2^{-k+1}}} |\psi_{\varepsilon,j,\gamma}(x)|^2\, dx \leqslant C\, 4^k \mathrm{cap}(e, \dot{H}^r).
$$

Now that we are assuming $d \geqslant 3$, we see that $\sum\limits_{k=1}^{\infty} 2^{-\frac{4d}{5}k+2k}$ converges. Thus, it follows that

$$\sum_{(j,\gamma)\in\mathbb{Z}\times\mathbb{Z}^d} |\lambda_{\varepsilon,j,\gamma}|^2 \int_{\mathbb{R}^d} |\psi_{\varepsilon,j,\gamma}(x)|^2 M[\chi_e](x)^{\frac{4}{5}} dx \leqslant C \sum_{k=0}^{\infty} 2^{-\frac{4d}{5}k+2k} \text{cap}\left(e, \dot{H}^r\right) = C \,\text{cap}\left(e, \dot{H}^r\right).$$

Therefore, the assertion that (1) implies (2) was proved. ■

Our main result relies also upon the following proposition :

Proposition 2.1 ([1, Section 3.2]). *Let $0 \leqslant r < \frac{d}{2}$. Then $f \in \dot{X}_r$ if and only if*

$$\sup_{e\subset\mathbb{R}^d \,:\, compact} \frac{\|f\|_{L^2(e)}}{\left(\text{cap}\left(e, \dot{H}^r\right)\right)^{\frac{1}{2}}} < \infty.$$

Furthermore, if this is the case, the following norm equivalence holds:

$$\|f\|_{\dot{X}_r} \sim \sup_{e\subset\mathbb{R}^d} \frac{\|f\|_{L^2(e)}}{\left(\text{cap}\left(e, \dot{H}^r\right)\right)^{\frac{1}{2}}}.$$

Now let us finish the proof of Theorem 1.1.

Begin with the “only if " part.

For notational convenience we shall write $\lambda_{\varepsilon,Q} = \lambda_{\varepsilon,j,\gamma} = \langle f, \psi_{\varepsilon,Q}\rangle$ for the wavelet coefficient of f associated with the wavelet $\psi_{\varepsilon,Q}$. Then we have the following decomposition for f :

$$f = \sum_{\varepsilon=1}^{2^d-1} \sum_{Q\in\mathcal{Q}} \lambda_{\varepsilon,Q}\psi_{\varepsilon,Q} = \sum_{\varepsilon=1}^{2^d-1} \sum_{j\in\mathbb{Z}} \sum_{\gamma\in\mathbb{Z}^d} \lambda_{\varepsilon,j,\gamma} 2^{j\frac{d}{2}} \psi_\varepsilon(2^j x - \gamma).$$

Let Λ be a fixed finite subset of $\mathbb{Z}^{d+1}$. For $j \in \mathbb{Z}$ and $\gamma \in \mathbb{Z}^d$ we shall write $(j,\gamma) = (j, \gamma_1, \gamma_2, ..., \gamma_d)$. For $\theta \in \{-1,1\}^\Lambda = \{\{\theta_{j,\gamma}\}_{j,\gamma\in\Lambda} : \theta_{j,\gamma} \in \{-1,1\}\}$, we define the operator $T_{\Lambda,\theta}$ by

$$T_{\Lambda,\theta} f(x) = \sum_{\varepsilon=1}^{2^d-1} \sum_{(j,\gamma)\in\Lambda} \theta_{j,\gamma} \langle f, \psi_{\varepsilon,j,\gamma}\rangle \psi_{\varepsilon,j,\gamma}(x).$$

The operator $T_{\Lambda,\theta}$ is actually an integral operator given by

$$T_{\Lambda,\theta} f(x) = \int_{\mathbb{R}^d} K_{\Lambda,\theta}(x,y) f(y) dy,$$

where the kernel is given by

$$K_{\Lambda,\theta}(x,y) = \sum_{\varepsilon=1}^{2^d-1} \sum_{(j,\gamma)\in\Lambda} \theta_{j,\gamma} \psi_{\varepsilon,j,\gamma}(x) \overline{\psi_{\varepsilon,j,\gamma}}(y).$$

Since the MRA is r-regular, we conclude that $\{T_{\Lambda,\theta}\}_{\Lambda,\theta}$ is a family of Calderón-Zygmund operators (see [4]). Then by a classical result of harmonic analysis we obtain, for some constant C independent of Λ,

$$\|T_{\Lambda,\theta} f\|_{\dot{X}_r} \leqslant C \|f\|_{\dot{X}_r}$$

for all $f \in \dot{X}_r$. Also denote by μ the measure on $\{0,1\}^{\mathbb{Z}\times\mathbb{Z}^d}$ generated by the coin toss. Let us set

$$\mathrm{I} = \int_e \left(\int_{\theta \in \{-1,1\}^{\mathbb{Z}^d \times \mathbb{Z}}} |T_{\Lambda,\theta} f(x)|^2 \, d\mu(\theta) \right) dx.$$

