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TWO EXCEPTIONAL CLASSES OF REAL NUMBERS

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Abstract: In a recent paper, Bugeaud and Dubickas have given an explicit characterisation of a rather remarkable class of transcendental numbers which are exceptional from the perspective of distribution of exponential sequences modulo 1. Much before, Helson and Kahane, from a completely different point-of-view had existentially exhibited another class of exceptional real numbers which conjecturally are either rational or transcendental. Wondering whether these two rather large class of real numbers overlap, we study their distribution functions and our investigation gives the first indication that these two interesting class of real numbers originating from different contexts are most likely different. We also frame a natural conjecture in this set up which would establish the above assertion. Our results can be regarded as the first step towards this conjecture.
Keywords: exponential sequences, distribution functions.

1. Introduction

Given any real number x, let $[x]$ denote its integral part while $\{x\}$ denote its fractional part.

Now given a sequence of real numbers (u_n) where $0 \leqslant u_n < 1$, for any interval I of length $l(I)$ contained in $[0,1]$, let $A(I,N)$ be the number of elements of (u_n) among its first N members which are in I, i.e.

$$A(I,N) = |\{n \mid n \leqslant N,\ u_n \in I\}|. \tag{1}$$

Then if for any such interval I, we have

$$\lim_{N\to\infty} \frac{A(I,N)}{N} = l(I),$$

we say that the given sequence (u_n) is *uniformly distributed*. We extend this notion for any arbitrary sequence of real numbers (u_n) by considering the sequence of their fractional parts $(\{u_n\})$ and we say that the sequence (u_n) is *uniformly distributed modulo 1* if the sequence of fractional parts $(\{u_n\})$ is uniformly distributed as defined above.

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We know that, for positive real numbers a and α , the sequence (an^{α}) is uniformly distributed modulo 1 if α is not an integer while the sequence $(\log n)$ is dense but not uniformly distributed modulo 1. Similarly, the sequence $(a \log^{\alpha} n)$ is uniformly distributed modulo 1 with $\alpha > 1$, it is dense but not uniformly distributed modulo 1 if $0 < \alpha \leqslant 1$ Finally, the sequence $(an^{\alpha} \log^{\beta} n)_{n>1}$ is uniformly distributed modulo 1 if α is not an integer and β is any positive real number, but the sequence $(a \log \log n)_{n>1}$ is not uniformly distributed modulo 1. We refer to [6] for an elaborated account of these results.

As we see, the sequences whose growth is slower than the sequence $(\log n)$ are not uniformly distributed modulo 1. On the other hand, sequences increasing faster than $(\log n)$, but not faster than a polynomial are uniformly distributed modulo 1.

However, almost nothing is known about the sequences whose growth is exponential. Let $\theta > 1$ and $\alpha \neq 0$ be real numbers. The distribution of the sequence (θ^n) or in general the sequence $(\alpha\theta^n)$ is one of the most intriguing questions in Number Theory about which the current state of knowledge is quite negligible. We refer to the survey article [1] for a more exhaustive report on the state-of-the-art. The algebraic nature of any real number x is encoded in its fractional part. There are very few tools to determine whether a given real number x is algebraic or transcendental. So a further motivation to study these exponential sequences of the form (θ^n) or in general the sequences $(\alpha\theta^n)$ apart from their intrinsic mystery is to explore the possibility of actually obtaining some information about the algebraic nature of θ and α by studying their distribution modulo 1, say from their set of limit points or whether they are dense modulo 1 or not (in other words, instead of studying the single number $\{x\}$, can we study the sequence $\{x^n\}$ and decipher $\{x\}$).

