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THE UNRESTRICTED VARIANT OF WARING'S PROBLEM IN FUNCTION FIELDS

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Abstract: Let $\mathbb{J}_q^k[t]$ denote the additive closure of the set of kth powers in the polynomial ring $\mathbb{F}_q[t]$, defined over the finite field $\mathbb{F}_q$ having q elements. We show that when $s \geqslant k+1$ and $q \geqslant k^{2k+2}$, then every polynomial in $\mathbb{J}_q^k[t]$ is the sum of at most s kth powers of polynomials from $\mathbb{F}_q[t]$. When k is large and $s \geqslant (\frac{4}{3}+o(1))k \log k$, the same conclusion holds without restriction on q. Refinements are offered that depend on the characteristic of $\mathbb{F}_q$.
Keywords: Waring's problem, function fields.

1. Introduction

Investigations concerning Waring's problem in function fields have focused on two variants, a *restricted* problem in which the degrees of the polynomials employed in the representation are confined to be as small as is possible, and the corresponding *unrestricted* problem in which no such constraints are imposed. Let $\mathbb{F}_q[t]$ denote the polynomial ring defined over the finite field $\mathbb{F}_q$ having q elements, and, when k is a natural number, define $\mathbb{J}_q^k[t]$ to be the additive closure of the set of kth powers in $\mathbb{F}_q[t]$. In 1933, Paley [8] considered the unrestricted variant of Waring's problem, showing that a natural number s exists with the property that every polynomial in $\mathbb{J}_q^k[t]$ may be represented as the sum of s kth powers of polynomials from $\mathbb{F}_q[t]$. Let $w_q(k)$ denote the least permissible choice for such a number s. In this paper we make progress on bounds for $w_q(k)$ in two directions. On one hand, we apply estimates stemming from Deligne's resolution of the Weil conjectures so as to obtain sharp bounds valid when q is sufficiently large in terms of k. On the other hand, making use of the Hardy-Littlewood method, we derive weaker bounds valid uniformly in q.

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Before proceeding further, we require some notation. When k is a natural number and p is a prime number, we define the integer k_p as follows. We write k in base p, say

$$k = a_0 + a_1 p + \ldots + a_n p^n, \tag{1}$$

where $0 \leqslant a_i < p$ $(0 \leqslant i \leqslant n)$, and then put $k_p = \prod_{i=0}^{n}(a_i + 1) - 1$. It is apparent that $k_p \leqslant k$ for every k, and that $k_p = k$ if and only if $k < p$, or else $k = p^m - 1$ for some natural number m. In §2 we derive the bound for $w_q(k)$ recorded in the following theorem.

Theorem 1. *Let k be a natural number, and suppose that $\mathbb{F}_q$ is a finite field of characteristic p. Then whenever $q \geqslant k^{2k_p} k_p^2$, one has $w_q(k) \leqslant k_p + 1$.*

For comparison, Theorems 1(iii) and 4(iii) of Vaserstein [12] show that when $q \geqslant k^4$, and in addition q exceeds a certain Ramsey number defined in terms of k, then $w_q(k) \leqslant 3k_p/2$. In addition to providing a sharper bound for $w_q(k)$, the conclusion of Theorem 1 has the merit of replacing the potentially astronomical Ramsey number in the condition on q by an explicit function of k of terrestrial magnitude. Now observe that -1 is a sum of kth powers in $\mathbb{F}_q$ (consider $q-1$ copies of 1^k for example), and so the set of polynomials that are the sum of some finite number of terms of the form $\pm x^k$, with $x \in \mathbb{F}_q[t]$, is equal to $\mathbb{J}_q^k[t]$. Let $v_q(k)$ denote the least natural number s with the property that, whenever $m \in \mathbb{J}_q^k[t]$, then m is the sum of at most s such terms. Theorem 1(iii) of Vaserstein [12] shows that when q is larger than a certain Ramsey number defined in terms of k, then $v_q(k) \leqslant k_p$. For odd k one has $w_q(k) = v_q(k)$, and so the latter conclusion supercedes the upper bound on $w_k(q)$ provided by Theorem 1 for odd k, albeit with a potentially severe constraint on q. On the other hand, the trivial relation $v_q(k) \leqslant w_q(k)$ leads from Theorem 1 to the bound $v_q(k) \leqslant k_p + 1$, provided only that $q \geqslant k^{2k_p} k_p^2$. See [10] and [11] for further bounds on $v_q(k)$ and $w_q(k)$ valid for intermediate ranges of q.

