Minimal Mahler Measures

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CONTENTS

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We determine the minimal Mahler measure of a primitive irreducible noncyclotomic polynomial with integer coefficients and fixed degree D, for each even degree $D \le 54$. We also compute all primitive irreducible noncyclotomic polynomials with measure less than 1.3 and degree at most 44.

1. INTRODUCTION

The *Mahler measure* of a polynomial $P \in \mathbb{C}[X]$, where

$$P(X) = c_0X^D + c_1X^{D-1} + \cdots + c_D = c_0 \prod_{k=1}^{D}(X - \alpha_k)$$

and $c_0 \neq 0$, is defined by

$$M(P) := |c_0| \prod_{k=1}^{D} \max(1, |\alpha_k|).$$

For an algebraic number α, we denote by $M(\alpha)$ the Mahler measure of its minimal polynomial in $\mathbb{Z}[X]$. For $P \in \mathbb{Z}[X]$, certainly $M(P) \ge 1$, and a well-known theorem of Kronecker implies that $M(P) = 1$ if and only if P is a product of cyclotomic polynomials and a power of X. In 1933, D. H. Lehmer [Lehmer 33] asked whether there exists a positive number ϵ such that if α is neither 0 nor a root of unity, then $M(\alpha) \ge 1 + \epsilon$. This is known as *Lehmer's question*, and it remains an open problem. This question arises in a variety of mathematical contexts, including number theory [Schinzel 00, Waldschmidt 00, Borwein 02], ergodic theory [Schmidt 95, Everest and Ward 99], knot theory [Hironaka 01, Silver and Williams 04], and the study of Coxeter systems [McMullen 02, Hironaka 03]. See [Ghate and Hironaka 01, Mossinghoff 07, Smyth 08] for additional references.

Lehmer's question has been resolved in some special cases. We say that a polynomial is *reciprocal* if $P(X) = X^DP(1/X)$, and that an algebraic number is reciprocal if its minimal polynomial is reciprocal. In 1951, Breusch [Breusch 51] proved that if α is a nonreciprocal algebraic number, and $\alpha \notin \{0, 1\}$, then $M(\alpha) >$

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$1.179652\ldots$. In 1971, Smyth [Smyth 71] determined the best possible lower bound in this case, showing that $M(\alpha) \geq \theta_0 := M(X^3 - X - 1) = 1.324717\ldots$, the smallest Pisot number. Lower bounds have also been established for $M(\alpha)$ when α is totally real [Schinzel 73, Höhn and Skoruppa 93] or totally p-adic [Bombieri and Zannier 01], when the splitting field of α over $\mathbb{Q}$ is abelian [Amoroso and Dvornicich 00] or dihedral [Garza 08] (and α is not a root of unity), and when the coefficients of the minimal polynomial of α satisfy certain arithmetic conditions [Borwein et al. 07, Dubickas and Mossinghoff 05].

In the general case, the best known explicit result is due to Voutier, who proved in 1996 [Voutier 96] that if α is algebraic of degree $D > 2$ and is not a root of unity, then

$$M(\alpha) > 1 + \frac{1}{4}\left(\frac{\log\log D}{\log D}\right)^3.$$

This is an explicit version of an inequality of Dobrowolski [Dobrowolski 79].

The smallest known Mahler measure larger than 1 was found by Lehmer in 1933: The polynomial

$$\ell(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

has just one root outside the unit disk, at the real number $\lambda_0 := 1.176280\ldots$ (in fact, λ_0 is the smallest known Salem number), so $M(\ell) = \lambda_0$. A number of prior computations have established lower bounds on the degree of an algebraic integer α with $1 < M(\alpha) < \lambda_0$, if such a number exists. Boyd [Boyd 80, Boyd 89] computed all irreducible noncyclotomic integer polynomials P with degree $D \leq 20$ having $M(P) < 1.3$, and the first author [Mossinghoff 98] used this same algorithm to extend the computation to $D \leq 24$. More recently, Flammang, the second author, and Sac-Épée [Flammang et al. 06] determined all irreducible noncyclotomic polynomials P with $M(P) < \theta_0$ and $D \leq 36$, and polynomials P with $M(P) < 1.31$ and $D = 38$ or 40. These computations employed a new method that uses a large family of explicit auxiliary functions to produce improved bounds on the coefficients of polynomials with small Mahler measure.