The distribution of exponential sequences of the form (θ^n) or $(\alpha\theta^n)$ modulo 1 however is quite enigmatic. As we have seen before, the rapidity of growth of a given sequence seems to play a crucial role in its distribution. So one would be tempted to expect, except for the degenerate cases, that the above exponential sequences would be uniformly distributed modulo 1. Interestingly, it is not so always. What is even more interesting is that the distribution of (θ^n) can be completely different from the distribution of $(\alpha\theta^n)$. The criteria of Weyl which involves exponential sums is not quite relevant in the context of studying the uniform distribution of these exponential sequences. So we do not have any general tool to check whether any given exponential sequence is uniformly distributed modulo 1 or not. Thus we do not know whether sequences as simple as e^n or $(3/2)^n$ are uniformly distributed modulo 1 or not. We do not even know whether they are dense modulo 1 in $[0, 1]$. But the following results which follows from a more general work of Koksma can be used to obtain some information about the distribution of these exponential sequences in general.

i) *Let $\theta > 1$ be a real number greater than 1; then the sequence $(\alpha\theta^n)$ is uniformly distributed modulo 1 for almost all real α.*

ii) *Let α be a non-zero real number ; then the sequence $(\alpha\theta^n)$ is uniformly distributed modulo 1 for almost all $\theta > 1$.*

Further, for almost all pairs (α, θ), $\theta > 1$, in the sense of planar Lebesgue measure, the sequence $(\alpha\theta^n)$ is uniformly distributed modulo 1 (see [6]). However it is quite paradoxical that we have very very few examples of pairs (α, θ) such that $(\alpha\theta^n)$ is uniformly distributed modulo 1, most of these examples are when θ is a natural number. The reason as we have mentioned before is that the very few existing criteria which we have for uniform distribution modulo 1 are not relevant in the context of such exponential sequences. Under these circumstances, it makes sense to turn in the other direction and try to study the numbers $\theta > 1$ or pairs (α, θ) for which the sequences (θ^n) or $(\alpha\theta^n)$ are not well distributed or even badly distributed. Let us refer to the real numbers θ or pairs (α, θ) for which the sequences (θ^n) or $(\alpha\theta^n)$ are not uniformly distributed modulo 1 as exceptional numbers or pairs. However, before setting out in pursuance of these exceptional numbers or pairs, it is worthwhile first to ensure that we actually have a non-trivial supply of such numbers/pairs, at least existentially. The fascinating results of Pollington are a confirmating pointer that these numbers/pairs which we hope to understand are actually plentiful. In a series of papers, Pollington studied these exceptional pairs extensively and computed their Hausdorff Dimension. In [9] [7], he proved the following theorem.

Theorem. *Given any non zero real number α and $0 < \delta < 1$, the set of real numbers $\theta > 1$ for which*

$$\{\alpha\theta^n\} \in [0, \delta] \qquad n = 1, 2, \cdots$$

has Hausdorff dimension 1.

In the other direction, Erdős [4] had asked whether given an increasing sequence of positive integers (n_k) such that the ratio $t_k = n_{k+1}/n_k > \rho > 1$, is it true that there always exists an irrational ξ for which the sequence $\{n_k\xi\}$ is not everywhere dense. This was settled by the following result of Pollington [8]:

Theorem. *If (x_k) is an increasing sequence of positive real numbers, such that $t_k = x_{k+1}/x_k > \rho > 1$, then the set of real numbers ξ such that $\{x_k\xi\}$ is not dense in the unit interval has Hausdorff dimension 1.*

Further, Pollington [9] also proved that the set of pairs (α, θ) with $\theta > 1$ for which $\{\alpha\theta^n\} \in [0, \delta]$ for all n has Hausdorff dimension 2.

In the light of the above theorems, we see that the sets of exceptional numbers θ and pairs (α, θ) for which the sequences (θ^n) or $(\alpha\theta^n)$ are not even dense modulo 1 are uncountable. Hence there exists transcendental numbers θ for which the sequence (θ^n) is not uniformly distributed modulo 1. Also it is possible to have pairs of transcendental numbers (α, θ) such that (θ^n) is uniformly distributed modulo 1, but $(\alpha\theta^n)$ is not even dense modulo 1. Interestingly, we are yet to find a transcendental number θ such that (θ^n) is not uniformly distributed modulo 1 although by the results of Pollington, there exists uncountably many such transcendental numbers.

