Functiones et Approximatio
40.2 (2009), 327–339

DIVISION ON A COMPLEX SPACE WITH ARBITRARY SINGULARITIES

Michał Jasiczak

Dedicated to Professor Bogdan Bojarski
on the occasion of his 75th birthday

Abstract: We study a division problem for holomorphic functions that vanish to sufficiently high order near the singularity of a singular complex space.

Keywords: Division problem, singular complex space.

1. Introduction

In [4] J. E. Fornaess, N. Øvrelid and S. Vassiliadou obtained existence result for $\overline{\partial}$-problem on a complex space with arbitrary singularity. The aim of this note is to show that the method used in *ibid.* (however, not the result itself) can be applied to obtain a solution to the division problem for holomorphic functions vanishing to high order near the singularity.

Before we present the results, we need to recall the setting. Namely, let X be a pure n-dimensional reduced Stein space, $A \supset X_{\text{sing}}$ a lower dimensional complex analytic subset with empty interior (we refer the Reader to [5] for background concerning Stein spaces). Let Ω be an open relatively compact Stein domain in X and $K = \widehat{\overline{\Omega}}$ be the holomorphic convex hull of the closure of Ω in X. K has a neighborhood basis of Oka-Weil domains in X and let $X_0 \subset X$ be such a neighborhood of K in X. Importantly, X_0 can be realized as a holomorphic subvariety of an open polydisk in $\mathbb{C}^N$ for some $N > 0$. Set $\Omega^* := \Omega \setminus A$. Observe that since Ω^* is embedded in the polydisk $\mathbb{P}^N \subset \mathbb{C}^N$, it can be equipped with the Hermitian metric, which is the restriction of the ambient space metric to Ω^*. This induces a norm $|\cdot|$ on $\Lambda\mathbb{C}T_z^*\Omega^*$ for $z \in \Omega^*$ and implies the existence of the volume element dV on Ω^*. Hence, for any $\Omega^{'} \subset \Omega$ and $N \in \mathbb{Z}$ we may define the following

2000 Mathematics Subject Classification: Primary 32B10, secondary: 32J25, 32W05, 14C30.

The author was supported in years 2007-2010 by The Ministry of Science and Higher Education, Poland, grant no. N N 201 2740 33 and in 2008 by the Foundation for Polish Science. The Author was supported in years 2008-2009 by the Foundation for Polish Science.

seminorms

$$\|u\|^2_{\Omega',N} := \int_{\Omega'} |u|^2 d_A^{-N} dV,$$
$$\|u\|^2_{\Omega,N} := \int_{\Omega} |u|^2 d_A^{-N} dV.$$

The symbol d_A stands for the distance to A. Our first result is the following theorem:

Theorem 1. *Let X, Ω be as above and assume that for each $\Omega' \subset\subset \Omega$ holomorphic functions $f_1, \ldots, f_m \in H(\Omega)$ satisfy the following condition*

$$\sup_{\Omega'} d_A^{\tilde{N}} \Big(\sum_{j=1}^{m} |f_j|^2 \Big)^{-1} < \infty \tag{1}$$

for some $\tilde{N} \in \mathbb{N}_0$.

For every $N_0 \geqslant 0$, there exists $N \geqslant 0$ such that if F is a holomorphic function in Ω with $\|F\|_{\Omega,N} < \infty$, then there exist functions $g_1, \ldots, g_m \in H(\Omega^*)$ such that $\|g_j\|_{\Omega',N_0} \leqslant C\|F\|_{\Omega,N}$ for any $\Omega' \subset\subset \Omega$ and

$$\sum_{j=1}^{m} f_j g_j = F \tag{2}$$

in Ω^*. The constant C depends on Ω', N, N_0 and $f_1, \ldots, f_m$.

Theorem 1 is proved by adapting the Koszul complex technique (cf. [8]) to sheaf cohomology argument based on a generalization of the result proved by Y. T. Siu in [10]. The result, which generalizes to lower order sheaf cohomology groups Theorem obtained by Y. T. Siu was proved by J. E. Fornaess, N. Øvrelid and S. Vassiliadou in [4].

Theorem 1 implies immediately the following fact.

Corollary 1. *Let X, Ω be as above and assume that for each $\Omega' \subset\subset \Omega$ functions $f_1, \ldots, f_m \in H(\Omega)$ satisfy condition (1). Furthermore, assume that X is normal.*

For every $N_0 \geqslant 0$, there exists $N \geqslant 0$ such that if F is a holomorphic function in Ω with $\|F\|_{\Omega,N} < \infty$, then there exist functions $g_1, \ldots, g_m \in H(\Omega)$, which satisfy the equation (2) and $\|g_j\|_{\Omega',N_0} \leqslant C\|F\|_{\Omega,N}, j = 1, \ldots, m$ for any $\Omega' \subset\subset \Omega$. The constant C depends on Ω', N, N_0 and $f_1, \ldots, f_m$.

Indeed, Corollary 1 is an immediate consequence of the first Riemann extension theorem, which holds on normal complex spaces (cf. [7]). Recall that a complex space X is normal at $x \in X$ if $\mathcal{O}_x$ is reduced and integrally closed in $\mathcal{M}_x$ – the field of germs of meromorphic functions at x. A complex space X is normal provided it is normal at each of its point. In particular if X is smooth, then x is normal.

The Authors in [4] were able to strengthen their result in case of isolated singularities. Namely, they proved that if $A \cap \bar{\Omega}$ is a finite subset of $\bar{\Omega}$ with $b\Omega \cap A = \emptyset$, then a weighted L^2 estimate on the whole Ω holds for the solution to the equation $\overline{\partial} u = f$.

Theorem 2 (Fornaess, Øvrelid, Vassiliadou). *Let X, Ω be as above and assume that $A \cap \bar{\Omega}$ is a finite subset of $\bar{\Omega}$ with $b\Omega \cap A = \emptyset$. Furthermore, assume that Ω is Stein and $\bar{\Omega}$ has a Stein neighbourhood.*

For each N_0 there exists N such that for every $\overline{\partial}$-closed (p,q)-form f with $\|f\|_{N,\Omega} < \infty$, there is a solution to $\overline{\partial} u = f$ such that $\|u\|_{\Omega,N_0} \leqslant c\|f\|_{\Omega,N}$ with a constant c independent of f.

