A Lower Bound for the Maximum Topological Entropy of $(4k + 2)$-Cycles

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CONTENTS

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For continuous interval maps we formulate a conjecture on the shape of the cycles of maximum topological entropy of period $4k + 2$. We also present numerical support for the conjecture. This numerical support is of two different kinds. For periods 6, 10, 14, and 18 we are able to compute the maximum-entropy cycles using nontrivial ad hoc numerical procedures and the known results of [Jungreis 91]. In fact, the conjecture we formulate is based on these results.

For periods $n = 22$, 26, and 30 we compute the maximum-entropy cycle of a restricted subfamily of cycles denoted by C_n^*. The obtained results agree with the conjectured ones. The conjecture that we can restrict our attention to C_n^* is motivated theoretically. On the other hand, it is worth noticing that the complexity of examining all cycles in C_{22}^*, C_{26}^*, and C_{30}^* is much less than the complexity of computing the entropy of each cycle of period 18 in order to determine those with maximal entropy, therefore making it a feasible problem.

1. INTRODUCTION

We embark on the final stages of the program of classification of maximum-entropy n-cycles and n-permutations. This problem has its genesis in Šarkovs′kiĭ's theorem [Šarkovs′kiĭ 64, Šarkovs′kiĭ 95], which describes an ordering of the set of possible periods of periodic points of a continuous map of an interval onto itself. If $f: I \to I$ is such a map and P is a finite, fully invariant set of f (that is, $f(P) = P$ and so P is a periodic orbit or union of periodic orbits), intrinsic information about the map is encoded in the set P.

We can think of the set P as a permutation θ induced by $f|_P$ in a natural way. If $S = \{p_1, p_2, \ldots, p_n\}$ with $p_1 < p_2 < \cdots < p_n$ is any finite, fully invariant set, then we define $\theta: \{1, \ldots, n\} \to \{1, \ldots, n\}$:

$$\theta(i) = j \iff f(p_i) = p_j.$$

The permutation θ is called the *type* of S.

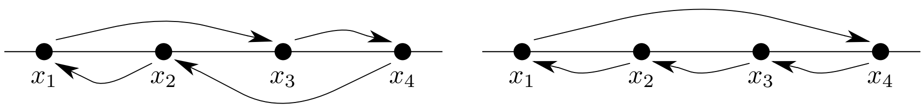

FIGURE 1. Two orbits of period 4 of different types.

In the early 1990s Misiurewicz and Nitecki [Misiurewicz and Nitecki 91], building on work of Baldwin [Baldwin 87], developed a more detailed description of the invariant sets of an interval map. The order that they described encompassed not only the period of the orbit but also its type; for example, the period-4 orbit $f(1) = 3$, $f(2) = 1$, $f(3) = 4$, $f(4) = 2$ has a different type from that of the period-4 orbit $g(1) = 4$, $g(2) = 1$, $g(3) = 2$, $g(4) = 3$ (see Figure 1).

A natural question arising from this work is this: for the sets P_n of permutations of length n and the sets C_n of cyclic permutations of length n ($n \in \mathbb{N}$), can we identify those elements that represent the periodic orbits and invariant sets (in general) that are the most complicated in terms of their dynamics? To answer this question, we can consider the topological entropy of these permutations, which gives us a numerical measure of the complexity for each permutation.

The topological entropy of a permutation θ, which will be denoted by $h(\theta)$, is defined as follows:

$$h(\theta) := \inf\{\, h(f) : f \text{ has an invariant set of type } \theta \,\},$$

where the topological entropy $h(f)$ of a map f, first defined in [Adler et al. 65], is a topological invariant that measures the dynamical complexity of f (for more information on the definition and basic properties of the topological entropy, see also [Alsedà et al. 00]).

Typically, computing the entropy of a map is difficult. However, the computation of the entropy of a permutation can be easily done using the following algebraic tools: if S is a finite, fully invariant set for f of type θ, then there is a unique map $f_\theta \colon [1, n] \to [1, n]$ that satisfies

(i) $f_\theta(i) = \theta(i)$, for $i \in \{1, \ldots, n\}$;

(ii) f_θ is affine on each interval $I_i = \{x \in \mathbb{R} : i \leq x \leq i + 1\}$ for each $i \in \{1, \ldots, n - 1\}$.

The map f_θ is known as the "connect-the-dots" map, and clearly it has an invariant set of type θ. From this map we can construct a matrix $M(\theta)$ with i, j entry given by

$$m_{ij} = \begin{cases} 1, & \text{if } f_\theta(I_i) \supset I_j, \\ 0, & \text{otherwise,} \end{cases}$$

for $i, j \in \{1, \ldots, n - 1\}$. It is well known (see [Block and Coppel 92, Proposition VIII.19]) that

$$h(\theta) = \log(\rho(M(\theta))) \geq 0,$$

where $\rho(M(\theta))$ is the spectral radius of $M(\theta)$.

In their paper, Misiurewicz and Nitecki obtain an asymptotic result that shows that the maximum entropy for n-cycles and n-permutations approaches $\log(2n/\pi)$ as $n \to \infty$. To prove this result, they constructed a family of cyclic permutations of period $n \equiv 1 \pmod 4$ that has the required asymptotic growth rate. Geller and Tolosa [Geller and Tolosa 92] extended this definition to a family of periodic orbits of period $n \equiv 3 \pmod 4$ and proved that this family in fact does have maximum entropy among all n-permutations.

This family was later shown to be unique [Geller and Weiss 95]. Since the family described is a family of cyclic permutations, the question of which n-cycles and n-permutations have maximum topological entropy for n odd has been completely answered.

For the case n even, the classification turns out to be somewhat more complicated, since the maximum-entropy n-permutations are acyclic. All maximum-entropy n-permutations for n even were described by King [King 97, King 97] and independently by Geller and Zhang [Geller and Zhang 98].

The remaining problem of classifying maximum-entropy n-cycles (n even) has been a much tougher nut to crack. While we can calculate the entropy of all n-cycles for a given n in the cases of n small, the number of n-cycles grows very fast, and so this quickly becomes an unrealistic approach to finding a solution. Despite these computational restrictions, two families of maximum-entropy $4k$-cycles have been described [King and Strantzen 01] and have recently been shown to be the only two families with maximum entropy (up to a reversal of orientation) [King and Strantzen 05].

The outstanding case in this classification problem is to classify the maximum-entropy $(4k + 2)$-cycles, which is the subject of our current investigation.

It is well known that the complexity of these sorts of combinatorial problems grows factorially, so a naive approach to this problem (generating all n-cycles and selecting those with maximum entropy) is infeasible from a computational point of view. Therefore, to carry on our investigation it is essential to find a valid means of restricting the number of cycles to be considered. The set C_n is endowed with a partial order, usually called the *forcing relation* (see [Jungreis 91] or [Misiurewicz and Nitecki 91] for details). It has been shown that topological entropy respects this partial order on C_n (see [Misiurewicz and Nitecki 91]), so that if ϕ is smaller than θ in the forcing relation, then $h(\phi) \leq h(\theta)$. As a consequence, any candidates for maximum-entropy cycles must be forcing-maximal in C_n. According to Jungreis, the forcing-maximal cycles satisfy the statements of Corollary 9.6 and Theorem 9.13 of [Jungreis 91]. We will call such cycles *Jungreis cycles* (see Section 2 for a precise definition). Therefore, any candidates for maximum-entropy cycles must be Jungreis.

Using appropriate numerical procedures, we have computed the topological entropy of all Jungreis cycles in C_n for $n \leq 17$, with the aim of both obtaining the unknown maximal entropy cycles for periods 6, 10, and 14 and testing the speed of this naive approach. Moreover, by developing nontrivial numerical procedures, which is one of the main issues of this paper, we have also identified the maximum-entropy cycle for period 18. So, we have obtained the maximum-entropy $(4k+2)$-cycles for $k = 1, 2, 3, 4$.

Performing the same kind of numerical exploration for $n \geq 22$ is beyond any current computer capabilities. However, by generalizing the results obtained for $n \leq 18$ we have defined three families of cycles (one for $n = 4k+2$, $k \geq 3$, k odd, and two for $n = 4k+2$, k even) with entropies that act as lower bounds for the maximum topological entropy in C_n for each respective case. Furthermore, we believe that these families are indeed those whose entropies are maximal in C_n.

This paper is organized as follows. In Section 2 we state Jungreis's results and define the notion of a Jungreis cycle. We split the set of all Jungreis cycles into two subsets C_n^0 and C_n^1, which are constructed and explored using two different computational approaches. The notation, tools, and algorithmic strategies to explore C_n^0 and C_n^1 are developed in Sections 3 and 4 respectively. These techniques have been used to systematically explore the case $n = 18$. The results obtained are also reported in these sections.

In Section 5 we introduce the families of $(4k+2)$-cycles that generalize the previous computational results. These families are candidates for maximum-entropy cycles. Finding the maximum-entropy n-cycle using the algorithm described in Sections 3 and 4 is not feasible in computational terms when $n > 18$. So, in Section 6 we study the problem of finding entropy-maximal $(4k+2)$-cycles, $k \geq 5$, on a restricted set of cycles that is a subclass of C_n^0. The validity of this restriction is motivated theoretically and justified numerically in the same section. Finally, in Section 7 we derive some conclusions and formulate the conjectures supported by the numerical experiments motivating the paper.

The C++ code of the programs that we have used to perform the computations in the paper, together with a file with brief instructions describing how to compile and use them, are available from http://www.mat.uab.cat/~alseda/research/.

2. JUNGREIS CYCLES

An n-permutation θ will be called *maximodal* if every point $1, 2, \ldots, n$ is either a local maximum or a local minimum for f_θ. An n-cycle θ will be called a *Jungreis cycle* if it is maximodal and f_θ satisfies one of the following conditions:

(J.i) all maximum values are above all minimum values;

(J.ii) exactly one maximum value is less than some minimum value and exactly one minimum value is greater than some maximum value.

