Explicit Construction of the $(13, 13)$-Regular Hypergraph

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CONTENTS

In this article we calculate explicitly the Ramanujan $(13, 13)$-regular hypergraph introduced in [Sarveniazi 07] using the computer algebra programs MAGMA and OCTAVE. This simple structure represents a nice and arithmetically very rich object in number theory, namely $\Gamma_f \setminus \Gamma(1)$.

1. INTRODUCTION

The global Jacquet–Langlands correspondence between automorphic representations of unit groups of certain division algebras and automorphic representations of GL_d in positive characteristic is a challenging problem. The cuspidality of representations of arithmetic groups arising from unit groups of division algebras is closely related to certain properties of adjacency matrices of Ramanujan hypergraphs.

This connection motivated our construction of such hypergraphs and their adjacency matrices. Indeed, the adjacency matrices $A^{(i)}$ are actually Hecke operators defined on the arithmetic group $\Gamma(1)$ at the place $p(t)$ (in our case $p(t) = 1 - t$) that transfer automorphic forms associated to the arithmetic subgroup Γ_f for $f(t) = 1+t$ (see Section 4 for the exact definition).

The cuspidality of the arising automorphic representation, which must be confirmed in this general case in positive characteristic, will be addressed in a forthcoming paper. Of course, the translation of such a deep result as the Jacquet–Langlands correspondence to the language of elementary graph theory is amazing, but in this note we are just trying to present a near-infrared object that is closely related to this deep result. For more details see [Sarveniazi 07] and references therein.

Here we give an explicit presentation of Ramanujan $(13, 13)$-regular hypergraphs introduced in [Sarveniazi 07] using the computer algebra programs MAGMA and OCTAVE. This is a class of examples of Morgenstern

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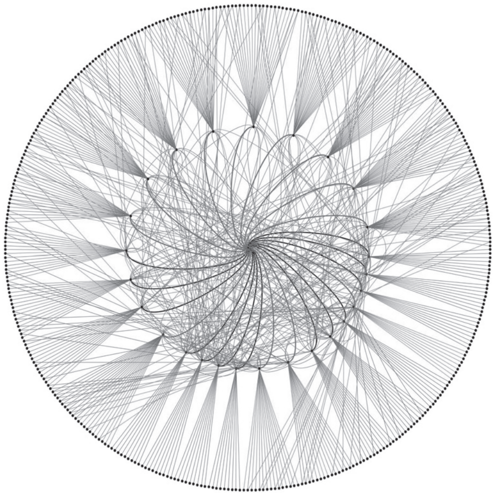

FIGURE 1. The graph of the first nearest neighbors of our $(13, 13)$-regular hypergraph.

Ramanujan graphs that we have generalized to higher dimensions, replacing quaternion algebras in Morgenstern's examples with quotient skew-fields of skew-polynomial rings.

We consider the skew-polynomial ring $\mathbb{F}_{q^d}\{\tau\}$ over the field $\mathbb{F}_{q^d}$ of q^d elements, where the indeterminate τ satisfies the rule $\tau \cdot \lambda = \lambda^q \cdot \tau$ for $\lambda \in \mathbb{F}_{q^d}$.

In Sections 2 and 3, we summarize some facts about Bruhat–Tits buildings, related hypergraphs, and the skew-polynomial rings. In Section 4 we study the structure of the arithmetic groups $\Gamma(1)$, $\Gamma(\tau)$, and $\Gamma_f(\tau)$. Finally, in Section 5, we present our explicit calculation in the smallest nontrivial case,

$$d = 3,\ q = 3,\ \text{and prime } f = 1 + t,$$

and we give a combinatorial description of our hypergraph as the Cayley graph of the group $\mathrm{PGL}(3, \mathbb{F}_3) = \mathrm{PSL}(3, \mathbb{F}_3)$ over the finite field $\mathbb{F}_3$.