This result can be used to obtain the following theorem:

Theorem 3. *Let X, Ω be as above. Assume additionally, that $A \cap \bar{\Omega}$ is a finite subset of Ω with $b\Omega \cap A = \emptyset$. Also, let Ω be Stein and assume that $\bar{\Omega}$ has a Stein neighbourhood.*

If $f_1, \dots, f_m \in H(\bar{\Omega})$ and there exist $\tilde{N}_1, \tilde{N}_2 \in \mathbb{Z}$ such that $\|f_j\|_{\tilde{N}_1,\Omega} < \infty$ for $j = 1, \dots, m$ and

$$\sup_{\Omega'} d_A^{\tilde{N}_2} \Big(\sum_{j=1}^{m} |f_j|^2 \Big)^{-1} < \infty \tag{3}$$

then for every N_0 there exists N such that for each F with $\|F\|_{\Omega,N} < \infty$ there exist $g_1, \dots, g_m$ such that (2) holds and

$$\|g_j\|_{\Omega,N_0} \leqslant C\|F\|_{\Omega,N}, \quad j = 1, \dots, m, \tag{4}$$

where C depends on N_0 only.

One comment is in order at this moment. Namely, in Theorem 3 we made the additional assumption that $f_1, \dots, f_m$ are holomorphic on $\bar{\Omega}$. The reason for this is, naturally, that we wanted to get rid of the impact of $b\Omega$ on solvability of the equation (2). Once we prove Theorem 1, The Reader will notice that Theorem 3 is an almost immediate consequence of Theorem 2. This is why we intend to present the proof of Theorem 1 only.

The division problem for holomorphic functions was studied extensively by many Authors. Among the manuscripts, which influenced our approach most, apart from [8], are also [1] and [2].

2. Proof of Theorem 1

There exists a proper, holomorphic surjection $\pi\colon \tilde{X} \to X$ with the following properties:

(i) $\tilde{X}$ is an n-dimensional complex manifold.
(ii) $\tilde{A} = \pi^{-1}(A)$ is a hypersurface in $\tilde{\Omega}$ with only normal crossing singularities.
(iii) $\pi\colon \tilde{X} \setminus \tilde{A} \to X \setminus A$ is a biholomorphism.

This follows from results proved in [3] and [6] – we refer the Reader to [4] for the corresponding argument.

Denote $\tilde{\Omega} := \pi^{-1}(\Omega)$. Following [4] we equip the complex manifold $\tilde{X}$ with a real analytic metric σ. The symbol $d\tilde{V}_{x,\sigma}$ (or $d\tilde{V}_{\sigma}$, or even $d\tilde{V}$) stands for the volume form for the metric σ at $x \in \tilde{X}$, while $d_{\tilde{A}}$ denotes the distance to the submanifold $\tilde{A}$, which corresponds to the metric σ. The choice of the metric σ induces also a norm on $\Lambda\mathbb{C}T^*_z\tilde{\Omega}$, $z \in \tilde{\Omega}$, which will be denoted by $|\cdot|_{z,\sigma}$, or simply $|\cdot|_z$, $|\cdot|_\sigma$.

We will use standard sheaf theoretical notation. Namely, let $\mathfrak{L}^{\mathrm{loc}}_{p,q}$ stand for the sheaf of locally square integrable measurable forms on $\tilde{X}$. Since, for each open set $U \subset \tilde{X}$ it holds $\mathfrak{L}^{\mathrm{loc}}_{p,q}(U) \subset \mathcal{D}'_{p,q}$, the operator $\overline{\partial}$ is well-defined on $\mathfrak{L}^{\mathrm{loc}}_{p,q}(U)$ in the sense of currents. Hence, we may consider its (maximal) domain

$$\mathrm{Dom}_{\overline{\partial}}(U) := \bigl\{u \in \mathfrak{L}^{\mathrm{loc}}_{p,q}(U) \colon \overline{\partial}u \in \mathfrak{L}^{\mathrm{loc}}_{p,q}(U)\bigr\}.$$

The symbol $\mathcal{L}_{p,q}$ stands for the sheaf $(\mathrm{Dom}_{\overline{\partial}}(U), r^U_V)$, where for any open $V \subset U$ the operator $r^U_V \colon \mathfrak{L}^{\mathrm{loc}}_{p,q}(U) \to \mathfrak{L}^{\mathrm{loc}}_{p,q}(V)$ is induced by restriction of forms defined on U to the set V. Let J stand for the ideal sheaf of $\tilde{A}$ in $\tilde{X}$ and Ω^p for the sheaf of holomorphic $(p,0)$-forms. We will consider the sheaf $J^k \cdot \mathcal{L}_{p,q}$. Recall that a germ of a differential form u belongs to $(J^k \cdot \mathcal{L}_{p,q})_x$, if it is locally of the form $h^k u_0$, where h generates J_x and $u_0 \in (\mathcal{L}_{p,q})_x$. The fact that $\tilde{A} = \pi^{-1}(A)$ is a hypersurface with only normal crossing singularities means that around each point $z \in \tilde{A}$ there are local holomorphic coordinates $(z_1, \ldots, z_n)$ in terms of which $\tilde{A}$ is given by $h(z) = z_1 \ldots z_m = 0$, where $1 \leqslant m \leqslant n$. This explains why J_x is a principal ideal.

We will repeatedly invoke the following fact, which was also used in [4] (cf. proof of Theorem 1.1 [4]). Namely, assume that u is a $\overline{\partial}$-closed differential form in $\tilde{\Omega} \setminus \tilde{A}$, which is locally square-integrable around each point $z \in \tilde{A}$. Then u extends to a $\overline{\partial}$-closed differential form in $\tilde{\Omega}$. Naturally, the extension is also locally square-integrable, since $\mathrm{int}\,\tilde{A} = \emptyset$. Also, the statement that u is $\overline{\partial}$-closed means that $\overline{\partial}u = 0$ in the sense of currents. When u is a holomorphic function, this is the first Riemann extension theorem.