The sets of Jungreis n-cycles satisfying (J.i) and (J.ii) will be respectively denoted by C_n^0 and C_n^1.

The following result is an immediate consequence of Corollary 9.6 and Theorem 9.13 of [Jungreis 91][1] together with the previously stated fact that topological entropy respects the forcing relation:

Theorem 2.1. *Each maximum-entropy cycle is a Jungreis cycle.*

Hence, to compute the maximum-entropy n-cycle, it is enough to explore the class of all Jungreis n-cycles. To have an idea of the computational complexity of this task, see in Table 1 the number of Jungreis n-cycles for each n between 4 and 17.

In the cases $n \leq 17$ (in particular, for $n = 4k+2$ for $k \leq 3$) we have calculated the entropies of all Jungreis n-cycles using a straightforward procedure. Specifically, we generate all maximodal n-cycles in lexicographic order and we discard those that are not Jungreis. For each

[1] In fact, Jungreis proves that the forcing-maximal cycles satisfy a third condition, which we do not consider here, since it is too difficult to implement algorithmically in an efficient way.

n	Card(C_n^0)	Card(C_n^1)	Total
4	2	0	2
5	2	1	3
6	7	5	12
7	24	15	39
8	72	105	177
9	288	561	849
10	1452	3228	4680
11	8640	20548	29188
12	43320	145572	188892
13	259200	1084512	1343712
14	1814760	8486268	10301028
15	14515200	73104480	87619680
16	101606400	636109560	737715960
17	812851200	5937577920	6750429120

TABLE 1. The number of elements in C_n^0 and C_n^1.

remaining cycle, we compute the Markov matrix and its spectral radius using the power method. The output of the program is the maximum spectral radius in this set.

This direct method has been implemented in C++ and when executed on a standard personal computer, gives the maximum entropy for cycles of period less than 14 in a matter of seconds. The case 14 takes a few minutes. For periods 15, 16, and 17, the program has to be executed on a more-powerful computer,[2] with running times of about half an hour, 5 hours, and 38 hours respectively.[3] In Table 2 we present the collection of maximum-entropy n-cycles for $n \in \{4, \dots, 17\}$ together with their respective entropies.

Of course, the results for periods 6, 10, and 14 are new and have been obtained using the method above. Since the maximum-entropy cycles for periods different from 6, 10, and 14 are already known, it was not strictly necessary for us to perform these lengthy computations in these cases. Our purpose in doing so was firstly, to gain an estimate of how long the execution of our method would take in each case; secondly, to determine how fast the execution time was increasing from one period to the next (to estimate the feasibility of the study of period 18 with the same techniques); and finally to verify that we had developed the procedure in a valid way by testing it in known situations.

In view of our previous discussion (and the reported execution times), extending the investigation to periods larger than 17 has proved challenging, since the number of Jungreis 18-cycles is already too large to be explored by this straightforward method. In Sections 3 and 4 we introduce some new tools to construct and explore the sets C_n^0 and C_n^1 (n even) in an efficient way. These tools have been shown to be powerful enough to test all Jungreis 18-cycles in a reasonable amount of time.

[2]In our case, a Dual Xeon at 2.66 GHz with hyperthreading.
[3]With a CPU usage higher than 95%.

3. EFFICIENT GENERATION OF CYCLES IN C_n^0

In this section, C^0 will stand for C_n^0. Each permutation $\theta \in P_n$ will be written as a sequence $(c_1, c_2, \dots, c_n)$, where $c_i = \theta(i)$. Hence the set P_n will be used to denote both the set of n-permutations and the set of sequences $\{(c_1, c_2, \dots, c_n) : \{c_1, c_2, \dots, c_n\} = \{1, 2, \dots, n\}\}$, without confusion.

For a sequence $\alpha = (c_1, c_2, \dots, c_n)$ with $1 \le c_i \le n$, the *dual* of α, denoted by $d(\alpha)$, is the sequence

$$(n+1-c_n, n+1-c_{n-1}, \dots, n+1-c_1).$$

Observe that $d(d(\alpha)) = \alpha$. It is well known that when $\theta \in P_n$, the entropies of θ and $d(\theta)$ are equal, since the corresponding connect-the-dots maps are topologically conjugated.

For the remainder of Sections 3 and 4, we will assume that $p \ge 3$ is an integer and that $n = 2p$. Also, N_p and Q_p will denote respectively the set of all sequences of p distinct integers $(c_1, c_2, \dots, c_p)$ such that $1 \le c_i \le n$ and the analogous set for $p+1 \le c_i \le n$, $i = 1, 2, \dots, p$. Note that $P_p, Q_p \subset N_p$.

In what follows we will also use the following three maps:

- $\widehat{\sigma}^+ \colon \{1, 2, \dots, p\} \to \{1, 2, \dots, p\}$ defined by

$$\widehat{\sigma}^+(a) := \begin{cases} a+1 & \text{if } a < p, \\ 1 & \text{if } a = p; \end{cases}$$

- $\widehat{\sigma}^- \colon \{p+1, p+2, \dots, n\} \to \{p+1, p+2, \dots, n\}$ defined by

$$\widehat{\sigma}^-(a) := \begin{cases} a-1 & \text{if } a > p+1, \\ n & \text{if } a = p+1; \end{cases}$$

- $\widehat{\delta} \colon \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$ defined by $\widehat{\delta}(a) := n+1-a$.

Observe that $\widehat{\delta}(\widehat{\sigma}^+(a)) = \widehat{\sigma}^-(\widehat{\delta}(a))$ for every $a \in \{1, 2, \dots, p\}$ and $\widehat{\delta}(\widehat{\sigma}^-(a)) = \widehat{\sigma}^+(\widehat{\delta}(a))$ for every $a \in \{p+1, p+2, \dots, n\}$.

n	Maximum-Entropy Cycles	Entropy
4	(2,4,1,3),	
	$(3, 1, 4, 2)$	$0.881373587\ldots$
5	$(2, 4, 1, 5, 3)$	$1.083936863\ldots$
6	$(3, 6, 2, 5, 1, 4)$	$1.256056722\ldots$
7	$(4, 6, 2, 7, 1, 5, 3)$	$1.454520522\ldots$
8	$(4, 6, 1, 8, 2, 7, 3, 5)$,	
	$(5, 3, 7, 2, 8, 1, 6, 4)$	$1.609651344\ldots$
9	$(4, 6, 2, 8, 1, 9, 3, 7, 5)$	$1.721042556\ldots$
10	$(6, 4, 9, 3, 8, 2, 10, 1, 7, 5)$	$1.815568127\ldots$
11	$(6, 8, 4, 10, 2, 11, 1, 9, 3, 7, 5)$	$1.929670502\ldots$
12	$(6, 8, 4, 10, 3, 11, 1, 12, 2, 9, 5, 7)$,	
	$(7, 5, 9, 2, 12, 1, 11, 3, 10, 4, 8, 6)$	$2.024121348\ldots$
13	$(6, 8, 4, 10, 2, 12, 1, 13, 3, 11, 5, 9, 7)$	$2.101379638\ldots$
14	$(7, 9, 4, 10, 1, 14, 2, 12, 3, 13, 5, 11, 6, 8)$	$2.169240867\ldots$
15	$(8, 10, 6, 12, 4, 14, 2, 15, 1, 13, 3, 11, 5, 9, 7)$	$2.247430219\ldots$
16	$(8, 10, 6, 12, 3, 15, 1, 16, 2, 14, 4, 13, 5, 11, 7, 9)$,	
	$(9, 7, 11, 5, 13, 4, 14, 2, 16, 1, 15, 3, 12, 6, 10, 8)$	$2.315471390\ldots$
17	$(8, 10, 6, 12, 4, 14, 2, 16, 1, 17, 3, 15, 5, 13, 7, 11, 9)$	$2.374577194\ldots$

TABLE 2. The maximum-entropy n-cycles for periods smaller than 18. The notation $(c_1, c_2, \ldots, c_n)$ for an n-cycle θ means that $c_i = \theta(i)$ for $1 \le i \le n$.

These three maps can be extended, in a straightforward way, to self-maps of N_p as follows. We define the map $\sigma^+\colon P_p \to P_p$ by setting

$$\sigma^+\big(a_1, a_2, \ldots, a_p\big) := \big(\widehat{\sigma}^+(a_1), \widehat{\sigma}^+(a_2), \ldots, \widehat{\sigma}^+(a_p)\big),$$

the map $\sigma^-\colon Q_p \to Q_p$ by

$$\sigma^-\big(a_1, a_2, \ldots, a_p\big) := \big(\widehat{\sigma}^-(a_1), \widehat{\sigma}^-(a_2), \ldots, \widehat{\sigma}^-(a_p)\big),$$

and finally, the map $\delta\colon N_p \to N_p$ by

$$\delta(a_1, a_2, \ldots, a_p) := \big(\widehat{\delta}(a_p), \widehat{\delta}(a_{p-1}), \ldots, \widehat{\delta}(a_1)\big).$$

The next result follows easily from the above definitions.

Lemma 3.1. *The following statements hold:*

1. *For every $\alpha \in N_p$ it follows that $\delta(\delta(\alpha)) = \alpha$.*
2. *σ^+ is a bijection from P_p onto P_p.*
3. *σ^- is a bijection from Q_p onto Q_p.*
4. *δ is a bijection between P_p and Q_p.*
5. *For every α in P_p it follows that $\delta(\sigma^+(\alpha)) = \sigma^-(\delta(\alpha))$.*
6. *For every α in Q_p it follows that $\delta(\sigma^-(\alpha)) = \sigma^+(\delta(\alpha))$.*

Definition 3.2. Let $\alpha = (a_1, a_2, \ldots, a_p)$ and $\beta = (b_1, b_2, \ldots, b_p)$ be sequences from N_p. Then, the sequence

$$(a_1, b_1, a_2, b_2, \ldots, a_p, b_p)$$

of length n will be denoted by $\alpha \oplus \beta$. Now we define the following two products:

cross product: For $\alpha, \beta \in N_p$ we define

$$\alpha \otimes \beta := \alpha \oplus \delta(\beta).$$

dot product: For $\alpha, \beta \in P_p$ we define

$$\alpha \odot \beta := \sigma^-(\delta(\alpha)) \oplus \sigma^+(\beta).$$

The proof of the next result is a simple exercise that follows directly from Lemma 3.1.