In the present work, for a fixed θ, we study the distribution of the sequences $\{\alpha\theta^n\}$ for various real values α. It turns out that when $\theta = b$ is a natural number, this opens up some avenues for investigating the distribution of the sequences $\{\alpha b^n\}$. It is because the b-ary expansion of α tends to reflect or manifest in the distribution $\{\alpha b^n\}$. Let the $b-ary$ expansion of α be

$$\alpha = [\alpha] + \frac{a_1}{b} + \frac{a_1}{b^2} + \cdots$$

Then the above expansion, associates to α, an infinite word $\mathbf{w} = a_1 a_2 \cdots$ with letters in $\{0, 1, \cdots b-1\}$. A very interesting recent result of Bugeaud and Dubickas [3] describes, in terms of complexity of $\mathbf{w}$, all irrational $\alpha > 0$ with the property that $\{\alpha b^n\}$ for all n belong to a semi-open or an open interval of length $1/b$. For an infinite word $\mathbf{w}$, let $P_{\mathbf{w}}(m)$ be its complexity function which is the number of distinct finite words of length m which occur in $\mathbf{w}$. An infinite word $\mathbf{w}$ is called Sturmian if its complexity function satisfies $p_{\mathbf{w}}(m) = m + 1$ for every natural number m. Bugeaud and Dubickas [3] have proved the following:

Theorem A. *Let $b \geqslant 2$ be an integer and α be an irrational number. Then the numbers $\{\alpha b^n\}$, $n \geqslant 0$, cannot all lie in an interval of length strictly smaller than $1/b$. On the other hand, the numbers $\{\alpha b^n\}$, $n \geqslant 0$ are all lying in a closed interval of length I of length $1/b$ if and only if*

$$\alpha = g + k/(b-1) + t_b(\mathbf{w}),$$

where g is an arbitrary integer, k is in $\{0, 1, \cdots, b-2\}$, and $\mathbf{w}$ is a Sturmian word on $\{0, 1\}$. If this is the case, then α is transcendental.

In particular, since there are uncountably many Sturmian words on $\{0, 1\}$, by the above theorem we get an uncountable supply of pairs (α, b), where α is irrational and $s \in (0, 1 - 1/b)$, such that $s < \{\alpha b^n\} < s + 1/b$ for every $n \geqslant 0$. Here it is worthwhile to mention that if there exist numbers α such that all the $\{\alpha b^n\}$, $n \geqslant 0$ lie in a closed interval of length I of length $1/b$, then one would expect them to be transcendental. For, it is a widely believed almost folklore conjecture that every irrational algebraic number is normal in base b. We recall that a real number α is normal in base b if, for any positive integer n, each one of the b^n blocks of length n on the alphabet $\{0, 1, \cdots, b-1\}$ occurs in the b-adic expansion of α with the same frequency $1/b^n$. Since α is normal in base b implies $\{\alpha b^n\}$ is uniformly distributed, it is to be expected that the numbers with the property as in the above theorem, if they exist, would be transcendental. However the conjecture on algebraic irrational numbers being normal in base b for every b appears to belong to the realms of distant future. We even do not know whether the digit 7 occurs infinitely often in the decimal expansion of $\sqrt{2}$. We must mention the very recent outstanding work of Adamczewski and Bugeaud in this context. In their recent work, Adamczewski and Bugeaud have proved the following remarkable theorem which is most certainly the most significant step towards the elusive Borel conjecture:

Theorem. *Let $b \geqslant 2$ be an integer. Then the complexity of the b-adic expansion of every irrational algebraic number satisfies the property that $\liminf_{n\to\infty} \frac{p(n)}{n} = +\infty$. Thus in particular, algebraic irrationals can not have sub-linear complexity.*

Now, from a rather different view point, a particular version of a result in Helson and Kahane [5] states the following:

Theorem B. *There exists an uncountable set of real numbers α such that $\{\alpha b^n\}$ does not have an asymptotic distribution function. Further, this set has Hausdorff dimension 1.*

We note that in particular for all these α, $\{\alpha b^n\}$ is not uniformly distributed. Now conjecturally, all such α's should be rational or transcendental. Note that the family of numbers obtained by Bugeaud and Dubickas and those obtained by Helson and Kahane are exceptional from the perspective of the distribution of $\{\alpha b^n\}$. But while the method of Bugeaud and Dubickas is constructive, that of Helson and Kahane is existential. So we wondered about the possibility that both might have stumbled upon the same class of exceptional real numbers. This made us investigate the distribution functions of the sequences $\{\alpha b^n\}$ for a general α. Our investigation gives the first indication that the numbers obtained by Bugeaud and Dubickas constructively and those obtained by Helson and Kahane existentially are *most likely different*. In the next section, we detail the required generalities on distribution functions and prove our results which suggests the above eventuality.