When $g \in \mathbb{F}_q[t]$, let ord g denote the degree of g. We say that m admits a *strict representation* as a sum of s kth powers when, for some $x_i \in \mathbb{F}_q[t]$ with ord $x_i \leqslant \lceil (\text{ord } m)/k \rceil$ $(1 \leqslant i \leqslant s)$, one has $m = x_1^k + \ldots + x_s^k$. Here, as usual, we write $\lceil \theta \rceil$ for the least integer greater than or equal to θ. When k and q are natural numbers exceeding 1, define $G_q(k)$ to be the least integer s with the property that, whenever $m \in \mathbb{J}_q^k[t]$ has degree sufficiently large in terms of k and q, then m admits a strict representation as the sum of s kth powers. By reference to the argument underlying Theorem 1.4(ii) of Gallardo and Vaserstein [3], we obtain the following direct consequence of Theorem 1 in §3.

Corollary 2. *Let k be an integer exceeding 3, and suppose that $\mathbb{F}_q$ is a finite field of characteristic p. Then whenever $q \geqslant k^{2k_p} k_p^2$, one has $G_q(k) < k \log k + k_p - \frac{1}{2} \log k + 4$.*

For comparison, Theorem 1.4(ii) of [3] shows that when $q \geqslant k^4$, one has $G_q(k) \leqslant k \log(k+1) + 2k + 1$. The conclusion of Corollary 2 is modestly sharper at the expense of requiring q to be rather larger.

For smaller values of q, the most problematic cases are those wherein the characteristic p of $\mathbb{F}_q$ is smaller than k. By employing work of Kubota [5,6], the paper of Chinburg [1] comes closest to providing bounds uniform in q, though the focus is on $v_q(k)$ rather than $w_q(k)$. In §4 we apply our recent work [7] to establish a uniform bound on $w_q(k)$. Define the integer $A = A_q(k)$ as follows. Let k_0 be the largest divisor of k coprime to q. Write k in base p as in (1), take $\gamma = a_0 + a_1 + \ldots + a_n$, and then set $A = (1 - 2^{-\gamma})^{-1}$ when $p < k_0$, and $A = 1$ when $p > k_0$. Finally, when x is a positive number, write $\mathrm{Log}\, x$ for $\max\{1, \log x\}$, and put

$$\widehat{G}_q(k) = Ak_0(\mathrm{Log}\, k_0 + \mathrm{Log}\,\mathrm{Log}\, k_0 + 2 + A\mathrm{Log}\,\mathrm{Log}\, k_0/\mathrm{Log}\, k_0).$$

Theorem 3. *There is a positive absolute constant C with the property that whenever k is a natural number and $\mathbb{F}_q$ is a finite field, then $w_q(k) \leqslant \widehat{G}_q(k) + Ck_0\sqrt{\mathrm{Log}\,\mathrm{Log}\, k_0}/\mathrm{Log}\, k_0$.*

The conclusion of Theorem 3 implies that the bound $w_q(k) \leqslant (\frac{4}{3}+o(1))k \log k$ holds uniformly in k and q. For a specific exponent k and finite field $\mathbb{F}_q$, moreover, the algorithm associated with Theorem 14.2 of [7] provides an explicit upper bound for $w_q(k)$. We avoid providing the lengthy details of this algorithm in the interests of concision. We note also that since $v_q(k) \leqslant w_q(k)$, the bound supplied by Theorem 3 for $w_q(k)$ applies also to $v_q(k)$.