A Magma program produces a complete factorization of $1 - \tau$, which is essential for our construction. The matrices $A^{(1)}$ and $A^{(2)}$ and their eigenvalues are calculated and the Ramanujan property (successfully) tested. A picture of the first nearest neighbors can be seen in Figure 1.

2. BUILDINGS AND HYPERGRAPHS

Let (F, ν) be a local nonarchimedean field with valuation ring $\mathcal{O}$, uniformizer π, and residue field $k := \mathcal{O}/(\pi)$.

Attached to the algebraic group $\mathrm{PGL}(d, F)$ for some fixed number $d > 1$ is the affine building $X_\bullet(F^d)$. It is defined as follows: The edges (0-simplices) are classes of $\mathcal{O}$-lattices $L \subset F^d$ of rank d, where

$$L \sim L' \Longleftrightarrow L' = \lambda L \text{ for some } \lambda \in F \setminus \{0\},$$

and r-simplices, i.e., simplices that include $r+1$ vertices, are defined by chains of lattices

$$\pi L \subsetneq L_1 \subsetneq \cdots \subsetneq L_r \subsetneq L.$$

It is easily seen that all maximal simplices (chambers) have cardinality d. Furthermore, we have a notion of apartments that satisfies the axioms of a building (see, for example, [Brown 89, Ronan 89]).

If we fix a lattice (i.e., a vertex) L, we define the *link* $\mathrm{lk}_{X_\bullet(F^d)}(L)$ as the simplicial complex given by all simplices $\Delta \in X_\bullet(F^d)$ such that $L \notin \Delta$, but $L \cup \Delta$ is a simplex in $X_\bullet(F^d)$. The simplicial complex $\mathrm{lk}_{X_\bullet(F^d)}(L)$ is isomorphic to the Tits building associated to the k-vector space $L/\pi L$ by the map

$$\mathrm{lk}_{X_\bullet(F^d)}(L) \to X_\bullet(L/\pi L),$$
$$\pi L \subsetneq L_1 \subsetneq \cdots \subsetneq L_r \subsetneq L \mapsto L_1/\pi L \subsetneq \cdots \subsetneq L_r/\pi L;$$

i.e., it is a building itself. Associated to the building $\mathrm{lk}_{X_\bullet(F^d)}(L)$ we have a canonical labeling with index set $\{1, \ldots, d-1\}$ defined as follows: If $\pi L \subsetneq L' \subsetneq L$ is an edge, then the label $t_L(L')$ is defined by $t_L(L') := \dim_k L'/\pi L$. If we regard $\mathrm{lk}_{X_\bullet(F^d)}(L)$ as a subcomplex of the whole building $X_\bullet(F^d)$, then the label of an edge L' evidently depends on the choice of the vertex L, but we have the following correspondences.

Lemma 2.1.

(1) *If L, L' is an edge in $X_\bullet(F^d)$, then*

$$t_L(L') \equiv -t_{L'}(L) \bmod d.$$

(2) *If L, L', L'' is a triangle (2-simplex) in $X_\bullet(F^d)$, then*

$$t_L(L') - t_L(L'') \equiv t_{L'}(L'') \bmod d.$$

Proof. 1. We have

$$\pi L \subsetneq L' \subsetneq L \Longrightarrow \pi L' \subsetneq \pi L \subsetneq L'.$$

From

$$\pi L/\pi L' \cong L/L' \cong (L/\pi L)/(L'/\pi L)$$

it follows that

$$t_{L'}(L) = d - t_L(L' \equiv -t_L(L')) \bmod d.$$

2. We have

$$\pi L \subsetneq L' \subsetneq L'' \subsetneq L \Longrightarrow \pi L'' \subsetneq L' \subsetneq L''.$$