2. Distribution Functions

One of the few tools which appears to hold some promise in the study of distribution of these elusive sequences $\{\xi\theta^n\}$ is the study of their distribution functions. In general, a *distribution function* $g(x)$ is just a real valued, non-decreasing function defined on $[0,1]$ with $g(0) = 0$ and $g(1) = 1$. Let us first clearly understand the notion of *distribution function of a sequence* in our context. From now onward for any real sequence (x_n), we will be looking at the sequence $(\{x_n\})$, so without loss of generality, all our sequences are assumed to be in $[0,1]$. Given any such sequence (x_n) , consider the following sequence of functions

$$f_N(x) =: \frac{|\{n \mid n \leqslant N,\ x_n \in [0,x)\}|}{N} \tag{2}$$

As we see, if the given sequence (x_n) is uniformly distributed, the above sequence of functions $(f_N(x))$ will converge to the function $I(x) = x$. In general, if these sequence of functions converge to a function $g(x)$, i.e. if :

$$\lim_{N\to\infty} \frac{|\{n \mid n \leqslant N,\ x_n \in [0,x)\}|}{N} = g(x), \qquad \forall x \in [0,1]$$

then the sequence (x_n) is said to have *the asymptotic distribution function g(x).* In general, however, these sequence of functions need not converge to any function, i.e. the sequence (x_n) may not have any asymptotic distribution function. But for any given x in $[0,1]$, the sequence $(f_N(x))$ being bounded $(0 \leqslant (f_N(x)) \leqslant 1)$, will have a convergent subsequence, where the subsequences will depend on the point x. However, if there exists a fixed subsequence which works for all x, i.e. if there exists an increasing sequence of natural numbers $N_1, N_2, \cdots$ such that

$$\lim_{i\to\infty} \frac{|\{n \mid n \leqslant N_i,\ x_n \in [0,x)\}|}{N_i} = g(x)\ , \qquad \forall x \in [0,1]$$

then $g(x)$ is called *a distribution function* of (x_n). If the above holds with $g(x) = x$. then the sequence (x_n) is called *almost uniformly distributed.* Clearly if $N_i = i$ for all i , then $g(x)$ is the asymptotic distribution function of (x_n). As mentioned before, a sequence in general, may not have an asymptotic distribution function. A little later we will give the example of such a sequence. However, every sequence (x_n) has at least one distribution function which follows from Helly's selection principle. Helly's principle asserts that any sequence of functions on $[0,1]$ which is uniformly bounded converges weakly to a monotonic function $g(x)$ (i.e. converges point wise at points where $g(x)$ is continuous).

We prove the following important theorem which gives the relation between distribution functions and asymptotic distribution functions.

Theorem 1. *The sequence (x_n) has the asymptotic distribution function $g(x)$ if and only if $g(x)$ is the only distribution function of (x_n).*

Proof. It is clear that if the sequence (x_n) has the asymptotic distribution function $g(x)$, then $g(x)$ is the only distribution function of (x_n). Conversely, suppose $g(x)$ is the only distribution function of (x_n). We claim that $g(x)$ is the asymptotic distribution function of (x_n). If not, then the sequence of functions

$$f_N(x) =: \frac{|\{n \mid n \leqslant N,\ x_n \in [0,x)\}|}{N} \tag{3}$$

do not converge to $g(x)$ for all $x \in [0,1]$. Hence there exists some $y \in [0,1]$ such that the sequence $(f_N(y))$ does not converge to $g(y)$. Since $(f_N(y))$ is bounded, it will have a subsequence, say $f_{N_1}(y), f_{N_2}(y), \cdots$ such that

$$\lim_{i\to\infty} f_{N_i}(y) = \gamma \neq g(y)$$

Now instead of the sequence of functions $(f_N(x))$, we use Helly's selection principle to the sequence of functions $(f_{N_i}(x))$ and get a distribution function say $h(x)$ for (x_n). But since $g(x)$ is the only distribution function of (x_n), $h(x) = g(x)$ for all $x \in [0,1]$. However $h(y) = \gamma \neq g(y)$. ■