The authors are grateful to Professors Gallardo and Vaserstein for making available their preprint [3], without which the conclusion recorded in Corollary 2 could not have been presented.

2. Methods applicable for larger *q*

In order to bound $w_q(k)$ for larger q, we consider the polynomial equation

$$x_1^k(t + y_1)^k + \ldots + x_s^k(t + y_s)^k = at + b. \tag{2}$$

For suitable elements $a, b \in \mathbb{F}_q$, with a non-zero, we seek a solution $\mathbf{x}, \mathbf{y} \in \mathbb{F}_q^s$ of the equation (2). It transpires that when q is sufficiently large, such a solution may be shown to exist when s is taken to be $k_p + 1$, which we henceforth assume. Fix any such solution of (2), and consider a given polynomial $m(t) \in \mathbb{F}_q[t]$. A representation of $m(t)$ as the sum of s kth powers of elements of $\mathbb{F}_q[t]$ is obtained by replacing t by $a^{-1}(m(t) - b)$ in (2), and thereby we confirm that $w_q(k) \leqslant s$. Considering the coefficients of powers of t in (2), we derive a system of equations over $\mathbb{F}_q$ which we investigate by means of Deligne's resolution of the Weil conjectures.

The proof of Theorem 1. Let k and q be natural numbers satisfying the hypotheses of the statement of Theorem 1, and let p be the characteristic of $\mathbb{F}_q$. Plainly, there is nothing to prove when $k = 1$. Moreover, when $p|k$ one has

$$x_1^k + \ldots + x_s^k = (x_1^{k/p} + \ldots + x_s^{k/p})^p \in \mathbb{F}_q[t^p].$$

Writing k_0 for the largest divisor of k coprime to q, we deduce that $w_q(k) = w_q(k_0)$. There is consequently no loss in supposing that $k \geqslant 2$ and $(k,p) = 1$, as we assume henceforth. Next write k in base p as in (1). We recall that the binomial coefficient $\binom{k}{r}$ is coprime to p if and only if the base p expansion of r takes the form $r = b_0 + b_1 p + \ldots + b_n p^n$, with $0 \leqslant b_i \leqslant a_i$ $(0 \leqslant i \leqslant n)$ (this follows from Lucas' criterion; see, for example, the argument of the proof of Lemma 8.1 of [7]). Write $\mathcal{R}$ for the set of integers r, with $0 \leqslant r \leqslant k$, for which $\binom{k}{r}$ is not divisible by p. Note that since $p \nmid k$, one has $k-1 \in \mathcal{R}$. We may suppose that $\mathcal{R} = \{r_1, r_2, \ldots, r_s\}$, with $0 = r_1 < r_2 < \ldots < r_s = k$. For the sake of concision, we write $\mathcal{R}_1$ for $\mathcal{R} \setminus \{k\}$, and $\mathcal{R}_2$ for $\mathcal{R} \setminus \{k-1, k\}$.

When ε is 1 or 2, and $\mathbf{y} \in \mathbb{F}_q^s$, we denote by $N_\varepsilon(\mathbf{y})$ the number of distinct $\mathbb{F}_q$-rational projective solutions $\mathbf{x}$ of the system

$$x_1^k y_1^r + \ldots + x_s^k y_s^r = 0 \quad (r \in \mathcal{R}_\varepsilon). \tag{3.ε}$$

Here, we interpret z^0 as unity for every z in $\mathbb{F}_q$. We seek to establish that $\mathbf{y}$ may be chosen from $\mathbb{F}_q^s$ in such a manner that $N_2(\mathbf{y}) > N_1(\mathbf{y})$. In such circumstances, a solution $\mathbf{x}$ of (3.2) necessarily exists for which the expression $x_1^k y_1^{k-1} + \ldots + x_s^k y_s^{k-1}$ is non-zero, and hence the equation (2) is satisfied with $a \neq 0$. From this, as we have already noted in the discussion following (2), the desired conclusion $w_q(k) \leqslant s$ follows at once.