Then, we have

$$\begin{aligned} \mathrm{I} &= \int_{\theta \in \{-1,1\}^{\mathbb{Z}^d \times \mathbb{Z}}} \left(\int_e |T_{\Lambda,\theta} f(x)|^2 \, dx \right) d\mu(\theta) \\ &\leqslant C \|f\|_{\dot{X}_r}^2 \operatorname{cap}\left(e, \dot{H}^r\right) \int_{\theta_{j,\gamma} \in \{-1,1\}^{\mathbb{Z}^d \times \mathbb{Z}}} d\mu(\theta) \\ &= C \|f\|_{\dot{X}_r}^2 \operatorname{cap}\left(e, \dot{H}^r\right). \end{aligned}$$

Meanwhile, if we use the Fubini theorem and write out $T_{\Lambda,\theta} f$, we obtain

$$\begin{aligned} \mathrm{I} &= \int_{\theta \in \{-1,1\}^{\mathbb{Z}^d \times \mathbb{Z}}} \|T_{\Lambda,\theta} f\|_{L^2(e)}^2 \, d\mu(\theta) \\ &= \int_{\theta \in \{-1,1\}^{\mathbb{Z}^d \times \mathbb{Z}}} \left\| \sum_{(j,\gamma)\in\Lambda} \left(\sum_{\varepsilon=1}^{2^d-1} \theta_{j,\gamma} \langle f, \psi_{\varepsilon,j,\gamma} \rangle \psi_{\varepsilon,j,\gamma} \right) \right\|_{L^2(e)}^2 d\mu(\theta). \end{aligned}$$

Moreover, using Khintchine' s inequality, we have

$$\begin{aligned} \mathrm{I} &\approx \sum_{\varepsilon=1}^{2^d-1} \left\| \sum_{(j,\gamma)\in\Lambda} |\langle f, \psi_{\varepsilon,j,\gamma} \rangle|^2 |\psi_{\varepsilon,j,\gamma}|^2 \right\|_{L^2(e)}^2 \\ &= \sum_{\varepsilon=1}^{2^d-1} \int_e \sum_{(j,\gamma)\in\Lambda} |\langle f, \psi_{\varepsilon,j,\gamma} \rangle|^2 |\psi_{\varepsilon,j,\gamma}(x)|^2 \, dx \\ &= \sum_{\varepsilon=1}^{2^d-1} \sum_{(j,\gamma)\in\Lambda} |\langle f, \psi_{\varepsilon,j,\gamma} \rangle|^2 \int_e |\psi_{\varepsilon,j,\gamma}(x)|^2 \, dx \\ &= \sum_{\varepsilon=1}^{2^d-1} \sum_{(j,\gamma)\in\mathbb{Z}\times\mathbb{Z}^d} |\lambda_{\varepsilon,j,\gamma}|^2 \int_e |\psi_{\varepsilon,j,\gamma}(x)|^2 \, dx. \end{aligned}$$

Thus for all $f \in \dot{X}_r$, we obtain

$$\sum_{\varepsilon=1}^{2^d-1} \sum_{(j,\gamma)\in\mathbb{Z}\times\mathbb{Z}^d} |\lambda_{\varepsilon,j,\gamma}|^2 \int_e |\psi_{j,\gamma}(x)|^2 \, dx \leqslant C \operatorname{cap}\left(e, \dot{H}^r\right).$$

As a consequence, we conclude that (i) implies (ii).

Let us show the proof of converse. We use once more the expansion:

$$f = \sum_{\varepsilon=1}^{2^d-1} \sum_{Q\in\mathcal{Q}} \lambda_{\varepsilon,Q}\psi_{\varepsilon,Q}.$$

It is well-known that $M[\chi_e]^{\frac{4}{5}}$ is an A_1-weight. Therefore, we are in the position of using the usual Calderón-Zygmund theory to conclude

$$\|f\|_{L^2(e)} \leqslant \left\| f \cdot M[\chi_e]^{\frac{2}{5}} \right\|_{L^2} \leqslant C \sum_{\varepsilon=1}^{2^d-1} \sum_{(j,\gamma)\in\mathbb{Z}\times\mathbb{Z}^d} |\lambda_{\varepsilon,j,\gamma}|^2 \int_{\mathbb{R}^d} |\psi_{\varepsilon,j,\gamma}(x)|^2 \, M[\chi_e](x)^{\frac{4}{5}} \, dx.$$

We remark that the proof is similar in spirit to the main theorem in [5]. If we use this inequality and Corollary 2.1, then we have

$$\begin{aligned} \|f\|_{\dot{X}^r} &\leqslant C \sup_{e\subset\mathbb{R}^d : compact} \frac{\|f\|_{L^2(e)}}{\operatorname{cap}\left(e, \dot{H}^r\right)} \\ &\leqslant C \sup_{e\subset\mathbb{R}^d : compact} \frac{1}{\operatorname{cap}\left(e, \dot{H}^r\right)} \\ &\quad \times \sum_{\varepsilon=1}^{2^d-1} \sum_{(j,\gamma)\in\mathbb{Z}\times\mathbb{Z}^d} |\lambda_{\varepsilon,j,\gamma}|^2 \int_{\mathbb{R}^d} |\psi_{\varepsilon,j,\gamma}(x)|^2 \, M[\chi_e](x)^{\frac{4}{5}} \, dx < \infty. \end{aligned}$$

This is the desired result.

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