At a conference in Bristol, UK, in April 2006, P. Borwein invited the authors to extend these computations to larger degrees on a cluster housed at the Centre for Interdisciplinary Research in the Mathematical and Computational Sciences (IRMACS) at Simon Fraser University, in British Columbia. Using otherwise idle cycles on the IRMACS workstations, over a period of four months we searched for polynomials with small Mahler measure, accumulating nearly six years of CPU time altogether.

With these computations, we establish the following theorem. Recall first that a polynomial $P(X)$ is *primitive* if it cannot be expressed as a polynomial in X^k, for some $k \geq 2$. It is easy to verify that $M(P(X^k)) = M(P(X))$ for all integers $k \geq 1$, so in the following statement we restrict to primitive polynomials.

Theorem 1.1. *If a polynomial $P \in \mathbb{Z}[X]$ satisfies $1 < M(P) < \lambda_0$, then $\deg P \geq 56$. Further, if P is an irreducible noncyclotomic primitive polynomial of even degree D, with $8 \leq D \leq 54$, then $M(P) \geq M_D$, where M_D is listed for each D in Table 1.*

Let $H(P)$ denote the *height* of P, defined as its largest coefficient in absolute value, so $H(P) = \max\{|c_k| : 0 \leq k \leq D\}$. Also, let $L(P)$ denote its *length*, defined by $L(P) = \sum_{k=0}^{D} |c_k|$.

The first author maintains a website [Mossinghoff 07] that contains all known irreducible polynomials $P \in \mathbb{Z}[X]$ with $\deg P \leq 180$ and $M(P) < 1.3$. This includes the polynomials found with the prior exhaustive searches, plus a large number of polynomials discovered using several heuristic searches. These heuristic methods include testing polynomials with height 1 and a fixed length [Mossinghoff 98, Lisonek 00], a numerical descent technique [Rhin and Sac-Épée 03], methods for generating polynomials whose coefficients are in a particular sense close to those of certain cyclotomic polynomials [Mossinghoff et al. 98, Rhin and Sac-Épée 03], and checking polynomials associated with certain small limit points of Mahler measures [Boyd and Mossinghoff 05].

Of these searches, the method of testing polynomials of a particular degree with height 1 and fixed length n has been remarkably successful. It is well known (see, for instance, [Mossinghoff 03]) that if $M(P) < 2$, then there exists a polynomial Q in $\mathbb{Z}[X]$ with $H(PQ) = 1$. It is not known whether one can always select Q with $M(Q) = 1$, but in practice this appears to be the case, at least for polynomials P with fairly small degree. Further, it seems that often one can select Q with very small degree. (It follows from [Bombieri and Vaaler 87] that if P is irreducible, $\deg P = D$, and $M(P) < 2$, then such a polynomial Q exists with $\deg Q = O(\frac{D \log D}{\log 2 - \log M(P)})$.)

Table 5, in Section 3, summarizes the search on height-1 polynomials performed in [Mossinghoff 98]. The recent article [Flammang et al. 06] verified that this search found all polynomials with measure less than 1.3 and degree at most 40. In addition, no heuristic search performed in the last decade found any additional polynomials with measure less than 1.3 and degree less than