From

$$L'/\pi L'' \cong (L'/\pi L')/(\pi L''/\pi L')$$

and

$$\pi L''/\pi L' \cong L''/L' \cong (L''/\pi L)/(L'/\pi L)$$

it follows that

$$\begin{aligned} t_{L''}(L') &= d - \dim(\pi L''/\pi L') = d - (t_L(L'') - t_L(L')) \\ &\equiv t_L(L') - t_L(L'') \bmod d. \end{aligned}$$

□

The group $\mathrm{PGL}(F, d)$ acts on the building $X_\bullet(F^d)$ by its natural action on the lattices. It is easily seen that this action is transitive and respects the simplicial structure. Since we are interested in some special quotients of the building by this action, we make the following definitions, due to [Sarveniazi 07].

Definition 2.2. (($n_1, \ldots, n_{d-1}$)-regular hypergraph.) [Sarveniazi 07, Definitions 29–31]

(1) An $(n_1, n_2, \ldots, n_{d-1})$-regular hypergraph is a simplicial chamber complex $X_\bullet$ equipped with a labeling function

$$\begin{aligned} t : X_0 \times X_0 &\to \{0\} \cup \{1, \ldots, d-1\}, \\ (L, L') &\mapsto t_L(L'), \end{aligned}$$

such that $t_L(L') = 0$ if $\{L, L'\} \notin X_1$, t satisfies the formulas of Lemma 2.1, and for $1 \leq i \leq d-1$ and all vertices L in $X_\bullet$, we have

$$\#\{L' \in \mathrm{lk}_{X_\bullet}(L) \mid t_L(L') = i\} = n_i.$$

(2) For $1 \leq i \leq d-1$ we define the ith adjacency matrix of $X_\bullet$ by

$$A^{(i)} := \left(\varepsilon^{(i)}(L, L')\right)_{L, L' \in X_0},$$

where

$$\varepsilon^{(i)}(L, L') := \begin{cases} 1 & \text{if } t_L(L') = i, \\ 0 & \text{otherwise.} \end{cases}$$

As in [Sarveniazi 07], we extend the usual notation of an expanding bound for regular graphs to hypergraphs.

Definition 2.3. (Upper bounds.) [Sarveniazi 07, Definition 32] Let $X_\bullet$ be an $(n_1, n_2, \ldots, n_{d-1})$-regular hypergraph and $A^{(1)}, \ldots, A^{(d-1)}$ its adjacency matrices.

(1) For $1 \leq i \leq d-1$ we define

$$\lambda^{(i)}(X_\bullet) := \max_{\substack{\lambda \text{ eigenvalue of } A^{(i)} \\ |\lambda| \neq n_i}} |\lambda|.$$

(2) We say that the hypergraph $X_\bullet$ is bounded above with bound $(c_1, \ldots, c_{d-1})$ for some real numbers $c_1, \ldots, c_{d-1} \in \mathbb{R}$ if $\lambda^{(i)}(X_\bullet) \leq c_i$ for $1 \leq i \leq d-1$.

Remark 2.4. Because of the regularity condition, it is easy to see that for $1 \leq i \leq d-1$,

$$\lambda^{(i)}(X_\bullet) < n_i.$$

3. THE SKEW-POLYNOMIAL RING OVER A FINITE FIELD

3.1 Skew-Polynomials and Division Rings

We will first recall some basic facts about the ring of skew-polynomials. Proofs of the following statements and further details can be found in [Jacobson 96, Section I].

The ring of skew-polynomials $\mathbb{F}_{q^d}\{\tau\}$ has a well-defined ring of quotients $\mathbb{F}_{q^d}(\tau)$, which is a skew-field. If we set $t := \tau^d$, then we can describe the center of $\mathbb{F}_{q^d}(\tau)$ to be the function field $\mathbb{F}_q(t)$. That is, $\mathbb{F}_{q^d}(\tau)$ is a finite division algebra over its center of dimension d^2.