The following Lemma was proved in [4].

Lemma 1 (Lemma 3.1 in [4]). *We have for $x \in \tilde{\Omega} \setminus \tilde{A}$ and $v \in \Lambda^r T_x(\tilde{\Omega})$*

$$c' d^t_{\tilde{A}}(x) \leqslant d_A(\pi(x)) \leqslant C' d_{\tilde{A}}(x),$$
$$c d^M_{\tilde{A}} |v|_{x,\sigma} \leqslant |\pi_* v|_{\pi(x)} \leqslant C|v|_{x,\sigma}$$

for some positive constants c', c, C', C, t, M, where c, C, M may depend on r.

For an r-form a in Ω^* set

$$|\pi^* a| := \max\bigl\{|\langle a_{\pi(x)}, \pi_* v\rangle| \colon |v|_{x,\sigma} \leqslant 1, v \in \Lambda^r T_x(\tilde{\Omega} \setminus \tilde{A})\bigr\},$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the pairing between an r-forms and a r-tangent vectors.

This implies

$$cd_{\tilde{A}}^{M}(x)|a|_{\pi(x)} \leqslant |\pi^* a|_{x,\sigma} \leqslant C|a|_{\pi(x)}$$

on $\tilde{\Omega}$*, for some constant* M*.*

The following estimates, or rather their versions for $(0,q)$-forms with $q > 0$, were used in [4].

Lemma 2. *Let* Ω*,* $\tilde{\Omega}$*,* A*,* $\tilde{A}$ *be as above.*

(i) *Assume that* F *is a function in* Ω^**. There exist constants* $M_1, c > 0$ *such that*

$$\int_{\tilde{\Omega}\setminus\tilde{A}} |F \circ \pi|^2 d_{\tilde{A}}^{M_1 - N} d\tilde{V} \leqslant c\|F\|_{N,\Omega}^2.$$

(ii) *Assume that* g *is a* $\overline{\partial}$*-closed* (p,q)*-form on* $\tilde{\Omega}$*. There exists a natural number* $M_2 \in \mathbb{N}$ *such that if for some* $N_1 \geqslant 0$

$$\int_{\tilde{\Omega}} |g|_\sigma^2 d_{\tilde{A}}^{-N_1} d\tilde{V}_\sigma < \infty,$$

then $g \in J^l \mathcal{L}_{p,q}(\tilde{\Omega})$ *provided* $l \leqslant \frac{N_1}{2M_2}$*.*

(iii) *For any* N_0 *there exists* $M_3 \in \mathbb{N}$ *such that for any* $\Omega' \subset\subset \Omega$ *there is a constant* $c > 0$ *such that for any function* h *on* $\tilde{\Omega}$

$$\int_{\Omega'} |h \circ \pi^{-1}|^2 d_A^{-N_0} dV \leqslant c \int_{\tilde{\Omega}'\setminus\tilde{A}} |h|^2 d_{\tilde{A}}^{-M_3} d\tilde{V}_\sigma,$$

where $\tilde{\Omega}' := \pi^{-1}(\Omega')$*.*

(iv) *If* $v \in J^k \cdot \mathcal{L}_{p,q}(\tilde{\Omega})$*, then for each* $\tilde{\Omega}' \subset\subset \tilde{\Omega}$

$$\int_{\tilde{\Omega}'} |v|_\sigma^2 d_{\tilde{A}}^{-2k} d\tilde{V}_\sigma < \infty.$$

Proof. In particular Lemma 1 implies that there exist c, C, M such that for $x \in \Omega \setminus A$

$$cd_{\tilde{A}}^{M} d\tilde{V}_{x,\sigma} \leqslant (\pi^* dV)_x \leqslant C_1 d\tilde{V}_{x,\sigma}.$$

This is the key fact, which suffices to prove (i) and (iii). Property (iv) is obvious. We sketch part (ii), which is not proved in [4]. Recall first the Łojasiewicz inequalities (cf. [9]). Assume that ϕ is a real valued, real analytic function defined in an open set $V \subset \mathbb{R}^d$ and let $Z_\phi = \{x \in V \colon \phi(x) = 0\}$. Then, for every compact set $K \subset V$, there exist positive constants c, m such that

$$|f(x)| \geqslant cd(x, Z_\phi)^m, \tag{5}$$

where $d(\cdot, Z_\phi)$ stands for the distance to Z_ϕ.

Recall that the fact that $\tilde{A} = \pi^{-1}(A)$ is a hypersurface with only normal crossing singularities means that around each point $z \in \tilde{A}$ there are local holomorphic coordinates $(z_1, \dots, z_n)$ in terms of which $\tilde{A}$ is given by $h(z) = z_1 \dots z_m = 0$, where $1 \leqslant m \leqslant n$. Choose a cover (V_α) of $\tilde{A}$ consisting of such charts and let h_α be the corresponding functions, which locally define $\tilde{A}$. Since $\bar{\Omega}$ is a compact subset of X and π is proper, the set $\pi^{-1}(\bar{\Omega})$ is compact. Since an analytic set is closed, the intersection $\tilde{A} \cap \bar{\tilde{\Omega}}$ is compact as a closed subset of a compact set.

Therefore, among (V_α) there exist charts $V_{\alpha_1}, \dots, V_{\alpha_\nu}$ such that

$$\tilde{A} \cap \tilde{\Omega} \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_\nu}.$$

Furthermore, we may assume that there exist sets K_{α_i}, $i = 1, \dots, \nu$ compactly contained in V_{α_i} such that

$$\tilde{A} \cap \tilde{\Omega} \subset \operatorname{int} K_{\alpha_1} \cup \cdots \cup \operatorname{int} K_{\alpha_\nu}.$$

In view of the Łojasiewicz inequality (5), there exist positive numbers m_i, c_i, $i = 1, \dots, \nu$ such that $|h_{\alpha_i}(z)| \geqslant c_i d(z, Z_{h_{\alpha_i}})^{m_i}$ for $z \in K_{\alpha_i}$, $i = 1, \dots, \nu$.