D	M_D	ν	Half of Coefficients
8	1.28063816	1	`1 0 0 1-1`
10	1.17628082	1	`1 1 0-1-1-1`
12	1.22778556	2	`1 1 1 0-1-1-1`
14	1.20002652	1	`1 0 0 1-1 0 0-1`
16	1.22427891	2	`1 1 0-1-1 0 1 1 1`
18	1.18836815	1	`1 1 1 1 0 0-1-1-1-1`
20	1.21282418	2	`1 1 0 0 1 1 0-1-1-1-1`
22	1.20501985	2	`1 0 1 0 0 1-1 1 0 0 1-1`
24	1.21885515	2	`1 0 0 0 0 1 0-1 0 0 0 0-1`
26	1.22377745	3	`1 1 1 0 0-1-1-1-1 0 0 1 0 1`
28	1.20795003	2	`1 1 1 1 0 0 0-1-1-1-1 0 0 0 1`
30	1.22561985	2	`1 0 1 0 0 0 0 0 0 0 0 0 0 1 0 1`
32	1.23608337	4	`1 1 1 1 0-1-1-2-1 0 0 1 1 0 0 0-1`
34	1.22028744	3	`1 0 1 0 0 1-1 1-1 0 0-1 1-1 0 0-1 1`
36	1.22649330	2	`1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0-1-1`
38	1.22344738	3	`1 1 0-1-1 0 0-1-1 0 1 1 0 0 1 1 0-1-1-1`
40	1.23624956	3	`1 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0 1 0 0 1`
42	1.23029547	4	`1 0 1 0 0 0-1 1-1 1 0 0 1-1 1-1 1 0 0 1-1 1`
44	1.23667481	4	`1 0 0 0 0 0 0 1 0 0 0 0 0 0 0-1 0 0 0 0 0 0-1`
46	1.23074301	3	`1 0 0 1 0 1 0 0 1 0 0 0 0 0 0-1 0 0-1 0-1 0 0-1`
48	1.23220295	4	`1 1 0 0 1 1 0-1-1 0 0-1-1 0 1 1 0 0 1 1 0-1-1-1-1`
50	1.24037907	3	`1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0-1-1`
52	1.23434837	4	`1 1 0-1-1 0 0-1-1 0 1 1 0 0 1 1 0-1-1 0 0-1-1 0 1 1 1`
54	1.23656692	5	`1 0 1 0 0 0-1 1-1 1 0 0 1-1 1-1 0 0-1 1-1 1-1 0 0-1 1-1`

TABLE 1. Smallest Mahler measures of primitive irreducible noncyclotomic polynomials by degree D. The column labeled ν indicates the number of roots outside the unit disk.

174. It is somewhat surprising, then, that the searches we describe here discovered three new polynomials with Mahler measure below 1.3 and degree $D \leq 52$. These polynomials are listed in Table 4, in Section 3.

Section 2 describes the search method, and Section 3 summarizes our computations and results.

2. THE ALGORITHM

For a fixed even integer $D = 2d$ and a fixed real number $M \in (1, \theta_0]$, we wish to determine the set of primitive irreducible noncyclotomic polynomials P with $\deg P = D$ and $M(P) < M$. We describe the algorithm of [Flammang et al. 06] for calculating this set of polynomials, and our description here follows that paper. The set is computed in three principal steps. First, we determine bounds on certain symmetric functions of the roots of a polynomial P satisfying $\deg P = D$ and $M(P) < M$. These inequalities determine bounds on the coefficients of an admissible polynomial P, and we let $E_1 = E_1(D, M)$ denote the set of polynomials whose coefficients satisfy these bounds. Second, we test each polynomial $P \in E_1$ against some additional necessary conditions, and the surviving polynomials form a set $E_2 = E_2(D, M)$. This step requires the great majority of the computation time. Third, we subject each polynomial $P \in E_2$ to further tests using Graeffe root squaring, and finally we compute the measures of the surviving polynomials. This produces the set we desire, which we denote by $E_3 = E_3(D, M)$.

2.1 Notation

By Smyth's theorem, a nonzero algebraic integer with measure less than θ_0 must be reciprocal. Let P denote its minimal polynomial, which must have even degree $2d$:

$$P(X) = X^{2d} + c_1 X^{2d-1} + \cdots + c_1 X + 1 = \prod_{i=1}^{2d} (X - \alpha_i).$$

We may suppose that $|\alpha_i| \geq 1$ and $\alpha_{d+i} = 1/\alpha_i$ for $1 \leq i \leq d$. We define a polynomial Q associated with the polynomial P by the formula $X^d Q(X + 1/X) = P(X)$. Thus, Q is a monic polynomial of degree d with integer coefficients,

$$Q(X) = X^d + b_1 X^{d-1} + \cdots + b_{d-1} X + b_d,$$

and its roots are $\gamma_i := \alpha_i + 1/\alpha_i$ for $1 \leq i \leq d$. For $k \geq 1$, let

$$\gamma_{i,k} := \alpha_i^k + \frac{1}{\alpha_i^k},$$

and let

$$s_k := \sum_{i=1}^{d} \gamma_{i,k} = \sum_{i=1}^{2d} \alpha_i^k.$$

If we suppose that $1 \leq |\alpha_i| \leq M^a$ with $0 < a \leq 1$, then γ_i lies inside the ellipse

$$\mathcal{E}_a := \left\{ z = x + iy : \left(\frac{x}{A}\right)^2 + \left(\frac{y}{B}\right)^2 \leq 1 \right\},$$

where $A = M^a + M^{-a}$ and $B = M^a - M^{-a}$. Last, we let $\mathcal{E}_0$ denote the real interval $[-2, 2]$.