Since $\mathbb{F}_{q^d}(t)$ is a cyclic Galois extension of $\mathbb{F}_q(t)$ of degree d whose Galois group is generated by the automorphism

$$\begin{aligned} \sigma : \mathbb{F}_{q^d}(t) &\to \mathbb{F}_{q^d}(t), \\ \lambda t &\mapsto \lambda^q t, \end{aligned}$$

we can describe the skew-field in terms of the cyclic algebra $(\mathbb{F}_{q^d}(t)/\mathbb{F}_q(t), \sigma, t)$, where we choose $1, \tau, \ldots, \tau^{d-1}$ to be an $\mathbb{F}_{q^d}(t)$ base of $\mathbb{F}_{q^d}(\tau)$.

Remark 3.1. It is easy to see that the cyclic extension of function fields $\mathbb{F}_{q^d}(t)/\mathbb{F}_q(t)$ is unramified at all primes (cf. [Rosen 02, Proposition 8.5]).

If v is a valuation of $\mathbb{F}_q(t)$ and w is a valuation of $\mathbb{F}_{q^d}(t)$ such that w lies over v, we get the field extension of the associated local fields $\mathbb{F}_{q^d}(t)_w/\mathbb{F}_q(t)_v$ of degree d_w. Furthermore, by Remark 3.1 we have $d_w = f_w$, where f_w is the degree of the extension of the corresponding residue fields and the numbers d_w and f_w are independent of the choice of the valuation above v, since the extension is Galois. We define $\mathbb{F}_{q^d}(\tau)_v := \mathbb{F}_{q^d}(\tau) \otimes_{\mathbb{F}_q(t)} \mathbb{F}_q(t)_v$.

Proposition 3.2.

(1) $\mathbb{F}_{q^d}(\tau)_v \cong (\mathbb{F}_{q^d}(t)_w/\mathbb{F}_q(t)_v, \sigma^{d/d_w}, t)$.

(2) *The division algebra* $\mathbb{F}_{q^d}(\tau)_v$ *splits completely if and only if* $v(t) = 0$.

(3) $\mathbb{F}_{q^d}(\tau)_v$ *splits completely if and only if* v *is not one of the valuations corresponding to the primes* t *and* $1/t$.

For a proof, see [Goss 96, Theorem 4.12.4, Corollary 4.12.5].

In the case that the skew-field splits, we can describe the splitting isomorphism explicitly. We follow [Reiner 75, Sections 29, 30]. The crucial point is that for $v(t) = 0$, we can find with the help of Hensel's lemma an element $T \in \mathbb{F}_{q^d}(t)_w$ such that $\mathrm{Norm}_{\mathbb{F}_{q^d}(t)_w/\mathbb{F}_q(t)_v}(T) = t$. For simplicity we will assume that $d_w = d$. Then $1, \tau, \ldots, \tau^{d-1}$ is still an $\mathbb{F}_{q^d}(t)_w$-base of $\mathbb{F}_{q^d}(\tau)_v$. We perform the base change

$$\begin{aligned} 1 &\mapsto 1, \\ \tau &\mapsto \tilde{\tau} := T^{-1}\tau, \\ \tau^2 &\mapsto (\tilde{\tau})^2 = T^{-1}\sigma(T^{-1})\tau^2 \\ &\vdots \\ \tau^{d-1} &\mapsto (\tilde{\tau})^{d-1} = T^{-1}\sigma(T^{-1})\cdots\sigma^{d-2}(T^{-1})\tau^{d-1}. \end{aligned}$$

Now we get an isomorphism of $\mathbb{F}_q(t)_v$-vector spaces:

$$\begin{aligned} &\mathbb{F}_{q^d}(\tau)_v \to \mathrm{End}_{\mathbb{F}_q(t)_v}(\mathbb{F}_{q^d}(t)_w), \\ &\quad \alpha_i\tilde{\tau}^i \mapsto \mathbb{F}_{q^d}(t)_w \to^{\sigma^i} \mathbb{F}_{q^d}(t)_w \to^{\cdot\alpha_i} \mathbb{F}_{q^d}(t)_w. \end{aligned}$$

If we choose an $\mathbb{F}_q(t)_v$-base of $\mathbb{F}_{q^d}(t)_w$, we can identify the endomorphism ring $\mathrm{End}_{\mathbb{F}_q(t)_v}(\mathbb{F}_{q^d}(t)_w)$ with the ring of matrices $\mathrm{M}_d(\mathbb{F}_q(t)_v)$.