In order to make a suitable choice for $\mathbf{y}$, we introduce for $\varepsilon = 1$ and 2 the determinant $\mathfrak{V}(\mathbf{z}; \mathcal{B}_\varepsilon, \mathcal{R}_\varepsilon)$, which we define for $(s-\varepsilon)$-element subsets $\mathcal{B}_\varepsilon$ of $\{1, 2, \ldots, s\}$ by $\mathfrak{V}(\mathbf{z}; \mathcal{B}_\varepsilon, \mathcal{R}_\varepsilon) = \det\big(z_i^{r_j}\big)$, where the entries are indexed by $i \in \mathcal{B}_\varepsilon$ and $j \in \{1, \ldots, s-\varepsilon\}$ (in numerically increasing order). Consider the polynomial $\mathfrak{F}(\mathbf{z})$ given by the product of the polynomials $\mathfrak{V}(\mathbf{z}; \mathcal{B}_\varepsilon, \mathcal{R}_\varepsilon)$ over all $(s-\varepsilon)$-element subsets $\mathcal{B}_\varepsilon$ of $\{1, \ldots, s\}$, for $\varepsilon = 1$ and 2. The degree of $\mathfrak{F}(\mathbf{z})$ is at most ks^3, and so it follows from Lemma 1 of Schmidt [9] that whenever $q > ks^3$, then a choice for $\mathbf{y} \in \mathbb{F}_q^s$ exists with the property that $\mathfrak{V}(\mathbf{y}; \mathcal{B}_\varepsilon, \mathcal{R}_\varepsilon) \neq 0$ for every $(s-\varepsilon)$-element subset $\mathcal{B}_\varepsilon$ of $\{1, \ldots, s\}$, for $\varepsilon = 1, 2$. We now fix a choice for $\mathbf{y}$ with the latter property, and we consider the system (3.2).

We claim that the complete intersection defined by (3.2) is non-singular. Suppose to the contrary that a singular solution $\mathbf{x}$ exists. Then whenever $\mathcal{B}_2 = \{u_1, u_2, \ldots, u_{s-2}\}$, with $1 \leqslant u_1 < u_2 < \ldots < u_{s-2} \leqslant s$, one must have $\det\big(k x_{u_i}^{k-1} y_{u_i}^{r_j}\big)_{1 \leqslant i,j \leqslant s-2} = 0$, whence

$$k^{s-2} (x_{u_1} x_{u_2} \ldots x_{u_{s-2}})^{k-1} \mathfrak{V}(\mathbf{y}; \mathcal{B}_2, \mathcal{R}_2) = 0.$$

But by hypothesis, one has $\mathfrak{V}(\mathbf{y}; \mathcal{B}_2, \mathcal{R}_2) \neq 0$, and so x_{u_i} must be zero for some index i with $1 \leqslant i \leqslant s-2$. Considering such implications as arise from all possible $(s-2)$-element subsets of $\{1, 2, \ldots, s\}$, we infer that x_j is necessarily zero for at least 3 distinct indices j with $1 \leqslant j \leqslant s$. Temporarily relabelling variables so that x_{s-1} and x_s are zero, we set $\mathcal{B}_2 = \{1, 2, \ldots, s-2\}$ and examine (3.2). Since $\mathbf{x}$ defines a projective solution of (3.2), the variables $x_1, \ldots, x_{s-2}$ cannot all be zero, and so one must have $\mathfrak{V}(\mathbf{y}; \mathcal{B}_2, \mathcal{R}_2) = 0$. But in view of our earlier choice of $\mathbf{y}$, this is impossible. We therefore arrive at a contradiction, and are forced to conclude that the variety X defined by the system (3.2) is non-singular.