2.2 Bounds on the s_k

To compute the bounds on the s_k, we use a family of auxiliary functions, each of the form

$$f(z) = \mathrm{Re}(z) - \sum_{j=1}^{J} e_j \log |Q_j(z)| \geq m. \qquad (2\text{–}1)$$

Here, z is a complex number in the ellipse $\mathcal{E}_a$, the e_j are positive real numbers, and each Q_j belongs to a particular finite set of integer polynomials. The numbers e_j are always chosen to obtain the best auxiliary function. Such an auxiliary function was produced by Smyth [Smyth 84] to study the absolute trace of totally positive algebraic integers.

Certainly, $\sum_{i=1}^{d} f(\gamma_i) \geq dm$, and so

$$s_1 \geq dm + \sum_{j=1}^{J} e_j \log \left| \prod_{i=1}^{d} Q_j(\gamma_i) \right|.$$

We assume now that the polynomial Q does not divide any polynomial Q_j for $1 \leq j \leq J$. It follows that $\prod_{i=1}^{d} Q_j(\gamma_i)$ is a nonzero rational integer, since it is the resultant of Q and Q_j, and therefore

$$s_1 \geq dm.$$

Since the numbers $-\gamma_i$ also lie inside the ellipse $\mathcal{E}_a$, we obtain the same lower bound for $-s_1$ if $Q(-X)$ satisfies the same condition as Q. The same method is used to obtain bounds for s_k for $2 \leq k \leq 40$.

If we replace the real numbers e_j by rational numbers in the auxiliary function (2–1), we may write

$$f(z) = \mathrm{Re}(z) - \frac{t}{h} \log |H(z)|,$$

where $H \in \mathbb{Z}[X]$ has degree h and t is a positive real number. We wish to find a function f whose minimum m in $\mathcal{E}_a$ is as large as possible. That is, we seek a polynomial $H \in \mathbb{Z}[X]$ such that

$$\sup_{z \in \mathcal{E}_a} |H(z)|^{t/h} e^{-\mathrm{Re}(z)} \leq e^{-m}.$$

Now, if we suppose that t is fixed, say $t = 1$, it is clear that we need an effective upper bound on the quantity

$$t_{\mathbb{Z},\varphi}(\mathcal{E}_a) := \liminf_{\substack{h \geq 1 \\ h \to \infty}} \inf_{\substack{H \in \mathbb{Z}[X] \\ \deg H = h}} \sup_{z \in \mathcal{E}_a} |H(z)|^{t/h} \varphi(z), \qquad (2\text{–}2)$$

where we use the weight $\varphi(z) = e^{-\mathrm{Re}(z)}$. To find an upper bound for $t_{\mathbb{Z},\varphi}(\mathcal{E}_a)$, it suffices to obtain an explicit polynomial $H \in \mathbb{Z}[X]$, and then use the sequence of the successive powers of H. It can be seen that in (2–2), if t is fixed, we have a generalization of the classical integer transfinite diameter.

Let K be a compact subset of $\mathbb{C}$. If $P \in \mathbb{C}[X]$, we put $|P|_{\infty,K} = \sup_{z \in K} |P(z)|$. Recall that the *integer transfinite diameter* of K is defined by

$$t_{\mathbb{Z}}(K) = \liminf_{\substack{n \geq 1 \\ n \to \infty}} \min_{\substack{P \in \mathbb{Z}[X] \\ \deg P = n}} |P|_{\infty,K}^{1/n}.$$

It is known that if $K = [a, b]$ is a real interval of length $b - a \geq 4$, then $t_{\mathbb{Z}}(K) = (b - a)/4$. However, if $b - a < 4$, then $t_{\mathbb{Z}}(K) < 1$, but in this case the exact value of $t_{\mathbb{Z}}(K)$ is unknown. We also recall that a polynomial P_n with integer coefficients and positive degree n is a *Chebyshev polynomial* if its supremum norm on K is smallest among all integer polynomials of the same degree, so

$$|P_n|_{\infty,K} = \min\{|P|_{\infty,K} : P \in \mathbb{Z}[X] \text{ and } \deg P = n\}.$$

(Note that P_n need not be unique.) For more details, see [Borwein and Erdélyi 96, Flammang et al. 97, Pritsker 05].