Lemma 3.3. *Let $\alpha, \beta \in N_p$. Then, $d(\alpha \oplus \beta) = \delta(\beta) \oplus \delta(\alpha)$, $d(\alpha \otimes \beta) = \beta \otimes \alpha$, and whenever $\alpha, \beta \in P_p$, $d(\alpha \odot \beta) = \beta \odot \alpha$.*

Remark 3.4. Clearly, each cycle $\theta \in C^0$ can be written as $\theta = \alpha \oplus \beta$ with $\alpha = (a_1, a_2, \ldots, a_p)$ and $\beta = (b_1, b_2, \ldots, b_p)$, where either

(i) $\alpha \in P_p$ and $\beta \in Q_p$, in which case the minimum values are $\{a_1, a_2, \ldots, a_p\}$ and the maximum values are $\{b_1, b_2, \ldots, b_p\}$, or

(ii) $\alpha \in Q_p$ and $\beta \in P_p$, in which case the maximum values are $\{a_1, a_2, \ldots, a_p\}$ and the minimum values are $\{b_1, b_2, \ldots, b_p\}$.

We define $C^{0,m}$ to be the subset of C^0 that contains all cycles for which $f_\theta(1)$ is a minimum, and $C^{0,M}$ to be the subset of C^0 that contains all cycles for which $f_\theta(1)$ is a maximum. We note that θ satisfies Remark 3.4(i) precisely when $\theta \in C^{0,m}$, and θ satisfies Remark 3.4(ii) precisely when $\theta \in C^{0,M}$.

By combining Remark 3.4 and Lemma 3.1 we easily obtain the following result, which says that every cycle in C^0 can be written either as a cross product or as a dot product of two permutations in P_p. This is the key result in this section and the one that motivated the definitions of cross and dot product.

Proposition 3.5. *If $\theta \in C^{0,m}$ then $\theta = \theta_1 \otimes \theta_2$ for some $\theta_1, \theta_2 \in P_p$. If $\theta \in C^{0,M}$, then $\theta = \theta_1 \odot \theta_2$ for some $\theta_1, \theta_2 \in P_p$.*

Remark 3.6. For every $\alpha, \beta \in P_p$ it follows that $\theta = \alpha \otimes \beta$ and $\theta' = \alpha \odot \beta$ are elements of P_n that are always maximodal and have all maxima above all minima. Moreover, $f_\theta(1)$ is a minimum, whereas $f_{\theta'}(1)$ is a maximum. Despite these facts, the converse of Proposition 3.5 does not hold, since in general, $\alpha \otimes \beta$ and $\alpha \odot \beta$ need not be cycles. To see this, consider the following examples: $(3, 1, 2) \otimes (1, 2, 3) = (3, 4, 1, 5, 2, 6)$, which is not a cycle because it contains the cycle $\{1, 3\}$, and $(2, 3, 1) \odot (1, 2, 3) = (5, 2, 6, 3, 4, 1)$, which has 2 as a fixed point.

In view of all we have said above, if the cross (respectively dot) product of two elements of P_p belongs to C_n, then it clearly belongs to $C^{0,m}$, respectively $C^{0,M}$.

3.1 Algorithmic Strategy to Generate C^0

We create a list, $\mathcal{A}$, consisting of all elements of P_p endowed with any order $\preceq$ (a natural candidate is the lexicographic order). In view of Proposition 3.5 and Remark 3.6, we have to compute all products $\alpha \otimes \beta$ and $\alpha \odot \beta$, for $\alpha, \beta \in \mathcal{A}$, and in each case, check whether the obtained permutation is a cycle. Note that since the entropies of a cycle and its dual are equal, Lemma 3.3 implies that it is enough to consider only those products $\alpha \otimes \beta$ and $\alpha \odot \beta$ for $\alpha, \beta \in \mathcal{A}$ such that $\alpha \preceq \beta$.

This algorithm is still inefficient, since we spend a lot of time performing products that do not produce cycles. For instance, there is a substantial proportion of permutations $\alpha \in P_p$ such that $\alpha \otimes \beta$ and $\alpha \odot \beta$ are not cycles for any β. To improve efficiency, these α's should be discarded from $\mathcal{A}$. Observe that neither condition can be derived from the other (for instance, $(2, 3, 1) \otimes \beta$ never gives a cycle, while $(2, 3, 1) \odot (3, 2, 1)$ is a cycle). Next we state and prove some results that allow us to decide whether a permutation α can be deleted from $\mathcal{A}$.

For $a, b \in \mathbb{N}$ with $1 \le a \le b$, we establish the following notation:

$$\begin{aligned} [a, b] &:= \{m \in \mathbb{N} : \ a \le m \le b\}; \\ O[a, b] &:= \{m \in \mathbb{N} : m \text{ is odd and } a \le m \le b\}; \\ E[a, b] &:= \{m \in \mathbb{N} : m \text{ is even and } a \le m \le b\}. \end{aligned}$$

For each $\alpha \in P_p$, we define two injective maps

$$\phi_\alpha \colon O[1, n] \to [1, p]$$

given by

$$\phi_\alpha(2i - 1) = \alpha(i),$$

for $1 \le i \le p$, and

$$\varphi_\alpha \colon O[1, n] \to [p + 1, n]$$

given by

$$\varphi_\alpha(n + 1 - 2i) = \begin{cases} n - \alpha(i) & \text{if } \alpha(i) \ne p, \\ n & \text{if } \alpha(i) = p, \end{cases}$$

for $1 \le i \le p$.

Lemma 3.7. *Let $\alpha \in P_p$. Then $\alpha \otimes \beta$ is not a cycle for any $\beta \in P_p$ if and only if ϕ_α has a cycle.*

To better understand the meaning of the above lemma and the definition of the map ϕ_α, we consider the following example.

Example 3.8. Let $\alpha = (3, 4, 2, 1, 5) \in P_5$. Then,

$$\phi_\alpha \colon \{1, 3, 5, 7, 9\} \to \{1, 2, 3, 4, 5\},$$

where $\phi_\alpha(1) = 3$, $\phi_\alpha(3) = 4$, $\phi_\alpha(5) = 2$, $\phi_\alpha(7) = 1$, and $\phi_\alpha(9) = 5$. Each point in $O[1, 10]$ has a finite ϕ_α-orbit that terminates with an even number or is a cycle of odd numbers. Here, $\mathrm{Orb}_{\phi_\alpha}(1) = \{1, 3, 4\}$, $\mathrm{Orb}_{\phi_\alpha}(3) = \{3, 4\}$, $\mathrm{Orb}_{\phi_\alpha}(5) = \{5, 2\}$, $\mathrm{Orb}_{\phi_\alpha}(7) = \{7, 1, 3, 4\}$, and $\mathrm{Orb}_{\phi_\alpha}(9) = \{9, 5, 2\}$, and ϕ_α has no cycles.

We note that when ϕ_α has no cycles,

(i) $\mathrm{Orb}_{\phi_\alpha}$ is a sequence of numbers that terminates exactly when an even number is reached;

(ii) $\bigcup_{i\in O[1,n]} \mathrm{Orb}_{\phi_\alpha}(i) = [1,p] \cup O[1,n]$. Hence, the elements of $E[p+1,n]$ are precisely those that do not appear in $\mathrm{Orb}_{\phi_\alpha}(i)$ for any $i \in O[1,n]$.

When ϕ_α has no cycles, there always exists a β such that $\theta := \alpha \otimes \beta$ is a cycle. To construct such a θ in the example above, we first take the ϕ_α-orbits of maximal length: $\mathrm{Orb}_{\phi_\alpha}(7) = \{7,1,3,4\}$ and $\mathrm{Orb}_{\phi_\alpha}(9) = \{9,5,2\}$. Then we set $\theta(i) = \phi_\alpha(i)$ for each $i \in O[1,n]$, $\theta(2) = 7$, $\theta(4) = 6$, $\theta(6) = 8$, $\theta(8) = 10$ and $\theta(10) = 9$. This gives a permutation $\theta = (3,7,4,6,2,8,1,10,5,9)$, which is clearly cyclic. In this case, $\theta = \alpha \otimes (2,1,3,5,4)$. This construction is not unique. Observe that we could just as easily have chosen $\theta(2) = 6$, $\theta(4) = 9$, $\theta(6) = 8$, $\theta(8) = 10$, and $\theta(10) = 7$, giving the cycle $\theta = (3,6,4,9,2,8,1,10,5,7) = \alpha \otimes (4,1,3,2,5)$.

Proof of Lemma 3.7: Assume that ϕ_α has an m-cycle X. In particular, each element of X is odd and not larger than p, so that $m < p/2+1$. Let $\beta \in P_p$ and $\theta := \alpha \otimes \beta \in P_n$. By the definition of the cross product we have $\alpha(i) = \theta(2i-1)$ for $1 \le i \le p$. So θ and ϕ_α take the same values in $O[1,n]$, and therefore, X is also an m-cycle of θ. Since $m < p/2+1 < n$, θ is not an n-cycle.