3.2 Skew-Polynomials and Endomorphisms

Let $\overline{\mathbb{F}}_q$ be an algebraic closure of $\mathbb{F}_q$. To an element $f := \sum_{i=0}^n \lambda_i\tau^i \in \mathbb{F}_{q^d}\{\tau\}$ of degree n we associate an $\mathbb{F}_q$-linear map

$$\varphi_f : \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q, \quad x \mapsto \sum_{i=0}^n \lambda_i x^{q^i}.$$

We define $V_f := \mathrm{Ker}(\varphi_f) \subseteq \overline{\mathbb{F}}_q$. It is a finite-dimensional $\mathbb{F}_q$-vector space. If φ_f is separable, that is, $\lambda_0 \neq 0$, then $\dim_{\mathbb{F}_q} V_f = n = \deg_\tau f$. We will call an element of $\mathbb{F}_{q^d}\{\tau\}$ *normalized* if its absolute coefficient is equal to 1.

Proposition 3.3. *Let* $f := \sum_{i=0}^n \lambda_i\tau^i \in \mathbb{F}_{q^d}\{\tau\}$ *be a normalized polynomial of degree* n. *Then we have the following correspondence between the two following sets of objects:*

$$f = f_r \cdots f_1,$$

which is a decomposition of f *into normalized skew-polynomials* $f_1, \ldots, f_r$ *such that* $\deg_\tau(f_i) \geq 1$ *for* $1 \leq i \leq r$, *and*

$$0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V_f,$$

which are flags of $t = \tau^d$*-stable* $\mathbb{F}_q$*-subvector spaces in* V_f.

We briefly describe the correspondence:

1. If $f_1, \ldots, f_r$ are given, then we define $V_i := V_{f_i\cdots f_1}$ for $1 \leq i < r$.
2. If $V_1, \ldots, V_r$ are given, then $\varphi_i := \prod_{\lambda\in V_i}(X - \lambda)$ is an $\mathbb{F}_q$-linear polynomial in $\mathbb{F}_{q^d}[X]$, that is, there exist elements $\lambda_{ij} \in \mathbb{F}_{q^d}$ such that
$$\varphi_i = \sum_{j=0}^{\dim_{\mathbb{F}_q}(V_i)} \lambda_{ij}X^{q^j}.$$
We define
$$f_1 := \sum_{j=0}^{\dim_{\mathbb{F}_q}(V_1)} \frac{\lambda_{1j}}{\lambda_{10}}\tau^j$$
and
$$f_i := \sum_{j=0}^{\dim_{\mathbb{F}_q}(V_i)} \frac{\lambda_{ij}}{\lambda_{i0}}\tau^j / f_{i-1}$$
for $1 < i \leq r$, where the division by f_{i-1} takes place inside $\mathbb{F}_{q^d}\{\tau\}$ from the right.

Application 3.4. If $f = 1 - t = 1 - \tau^d$, then $V_{1-t} = \mathbb{F}_{q^d} \subseteq \overline{\mathbb{F}}_q$ and all $\mathbb{F}_q$-subspaces of V_{1-t} are τ^d $(= t)$ stable. We get the following correspondence between the two following sets of objects:

$$1 - \tau^d = (1 + \lambda_d\tau)\cdots(1 + \lambda_1\tau),$$

decompositions of $1 - \tau^d$ into normalized skew-polynomials of degree 1, and

$$0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_d = V_{1-t},$$

maximal flags of $\mathbb{F}_q$-subvector spaces in V_{1-t}.