In order to obtain a good upper bound on $t_{\mathbb{Z},\varphi}(\mathcal{E}_a)$ when the parameter t in (2–2) is fixed, in general one needs a polynomial H of rather large degree—about 10^8. However, it is not possible to compute a Chebyshev polynomial of such a large degree. Instead, we employ the third author's algorithm [Wu 03] to compute Chebyshev polynomials, or at least polynomials whose supremum norm is close to minimal, of degree less than 40, then use factors of these polynomials as the Q_j in the auxiliary function (2–1). More generally, when t varies, we first select an initial value of t (say $t_0 = 1$). We compute a polynomial H_1 of small degree (usually at most 5), let Q_1 be an irreducible factor of H_1, and select a positive real number e_1 to optimize our auxiliary function f_1. We deduce from this the value of $t = t_1$, compute a new polynomial H_2 and irreducible factor Q_2, then optimize f_2 with respect to the two factors Q_1 and Q_2. This process is continued while $\deg(H_i) \leq 40$. We use Smyth's semi-infinite linear programming method [Smyth 84] to optimize the auxiliary function at each stage.

As in [Flammang et al. 06], we also obtain additional bounds on the s_k by incorporating values of the function $g(a) := \min_{z\in\mathcal{E}_a} f(z)$. In fact, we derive seven different sets of bounds for the s_k by considering the value of the root α_1 of P of largest modulus:

- In the first six cases, we suppose that $|\alpha_1| > \sqrt{M}$. Thus, α_1 is necessarily real, and we may assume that $\alpha_1 > 0$. The six subcases arise by selecting an integer j between 1 and 6 such that
$$t_1(j) = \frac{1}{2} + \frac{j-1}{12} \le \frac{\log|\alpha_1|}{\log M} < \frac{1}{2} + \frac{j}{12} = t_2(j).$$
In case j, the numbers $\gamma_2, \ldots, \gamma_d$ all lie in the ellipse $\mathcal{E}_{\frac{1}{2}-\frac{(j-1)}{12}}$, and are therefore close to the real axis. In this case, the worst situation occurs when P has $2d-2$ roots of modulus 1 (i.e., the corresponding numbers $\gamma_{i,k}$ lie in $\mathcal{E}_0$), one root has modulus $M^{\frac{1}{2}-\frac{(j-1)}{12}}$, and the remaining root has modulus $1/M^{\frac{1}{2}-\frac{(j-1)}{12}}$.

- In the seventh case, we assume that all the roots of P have modulus at most $\sqrt{M}$. In this case, the worst situation occurs when $2d-4$ roots α_i have modulus 1, two have modulus $\sqrt{M}$, and two have modulus $1/\sqrt{M}$. Here, we may assume that the first nonzero coefficient b_i of Q with i odd satisfies $b_i > 0$.

Note that in any case, in the worst situation we have at least $d-2$ roots of Q lying in the real interval $(-2, 2)$. This is a favorable situation, since our auxiliary functions are most efficient on the real axis.

We describe a further improvement in the bounds for s_{2k} for $1 \le k \le 20$. Since the seventh case above produces the most important contribution to E_1, we study some relations between s_k and s_{2k} in this case. The heuristic idea is that s_k and s_{2k} cannot simultaneously lie too close to the respective bounds we computed earlier. We consider an auxiliary function f of the following type:

$$f(z) = \mathrm{Re}(z^2-2) + e_0 \,\mathrm{Re}(z) - \sum_{1\le j\le J} e_j \log|Q_j(z)|\,, \quad (2\text{–}3)$$

with the same conditions as before for the numbers e_j for $0 \le j \le J$ and for the polynomials Q_j. Since $\gamma_{i,2} = \alpha_i^2 + 1/\alpha_i^2 = (\alpha_i + 1/\alpha_i)^2 - 2 = \gamma_i^2 - 2$, we find that $s_2 + e_0 s_1 \ge m$. If we assume that s_1 has the value σ, then