Now assume that ϕ_α has no cycles. We will show that $\alpha \otimes \beta$ is an n-cycle for some $\beta \in P_p$. Observe that

$$\begin{aligned}\theta &:= \alpha \otimes \beta \\ &= (\alpha_1, n+1-\beta_p, \alpha_2, n+1-\beta_{p-1}, \dots, \alpha_p, n+1-\beta_1) \\ &= (\phi_\alpha(1), \gamma_2, \phi_\alpha(3), \gamma_4, \dots, \phi_\alpha(n-1), \gamma_n),\end{aligned}$$

where γ_{2i} denotes $n+1-\beta_{p+1-i}$ for $1 \le i \le p$. To end the proof of the lemma it is enough to choose each γ_{2i} in such a way that the resulting $\alpha\otimes\beta$ is a cycle. To do this, we proceed as follows: Recall that ϕ_α takes values in $[1,p]$ and that $\phi_\alpha^{-1}([1,p]) = O[1,n]$. Therefore, by backward iteration of ϕ_α, for each $l \in E[1,p]$ we can construct a sequence $\{i_l^0, i_l^1, \dots, i_l^{m_l-1}\}$ such that

1. $m_l \ge 1$,
2. $i_l^0 \in O[p+1,n]$,
3. $i_l^j \in O[1,p]$ for $1 \le j \le m_l - 1$,
4. $\phi_\alpha\left(i_l^j\right) = i_l^{j+1}$ for $0 \le j \le m_l-2$ and $\phi_\alpha\left(i_l^{m_l-1}\right) = l$.

Moreover, since we are assuming that ϕ_α has no cycles, it follows that for each $i \in O[1,n]$, some ϕ_α-iterate of i belongs to $E[1,p]$. Hence,

$$\bigcup_{l\in E[1,p]} \{i_l^0, i_l^1, \dots, i_l^{m_l-1}\} = O[1,n].$$

Let m be the cardinality of $E[1,p]$ (so m equals $p/2$ if p is even and $(p-1)/2$ if p is odd). Now we define

$$\gamma_{2j} = \theta(2j) := \begin{cases} i_{2j+2}^0, & \text{for } 1 \le j < m, \\ 2j+2, & \text{for } m \le j \le p-1, \\ i_2^0, & \text{if } j = p. \end{cases}$$

Then, the permutation $\alpha \otimes \beta$ is the cycle

$$i_2^0, i_2^1, \dots, 2, i_4^0, i_4^1, \dots, 4, \dots, i_{2m}^0, i_{2m}^1, \dots, 2m,$$
$$2m+2, 2m+4, \dots, n$$

(where the successive iterates of i_2^0 are consecutively written from left to right). □

The proof of the next result is analogous to that of Lemma 3.7, and hence it is omitted:

Lemma 3.9. *Let $\alpha \in P_p$. Then $\alpha \odot \beta$ is not a cycle for any $\beta \in P_p$ if and only if φ_α has a cycle.*

Corollary 3.10. *Let $\alpha \in P_p$. If ϕ_α has a cycle that does not contain p then $\alpha\otimes\beta$ and $\alpha\odot\beta$ are not cycles for any $\beta \in P_p$.*

Proof: By Lemma 3.7, we have only to prove that $\alpha \odot \beta$ is not a cycle for any $\beta \in P_p$. By Lemma 3.9, it is enough to show that φ_α has a cycle. Let $X = \{x, \phi_\alpha(x), \dots, \phi_\alpha^{m-1}(x)\}$ be a cycle of ϕ_α not containing p and consider the map $\overline{\varphi}\colon O[1,n] \to [p+1,n]$ given by $\overline{\varphi}(n+1-2i) = n - \alpha(i)$ for $1 \le i \le p$. Since X is a cycle of the map $2i-1 \to \alpha(i)$, obviously $n - X := \{n-x, n-\phi_\alpha(x), \dots, n-\phi_\alpha^{m-1}(x)\}$ is a cycle of $\overline{\varphi}$. Moreover, since $p \notin X$, then $p = n-p \notin n-X$. So, from the definition of the maps φ_α and $\overline{\varphi}$ it follows that they take equal values over $n-X$. Therefore, $n-X$ is also a cycle of φ_α. □

In view of Corollary 3.10, each permutation $\alpha \in P_p$ such that ϕ_α has a cycle not containing p can be deleted from $\mathcal{A}$. This trick has allowed us to significantly shorten the length of $\mathcal{A}$, thus reducing the total number of cross and dot products performed, and hence the combinatorial complexity of the task.

The task of generating the reduced list $\mathcal{A}$ taking into account Corollary 3.10, performing all the products, and computing the entropy of each product that gives rise to a cycle has been implemented in C++ and executed for period 18 in eight separate parallel jobs (four dealing with each kind of product, cross and dot) on a cluster of Dual Xeon computers at 2.66 GHz with hyperthreading, with an execution time of about 6.5 hours.[4] This procedure has given

$$\begin{aligned}(10, 8, 12, 5, 13, 3, 15, 4, 16, 2, 18, 1, 17, 6, 14, 7, 11, 9)\\ = (7, 4, 1, 9, 2, 3, 5, 6, 8) \odot (7, 4, 2, 3, 1, 9, 5, 6, 8)\end{aligned}$$

as the maximum-entropy cycle in C^0_{18}, with entropy $\log(11.33428901405\ldots)$.

4. EFFICIENT GENERATION OF CYCLES IN C^1_n

In this section, C^1 will stand for C^1_n. Also, recall that $p \geq 3$ is an integer and $n = 2p$. Let $A^- \subset N_p$ be the set of sequences $(a_1, a_2, \ldots, a_p)$ such that $a_i > p$ for a unique $i \in [1, p]$ and let $A^+ \subset N_p$ be the set of sequences $(a_1, a_2, \ldots, a_p)$ such that $a_i \leq p$ for a unique $i \in [1, p]$.

The next lemma is an immediate consequence of Lemma 3.1(1) and the definitions of A^- and A^+:

Lemma 4.1. $\delta(A^+) = A^-$ *and* $\delta(A^-) = A^+$.

Remark 4.2. Clearly, each cycle $\theta \in C^1$ can be written as $\theta = \alpha \oplus \beta$ with $\alpha = (a_1, a_2, \ldots, a_p)$ and $\beta = (b_1, b_2, \ldots, b_p)$, where either

(i) $\alpha \in A^-$ and $\beta \in A^+$, in which case the minimum values are $\{a_1, a_2, \ldots, a_p\}$ and the maximum values are $\{b_1, b_2, \ldots, b_p\}$, or

(ii) $\alpha \in A^+$ and $\beta \in A^-$, in which case the maximum values are $\{a_1, a_2, \ldots, a_p\}$ and the minimum values are $\{b_1, b_2, \ldots, b_p\}$.

It is worth noticing that, in both cases, $a_i \neq b_j$ for any i, j.

We define $C^{1,m}$ to be the subset of C^1 that contains all cycles for which $f_\theta(1)$ is a minimum, and $C^{1,M}$ to be the subset of C^1 that contains all cycles for which $f_\theta(1)$ is a maximum. We note that θ satisfies Remark 4.2(i) precisely when $\theta \in C^{1,m}$, and θ satisfies Remark 4.2(ii) precisely when $\theta \in C^{1,M}$.

[4] With a CPU usage higher than 95% for each job.

We will next show that any cycle in $C^{1,m}$ can be written as a cross product of two elements of A^- satisfying certain properties that will be characterized in detail.

Lemma 4.3. *Assume that* $\theta \in C^{1,m}$ *and let* $\alpha = (a_1, a_2, \ldots, a_p) \in A^-$ *and* $\beta = (b_1, b_2, \ldots, b_p) \in A^+$ *be such that* $\theta = \alpha \oplus \beta$. *Then:*

i.m: $a_i \neq n$ *for* $1 \leq i \leq p$;

i.M: $b_i \neq 1$ *for* $1 \leq i \leq p$;

ii.m: *there is an* $i \in [1, p]$ *such that* $a_i = 1$;

ii.M: *there is an* $i \in [1, p]$ *such that* $b_i = n$;

iii.m: *if* $a_i = n - 1$ *for some* i *then* $i = 1$;

iii.M: *if* $b_i = 2$ *for some* i *then* $i = p$;

iv.m: $a_1 \neq 1$;

iv.M: $b_p \neq n$.

Proof: Since $\theta \in C^{1,m}$, the minimum values are $\{a_1, a_2, \ldots, a_p\}$ and the maximum values are $\{b_1, b_2, \ldots, b_p\}$. Set $\theta = (c_1, c_2, \ldots, c_n)$. Note that $a_i = c_{2i-1}$ and $b_i = c_{2i}$ for $1 \leq i \leq p$. Statement i.m follows from the fact that n cannot be a minimum value.

Statement i.M follows from the fact that 1 cannot be a maximum value. Statement ii.m follows directly from i.M. Statement ii.M follows directly from i.m. For statement iii.m, in a maximodal n-permutation, if $n-1$ is a minimum value, then it has to be the image of either 1 or n (since there is only one integer in the range $[1, n]$ larger than $n-1$). Hence, either $c_1 = n-1$ or $c_n = n-1$. But $c_n = b_p \in \beta$; hence $c_n \neq n-1$, since we have assumed that $a_i = n-1 \in \alpha$. Statement iii.M follows analogously. To prove statement iv.m, if $a_1 = 1$, then $c_1 = 1$ and θ is not a cycle. Statement iv.M follows from the fact that if $b_p = n$, then $c_n = n$ and θ is not a cycle. □

The proof of the following lemma follows directly from the definition of d.

Lemma 4.4. *If* $\gamma \in A^-$ *satisfies properties* i.m *through* iv.m, *then* $\delta(\gamma)$ *satisfies properties* i.M through iv.M.

The next result states that each cycle in $C^{1,m}$ can be written as a cross product of two elements of A^-.

Lemma 4.5. *If* $\theta \in C^{1,m}$, *then* $\theta = \theta_1 \otimes \theta_2$ *for some* $\theta_1, \theta_2 \in A^-$ *satisfying properties* i.m *through* iv.m.