In order to complete the proof that $g \in J^l \cdot \mathcal{L}_{p,q}(\tilde{\Omega})$, it suffices to show that around each point $z \in \tilde{A}$ the form g may be represented as $h^l u_0$, where h generates J_z and $u_0 \in (\mathcal{L}_{p,q})_z$. This follows from the fact that in $K_{\alpha_i} \setminus \tilde{A}$ we may simply write

$$g = h^l_{\alpha_i} \frac{g}{h^l_{\alpha_i}},$$

Indeed, if we set $c := \max\{c_i^{-1} \colon i = 1, \dots, \nu\}$ and $M_1 := \max\{m_i \colon i = 1, \dots, \nu\}$, then

$$\begin{aligned} \int_{K_{\alpha_i}} \left| \frac{g}{h^l_{\alpha_i}} \right|^2_\sigma d\tilde{V}_\sigma &\leqslant c_i^{-1} \int_{K_{\alpha_i}} |g|^2_\sigma d_{\tilde{A}}^{-2lm_i} d\tilde{V}_\sigma \\ &\lesssim c \int_{\tilde{\Omega}} |g|^2_\sigma d_{\tilde{A}}^{-N_1} d\tilde{V}_\sigma, \end{aligned} \tag{6}$$

provided $2lM_1 \leqslant N_1$. Hence, $g/h^l_{\alpha_i}$ is well-defined as a square integrable form on $\operatorname{int} K_{\alpha_i}$, not only on $\operatorname{int} K_{\alpha_i} \setminus \tilde{A}$, since $\operatorname{int} \tilde{A}$ has empty interior. This argument implies also that $g/h^l_{\alpha_i}$, a priori defined on $\operatorname{int} K_{\alpha_i} \setminus \tilde{A}$, extends to a $\overline{\partial}$-closed form in $\operatorname{int} K_{\alpha_i}$. Hence, it belongs to $\mathcal{L}_{p,q}(\operatorname{int} K_{\alpha_i})$ but this means precisely that $g \in J^l \cdot \mathcal{L}_{p,q}(\tilde{\Omega})$. ■

Denote by Λ the exterior algebra over $\mathbb{C}$ generated by $e_1, \dots, e_m$ and by Λ_l its subspace spanned by $e^I := e_{i_1} \wedge \cdots \wedge e_{i_l}$ with $I = (i_1, \dots, i_l)$. The assumption that e^I with $I = \{e_{i_1} < \cdots < e_{i_{|I|}}\} \subset \{1, \dots, m\}$ are orthonormal turns Λ into 2^m-dimensional Hilbert space. For any given $\mathbf{e} \in \Lambda$ we use the symbol $\mathbf{e} \vee \bullet \colon \Lambda \to \Lambda$ in order to denote the adjoint operator, in the Hilbert space sense, of right multiplication in Λ by $\mathbf{e}$.

For each $k \in \mathbb{N}_0$ we consider now the sheaf $\left(J^k \cdot \mathcal{L}\right) \otimes_{\mathbb{C}} \Lambda$ of $\mathcal{E}_{\tilde{X}} \otimes_{\mathbb{C}} \Lambda$-modules and for each (p, q) and $0 \leqslant l \leqslant m$ its subsheaf of linear spaces $\left(J^k \cdot \mathcal{L}_{p,q}\right) \otimes_{\mathbb{C}} \Lambda_l$.

The operator $\overline{\partial}$ is extended in a canonical way to a $\mathcal{O}_{\tilde{X}} \otimes_{\mathbb{C}} \Lambda$-sheaf homomorphism between $\mathcal{L}_{p,q} \otimes_{\mathbb{C}} \Lambda_l$ and $\mathcal{L}_{p,q+1} \otimes_{\mathbb{C}} \Lambda_l$. The latter statement means that for each open set $V \subset \tilde{X}$, section $u_V \in \mathcal{L}_{p,q}(V)$ and $e \in \Lambda$

$$\overline{\partial}\big(u_V \otimes e\big) := \big(\overline{\partial} u_V\big) \otimes e.$$

Similarly, we extend the operation $\vee$ to $(J^k \cdot \mathcal{L}) \otimes_{\mathbb{C}} \Lambda$. Namely, let $\mathfrak{s}$ be a global section of $\mathfrak{L} \otimes \Lambda$, i.e.

$$\mathfrak{s} = \sum_I s_I \otimes e^I,$$

with $s^I \in \mathfrak{L}(\tilde{X})$. Furthermore, assume that for any open set $U \subset \tilde{X}$ and $s \in \mathfrak{L}(U)$ it holds $s_I \wedge s \in \mathfrak{L}(U)$. Then the adjoint of $s \wedge \bullet \colon \mathfrak{L}(U) \to \mathfrak{L}(U)$ is well-defined. As a consequence, we obtain the sheaf morphism $\mathfrak{s} \vee \bullet \colon \mathfrak{L} \otimes_{\mathbb{C}} \Lambda \to \mathfrak{L} \otimes_{\mathbb{C}} \Lambda$ once we set

$$\begin{aligned} \mathfrak{s} \vee \big(s_U \otimes e\big) &= \Big(\sum_I s_I \otimes e^I\Big) \vee \big(s_U \otimes e\big) \\ &:= \sum_I \big(s_I \vee s_U\big) \otimes \big(e^I \vee e\big). \end{aligned}$$