The projective non-singular complete intersection (3.2) is defined by $s-2$ equations of degree k in s variables, so the components of X each have dimension 1. Since we have established that this variety is non-singular, it follows that X is regular in codimension one, and hence irreducible (see, for example, the preamble to Corollary 6.2 of [4]). We therefore deduce from Theorem 6.1 of Ghorpade and Lachaud [4] that $|N_2(\mathbf{y})-(q+1)|\leqslant b_1\sqrt{q}$, where, in view of Example 4.3(ii) of [4], the Betti number b_1 is equal to $k^{s-2}(k(s-2)-s)+2$ (see also Theorem 8.1 of Deligne [2]). We may conclude thus far, therefore, that

$$N_2(\mathbf{y})\geqslant q+1-k^{s-1}(s-2)\sqrt{q}. \tag{4}$$

Next we consider the system (3.1). Set $\mathcal{B}_1=\{1,2,\dots,s-1\}$. In view of our choice for $\mathbf{y}$, one has $\mathfrak{V}(\mathbf{y};\mathcal{B}_1,\mathcal{R}_1)\neq 0$. Therefore, if we fix any non-zero choice for x_s, we deduce that the system

$$x_1^ky_1^r+\dots+x_{s-1}^ky_{s-1}^r=-x_s^ky_s^r\quad (r\in\mathcal{R}_1),$$

uniquely determines $(x_1^k,\dots,x_{s-1}^k)$. There are consequently at most k^{s-1} possible such choices for $(x_1,\dots,x_{s-1})$. When $x_s=0$, meanwhile, the same argument shows that $(x_1,\dots,x_{s-1})=\mathbf{0}$. We therefore deduce that the number of projective solutions of the system (3.1) counted by $N_1(\mathbf{y})$ is at most k^{s-1}. On combining the latter estimate with (4), we find that

$$N_2(\mathbf{y})-N_1(\mathbf{y})\geqslant q+1-k^{s-1}(s-2)\sqrt{q}-k^{s-1}.$$

But by hypothesis, we may suppose that $q\geqslant k^{2s-2}(s-1)^2$, and thus we conclude that $N_2(\mathbf{y})>N_1(\mathbf{y})$. In view of the discussion following equation $(3.\varepsilon)$ above, we infer that $w_q(k)\leqslant s$, and this completes the proof of Theorem 1. ■

3. The method of Gallardo and Vaserstein

The conclusion of Corollary 2 follows from Theorem 1 by means of a direct application of the methods of Gallardo and Vaserstein [3], additional refinements stemming only from careful book-keeping. Consider a polynomial $m\in\mathbb{J}_q^k[t]$ of sufficiently large degree d. As reported in [3], it is a consequence of work of Weil [13] that when $q\geqslant k^4$, then every element of $\mathbb{F}_q$ is the sum of two kth powers from $\mathbb{F}_q$. Let

$$n=\left\lceil\frac{\log k}{\log(k/(k-1))}\right\rceil+2. \tag{5}$$

Then an inspection of the argument of the proof of Proposition 3.5 of [3] reveals that there exist polynomials $x_i\in\mathbb{F}_q[t]$ $(1\leqslant i\leqslant n)$, each of degree not exceeding

$\lceil(\text{ord } m)/k\rceil$, with the property that the polynomial $m_0 = m - x_1^k - \ldots - x_n^k$ has degree at most D, where $D = k\lceil d/k\rceil(1-1/k)^{n-2} + k(k-1)$. When $k > 2$, the quotient $(\log k)/(\log(k/(k-1)))$ is not an integer, and hence there is a positive number $\delta = \delta_k$ for which $D \leqslant (1-\delta)d/k + k^2$. We note that the latter is at most d/k whenever d is sufficiently large in terms of k. It follows from (5), moreover, that for $k \geqslant 4$ one has $n < k\log k - \frac{1}{2}\log k + 3$.