$$s_2 \ge m - e_0\sigma. \quad (2\text{–}4)$$

We maximize the right-hand side of (2–4), which is linear in the numbers e_j, and obtain a lower bound for s_2 depending on the value of $s_1 = \sigma$. When σ increases from the lower bound of s_1 computed above, say $\sigma = -B_1$, the bound for s_2 decreases. We stop when this lower bound is less than $-B_2$, the lower bound for s_2 determined earlier. Since the ellipse $\mathcal{E}_a$ is symmetric, we may replace the numbers γ_i by the numbers $-\gamma_i$. If we replace $e_0\,\mathrm{Re}(z)$ by $-e_0\,\mathrm{Re}(z)$ in (2–3), we get the same lower bound for s_2 when s_1 takes the value $-\sigma$. We may also replace $\mathrm{Re}(z^2-2)$ by $-\mathrm{Re}(z^2-2)$ in (2–3). Then we obtain an upper bound for s_2 depending on the value of $|s_1|$. We obtain bounds on s_{2k} depending on $|s_k|$ in the same way for $2 \le k \le 20$, replacing γ_i by $\pm\gamma_{i,k}$ and M by M^k.

2.3 Computing E_1, E_2, and E_3

Using the bounds on the numbers s_k for $1 \le k \le 40$ determined in Section 2.2, we inductively obtain bounds on the coefficients c_k for $1 \le k \le d$ using Newton's formula,

$$s_k c_0 + s_{k-1}c_1 + \cdots + s_1 c_{k-1} + k c_k = 0.$$

These bounds on the c_k determine our set of reciprocal polynomials E_1.

To calculate E_2, we employ a Pascal program (in double precision) that checks additional constraints on the values of the s_k for each polynomial in E_1, and rejects any for which a required inequality fails. This program enumerates the polynomials in E_1 in each of the seven cases described in Section 2.2 separately, since each case is equipped with its own bounds on the s_k. It checks that the bounds on the s_k for $d+1 \le k \le 40$ are satisfied, and verifies any additional upper and lower bounds for each s_{2k}, relative to the value of s_k. In addition, we perform the following test. For polynomials generated in one of the first six cases (where $\alpha_1 > \sqrt{M}$), we note that the function $P(x)$ is convex for $x > t_1(j)$, with j as in Section 2.2. Thus, the line joining the points $(t_1(j), P(t_1(j)))$ and $(t_2(j), P(t_2(j)))$ intersects the real axis at a value $t_3 \le \alpha_1$. Then all the other roots of P lie in the disk of radius $M_1 = M/t_3$. In all cases, we use the Schur–Cohn algorithm [Marden 66] to compute the number of roots of P that lie outside a sequence of disks of decreasing radius. From this we obtain a lower bound on $M(P)$, and we check that this bound does not exceed M. The surviving polynomials form the set E_2, which we save to a file for the third phase of the search.

In the third phase, we use PARI [Batut et al. 02] to implement the modified Graeffe algorithm described in [Flammang et al. 06], and use this to obtain a lower bound on the measure of each polynomial in the set E_2. We reject any polynomial for which we determine that $M(P) > M$. Then we compute the Mahler measure of each of the surviving polynomials to obtain the set E_3.

D	M	$\|E_1\|$	$\|E_2\|$	$\|E_3\|$	Jobs	CPU Time
42	1.2400	29 648 652 246	2 606	3	12	0.2 day
44	1.2367	91 026 218 551	3 456	1	11	0.7 day
46	1.2308	260 567 756 447	4 170	1	11	1.8 day
48	1.2323	1 284 442 516 789	6 207	1	12	8.6 days
50	1.2420	14 890 559 906 854	16 295	2	100	99.1 days
52	1.2350	37 543 959 636 961	17 155	1	128	246.5 days
54	1.2366	265 346 561 290 684	29 685	1	757	1668.5 days

TABLE 2. Summary of search for minimal measures to degree 54.