We define

$$\begin{aligned} \mathcal{L}_{1-t} := \big\{(\lambda_1, \ldots, \lambda_d) \in (\mathbb{F}_{q^d})^d \mid 1 - \tau^d \\ = (1 + \lambda_d\tau)\cdots(1 + \lambda_1\tau)\big\}. \end{aligned}$$

4. ARITHMETIC SUBGROUPS OF $\mathbb{F}_{q^d}\{\tau\}$

In this section we fix the irreducible polynomial $p := 1-t \in \mathbb{F}_q[t]$. Let $f \in \mathbb{F}_q[t]$ be another irreducible polynomial not equal to t and $1-t$. By abuse of language we will not distinguish between the irreducible polynomial f and its corresponding valuation. As done in [Sarveniazi 07] we define the following groups:

(1) $\Gamma(1) := \mathbb{F}_{q^d}\{\tau\}[\frac{1}{p}]^\times/Z$, where $Z = \mathbb{F}_q[t,\frac{1}{p}]^\times$ is the center;

(2) $\Gamma(\tau) := \mathrm{Ker}(\mathbb{F}_{q^d}\{\tau\}[\frac{1}{p}]^\times/Z \to^{\tau=0} \mathbb{F}_{q^d}^\times/\mathbb{F}_q^\times)$;

(3) $\Gamma_f(\tau) := \{g \in \Gamma(\tau) \mid g \equiv 1 \bmod f\}$. Here $g \equiv 1 \bmod f$ means $g-1 \in f\,\mathbb{F}_{q^d}\{\tau\}[\frac{1}{p}]$.

According to Proposition 3.2, the prime f splits completely $\mathbb{F}_{q^d}\{\tau\}$, and the composition of the maps

$$\begin{aligned}\mathbb{F}_{q^d}\{\tau\}\left[\frac{1}{p}\right] &\longrightarrow \mathbb{F}_{q^d}\{\tau\}\left[\frac{1}{p}\right]\otimes_{\mathbb{F}_q[t]}\mathbb{F}_q[t]/(f)\\ &\xrightarrow{\cong} \mathbb{F}_{q^d}\{\tau\}\left[\frac{1}{p}\right]\otimes_{\mathbb{F}_q[t]}\mathbb{F}_q[t]_f\otimes_{\mathbb{F}_q[t]_f}\mathbb{F}_q[t]_f/(f)\\ &\xrightarrow{\cong} \mathrm{M}_d(\mathbb{F}_q[t]/(f))\end{aligned}$$

induces a representation of the quotient group

$$\begin{aligned}\rho : \Gamma(\tau)/\Gamma_f(\tau) \hookrightarrow \left(\mathbb{F}_{q^d}\{\tau\}\left[\frac{1}{p}\right]/(f)\right)^\times/\text{center}\\ \cong \mathrm{PGL}_d(\mathbb{F}_q[t]/(f)).\end{aligned}$$

We use the representation ρ to define the set of matrices

$$M_i^{\boldsymbol{\lambda}} := \rho((1-\lambda_i\tau)\cdots(1-\lambda_1\tau))$$

for $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_d) \in \mathcal{L}_{1-t}$ and define

$$\mathcal{M}_i := \{M_i^{\boldsymbol{\lambda}} \mid \boldsymbol{\lambda} \in \mathcal{L}_{1-t}\}$$

for $1 \le i < d$. Furthermore, we define $\mathcal{M} := \bigcup_{i=1}^{d-1}\mathcal{M}_i$.