Proof: By Lemma 4.3, $\theta = \alpha \oplus \beta$ with $\alpha \in A^-$ satisfying i.m through iv.m and $\beta \in A^+$ satisfying i.M through iv.M. From Lemma 4.1 we have $\delta(\beta) \in A^-$, and from Lemma 4.4, it satisfies i.m through iv.m. By the definition of the cross product and using the fact that $\delta(\delta(\beta)) = \beta$, we can write $\theta = \alpha \otimes \delta(\beta)$. So we are done by taking $\theta_1 = \alpha$ and $\theta_2 = \delta(\beta)$. □

Remark 4.6. Obviously, the converse of Lemma 4.5 does not hold. The cross product of two elements from A^- that satisfy properties i.m through iv.m may not give an element of $C^{1,m}$ for two reasons. The first one is that such a product may fail to give a cyclic permutation as shown in the following example (here $p = 9$):

$$(10,1,3,4,5,6,7,8,9) \otimes (2,5,8,1,3,9,6,12,4)$$
$$= (10,15,1,7,3,13,4,10,5,16,6,18,7,11,8,14,9,17).$$

(The result of the above product is not a permutation since 10 and 7 appear twice while 2 and 12 are omitted and, furthermore, it contains the 2-cycle $\{7,4\}$.)

The second problem we may have is that even when the product of two elements from A^- gives a cycle, we cannot guarantee in advance that this cycle will be maximodal:

$$(10,1,3,4,5,6,7,8,9) \otimes (17,8,7,6,5,3,2,1,4)$$
$$= (10,15,1,18,3,17,4,16,5,14,6,13,7,12,8,11,9,2).$$

What is clear from the construction is that when the product of two elements from A^- satisfying properties i.m through iv.m gives a maximodal cycle, then this cycle must belong to $C^{1,m}$.

Our next step is to show that in a similar way as before, every element of $C^{1,M}$ can be written as a cross product of two elements of A^+ satisfying certain properties that will be characterized in detail.

Lemma 4.7. *Assume that $\theta \in C^{1,M}$ and let $\alpha = (a_1, a_2, \ldots, a_p) \in A^+$ and $\beta = (b_1, b_2, \ldots, b_p) \in A^-$ be such that $\theta = \alpha \oplus \beta$. Then*

I.M: *$a_i \neq 1$ for $1 \le i \le p$;*

I.m: *$b_i \neq n$ for $1 \le i \le p$;*

II.M: *there is an $i \in [1,p]$ such that $a_i = n$;*

II.m: *there is an $i \in [1,p]$ such that $b_i = 1$;*

III.M: *$a_i \neq 2$ for $1 \le i \le p$;*

III.m: *$b_i \neq n-1$ for $1 \le i \le p$.*

Proof: Since $\theta \in C^{1,M}$, the maximum values are $\{a_1, a_2, \ldots, a_p\}$ and the minimum values are $\{b_1, b_2, \ldots, b_p\}$. Set $\theta = (c_1, c_2, \ldots, c_n)$. Note that $a_i = c_{2i-1}$ and $b_i = c_{2i}$ for $1 \le i \le p$. Statement I.M follows from the fact that 1 cannot be a maximum value. Statement I.m follows from the fact that n cannot be a minimum value.

Statement II.M follows directly from I.m. Statement II.m follows directly from I.M.

For statement III.M, assume that $a_i = 2$ for some i. In a maximodal n-permutation, if 2 is a maximum value then it has to be the image of either 1 or n (since there is only one integer from 1 to n smaller than 2). Hence, either $c_1 = 2$ or $c_n = 2$. But $c_n = b_p \in \beta$, and hence $c_n \neq 2$, since we have assumed that $a_i = 2 \in \alpha$. Therefore, $2 = c_1 = a_1$. But since 1 is a maximum of θ, $c_2 = 1$. Thus $\{1,2\}$ is a 2-periodic orbit of θ, and θ is not a cycle, a contradiction.

For statement III.m, assume that $b_i = n-1$ for some i. In a maximodal n-permutation, if $n-1$ is a minimum value, then it has to be the image of either 1 or n (since there is only one integer from 1 to n larger than $n-1$). Hence either $c_1 = n-1$ or $c_n = n-1$. But $c_1 = a_1 \in \alpha$, and hence $c_1 \neq n-1$, since we have assumed that $b_i = n-1 \in \beta$. Therefore, $n-1 = c_n = b_p$. But since n is a minimum of θ, $c_{n-1} = n$. Thus $\{n-1, n\}$ is a 2-periodic orbit of θ, and θ is not a cycle, a contradiction. □

The proof of the following lemma is straightforward.

Lemma 4.8. *If $\gamma \in A^-$ satisfies properties* I.m *through* III.m*, then $\delta(\gamma)$ satisfies properties* I.m *through* III.M.

Finally, the proof of the next lemma is analogous to that of Lemma 4.5 by replacing Lemmas 4.3 and 4.4 by Lemmas 4.7 and 4.8, respectively.

Lemma 4.9. *If $\theta \in C^{1,M}$, then $\theta = \theta_1 \otimes \theta_2$ for some $\theta_1, \theta_2 \in A^+$ satisfying properties* I.M *through* III.M.

Remark 4.10. As in the case of Lemma 4.5, the converse of Lemma 4.9 does not hold for similar reasons. The examples (also for the case $p = 9$) are

$$(13,18,4,14,15,16,12,10,11)$$
$$\otimes (14,10,11,8,17,18,15,12,16)$$
$$= (13,3,18,7,4,4,14,1,15,2,16,11,12,8,10,9,11,5)$$

(where in this case the result of the above product is not a permutation because 4 and 11 appear twice and 5 and 6 are both mapped to 4) and

$$\begin{aligned}&(9,11,12,13,14,15,16,17,18)\\&\quad\otimes(15,18,17,16,14,13,12,11,9)\\&\qquad=(9,10,11,8,12,7,13,6,14,5,15,3,16,2,\\&\qquad\qquad 17,1,18,4)\end{aligned}$$

which is clearly not maximodal.

Also, as in the previous case, when the product of two elements from A^+ satisfying properties I.M through III.M gives a maximodal cycle, this cycle must belong to $C^{1,M}$.

4.1 Algorithmic Strategy to Generate C^1

We will generate separately the elements of $C^{1,m}$ and those of $C^{1,M}$. To generate $C^{1,m}$ we need to create the list $\mathcal{Y}$ of all the elements of A^- satisfying properties i.m through iv.m, endowed with any order $\preceq$ (a natural candidate is the lexicographic order).

In view of Lemma 4.5 and Remark 4.6, we have to compute all the products $\alpha \otimes \beta$ for $\alpha, \beta \in \mathcal{Y}$, and in each case, check whether the obtained permutation is a maximodal cycle. Observe that since the entropies of a cycle and its dual are equal, in view of Lemma 3.3 it is enough to consider only the products $\alpha \otimes \beta$ for $\alpha, \beta \in \mathcal{Y}$ such that $\alpha \preceq \beta$.

Analogously, to generate $C^{1,M}$ we need to create the list $\mathcal{Z}$ of all the elements of A^- satisfying the properties I.M through III.M, endowed with an order $\preceq$. As above, using Lemmas 4.9 and 3.3 it is enough to perform all products $\alpha \otimes \beta$ for $\alpha, \beta \in \mathcal{Z}$ such that $\alpha \preceq \beta$.

Again, by Remark 4.10, in each case we have to check whether the product gives rise to a maximodal cycle.

The method of generating the two lists $\mathcal{Y}$ and $\mathcal{Z}$, performing all the products, and computing the entropy of each product that gives rise to a cycle has been implemented in C++. This program has been used for period 18, splitting the task into 16 subtasks (four dealing with each list $\mathcal{Y}$ and $\mathcal{Z}$) that have been executed in parallel on a cluster of Dual Xeon computers at 2.66 GHz with hyperthreading, with an execution time of about three months.[5] This procedure has given

$$\begin{aligned}&(9,8,12,6,13,3,15,4,16,2,18,1,17,5,14,7,11,10)\\&\quad=(9,12,13,15,16,18,17,14,11)\\&\qquad\otimes(9,12,14,18,17,15,16,13,11)\end{aligned}$$

and

$$\begin{aligned}&(9,8,12,6,13,2,18,1,17,3,15,4,16,5,14,7,11,10)\\&\quad=(9,12,13,18,17,15,16,14,11)\\&\qquad\otimes(9,12,14,15,16,18,17,13,11)\end{aligned}$$

as the maximum-entropy cycles in C^1_{18}, with entropy $\log(11.321231505957\ldots)$.

As a consequence of this together with the numerical results described at the end of Section 3, the maximum-entropy cycle for period 18 is the one in C^0_{18} reported there.

It is clear that this method cannot be extended to periods larger than 18, since the execution time is prohibitive. To continue our investigation to higher periods, it has been necessary to focus our attention on a restricted set of cycles, one that is most likely to include those of highest entropy.

5. THREE CONJECTURED FAMILIES OF CYCLES

In this section we introduce three families of $(4k+2)$-cycles (one for k odd and two for k even). They have been obtained by generalizing the computational results that have been reported in the previous sections. The topological entropy of the cycles generated by these families is monotonically increasing as $k \to \infty$.