D	M	$\|E_1\|$	$\|E_2\|$	$\|E_3\|$	Jobs	CPU Time
42	1.3	1 803 973 627 644	31 318	57	85	14.8 days
44	1.3	13 329 991 800 456	53 185	77	101	125.6 days

TABLE 3. Summary of search for measures less than 1.3 to degree 44.

D	Measure	ν	Half of Coefficients
46	1.2766661568	5	1 2 2 1-1-2-2-2-1 0 1 1 0-1-2-2-1 0 1 1 1 1 1 1
48	1.2960624293	6	1 2 2 1-1-2-2-2-1 0 1 1 0-1-2-2-1 0 1 1 1 1 1 1 1
52	1.2990402520	8	1 2 2 1 0-1-1-1-2-3-3-3-3-2-1 0 1 1 0 0 1 2 3 3 2 1 1

TABLE 4. New polynomials with Mahler measure less than 1.3.

3. RESULTS

We used this method to perform two searches for polynomials with small Mahler measure. First, we searched for polynomials with especially small measure by setting the value of M for each fixed degree to a real number that is slightly larger than the smallest known Mahler measure of a primitive irreducible noncyclotomic integer polynomial of that degree. Table 2 shows the value of M selected in each case in these computations, which were completed through degree 54. Second, we determined all polynomials with measure less than 1.3 and degree 42 or 44. Table 3 summarizes these calculations.

Both tables exhibit the size of each set E_1, E_2, and E_3 constructed by the method, the number of jobs used to compute E_2, and the total computation time to construct E_2, given the bounds on the s_k. These computations were distributed across multiple processors in a simple way. For smaller degrees, we allowed one processor to handle one of the cases described in Section 2.2 (although case 7 was always split over several computers). For larger degrees, each case was split among several processors by prescribing up to three initial coefficients (besides $c_0 = 1$) of the polynomials examined in a particular job.

The first set of computations verified that each of the polynomials listed in Table 1 indeed has the minimal Mahler measure among the primitive irreducible noncyclotomic polynomials of the same degree. The second set verified that the lists of polynomials in [Mossinghoff 07] with measure less than 1.3 and degree 42 or 44 are complete. Incidentally, it also established that the list of known Salem numbers less than 1.3 shown in [Mossinghoff 07] is also complete to degree 44, extending the result of [Flammang et al. 99], where this was verified up to degree 40.

Our computations also uncovered three new polynomials with measure less than 1.3. These polynomials are exhibited in Table 4. It is interesting that these polynomials were not discovered in prior heuristic searches, considering that the method of searching sparse polynomials with height 1 was so successful in identifying polynomials with small Mahler measure. However, using lattice reduction we may compute sparse height-1 multiples of each of the new polynomials, and with this we see how the prior searches missed these three examples. For example, multiplying the new degree-46 polynomial by the cyclotomic product $\Phi_1\Phi_2^2\Phi_4\Phi_6\Phi_8\Phi_{12}\Phi_{22}\Phi_{26}$, where Φ_n denotes the nth cyclotomic polynomial, produces a polynomial with height 1 and length 12:

$$X^{83} - X^{76} - X^{66} + X^{50} + X^{44} - X^{43} + X^{40} - X^{39} - X^{33} + X^{17} + X^7 - 1.$$

However, the search of twelve-term height-1 polynomials in [Mossinghoff 98] checked only up to degree 75. We also find height-1 multiples of this polynomial with the same measure of degree 70 and length 14 (using $\Phi_1\Phi_2\Phi_4\Phi_6\Phi_8\Phi_{12}\Phi_{22}$), degree 53 and length 16 (using

n	D
5, 6, 7	181
8, 9	131
10, 11	101
12, 13	75
14, 15	55
16, 17	47
18, 19	43

TABLE 5. Maximum degree D checked in search of polynomials with height 1 and length n, from [Mossinghoff 98].

$\Phi_1\Phi_6\Phi_8$), and degree 49 and length 18 (using $\Phi_1\Phi_6$). Table 5 shows that none of these was covered by the earlier searches of sparse polynomials. Similar phenomena occur for the other two new polynomials. In fact, the best sparse multiple of the degree-52 polynomial we find with height 1 and the same measure has degree 73 and length 18, achieved by multiplying by the cyclotomic product $\Phi_1\Phi_3\Phi_6\Phi_{15}\Phi_{24}$.

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