For any such $\mathfrak{s}$ the sheaf map $\big(\mathfrak{s} \vee \bullet\big) \colon \mathfrak{L} \otimes \Lambda \to \mathfrak{L} \otimes \Lambda$ is a $\mathcal{E} \otimes_{\mathbb{C}} 1$-morphism of sheaves. Let δ stand for sheaf morphism

$$\Big(\sum_{j=1}^{m} \tilde{f}_j \otimes e_j\Big) \vee \bullet,$$

where $\tilde{f}_j := f_j \circ \pi$. It follows from the definition that $\delta^2 = 0$ and the sheaf morphisms $\overline{\partial}$ and δ commute. Also, it is a consequence of the assumption that

$$\delta \colon J^k \cdot \mathcal{L}_{p,q} \otimes_{\mathbb{C}} \Lambda_l \to J^k \cdot \mathcal{L}_{p,q} \otimes_{\mathbb{C}} \Lambda_{l-1}$$

if $l \geqslant 1$ and $\delta|_{J^k \cdot \mathcal{L}_{p,q} \otimes_{\mathbb{C}} \Lambda_0} = 0$. Hence, we have for each $k \in \mathbb{N}_0$ the following commuting diagram of sheaf morphisms

$$
\begin{tikzcd}
 & 0 & 0 & 0 & 0 & \\
0 \arrow[r] & J^k\Omega^p \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & J^k\mathcal{L}_{p,0} \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & \cdots \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & J^k\mathcal{L}_{p,n} \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & 0 \\
0 \arrow[r] & J^k\Omega^p \otimes \Lambda_1 \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & J^k\mathcal{L}_{p,0} \otimes \Lambda_1 \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & \cdots \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & J^k\mathcal{L}_{p,n} \otimes \Lambda_1 \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & 0 \\
0 \arrow[r] & \cdots \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & \cdots \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & \cdots \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & \cdots \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & 0 \\
0 \arrow[r] & J^k\Omega \otimes \Lambda_l \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & J^k\mathcal{L}_{p,0} \otimes \Lambda_l \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & \cdots \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & J^k\mathcal{L}_{p,n} \otimes \Lambda_l \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & 0 \\
0 \arrow[r] & \cdots \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & \cdots \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & \cdots \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & \cdots \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & 0 \\
0 \arrow[r] & J^k\Omega^p \otimes \Lambda_m \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & J^k\mathcal{L}_{p,0} \otimes \Lambda_m \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & \cdots \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & J^k\mathcal{L}_{p,n} \otimes \Lambda_m \arrow[r, "\overline{\partial}"] \arrow[u, "\delta"'] & 0
\end{tikzcd}
$$

with exact rows. The latter statement is a consequence of the Poincaré lemma. If we denote this diagram by $\mathfrak{J}^k$, then the following inclusions hold

$$\mathfrak{J}^0 \hookleftarrow \mathfrak{J}^1 \hookleftarrow \ldots \hookleftarrow \mathfrak{J}^k \hookleftarrow \ldots$$

Furthermore, the corresponding row in the diagram is a fine resolution of $J^k\Omega^p \otimes \Lambda_l$. Therefore,

$$H^q(\tilde{\Omega}, J^k \cdot \Omega^p \otimes \Lambda_l) \cong \frac{\ker\bigl(\overline{\partial}\colon J^k \cdot \mathcal{L}_{p,q}(\tilde{\Omega}) \otimes \Lambda_l \to J^k \cdot \mathcal{L}_{p,q+1}(\tilde{\Omega}) \otimes \Lambda_l\bigr)}{\overline{\partial}\bigl(J^k \cdot \mathcal{L}_{p,q-1}(\tilde{\Omega}) \otimes \Lambda_l\bigr)}. \tag{7}$$

The key fact proved in [4], which we shall refer to, is the next Proposition. The case $q = n$ was proved by Y. T. Siu in [10].

Proposition 1 (Proposition 1.3 [4]). *For $q > 0$ and $k \geqslant 0$ given, there exists a natural number $l, l \geqslant k$ such that the map*

$$i_*\colon H^q(\tilde{\Omega}, J^l \cdot \Omega^p) \to H^q(\tilde{\Omega}, J^k \cdot \Omega^p)$$

induced by the inclusion $i\colon J^l \cdot \Omega^p \to J^k \cdot \Omega^p$, is the zero map.

Since

$$H^q(\tilde{\Omega}, J^k \cdot \Omega^p \otimes \Lambda_l) \cong H^q(\tilde{\Omega}, J^k \cdot \Omega^p) \otimes \Lambda_l$$

we have also that for each k and q there exists l such that the map

$$(i \otimes \mathrm{id})_*\colon H^q(\tilde{\Omega}, J^l \cdot \Omega^p \otimes \Lambda) \to H^q(\tilde{\Omega}, J^k \cdot \Omega^p \otimes \Lambda), \tag{8}$$

induced on the sheaf cohomology by $i \otimes \mathrm{id}$, is the zero map. Naturally, we will be concerned with the case $p = 0$, when Ω^p is just equal to $\mathcal{O}_{\tilde{\Omega}}$ – the sheaf of holomorphic functions on $\tilde{\Omega}$.

Define

$$\gamma = \sum_{j=1}^{m} \gamma_j e_j := \sum_{j=1}^{m} \frac{\overline{\tilde{f}}_j}{\sum_{i=1}^{m} |\tilde{f}_i|^2} e_j,$$

where as before $\tilde{f}_j := f_j \circ \pi$. Naturally, it follows from the assumptions that $\gamma \in \mathcal{E}(\tilde{\Omega} \setminus \tilde{A}) \otimes \Lambda_1$.

Naturally, the norm $|\cdot|_{z,\sigma}$ on $\Lambda \mathbb{C} T_z^* \tilde{\Omega}$ can be extended in a canonical way to a norm on $\Lambda \mathbb{C} T_z^* \tilde{\Omega} \otimes \Lambda \cong \left(\Lambda \mathbb{C} T^* \tilde{\Omega} \otimes \Lambda\right)_z$. Namely, one sets

$$\Big|\sum_I u_I e^I\Big|_{z,\sigma}^2 := \sum_I \big|u_I\big|_{z,\sigma}^2.$$

Lemma 3. *Denote $\tilde{F} := F \circ \pi$. Under assumption* (1) *for each $M \in \mathbb{N}_0$ and $\tilde{k} \in \mathbb{N}$ there exists k such that if for each $\tilde{\Omega}' \subset\subset \tilde{\Omega}$*

$$\int_{\tilde{\Omega}'} |\tilde{F}|^2 d_{\tilde{A}}^{-k} d\tilde{V}_\sigma < \infty,$$

then for each $\tilde{\Omega}' \subset\subset \tilde{\Omega}$

$$\int_{\tilde{\Omega}'} \big|\gamma \wedge \left(\overline{\partial}\gamma\right)^M \wedge (\tilde{F} \otimes 1)\big|_\sigma^2 d_{\tilde{A}}^{-\tilde{k}} d\tilde{V}_\sigma < \infty. \tag{9}$$

Before we prove this fact notice that in (9) we may integrate over $\tilde{\Omega}'$ since $\tilde{A}$ has empty interior.