A little reflection on the above proof shows that we have proved the following lemma:

Lemma 1. *If for some $y \in [0,1]$, there exists a subsequence (N_k) of natural numbers such that $\lim_{k\to\infty} f_{N_k}(y) = \beta$, then there exists a distribution function $g(x)$ of (x_n) such that $g(y) = \beta$.*

Using this and Theorem 1, we immediately see that the following sequence does not have an asymptotic distribution function: For any natural number n, let $l(n)$ be the unique integer such that $2^{l(n)} < n \leqslant 2^{l(n)+1}$. Then we define (x_n) to be equal to $1/3$ if $l(n)$ is even and equal to $1/4$ if $l(n)$ is odd.

Let $g(x)$ be a distribution function of (x_n). Let $\varphi : [0,1] \to [0,1]$ be any function. Then a distribution function of the altered sequence $(\varphi(x_n))$, in general, may not have any relation with $g(x)$. However, in certain cases, distribution functions of the sequences (x_n) and $(\varphi(x_n))$ are related as we see now. Let $\varphi : [0,1] \to [0,1]$ be such that for every $x \in [0,1]$, $\varphi^{-1}([0,x))$ is expressible as the union of finitely many disjoint subintervals $I_i(x)$ of $[0,1]$ with endpoints $\alpha_i(x) \leqslant \beta_i(x)$. For any distribution function $g(x)$ we put

$$g_\varphi(x) = \sum_i (g(\beta_i(x)) - g(\alpha_i(x))).$$

For any sequence $\Delta = \{x_n\}_{n=1}^\infty$, $x_n \in [0,1]$ and $\varphi : [0,1] \to [0,1]$ as above, if $\varphi(\Delta)$ denotes the sequence $\{\varphi(x_n)\}_{n=1}^\infty$, then we have the following theorem due to Strauch [10]:

Theorem (Strauch). *Let $g(x)$ be a distribution function of Δ associated with the sequence of indices $N_1, N_2, \cdots$. Suppose each term x_n is repeated only finitely many times. Then $\varphi(\Delta)$ has the distribution function g_φ for the same sequence of indices $N_1, N_2, \cdots$. Further, every distribution function of $\varphi(\Delta)$ has this form.*

Taking $\varphi(x) = \varphi_t(x) = \{tx\}$ with t a positive integer, we have the following theorem which will be important for us:

Theorem 2. *Every distribution function g of $\{\xi(p/q)^n\}$ with co-prime integers $p > q \geqslant 1$ satisfies $g_{\varphi_p}(x) = g_{\varphi_q}(x)$ for $x \in [0,1]$.*

Proof. We have $\{q\{x\}\} = \{qx\}$. Hence

$$\{q\{\xi(p/q)^n\}\} = \{\xi(p^n/q^{n-1})\} = \{p\xi(p/q)^{n-1}\} = \{p\{\xi(p/q)^{n-1}\}\}$$

Thus $\varphi_q(\{\xi(p/q)^n\})$ and $\varphi_p(\{\xi(p/q)^{n-1}\})$ form the same sequence and the conclusion follows by the above theorem. ∎