Definition 5.1. For $n = 4k+2$, $k > 1$, we denote by θ_n the element of C^0_n that is given as follows:

- If $k = 2p$ with p odd, then

$$\theta_n : j \to \begin{cases} 4p+1+j, & \text{if } j \in O[1,p],\\ 4p+2+j, & \text{if } j \in O[p+2,3p],\\ 4p-1+j, & \text{if } j \in O[3p+2,4p+3],\\ 12p+6-j, & \text{if } j \in O[4p+5,5p+2],\\ 12p+3-j, & \text{if } j \in O[5p+4,7p],\\ 12p+4-j, & \text{if } j \in O[7p+2,n-1],\\ 4p+2-j, & \text{if } j \in E[2,p+1],\\ 4p+3-j, & \text{if } j \in E[p+3,3p+1],\\ 4p+4-j, & \text{if } j \in E[3p+3,4p+2],\\ j-4p-3, & \text{if } j \in E[4p+4,5p+3],\\ j-4p-2, & \text{if } j \in E[5p+5,7p+1],\\ j-4p-1, & \text{if } j \in E[7p+3,n]. \end{cases}$$

[5] With a typical CPU usage higher than 45% for each job.

$$
\begin{aligned}
&\Big(1\ \overbrace{4p+4-2i\ 2i\ \ldots\ 3p+3\ p+1}^{1\leq i\leq \frac{p+1}{2}}\\
&\qquad \overbrace{3p+5-4i\ p-2+4i\ 5p+4i\ 7p+3-4i\ \ldots\ p+7\ 3p-4\ 7p-2\ 5p+5}^{1\leq i\leq \frac{p-1}{2}}\\
&\qquad p+3\ 3p\ \overbrace{7p+2i\ 5p+4-2i\ \ldots\ n-3\ 4p+5}^{1\leq i\leq \frac{p-1}{2}}\ n-1\\
&\qquad \overbrace{4p+5-2i\ 8p+4-2i\ \ldots\ 3p+4\ 7p+3}^{1\leq i\leq \frac{p+1}{2}}\ 3p+2\\
&\qquad \overbrace{7p+5-4i\ 3p+3-4i\ p+4i\ 5p+2+4i\ldots\ 5p+7\ p+5\ 3p-2\ 7p}^{1\leq i\leq \frac{p-1}{2}}\\
&\qquad \overbrace{5p+5-2i\ p+2-2i\ \ldots\ 4p+4\ 1}^{1\leq i\leq \frac{p+1}{2}}\Big).
\end{aligned}
$$

FIGURE 2. The cycle θ_n for the case of $n=8p+2$, p odd.

- If $k=2p$ with p even, then

$$
\theta_n : j \rightarrow \begin{cases}
4p+1+j, & \text{if } j\in O[1,p+1],\\
4p+j, & \text{if } j\in O[p+3,3p+1],\\
4p-1+j, & \text{if } j\in O[3p+3,4p+3],\\
12p+6-j, & \text{if } j\in O[4p+5,5p+3],\\
12p+5-j, & \text{if } j\in O[5p+5,7p+1],\\
12p+4-j, & \text{if } j\in O[7p+3,n-1],\\
4p+2-j, & \text{if } j\in E[2,p],\\
4p+1-j, & \text{if } j\in E[p+2,3p],\\
4p+4-j, & \text{if } j\in E[3p+2,4p+2],\\
j-4p-3, & \text{if } j\in E[4p+4,5p+2],\\
j-4p, & \text{if } j\in E[5p+4,7p],\\
j-4p-1, & \text{if } j\in E[7p+2,n].
\end{cases}
$$

- If $k\geq 3$ is odd, then

$$
\theta_n : j \rightarrow \begin{cases}
2k-j+2, & \text{if } j\in O[1,k-2],\\
k+1, & \text{if } j=k,\\
2k-j, & \text{if } j\in O[k+2,2k-1],\\
j-2k+1, & \text{if } j\in O[2k+1,3k-2],\\
j-2k, & \text{if } j\in O[3k,3k+2],\\
j-2k-1, & \text{if } j\in O[3k+4,n-1],\\
2k+1+j, & \text{if } j\in E[2,k-1],\\
3k+1, & \text{if } j=k+1,\\
2k+3+j, & \text{if } j\in E[k+3,2k-2],\\
6k+2-j, & \text{if } j\in E[2k,3k-1],\\
6k+5-j, & \text{if } j\in E[3k+1,3k+3],\\
6k+4-j, & \text{if } j\in E[3k+5,n].
\end{cases}
$$

For example, for $n=8p+2$, p odd, it can be shown that θ_n is the cycle shown in Figure 2.

We also note the following general features of f_{θ_n}:

1. For $n=4k+2$, k even, the map f_{θ_n} has a local maximum at $j=1$, while for $n=4k+2$, k odd, the map f_{θ_n} has a local minimum at $j=1$.

2. Each cycle θ_n is maximodal, and f_{θ_n} has all maximum values greater than all minimum values (that is, $\theta_n\in C_n^0$).

3. For $n=4k+2$, k even, f_{θ_n} has a global minimum at $j=2k+4$, while for $n=4k+2$, k odd, f_{θ_n} has a global minimum at $j=2k-1$.

4. For $n=4k+2$, k even, f_{θ_n} has a global maximum at $j=2k+3$, while for $n=4k+2$, k odd, f_{θ_n} has a global maximum at $j=2k$.

Note that the entropy-maximal 6-cycle is not generated by the formulas given in Definition 5.1. However, we have computed it to be the cycle $\theta_6(1)=3,\ \theta_6(2)=6,\ \theta_6(3)=2,\ \theta_6(4)=5,\ \theta_6(5)=1,\ \theta_6(6)=4$. Moreover, all other 6-cycles (up to a reversal of orientation) have entropy strictly smaller than $h(\theta_6)$.

Figure 3 shows the asymptotic behavior of the entropies of the cycles in the conjectured families, together with those of the maximum-entropy cycles of period $4k$, compared to the Misiurewicz–Nitecki bound $\log(2n/\pi)$.

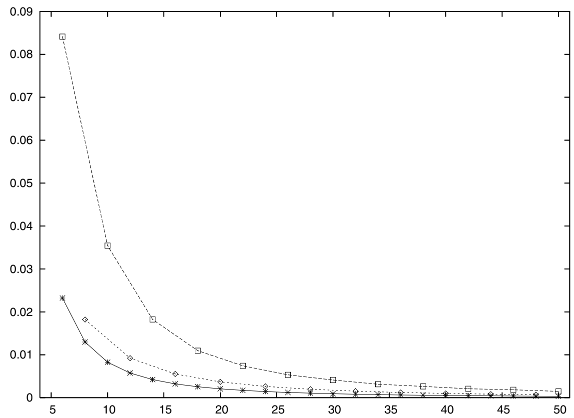

FIGURE 3. The three curves in the figure represent the difference between the Misiurewicz–Nitecki bound $\log(2n/\pi)$ and the entropies of (i) the maximum-entropy n-permutation, for $n \in E[6, 50]$ (lower curve), (ii) the maximum-entropy $4k$-cycle, for $k \in [2, 12]$ (center curve), (iii) the cycle θ_{4k+2}, for $k \in [1, 12]$ (upper curve).

6. FURTHER RESTRICTIONS FOR $n = 22$ AND BEYOND

Finding the maximum-entropy n-cycle using the algorithm described in Sections 3 and 4 is infeasible in computational terms when $n > 18$. Instead, it is necessary to make further restrictions on the subclass of cycles to be explored. For $n = 22$, we have considered the subclass C^*_{22} of 22-cycles satisfying that for each $\phi \in C^*_{22}$,

(i) $\phi \in C^0_{22}$,

(ii) $\phi(i) = \psi_{22}(i)$, for $i \in \{1, 2, 21, 22\}$,

where ψ_{22} is the entropy-maximal 22-permutation described in [King 97] (see also Table 3). We remark that θ_{22} belongs to C^*_{22}, and precisely, we have found that it is the maximum-entropy cycle in C^*_{22}. Moreover, the entropy of any other cycle in the class is strictly smaller than $h(\theta_{22})$ (up to duality).

Clearly, using this procedure, we have not calculated the entropies of a large number of 22-cycles that potentially have larger entropy than θ_{22}. However, based on preliminary results (again see Table 3), we believe that

(i) for $k \in \mathbb{N}$, the entropy-maximal $(4k + 2)$-cycles will have all maximum values above all minimum values;

(ii) for $k \geq 2$ and $n = 4k + 2$, if ψ_n is the entropy-maximal n-permutation as defined in [King 97], the maximum-entropy cycle ϕ_n will be such that $\phi_n(i) = \psi_n(n+1-i)$ for all $i \in \left[1, \frac{k}{2}+1\right] \cup \left[\frac{7k}{2}+2, n\right]$, k even, or $\phi_n(i) = \psi_n(i)$ for all $i \in [1, k-1] \cup [3k+4, n]$, k odd.

To support these claims we will now briefly explain why we think that the maximum-entropy cycles have this structure.

6.1 All Maximum Values Are above All Minimum Values

Consider the induced matrix of a maximodal $(4k + 2)$-cyclic permutation θ that has all maximum values above all minimum values. Without loss of generality we will assume that f_θ has a minimum at 1.

It is known that the jth column sum of the induced matrix $M(\theta)$ is bounded above by the minimum of $\{2j, 2(n - j)\}$ [Misiurewicz and Nitecki 91]. In this case, the upper bound is achieved for each $j \in [1, 4k+1]$. This means, for example, that column $2k+1$ (the central column of $M(\theta)$) will consist entirely of 1's. This is because $[2k + 1, 2k + 2] \subseteq [f_\theta(i), f_\theta(i + 1)]$ for all $i \in [1, 4k + 1]$. As a consequence, $\|M(\theta)\|$ is maximal, where for any nonnegative matrix A, $\|A\|$ denotes the sum of all of its entries. This is clearly an important factor in identifying maximum-entropy permutations, since

$$\rho(M(\theta)) = \lim_{m\to\infty} \|M(\theta)^m\|^{1/m}$$

(see [Seneta 81]).

However, for a maximodal cyclic permutation ϕ with one maximum value less than or equal to $2k + 1$ (and hence one minimum value greater than $2k+1$), the upper bound is not achieved in at least the $(2k + 1)$st column, since for some i odd, $[f_\phi(i), f_\phi(i + 1)] \subseteq [2k + 2, 4k + 2]$, and for some i even, $[f_\phi(i - 1), f_\phi(i)] \subseteq [1, 2k + 1]$. Thus $\left|M^{(2k+1)}\right| \in \{4k - 1, 4k - 2, 4k - 3\}$; that is, the column sum is reduced by 2, 3, or 4. It is also possible that other column sums are less than $\min\{2j, 2(n - j)\}$, reducing the value of $\|M(\theta)\|$ even further. Consequently, $\|M(\theta)\|$ is not maximal. However, it should be noted that for permutations ϕ and θ, $\|M(\phi)\| < \|M(\theta)\|$ does not necessarily imply that $h(\phi) < h(\theta)$.