We use these matrices to define the Cayley graph

$$\mathrm{Hyp}_f(1-t) := \begin{cases}\mathcal{C}(\mathrm{PSL}_d(\mathbb{F}_q[t]/(f),\mathcal{M}),\\ \quad \text{if } 1-t \text{ is a } d\text{th power in } \mathbb{F}_q[t]/(f);\\ \mathcal{C}(\mathrm{PGL}_d(\mathbb{F}_q[t]/(f),\mathcal{M}),\\ \quad \text{if } 1-t \text{ is not a power in } \mathbb{F}_q[t]/(f).\end{cases}$$

We have the following result [Sarveniazi 07, Theorem 33].

Theorem 4.1. *The Cayley graph $\mathrm{Hyp}_f(1-t)$ is a Ramanujan $(n_1,\ldots,n_{d-1})$-regular hypergraph with bounds $(c_1,\ldots,c_{d-1})$, where for $1 \le i < d$,*

(1) *$n_i :=$ number of i-dimensional $\mathbb{F}_q$-subspaces of $\mathbb{F}_{q^d}$;*

(2) *$c_i := \binom{d}{i}q^{(d-i)i/2}$;*

(3) *the local labeling function for all vertices x,y of $\mathrm{Hyp}_f(1-t)$ is given by*

$$t_x(y) := \begin{cases} i & \textit{iff } \exists\, g \in \mathcal{M}_i \textit{ such that } gx = y;\\ 0 & \textit{otherwise.}\end{cases}$$

5. EXPLICIT CALCULATION OF THE $(13,13)$-REGULAR HYPERGRAPH

In this section we present the results of an explicit calculation done with the computer algebra programs Magma and Octave. We set the parameters as small as possible, which means that $q=3$, $d=3$, $p=1-t$, $f=1+t$.

We calculate the elements of $\mathcal{L}_{1-t}$:

```
{
 <w,w,w>,<w,w^9,w^3>,<w,w^21,w^19>,
 <w,w^25,w^7>,<w^3,w,w^9>,<w^3,w^3,w^3>,
 <w^3,w^11,w^5>,<w^3,w^23,w^21>,<w^5,w^3,w^11>,
 <w^5,w^5,w^5>,<w^5,2,w^7>,<w^5,w^25,w^23>,
 <w^7,w,w^25>,<w^7,w^5,2>,<w^7,w^7,w^7>,
 <w^7,w^15,w^9>,<w^9,w^3,w>,<w^9,w^7,w^15>,
 <w^9,w^9,w^9>,<w^9,w^17,w^11>,<w^11,w^5,w^3>,
 <w^11,w^9,w^17>,<w^11,w^11,w^11>,<w^11,w^19,2>,
 <2,w^7,w^5>,<2,w^11,w^19>,<2,2,2>,
 <2,w^21,w^15>,<w^15,w^9,w^7>,<w^15,2,w^21>,
 <w^15,w^15,w^15>,<w^15,w^23,w^17>,<w^17,w^11,w^9>,
 <w^17,w^15,w^23>,<w^17,w^17,w^17>,<w^17,w^25,w^19>,
 <w^19,w,w^21>,<w^19,2,w^11>,<w^19,w^17,w^25>,
 <w^19,w^19,w^19>,<w^21,w^3,w^23>,<w^21,w^15,2>,
 <w^21,w^19,w>,<w^21,w^21,w^21>,<w^23,w^5,w^25>,
 <w^23,w^17,w^15>,<w^23,w^21,w^3>,<w^23,w^23,w^23>,
 <w^25,w^7,w>,<w^25,w^19,w^17>,<w^25,w^23,w^5>,
 <w^25,w^25,w^25>
}
```

The elements of $1+\lambda_1\tau$ for $(\lambda_1,\lambda_2,\lambda_3) \in \mathcal{L}_{1-t}$:

```
{
 w*tau+1, w^17*tau+1, w^15*tau+1, 2*tau+1,
 w^7*tau+1, w^3*tau+1, w^5*tau+1, w^9*tau+1,
 w^11*tau+1, w^19*tau+1, w^21*tau+1, w^23*tau+1,
 w^25*tau+1
}
```