In the light of theorem 1, it is apparent that the more uniform is the distribution of a sequence (x_n) in the unit interval, the fewer will be its distribution functions. Hence the concept of *sets of uniqueness* of distribution functions is of relevance. Given a sequence (x_n) in $[0,1]$, a subset X of $[0,1]$ is said to be a set of uniqueness of the distribution functions of (x_n) if, for any two distribution functions $g_1(x)$ and $g_2(x)$ of (x_n), $g_1(x) = g_2(x)$ on X implies $g_1(x) = g_2(x)$ on $[0,1]$. Hence the distribution functions of (x_n) are determined by their values on X. While studying the distribution of a sequence (x_n), ideally one would like to have sets of uniqueness of smaller size/measure. It would seem that the smaller the measure of sets of uniqueness of distribution functions, the fewer would be the number of distribution functions and hence more uniform would be the distribution of the sequence. For instance, in the ideal scenario when the sequence (x_n) is uniformly distributed, any point in $[0,1]$ is a set of uniqueness. It is worthwhile to note that the sets of uniqueness of distribution functions of a sequence in general are not translation invariant. For instance, let us consider again the following sequence: For any natural number n, let $l(n)$ be the unique integer such that $2^{l(n)} < n \leqslant 2^{l(n)+1}$. Then we define (x_n) to be equal to $1/3$ if $l(n)$ is even and equal to $1/4$ if $l(n)$ is odd. It is clear that $X = [0,1/3]$ is a set of uniqueness. However neither $[1/3,2/3]$ nor $[2/3,1]$ is a set of uniqueness. For, any distribution function of the given sequence takes the vales 1 on either of these sets and hence if they were sets of uniqueness of the given sequence, the sequence will have only one distribution function. But as we have noted before, Lemma 1 implies that the given sequence has at least two distribution functions. We prove the following theorem:

Theorem 3. *Suppose g is a distribution function of $\{\alpha b^n\}$ with $b \in \mathbb{N}$. Then any interval $[a, a+\frac{b-1}{b}] \subset [0,1]$ of length $\frac{b-1}{b}$ is a set of uniqueness of g.*

Proof. By taking $\varphi(x) = \{x\}$ and $\varphi(x) = \{bx\}$ in Lemma 1, we see that any distribution function g of $\{\alpha b^n\}$ with $b \in \mathbb{N}$ satisfies

$$g(x) = \sum_{i=0}^{b-1} g\left(\frac{x+i}{b}\right) - \sum_{i=1}^{b-1} g\left(\frac{i}{b}\right) \tag{4}$$

Suppose $g(x)$ is known for $x \in [a, a+\frac{p-1}{p}]$. We have,

$$0 \leqslant a \leqslant \frac{1}{b} < \frac{b-1}{b} \leqslant a + \frac{b-1}{b}.$$

For all $x \in [0,1]$, since

$$\frac{i}{b} \leqslant \frac{x+i}{b} \leqslant \frac{i+1}{b},$$

$g(\frac{x+i}{b})$ is known for $1 \leqslant i \leqslant b-2$. Also the value of $g(\frac{i}{b})$ is known for all $i = 1, 2, \cdots, b-1$. Let

$$x \in A_1 := [a, ba].$$

Then, for such an x,

$$a \leqslant x \leqslant \frac{x+b-1}{b} \leqslant a + \frac{b-1}{b}.$$

Hence for $x \in [a, ba]$, both $g(x)$ and $g(\frac{x+b-1}{b})$ are known. Thus, for $x \in [a, ba]$, all the entries in (4) are known except for $g(\frac{x}{b})$. Hence using (4), $g(\frac{x}{b})$ gets known when $x \in [a, ba]$. But $x \in [a, ba]$ implies $\frac{x}{b} \in [\frac{a}{b}, a]$. Thus $g(x)$ is now known in the interval $B_1 := [\frac{a}{b}, a + \frac{b-1}{b}]$. Recursively, after n steps, taking $x \in A_n := [(\frac{1}{b})^{n-1}a, ba]$, $g(x)$ gets known for any x in the interval $B_n = [(\frac{1}{b})^n a, a + \frac{b-1}{b}]$. Since $(\frac{1}{b})^n a \to 0$ as $n \to \infty$, we see that by this process $g(x)$ gets known over the interval $[0, a + \frac{b-1}{b}]$. Now for all $x \in [0, a + \frac{b-1}{b}]$, all the entries in (4) are known except for $g(\frac{x+b-1}{b})$. Hence $g(\frac{x+b-1}{b})$ gets known for all $x \in [0, a + \frac{b-1}{b}]$. But $x \in [0, a + \frac{b-1}{b}]$ implies $\frac{x+b-1}{b} \in [\frac{b-1}{b}, \frac{a}{b} + \frac{b-1}{b} + \frac{b-1}{b^2}]$. Thus $g(x)$ gets known over the interval $[0, \frac{a}{b} + \frac{b-1}{b} + \frac{b-1}{b^2}]$. Recursively, after n steps, taking x in the interval $[0, \frac{a}{b^{n-1}} + \frac{b-1}{b} + \frac{b-1}{b^2} + \cdots + \frac{b-1}{b^n}]$, $g(x)$ gets known over the interval $[0, \frac{a}{b^n} + \frac{b-1}{b} + \frac{b-1}{b^2} + \cdots + \frac{b-1}{b^{n+1}}]$. Since $\frac{a}{b^n} + \frac{b-1}{b} + \frac{b-1}{b^2} + \cdots + \frac{b-1}{b^{n+1}} \to 1$ as $n \to \infty$, $g(x)$ gets known in $[0, 1]$. ■