The above discussion leads us to the following conjecture:

Conjecture 6.1. *For each period n, the maximum-entropy n-cycle belongs to C^0_n.*

Table 4 provides numerical evidence supporting Conjecture 6.1. Indeed, comparing the maximum entropies in C^1_n shown in this table with those in C^0_n from Table 2 (together with C^0_{18}) confirms that Conjecture 6.1 holds for $4 \leq n \leq 18$.

Period $\boldsymbol{n}$	**Max Permutation** $\boldsymbol{\psi_n}$	**Entropy**	**Max Cycle** $\boldsymbol{\theta_n}$	**Entropy**
10	$(5,7,3,9,1,10,2,8,4,6)$	$1.8427299\ldots$	$(6,4,9,3,8,2,10,1,7,5)$	$1.8155681\ldots$
14	$(7,9,5,11,3,13,1,$ $14,2,12,4,10,6,8)$	$2.1832659\ldots$	$(7,9,4,10,1,14,2,$ $12,3,13,5,11,6,8)$	$2.1692408\ldots$
18	$(9,11,7,13,5,15,3,17,1,$ $18,2,16,4,14,6,12,8,10)$	$2.4362460\ldots$	$(10,8,12,5,13,3,15,4,16,$ $2,18,1,17,6,14,7,11,9)$	$2.4278325\ldots$
22	$(11,13,9,15,7,17,5,$ $19,3,21,1,22,2,4,$ $18,6,16,8,14,10,12)$	$2.6377584\ldots$	$(11,13,9,15,6,16,3,$ $21,1,22,2,20,4,18,5,$ $19,7,17,8,14,10,12)$	$2.6320413\ldots$
26	$(13,15,11,17,9,19,7,21,5,$ $23,3,25,1,26,2,24,4,22,$ $6,20,8,18,10,16,12,14)$	$2.8052961\ldots$	$(14,12,16,10,19,9,21,7,$ $23,5,22,4,24,2,26,1,25,$ $3,20,6,18,8,17,11,15,13)$	$2.8011896\ldots$
30	$(15,17,13,19,11,21,9,$ $23,7,25,5,27,3,29,1,$ $30,2,28,4,26,6,24,8,$ $22,10,20,12,18,14,16)$	$2.9487002\ldots$	$(15,17,13,19,11,21,8,$ $22,5,27,3,29,1,30,2,$ $28,4,26,6,24,7,25,9,$ $23,10,20,12,18,14,16)$	$2.9454988\ldots$
34	$(17,19,15,21,13,23,11,25,$ $9,27,7,29,5,31,3,33,1,$ $34,2,32,4,30,6,28,8,26,$ $10,24,12,22,14,20,16,18)$	$3.0740659\ldots$	$(18,16,20,14,22,11,23,9,$ $25,7,27,5,29,6,30,4,32,$ $2,34,1,33,3,31,8,28,10,$ $26,12,24,13,21,15,19,17)$	$3.0716352\ldots$

TABLE 3. In column 2, we list the maximum-entropy permutations ψ_n defined in [King 97]. In column 4, we list the maximum-entropy cycles (obtained by numerical exploration for $n = 10, 14, 18$ and conjectured for $n = 22, 26, 30, 34$).

6.2 Positions Fixed for Certain Values

The asymptotic estimate of the upper bound for the topological entropy of a permutation obtained by Misiurewicz and Nitecki [Misiurewicz and Nitecki 91] depends on an $(n \times n)$, 0-1 matrix $\diamondsuit_n$, n even, in which the 1's form a diamond pattern. Although this matrix is not the matrix of any permutation, the matrix of the entropy-maximal $(n{+}1)$-permutation can be obtained from it with minimal changes.

More specifically, in the induced matrix of a permutation, no two columns can be identical. For the matrix $\diamondsuit_n$, there are $n/2$ pairs of identical columns (columns i and $n-i$ for $i \in [1, n/2]$). We select one column from each pair (except the central pair) and shift the 1's in that column either up or down by 1; that is, if the selected column j has 1's in rows $a, \ldots, b$, then our new column j will have 1's in rows $a+1, \ldots, b+1$ or $a-1, \ldots, b-1$. In the case of the central pair of columns, both of which contain all 1's, we simply delete a 1 in either row 1 or row n in one of the columns, as indicated:

$$\diamondsuit_8 = \begin{bmatrix} 0&0&0&1&1&0&0&0\\ 0&0&1&1&1&1&0&0\\ 0&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 0&1&1&1&1&1&1&0\\ 0&0&1&1&1&1&0&0\\ 0&0&0&1&1&0&0&0 \end{bmatrix},$$

$$M(\theta_9) = \begin{bmatrix} 0&0&0&1&1&0&0&0\\ 0&1&1&1&1&0&0&0\\ 0&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1&0\\ 1&1&1&1&1&1&1&1\\ 0&0&1&1&1&1&1&1\\ 0&0&1&1&1&1&0&0\\ 0&0&0&0&1&1&0&0 \end{bmatrix}.$$

It is natural to assume that the induced matrices of the maximum-entropy n-cycles will also retain the basic shape of $\diamondsuit_n$, with minimal variations occurring due to

n	**Max Cycles in C_n^1**	**Entropy**
5	$(4,5,2,3,1)$	$0.739693870\dots$
6	$(4,5,2,6,1,3)$	$1.209787792\dots$
7	$(5,4,6,1,7,2,3)$	$1.335115163\dots$
8	$(4,3,8,2,7,1,6,5)$,	
	$(5,6,1,7,2,8,3,4)$	$1.572942040\dots$
9	$(4,3,7,2,9,1,8,5,6)$	$1.667923417\dots$
10	$(5,4,8,1,9,3,10,2,7,6)$,	
	$(5,4,8,3,9,2,10,1,7,6)$	$1.812929625\dots$
11	$(7,6,9,3,10,1,11,2,8,4,5)$	$1.900790891\dots$
12	$(6,5,9,1,12,2,11,3,10,4,8,7)$,	
	$(6,5,9,2,12,3,11,1,10,4,8,7)$,	
	$(7,8,4,10,1,11,3,12,2,9,5,6)$,	
	$(7,8,4,10,3,11,2,12,1,9,5,6)$	$2.012291633\dots$
13	$(6,5,9,3,11,2,13,1,12,4,10,7,8)$	$2.083877398\dots$
14	$(8,9,5,11,1,14,2,12,4,13,3,10,6,7)$,	
	$(8,9,5,11,4,12,3,13,1,14,2,10,6,7)$	$2.167815951\dots$
15	$(9,8,11,5,13,3,14,1,15,2,12,4,10,6,7)$	$2.236009214\dots$
16	$(8,7,11,5,13,4,14,2,15,1,16,3,12,6,10,9)$,	
	$(8,7,11,5,13,4,16,2,14,1,15,3,12,6,10,9)$,	
	$(9,10,6,12,3,15,1,14,2,16,4,13,5,11,7,8)$,	
	$(9,10,6,12,3,16,1,15,2,14,4,13,5,11,7,8)$	$2.310571525\dots$
17	$(8,7,11,5,13,3,15,2,17,1,16,4,14,6,12,9,10)$	$2.366709736\dots$
18	$(9,8,12,6,13,3,15,4,16,2,18,1,17,5,14,7,11,10)$,	
	$(9,8,12,6,13,2,18,1,17,3,15,4,16,5,14,7,11,10)$	$2.426679857\dots$

TABLE 4. The maximum-entropy cycles in C_n^1.

cyclicity conditions. Changing the positions of the 1's in the columns of this matrix to form the matrix of a permutation cannot increase the spectral radius.

It is worth noting that in the case $n = 22$, we imposed the constraint $\phi(i) = \psi_{22}(i)$ for $i \in \{1,2,21,22\}$ purely for computational reasons. However, it turns out that the cycle of highest entropy that we found with this restriction, that is, θ_{22}, satisfies $\theta_{22}(i) = \psi_{22}(i)$ for $i \in \{1,2,3,4,19,20,21,22\}$, which gives numerical support to our argument above.

An n-permutation ϕ that satisfies conditions 1 and 2 above will have an induced matrix M such that

(i) $\langle M^{(2k+1)}\rangle = [1,n-1]$,

(ii) $\langle M_{(1)}\rangle = [2k+1,2k+2]$,

(iii) $\langle M_{(2)}\rangle = [a,2k+2]$, where $a < 2k+1$,

(iv) $\langle M_{(n-2)}\rangle = [2k,b]$, where $b > 2k+1$,

(v) $\langle M_{(n-1)}\rangle = [2k,2k+1]$,

(vi) $|M^{(j)}| = \min\{2j, 2(n-j)\}$ for all $j \in [1,n-1]$,

where $M^{(j)}$ denotes the jth column of M, $M_{(i)}$ denotes the ith row of M and $\langle M^{(j)}\rangle$ (respectively $\langle M_{(i)}\rangle$) denotes the set $\{i : a_{ij} = 1\}$ (respectively $\{j : a_{ij} = 1\}$). These six conditions imply that

(vii) $\langle M^{(2k)}\rangle = [2,n-1]$,

(viii) $\langle M^{(2k+1)}\rangle = [1,n-1]$,

(ix) $\langle M^{(2k+2)}\rangle = [1,n-2]$,

(x) $\langle M^{(j)}\rangle \subseteq [2,n-3], \forall j \le 2k-1$,

(xi) $\langle M^{(j)}\rangle \subseteq [3,n-2], \forall j \ge 2k+3$,

(xii) $\|M\|$ is maximal.

As a consequence, the 1's in the matrix will retain the approximate diamond shape.