The elements of $(1+\lambda_2\tau)(1+\lambda_1\tau)$ for $(\lambda_1,\lambda_2,\lambda_3) \in \mathcal{L}_{1-t}$:

```
{
 w^4*tau^2 +w^14*tau+1, w^14*tau^2+w^10*tau+1,
 w^18*tau^2+w^24*tau+1, w^6*tau^2+w^8*tau+1,
 w^2*tau^2+w^20*tau+1, w^12*tau^2+w^16*tau+1,
 w^24*tau^2+w^6*tau+1, w^10*tau^2+w^22*tau+1,
 w^20*tau^2+w^18*tau+1, w^22*tau^2+w^12*tau+1,
 tau^2+tau+1, w^8*tau^2+w^2*tau+1,
 w^16*tau^2+w^4*tau+1
}
```

The elements of $\mathcal{M}_1$:

```
[
  [2 2 1]  [2 2 2]  [1 0 1]  [0 0 1]  [2 1 2]
  [0 2 1]  [2 0 2]  [2 2 1]  [0 1 2]  [1 2 0]
  [0 0 2], [0 2 1], [0 2 0], [1 2 2], [1 0 2],

  [1 2 2]  [1 2 0]  [0 2 0]  [1 2 1]  [2 1 1]
  [0 2 0]  [2 0 1]  [0 2 2]  [1 1 2]  [2 1 2]
  [1 2 0], [1 1 2], [2 1 1], [1 0 1], [1 1 0],

  [2 0 1]  [0 1 0]  [2 0 0]
  [1 0 0]  [2 1 0]  [2 2 2]
  [2 2 1], [0 2 2], [2 0 2]
]
```

The elements of $\mathcal{M}_2$:

```
[
 [1 2 0]  [2 1 0]  [1 1 0]  [0 2 0]  [1 1 2]
 [0 1 1]  [2 0 0]  [1 0 2]  [2 0 1]  [1 2 2]
 [0 0 1], [1 0 1], [2 2 2], [2 2 0], [1 1 0],

 [0 1 2]  [1 1 1]  [0 1 1]  [2 2 2]  [1 0 0]
 [0 1 0]  [2 2 1]  [1 0 0]  [0 2 2]  [0 1 2]
 [1 0 2], [1 2 0], [2 1 0], [2 1 2], [2 0 1],

 [1 2 2]  [1 2 1]  [2 2 1]
 [2 2 0]  [0 0 1]  [1 2 0]
 [2 0 0], [1 1 2], [1 2 2]
]
```

We calculate the number of elements of $\mathrm{PGL}(3,\mathbb{F}_3) = \mathrm{PSL}(3,\mathbb{F}_3) = 5616$ and check that the matrices above generate this group. Then we calculate the adjacency matrices $A^{(1)}$, $A^{(2)}$ and calculate the eigenvalues with the help of OCTAVE. Since $A^{(2)}$ is the transpose of $A^{(1)}$, we calculate only the eigenvalues of $A^{(1)}$.

Eigenvalues of $A^{(1)}$ (and eigenvalues of $A^{(2)}$):

```
# Created by Octave 3.0.0,
# rows: 5616
# columns: 1
 (13.00000000000001,0)
 (7.126860683941947,0)
 (-3.417234277700118,5.507744181041602)
 (-3.417234277700118,-5.507744181041602)
      ...
```

One sees that 13 is an eigenvalue, as expected. We check that the upper-bound condition predicted above is true, that is, $c_1 = c_2 = 9$ and

$$\lambda^{(1)}(X_\bullet) = \lambda^{(2)}(X_\bullet) \approx 7.126860683942489 \leq 9.$$

Finally, we calculate that the graph, regarded as a 26-regular graph, is not a Ramanujan graph:

$$\lambda(X_\bullet) \approx 14.25372136788398 > 2\sqrt{26-1} = 10.$$

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