Proof. Functions $\tilde{f}_1, \dots, \tilde{f}_m$ are holomorphic in $\tilde{\Omega}$ as composition of holomorphic maps. A holomorphic function is continuous and, hence, locally bounded and locally square integrable. Also, it follows from (1) that there exists $t \in \mathbb{R}_{>0}$ such that for each $\tilde{\Omega}' \subset\subset \tilde{\Omega}$ there exists $C_{\tilde{\Omega}'}$ such that for $z \in \tilde{\Omega}'$

$$\Big(\sum_{j=1}^{m} |\tilde{f}_j(z)|^2\Big)^{-1} = \Big(\sum_{j=1}^{m} |f_j(\pi(z))|^2\Big)^{-1} \leqslant C_{\tilde{\Omega}'} d_A^{-\tilde{N}}(\pi(z)) \leqslant C' C_{\tilde{\Omega}'} d_{\tilde{A}}^{-t\tilde{N}}(z).$$

The last estimate is proved as Lemma 3.1 in [4] and is a consequence of the Łojasiewicz inequality (we recalled it above as Lemma 1). This implies that there exists $n_M \in \mathbb{N}$ such that for each $\tilde{\Omega}' \subset\subset \tilde{\Omega}$ there exists a constant $C_{\tilde{\Omega}'}$ such that for each $z \in \tilde{\Omega}' \setminus \tilde{A}$

$$\big|\gamma \wedge \left(\overline{\partial}\gamma\right)^M\big|_{z,\sigma}^2 \leqslant C_{\tilde{\Omega}'} d_{\tilde{A}}^{-n_M}(z).$$

Observe that $n_M \leqslant n_{M+1}$. Naturally,

$$\int_{\tilde{\Omega}'} \big|\gamma \wedge \left(\overline{\partial}\gamma\right)^M \wedge (\tilde{F} \otimes 1)\big|_\sigma^2 d_{\tilde{A}}^{-\tilde{k}} d\tilde{V} \leqslant C_{\tilde{\Omega}'} \int_{\tilde{\Omega}'} |\tilde{F}|^2 d_{\tilde{A}}^{-\tilde{k}-n_M} d\tilde{V},$$

which completes the proof if we simply put $k = n_M + \tilde{k}$. ■

Fix N_0 and define $\tilde{F} := F \circ \pi$, where $F \in H(\Omega)$ is the function on the right-hand side of (2). There exists $M \in \mathbb{N}$ such that $\gamma \wedge (\overline{\partial}\gamma)^M \wedge (\tilde{F} \otimes 1)$ is $\overline{\partial}$-closed in $\pi^{-1}(\Omega^*)$. Notice that

$$\overline{\partial}\Big(\sum_I u_I e^I\Big) = 0 \Longleftrightarrow \forall_I\ \overline{\partial} u_I = 0.$$

Hence, if for each $\tilde{\Omega}' \subset\subset \tilde{\Omega}$

$$\int_{\tilde{\Omega}'} \big|\gamma \wedge \big(\overline{\partial}\gamma\big)^M \wedge (\tilde{F} \otimes 1)\big|_\sigma d\tilde{V} < \infty,$$

then we may treat $\gamma \wedge \big(\overline{\partial}\gamma\big)^M \wedge (\tilde{F} \otimes 1)$ as $\overline{\partial}$-closed in the sense of currents in $\tilde{\Omega}$, not only in $\pi^{-1}(\Omega^*)$. Since F is a function it holds $M \leqslant n$ and $M + 1 \leqslant m$. We recall the Reader at this moment that n is the dimension of the manifold and m is the number of functions in (2). The fact that $M \leqslant n$ is obvious, while the second inequality follows from the fact $\tilde{f}_1\gamma_1 + \cdots + \tilde{f}_m\gamma_m = 1$ in $\pi^{-1}(\Omega^*)$. This means that in $\tilde{\Omega} \setminus \tilde{A}$

$$\tilde{f}_1\overline{\partial}\gamma_1 + \cdots + f_m\overline{\partial}\gamma_m = 0$$

and, consequently, in this set $\overline{\partial}\gamma_1 \wedge \cdots \wedge \overline{\partial}\gamma_m = 0$. This implies, under a suitable assumption concerning order of vanishing of F, that $\overline{\partial}\gamma_1 \wedge \cdots \wedge \overline{\partial}\gamma_m = 0$ in $\tilde{\Omega}$. We may, therefore, assume that $M = \min\{n, m-1\}$.

Assume that we managed to solve the equation

$$\overline{\partial} v_M = \gamma \wedge \big(\overline{\partial}\gamma\big)^M \wedge (\tilde{F} \otimes 1). \tag{10}$$

Then, for a fixed $k \in \mathbb{N}$

$$\begin{aligned}\int_{\tilde{\Omega}'} \big|\gamma \wedge \big(\overline{\partial}\gamma\big)^{M-1} &\wedge (\tilde{F} \otimes 1) - \delta v_M\big|_\sigma d_{\tilde{A}}^{-2M_2k}\tilde{V}_\sigma \\ &\leqslant C_{\tilde{\Omega}'} \int_{\tilde{\Omega}'} |F|^2 d_{\tilde{A}}^{-2M_2k - n_{M-1}} d\tilde{V}_\sigma + C_{\tilde{\Omega}'} \int_{\tilde{\Omega}'} |v_M|_\sigma^2 d_{\tilde{A}}^{-2M_2k} d\tilde{V}_\sigma.\end{aligned} \tag{11}$$