For $n \in \{26,30\}$ we have restricted our computations to specific classes of maximodal n-cycles, in line with the discussion above. In particular, we consider the classes $C_{26}^* \subset C_{26}^0$ of cycles ϕ such that $\phi(i) = \psi_{26}(27-i)$, and $C_{30}^* \subset C_{30}^0$ of cycles ϕ such that $\phi(i) = \psi_{30}(i)$, for all $i \in \{1,2,\dots,2k-8,2k+11,\dots,n-1,n\}$. Note that as remarked in the case $n = 22$, $\theta_{26} \in C_{26}^*$ and $\theta_{30} \in C_{30}^*$. Observe also that in the cases $n \in \{22,26,30\}$, this restriction leaves $\phi \in C_n^*$ with $n-18$ positions fixed according to ψ_n, while for periods $n = 4k+2 \geq 34$ this is no longer true. Indeed, for period 34, the conjectured maximum-entropy cycle θ_{34} agrees with ψ_{34} in only 10 positions, thus leaving 24 free positions.

Thus, the task of finding the highest-entropy cycle in C_{22}^*, C_{26}^*, and C_{30}^* has a computational complexity not larger than that of finding the highest-entropy cycle in C_{18}^0, while this problem is unsolvable with the conjectures and techniques devised in this paper for periods $n = 4k+2 \geq 34$.

6.3 Algorithmic Strategy to Generate C_n^*, $n \in \{22, 26, 30\}$

As already defined, the elements of C_n^* are the maximodal cycles of period $n = 4k+2$ that have a certain pattern at the beginning and at the end of the cycle, which is determined by ψ_n. As Table 5 already shows, the parity of k completely determines the structure of C_n^*, and therefore we will consider these two cases separately. However, before continuing our discussion we should bear in mind a crucial property that influences the whole strategy of this computation: *the permutations ψ_n are self-dual* (this can be checked directly from the definition of ψ_n; see also [King 97]).

6.4 The Case k Odd

All cycles $\theta \in C_n^*$ have $f_\theta(1)$ as a minimum. Hence, in view of Proposition 3.5, $\theta = \theta_1 \otimes \theta_2$ for some $\theta_1, \theta_2 \in P_{2k+1}$. We use a method similar to that used to generate C_n to generate all cycles in C_n^*; that is, we create a list $\mathcal{A}^* \subset P_{2k+1}$ such that each $\alpha \in \mathcal{A}^*$ satisfies $\alpha(i) = \psi_n(2i-1)$ for $i \in \{1, 2, \ldots, k-4, k+6, \ldots, 2k+1\}$, and ϕ_α has no cycle not containing $2k+1$ (that is, we use Corollary 3.10 to discard those permutations α such that $\alpha \otimes \beta$ is not a cycle for any $\beta \in P_{2k+1}$). Also, as usual, we endow the list $\mathcal{A}^*$ with any order $\preceq$ (a natural candidate is the lexicographic order).

Now observe that for every $\alpha, \beta \in \mathcal{A}^*$, from the definition of the cross product and the fact that ψ_n is self-dual, we have that $\alpha \otimes \beta$ takes the form

$$\Big(\alpha(1), \widehat{\delta}(\beta(2k+1)), \alpha(2), \widehat{\delta}(\beta(2k)), \ldots, \alpha(k-4), \widehat{\delta}(\beta(k+6)), \underline{\star}_{18}, \alpha(k+6), \widehat{\delta}(\beta(k-4)), \alpha(k+7), \widehat{\delta}(\beta(k-5)), \ldots, \alpha(2k+1), \widehat{\delta}(\beta(1))\Big)$$

$$= \Big(\psi_n(1), \widehat{\delta}(\psi_n(n-1)), \psi_n(3), \widehat{\delta}(\psi_n(n-3)), \ldots, \psi_n(2k-9), \widehat{\delta}(\psi_n(2k+11)), \underline{\star}_{18}, \psi_n(2k+11), \widehat{\delta}(\psi_n(2k-9)), \psi_n(2k+13), \widehat{\delta}(\psi_n(2k-11)), \ldots, \psi_n(n-1), \widehat{\delta}(\psi_n(1))\Big)$$

$$= \Big(\psi_n(1), \psi_n(2), \ldots, \psi_n(2k-8), \underline{\star}_{18}, \psi_n(2k+11), \psi_n(2k+12), \ldots, \psi_n(n)\Big) \in C_n^*,$$

where $\underline{\star}_{18}$ denotes an undetermined sequence in P_{18}. Therefore, by Remark 3.6, to generate all elements from C_n^* we have to compute all products $\alpha \otimes \beta$ for $\alpha, \beta \in \mathcal{A}^*$, and in each case, check whether the obtained permutation is a cycle. Note that since the entropies of a cycle and its dual are equal, Lemma 3.3 implies that it is enough to consider only those products $\alpha \otimes \beta$ for $\alpha, \beta \in \mathcal{A}^*$ such that $\alpha \preceq \beta$.

6.5 The Case k Even

In the case of k even, all cycles $\theta \in C_n^*$ have $f_\theta(1)$ as a maximum and hence $\theta = \theta_1 \odot \theta_2$ for some $\theta_1, \theta_2 \in P_{2k+1}$, again by Proposition 3.5. The list $\mathcal{A}^* \subset P_{2k+1}$ is generated in exactly the same manner as in the case k odd, except that in this case we fix $\alpha(i) = (\widehat{\sigma}^+)^{-1}(\psi_n(n+1-2i))$ for $i \in \{1, 2, \ldots, k-4, k+6, \ldots, 2k+1\}$. For every $\alpha, \beta \in \mathcal{A}^*$, the product $\alpha \odot \beta$ takes the form

$$\Big(\widehat{\sigma}^-(\widehat{\delta}(\alpha(2k+1))), \widehat{\sigma}^+(\beta(1)), \widehat{\sigma}^-(\widehat{\delta}(\alpha(2k))), \widehat{\sigma}^+(\beta(2)), \ldots, \widehat{\sigma}^-(\widehat{\delta}(\alpha(k+6))), \widehat{\sigma}^+(\beta(k-4)), \underline{\star}_{18}, \widehat{\sigma}^-(\widehat{\delta}(\alpha(k-4))), \widehat{\sigma}^+(\beta(k+6)), \widehat{\sigma}^-(\widehat{\delta}(\alpha(k-5))), \widehat{\sigma}^+(\beta(k+7)), \ldots, \widehat{\sigma}^-(\widehat{\delta}(\alpha(1))), \widehat{\sigma}^+(\beta(2k+1))\Big)$$

$$= \Big(\widehat{\delta}(\psi_n(1)), \psi_n(n-1), \widehat{\delta}(\psi_n(3)), \psi_n(n-3), \ldots, \widehat{\delta}(\psi_n(2k-9)), \psi_n(2k+11), \underline{\star}_{18}, \widehat{\delta}(\psi_n(2k+11)), \psi_n(2k-9), \widehat{\delta}(\psi_n(2k+13)), \psi_n(2k-11), \ldots, \widehat{\delta}(\psi_n(n-1)), \psi_n(1)\Big)$$

$$= \Big(\psi_n(n), \psi_n(n-1), \ldots, \psi_n(2k+11), \underline{\star}_{18}, \psi_n(2k-8), \psi_n(2k-9), \ldots, \psi_n(1)\Big) \in C_n^*,$$

where $\underline{\star}_{18}$ denotes an undetermined sequence in P_{18}. Then we can generate all relevant elements in C_n^* as in the previous case: we have to compute all products $\alpha \odot \beta$ for $\alpha, \beta \in \mathcal{A}^*$ such that $\alpha \preceq \beta$ and in each case, check whether the obtained permutation is a cycle.

In Table 5 we summarize all the above information (namely the type of product to use and the structure

n	**Structure of the Restricted $\mathcal{A}$-List**	**Product**
22	$(11; \boldsymbol{P}_9; 10)$	$\otimes$
26	$(11, 9; \boldsymbol{P}_9^{13}; 10, 12)$	$\odot$
30	$(15, 13, 11; \boldsymbol{P}_9; 10, 12, 14)$	$\otimes$

TABLE 5. The structure of the $\mathcal{A}^*$-lists and the type of products to consider to compute the maximum-entropy cycle in C_n^*.

of the $\mathcal{A}^*$-list for each period) for the particular case of periods $n = 22$, 26, 30. When we write $\boldsymbol{P}_9^p$, we mean that the list is generated by successively inserting in the corresponding place each permutation $\alpha \in P_9$ and then replacing 9 by p (when we omit the superscript we mean that the last step, replacing 9 by p, is omitted). Of course, any permutation $\alpha \in P_{n/2}$ such that ϕ_α has a cycle not containing $n/2$ can be discarded from $\mathcal{A}^*$ (see Corollary 3.10).

Using the above strategy, we have found numerically[6] that the maximum-entropy cycle in C_n^* is θ_n for $n = 26, 30$, as well as for $n = 22$. Moreover, the entropy of any other cycle in the class is strictly smaller than $h(\theta_n)$ (up to duality). These results are summarized in Table 3.

7. CONCLUSIONS: FAMILIES AS LOWER BOUNDS

The families of cycles that we have described in Section 5 provide a good lower bound on the maximum topological entropy of cycles in C_{4k+2}. Indeed, the sequence of topological entropies of the cycles generated by each family is monotonically increasing as $k \to \infty$, and furthermore, if we combine the three sequences into a single sequence, the new sequence obtained is also monotonically increasing as $k \to \infty$ (see Figure 3 and Tables 2 and 3).

However, a degree of caution should be taken: as remarked by a referee, in the search for entropy-maximizing cycles of order n, first there was a distinction between n odd and $n = 2m$; then between m odd and $m = 2k$; and now, conjecturally, between k odd and $k = 2\ell$.

In this situation we might think that we are facing an infinite cascade of such distinctions. However, since we have found no single cycle among those generated with entropy larger than θ_n, we believe that the following conjectures are reasonable.

[6] Of course, the execution times of these computations are at most half of the necessary time to compute the maximum-entropy cycle in C_{18}^0, since here for each period we have to consider only one kind of product.

Conjecture 7.1. *If θ_n is an n-cycle as described in Definition 5.1, then θ_n has maximum entropy in C_n.*

Conjecture 7.2. *If θ_n is an n-cycle as described in Definition 5.1, then θ_n is the unique entropy-maximal element of C_n, up to duality.*

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