We next recall that since -1 is a sum of kth powers in $\mathbb{F}_q$, then m_0 is the sum of some number of kth powers from $\mathbb{F}_q[t]$, that is $m_0 \in \mathbb{J}_q^k[t]$. Consequently, when $q \geqslant k^{2k_p}k_p^2$ and $u \geqslant k_p + 1$, the conclusion of Theorem 1 demonstrates that the polynomial m_0 is represented in the shape $m_0 = y_1^k + \ldots + y_u^k$, with $y_i \in \mathbb{F}_q[t]$ $(1 \leqslant i \leqslant u)$. An inspection of the proof of Theorem 1 in §2, moreover, confirms that one may constrain the polynomials y_i $(1 \leqslant i \leqslant u)$ employed in the latter representation to have degree at most that of m_0, namely $D \leqslant d/k$. We conclude that m possesses the representation $m = x_1^k + \ldots + x_n^k + y_1^k + \ldots + y_u^k$, with $x_i \in \mathbb{F}_q[t]$ $(1 \leqslant i \leqslant n)$ each of degree at most $\lceil(\text{ord } m)/k\rceil$, and with $y_j \in \mathbb{F}_q[t]$ $(1 \leqslant j \leqslant u)$ each of degree $d/k \leqslant \lceil(\text{ord } m)/k\rceil$. In particular, the polynomial m has a restricted representation as the sum of $(n+u)$ kth powers of polynomials from $\mathbb{F}_q[t]$. We conclude that $G_q(k) \leqslant n+u$, and so on recalling our upper bound on n, we find that

$$G_q(k) < (k\log k - \tfrac{1}{2}\log k + 3) + (k_p + 1).$$

This completes the proof of Corollary 2.

4. Methods applicable for smaller *q*

The upper bound presented in Theorem 3 may be established cheaply by making use of our recent work [7] concerning the restricted variant of Waring's problem. The argument is familiar, but we provide details for the sake of completeness. Observe first that when the characteristic of $\mathbb{F}_q$ divides k, one has $w_q(k) = w_q(k_0)$, in which k_0 is the largest divisor of k coprime to q. It therefore suffices to bound $w_q(k)$ for $(k,q) = 1$, as we henceforth assume. Suppose that $m \in \mathbb{J}_q^k[t]$, so that m is the sum of some number of kth powers from $\mathbb{F}_q[t]$. Let x_0 be an element of $\mathbb{F}_q[t]$ of degree sufficiently large in the context of the methods of [7], and consider the polynomial $m_0 = m - x_0^k$. In accordance with our opening observation in the final paragraph of §3, one has $m_0 \in \mathbb{J}_q^k[t]$. Let C_0 be a suitable positive absolute constant, and write $v = [\widehat{G}_q(k) + C_0 k\sqrt{\text{Log Log } k}/\text{Log } k]$. Then since m_0 has sufficiently large degree, the conclusion of Theorem 1.1 of [7] ensures that m_0 is the sum of at most v kth powers from $\mathbb{F}_q[t]$, say $m_0 = x_1^k + \ldots + x_v^k$. But then one has $m = x_0^k + x_1^k + \ldots + x_v^k$, whence m is the sum of at most $v+1$ kth powers from $\mathbb{F}_q[t]$. This completes the proof of Theorem 3.

We remark that the methods of §§2–14 of [7] may be used to count the number of solutions of the equation $m = x_1^k + \ldots + x_u^k$, with $x_i \in \mathbb{F}_q[t]$ of degree B sufficiently large in terms of k $(1 \leqslant i \leqslant u)$. When u is at least as large as the integer v above, for a suitable absolute constant C_0, an asymptotic lower bound for the number of solutions may be obtained which confirms that m has infinitely many representations as the sum of u kth powers whenever $m \in \mathbb{J}_q^k[t]$. In some sense, therefore, the additional kth power employed in the first paragraph is redundant, and may be eliminated in a more refined analysis of this problem.

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