It is a consequence of Proposition 1 that for $\tilde{k} := 2M_2k$ there exists $l = l(n, \tilde{k})$ such that

$$(i \otimes \mathrm{id})_*\colon H^n(\tilde{\Omega}, J^l \cdot \Omega^p \otimes \Lambda_{n+1}) \to H^n(\tilde{\Omega}, J^{\tilde{k}} \cdot \Omega^p \otimes \Lambda_{n+1}) \tag{12}$$

is the zero map. Furthermore, as we have already noticed

$$H^q(\tilde{\Omega}, J^k \cdot \Omega^p \otimes \Lambda_l) \cong \frac{\ker\big(\overline{\partial}\colon J^k \cdot \mathcal{L}_{p,q}(\tilde{\Omega}) \otimes \Lambda_l \to J^k \cdot \mathcal{L}_{p,q+1}(\tilde{\Omega}) \otimes \Lambda_l\big)}{\overline{\partial}\big(J^k \cdot \mathcal{L}_{p,q-1}(\tilde{\Omega}) \otimes \Lambda_l\big)}.$$

From Lemma 2 it follows that there exists $\kappa_1 = \kappa_1(M, \tilde{k})$ such that if for each $\tilde{\Omega}' \subset\subset \tilde{\Omega}$

$$\int_{\tilde{\Omega}'} |\tilde{F}|^2 d_{\tilde{A}}^{-\kappa_1} d\tilde{V} < \infty, \tag{13}$$

then for each $\tilde{\Omega}' \subset\subset \tilde{\Omega}$

$$\int_{\tilde{\Omega}'} \left|\gamma \wedge \left(\overline{\partial}\gamma\right)^M \wedge (\tilde{F} \otimes 1)\right|_\sigma^2 d_{\tilde{A}}^{-2M_2 l(n,\tilde{k})} d\tilde{V} < \infty.$$

Lemma 2 implies now that $\gamma \wedge \left(\overline{\partial}\gamma\right)^M \wedge (\tilde{F} \otimes 1)$ is a $\overline{\partial}$-closed element of $J^{l(n,\tilde{k})} \cdot \mathcal{L}(\tilde{\Omega}) \otimes \Lambda$ and, as a consequence in view of Proposition 1 in concert with (7) and (8), there exists $v_M \in J^{\tilde{k}} \cdot \mathcal{L}(\tilde{\Omega}) \otimes \Lambda$ such that equation (10) is, indeed, satisfied in $\tilde{\Omega}$.

Consider now the expression $\gamma \wedge (\overline{\partial}\gamma)^{M-1} \wedge \tilde{F} - \delta v_M$ and observe that

$$\overline{\partial}\big[\gamma \wedge (\overline{\partial}\gamma)^{M-1} \wedge \tilde{F} - \delta v_M\big] = (\overline{\partial}\gamma)^M \wedge \tilde{F} - \delta\overline{\partial} v_M = 0, \tag{14}$$

since $\delta\big[\gamma \wedge (\overline{\partial}\gamma)^M \wedge F\big] = (\overline{\partial}\gamma)^M \wedge F$.

In order to sum up the argument set

$$\tilde{\kappa}_2(M,k) := \max\{2M_2k, \kappa_1(M, 2M_2k)\}.$$

We have shown so far that if for any $\tilde{\Omega}' \subset\subset \tilde{\Omega}$

$$\int_{\tilde{\Omega}'} |\tilde{F}|^2 d_{\tilde{A}}^{-\tilde{\kappa}_2(M,k)} d\tilde{V}_\sigma < \infty,$$

then for each $\tilde{\Omega}' \subset\subset \tilde{\Omega}$

$$\int_{\tilde{\Omega}'} \left|\gamma \wedge \left(\overline{\partial}\gamma\right)^{M-1} \wedge (\tilde{F} \otimes 1) - \delta v_M\right|_\sigma d_{\tilde{A}}^{-2M_2k} \tilde{V}_\sigma < \infty.$$

Lemma 2 implies now, in view of (14), that

$$\gamma \wedge \left(\overline{\partial}\gamma\right)^{M-1} \wedge (\tilde{F} \otimes 1) - \delta v_M \in J^k \cdot \mathcal{L}(\tilde{\Omega}) \otimes \Lambda.$$

Lemma 4. *For each $k \in \mathbb{N}$ there exists $\kappa_2(k) \in \mathbb{N}$ such that if F is a holomorphic function in Ω with*

$$\int_{\Omega^*} |F|^2 d_A^{-\kappa_2} dV < \infty$$

then, there exist $v_M, \ldots, v_1 \in \mathcal{L}(\tilde{\Omega}) \otimes \Lambda$ such that

- *(i)* $\gamma \wedge (\tilde{F} \otimes 1) - \delta v_1 \in J^k \cdot \mathcal{L}(\tilde{\Omega}) \otimes \Lambda$
- *(ii)* $\overline{\partial} v_{M-j} = \gamma \wedge \left(\overline{\partial}\gamma\right)^{M-j} \wedge (\tilde{F} \otimes 1) - \delta v_{M-j+1}$, $j = 0, \ldots, M-1$, *where* $\tilde{F} = F \circ \pi$ *and we put* $v_{M+1} = 0$.

Furthermore,

$$\begin{aligned} \delta\big[\gamma \wedge (\tilde{F} \otimes 1) - \delta v_1\big] &= \tilde{F}, \\ \overline{\partial}\big[\gamma \wedge (\tilde{F} \otimes 1) - \delta v_1\big] &= 0. \end{aligned} \tag{15}$$

Proof. Set $v_{M+1}=0$ and consider the following property:

For a fixed $k\in\mathbb{N}$ and $i=0,\dots,M-1$ there exists $\kappa_3=\kappa_3(k,i)$ such that if for each $\tilde{\Omega}^{'}\subset\subset\tilde{\Omega}$

$$\int_{\tilde{\Omega}'}|\tilde{F}|^2 d_{\tilde{A}}^{-\kappa_3}d\tilde{V}<\infty,$$

then there exist $v_M,\dots,v_{M-j}\in\mathcal{L}(\tilde{\Omega})\otimes\Lambda$ such that

(i) $\gamma\wedge\left(\overline{\partial}\gamma\right)^{M-i-1}\wedge(\tilde{F}\otimes 1)-\delta v_{M-i}\in J^k\cdot\mathcal{L}(\tilde{\Omega})\otimes\Lambda$,

(ii) $\overline{\partial}v_{M-j}=\gamma\wedge\left(\overline{\partial}\gamma\right)^{M-j}\wedge(\tilde{F}\otimes 1)-\delta v_{M-j+1}$, where $j=0,\dots,i$.