Arguing along similar lines and using the methodology adopted in [2], we can prove the following the proof of which we omit:

Theorem 4. *Suppose g is a distribution function of $\{\alpha b^n\}$ with $b \in \mathbb{N}$. Then complement of any interval of the form $[j/b, j/b + 1/b] \subset [0, 1]$ where $j = 0, 1, \cdots, b-1$, is a set of uniqueness of g.*

3. Remarks and Consequences

We begin by remarking that for sequences of the form $\{\alpha b^n\}$ where b is a natural number, we have shown that any interval of length $(b-1)/b$ is a set of uniqueness. But we could not lower the interval size here . But in this context, result of Bugeaud and Dubickas shows that

$$\limsup_{n\to\infty}\{\alpha b^n\} - \liminf_{n\to\infty}\{\alpha b^n\} = \frac{1}{b}.$$

So we feel that exhibiting an interval of length less than $(b-1)/b$ (or proving their non-existence) which is a set of uniqueness for the sequences $\{\alpha b^n\}$ for all $\alpha > 0$ will be of considerable interest (and difficulty).

Determining the existence of asymptotic distribution function of sequences of the form $\{\alpha b^n\}$ in general is rather difficult. As we have mentioned before, result of Koksma establishes that for almost all α, $\{\alpha b^n\}$ is uniformly distributed in $[0, 1]$ and hence has the asymptotic distribution function $g(x) = x$. In the other direction, Theorem B due to Helson and Kahane established the existence of uncountably many α such that $\{\alpha b^n\}$ does not have an asymptotic distribution function. In their theorem A, Bugeaud and Dubickas constructed an uncountable family of α such that $\{\alpha b^n\}$ lie in an interval I of length $1/b$. Suppose the end points of such an interval I were of the form j/b and $j/b + 1/b$. Then note that any two distribution functions of such $\{\alpha b^n\}$ will agree in the complement of I. Then by our theorem 4, they will agree everywhere. Thus such a sequence has only one distribution function and hence by theorem 1 will necessarily have an asymptotic distribution function. But the sequences exhibited by Helson and Kahane are characterised by having no asymptotic distribution function and hence will be different. But it is remarkable (pointed out to the author by Bugeaud) that while it is not possible in general to describe the possible intervals of length $1/b$ where the sequences $\{\alpha b^n\}$ can lie, the end points are necessarily transcendental. So our Theorem 4 is not able to establish that the real numbers described in the theorems A and B are different. However, we note that the counterpart of a natural conjecture mentioned in [2] in our set up will suggest the following:

Conjecture. *Every measurable set $X \subset [0,1]$ with Lebesgue measure at least $(b-1)/b$ is a set of uniqueness of any distribution function of $\{\alpha b^n\}$ for any $\alpha > 0$.*

Our theorems 2 and 3 can be regarded as the first step towards this. Note that this will clearly prove that the real numbers exhibited by Bugeaud and Dubickas are different from those exhibited by Helson and Kahane. However, we suspect proving this in generality could be rather difficult. We believe it is more realistic to strive for a refinement of our Theorem 4 to prove that the complement of any interval of length $1/b$ is a set of uniqueness of distribution functions of $\{\alpha b^n\}$. This will be enough to establish that the two uncountable class real numbers constructed by Bugeaud and Dubickas and those exhibited by Helson and Kahane are necessarily different.

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