Denote this property by $\mathfrak{S}(k,i)$. We have already proved that the property $\mathfrak{S}(k,0)$ holds for each $k\in\mathbb{N}$.

Fix $\hat{k}\in\mathbb{N}$. Notice that $\gamma\wedge\left(\overline{\partial}\gamma\right)^{M-i}\wedge(\tilde{F}\otimes 1)-\delta v_{M-i+1}$ is $\overline{\partial}$-closed in $\tilde{\Omega}$. Therefore, it is a consequence of Proposition 1, (7) and (8), that there exists $l=l(\hat{k})$ such that a solution to the equation

$$\overline{\partial}v_{M-i}=\gamma\wedge\left(\overline{\partial}\gamma\right)^{M-i}\wedge(\tilde{F}\otimes 1)-\delta v_{M-i+1}$$

exists in $J^{\hat{k}}\cdot\mathcal{L}(\tilde{\Omega})\otimes\Lambda$ if $\gamma\wedge\left(\overline{\partial}\gamma\right)^{M-i}\wedge(\tilde{F}\otimes 1)-\delta v_{M-i+1}\in J^{l(\hat{k})}\cdot\mathcal{L}(\tilde{\Omega})\otimes\Lambda$. Hence, for the fixed $\hat{k}\in\mathbb{N}$ if $\mathfrak{S}(k,i)$ folds for each $k\in\mathbb{N}$, then the property $\mathfrak{S}(\hat{k},i+1)$ holds true as well. This completes the induction argument, which allows us to infer that the property $\mathfrak{S}(k,i)$ holds for each $k\in\mathbb{N}$ and $i=0,\dots,M-1$.

In particular, property $\mathfrak{S}(k,M-1)$ holds true. According to Lemma 2, there exists $M_1\in\mathbb{N}$ and $c>0$ such that for each $N\in\mathbb{N}$

$$\int_{\tilde{\Omega}}|\tilde{F}|^2 d_{\tilde{A}}^{M_1-N}d\tilde{V}\leqslant c\int_{\Omega}|F|^2 d_A^{-N}dV.$$

Therefore, it suffices to define $\kappa_2(k):=\kappa_3(k,M-1)+M_1$. One easily checks that equations (15) are also satisfied. ∎

Proof of Theorem 1. Fix N_0 as in Theorem 1. We intend to show that there exists a natural number N and functions $g_1,\dots,g_m\in H(\Omega^*)$ with $\|g_j\|_{\Omega',N_0}\leqslant C\|F\|_{\Omega,N}$ for any $\Omega^{'}\subset\subset\Omega$ such that

$$\sum_{j=1}^m f_j g_j=F$$

in Ω^*. First choose M_3 for N_0 according to (*iii*) of Lemma 2 and let $k=\left\lceil\frac{M_3}{2}\right\rceil$. It follows from Lemma 4 that if

$$\int_{\Omega^*}|F|^2 d_A^{-\kappa_2(k)}dV<\infty$$

then there exists v_1 such that

$$\sum_{j=1}^m\tilde{g}_j e_j:=\gamma\wedge(\tilde{F}\otimes 1)-\delta v_1$$

belongs to $J^k \cdot \mathcal{L}(\tilde{\Omega}) \otimes \Lambda$ and satisfies (15). Set $g_j := \tilde{g}_j \circ \pi^{-1}$ and notice that Lemma 2 implies that for any $\Omega' \subset\subset \Omega$ it holds

$$\int_{\Omega'} |g_j|^2 d_A^{-N_0} dV < \infty \tag{16}$$

for $j = 1, \ldots, m$. Obviously, functions g_j are holomorphic in Ω^* and in Ω^* satisfy the condition

$$\sum_{j=1}^{m} g_j f_j = \Big(\sum_{j=1}^{m} \tilde{g}_j \tilde{f}_j\Big) \circ \pi^{-1} = \tilde{F} \circ \pi^{-1} = F.$$

This completes the proof with $N := \kappa_2\big(\lceil \frac{M_3(N_0)}{2} \rceil\big)$. ■

References

[1] M. Andersson, *Estimates of solutions of the H^p and BMOA corona problem*, Math. Ann. **316** (2000), 83–102.

[2] M. Andersson, *Integral representation with weights. II. Division and interpolation*, Math. Z. **254** (2006), 315–332.

[3] E. Bierstone, P. Milman, *Canonical desingularization in characteristic zero by blowing-up the maximum strata of a local invariant*, Invent. Math. **128** (1997), 207–302.

[4] J. E. Fornaess, N. Øvrelid, S. Vasiliadou, *Semiglobal results for $\overline{\partial}$ on a complex space with arbitrary singularities*, Proc. Amer. Math. Soc. **133** (2005), 2377–2386.

[5] H. Grauert, R. Remmert, *Theory of Stein spaces*, Translated from the German by Alan Huckleberry. Reprint of the 1979 translation. Classics in Mathematics. Springer-Verlag, Berlin, 2004.

[6] H. Grauert, *Über Modificationen und exzeptionelle analytische Mengen*, Math. Ann. **146** (1962), 331–368.

[7] G.-M. Greuel, C. Lossen, E. Shustin, *Introduction to singularities and deformations*, Springer Monographs in Mathematics. Springer, Berlin, 2007.

[8] L. Hörmander, *Generators for some rings of analytic functions*, Bull. Amer. Math. Soc. **73** (1967), 943–949.

[9] S. Łojasiewicz, *Sur le probléme de la division*, Studia Math. **8** (1959), 87–136.

[10] Y. T. Siu, *Analytic sheaf cohomology groups of dimension n of n-dimensional non-compact complex manifolds*, Pacific J. Math. **28** (1969), 407–411.

Address: M. Jasiczak: Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warszawa, Poland;

E-mail: mjk@amu.edu.pl

Received: 11 January 2009; **revised:** 27 March 2009