Fold Maps from the Sphere to the Plane

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Any stable map from a surface to the plane has an associated graph. In the case of the sphere, such graphs are of tree type [Hacon et al. 03]. We characterize the trees that can occur as graphs of fold maps from the sphere to the plane. In order to do so, we first determine the sets of integers that may occur as winding numbers for the branch sets of these maps.

1. INTRODUCTION

By a well-known theorem of Whitney, critical points of generic smooth maps from a smooth surface to the plane are either fold points or (isolated) cusp points [Golubitsky and Guillemin 76]. The critical set of any stable map from a closed smooth surface to the plane consists of finitely many disjoint simple closed curves in the surface (the critical curves) whose image (the branch set) consists of closed curves with transversal intersections and (isolated) cusps. With a view to classifying stable maps modulo, for example, the action of diffeomorphisms of the surface and the plane, one is interested in the existence of relevant invariants.

For example, any stable map defines a graph whose edges (respectively vertices) correspond to the critical curves (respectively components of the complement of the critical set). In the case of the sphere, the graph is a tree. In [Hacon et al. 03] it is shown that any tree occurs as the graph of a stable map of the sphere into the plane. In the present work we study cuspless stable maps, classically known as fold maps (i.e., maps whose critical set is immersed in the plane). Such maps determine a particular class of stable maps between smooth submanifolds, whose study and classification, as well as the possible related homotopy principles, have captured the attention of several authors [Ando 04, Ando 02, Eliashberg 70, Saeki 03, Sakuma 94].

In the case of fold maps of closed surfaces into the plane, this study is closely related to that of immersions

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of surfaces with boundary into the plane, whose classification up to image isotopy has been studied by L. Kauffman [Kauffman 79] and M. Yamamoto [Yamamoto 03]. In fact, for a given stable map, the surface is the union of regions (a region being the closure of a connected component of the complement of the critical curves and hence a surface whose (nonempty) boundary consists of a number of critical curves).

Since the map immerses the complement of the critical set, fold maps may be constructed by first immersing the critical curves and then extending to immersions of the regions, which, after smoothing along the critical curves, piece together to give the desired map. Whether a set of immersed curves bounds an immersed region depends on the so-called Blank words associated with the branch curves; see [Malta et al. 93], where an exposition of Blank and Troyer's results [Blank 67, Troyer 73] is given.

However, instead of using this criterion, it is usually possible to assemble the immersed region directly as the union of simpler immersed regions. For example, in the proof of Theorem 4.3 (which characterizes the sets of integers that can occur as the winding numbers [Whitney 37] of a fold map of the sphere), the necessary immersed regions are constructed explicitly as unions of immersed annuli. On the other hand, to produce stable maps with certain prescribed branch sets, as illustrated in Figure 6, the required immersed regions may be constructed as unions of immersed disks.

Here we are interested mainly in the specific case of the sphere and, in particular, in which sets of integers can occur as winding numbers of the (immersed) curves of the branch set. Observe that each branch curve receives an orientation induced by the map from the chosen orientations of the sphere and the plane. The set of winding numbers is, in general, invariant only by orientation-preserving diffeomorphisms (of the sphere and the plane).

This procedure enables one to characterize those trees that occur as graphs of fold maps (Theorem 4.3). The resulting list of realizable pairs (graph, branch set) provides a first step toward the classification of fold maps of the sphere (modulo diffeomorphisms of the sphere and the plane). Classification entails first the determination of all possible pairs (graph, branch set) and, for each such pair, all collections of immersed planar regions fitting together along the branch curves as above.

2. STABLE MAPS AND THEIR INVARIANTS

We begin by recalling some definitions and basic results. Two smooth maps f and g from a surface M to the plane are said to be $\mathcal{A}$-*equivalent* if there are orientation-preserving diffeomorphisms, l and k, such that $l \circ f = g \circ k$. If both l and k are isotopic to the identity, f and g are said to be isotopic. A smooth map f is said to be *stable* if all maps sufficiently close to f (in the Whitney C^∞-topology) are isotopic to f.

We denote by $\mathcal{E}(M, \mathbb{R}^2)$ the space of stable maps from a surface M to the plane. As mentioned above, the critical set of a stable map consists of a finite number of disjoint embedded (critical) curves. Each curve consists of fold points together with finitely many cusp points (i.e., points whose image is a cusp point of the branch set). If f and g are $\mathcal{A}$-equivalent, then there is a diffeomorphism of the surface carrying the critical set of f to the critical set of g and similarly for the branch sets of f and g.

There is also the corresponding notion of isotopy. Clearly, any diffeomorphism invariant of the critical set or of the branch set will be an $\mathcal{A}$-invariant of f. The number of connected components of the singular set and the topological type of its complement are invariants. This information can be encoded in a weighted graph from which the surface and the critical set may be reconstructed (up to equivalence) as shown in [Hacon et al. 03].

The graph of a stable map is defined as follows: each region (i.e., closure of a connected component of the complement of the critical set) is represented by a vertex labeled by the genus of the region (in the case of a sphere the labels are superfluous, since all the regions are planar, i.e., of genus zero) and each critical curve is represented by an edge joining the two vertices corresponding to the (two) regions containing the critical curve in question.

Since the surface and the plane are both assumed to be oriented, the graph is bipartite: there are two types of vertices, corresponding to positive (respectively negative) regions whose orientation agrees (respectively does not agree) with the orientation induced by the map from the orientation of the plane. Any edge has two vertices, one of each type, corresponding to the fact that any critical curve lies in the boundary of one positive and one negative region. Thus the graph is bipartite.

Each critical curve is oriented positively (i.e., inherits an orientation from the positive region to whose boundary it belongs). Thus the branch set is oriented via the map.

In [Hacon et al. 03] it was shown that the graph of any stable map from the sphere to the plane is a tree. Moreover, any tree may be realized as the graph of a stable map from the sphere into the plane. The proof is by induction on the number of edges of the graph, and the inductive step consists in introducing an extra branch curve by means of a transition of "lips" type. We point out that this transition introduces two cusps, so the resulting stable map is not cuspless.

We end this section by observing that there are three obvious invariants: μ, C, and D (respectively the number of components of the singular set, the number of cusps, and the number of double points). The invariants C, D together with a fourth, less obvious, invariant, F, defined by Aicardi and Ohmoto, provide a set of generators for the cohomology group $H^0(\mathcal{E}(M, \mathbb{R}^2); \mathbb{Z})$ of first-order invariants of Vassiliev type [Ohmoto and Aicardi 06]. Such invariants are defined in terms of the codimension-one transitions in the branch set.

Vassiliev's technique is based on the consideration of a convenient stratification on the discriminant set defined by the complement of the stable maps in the set of all smooth maps from a surface M to the plane and assignment of (integer) indices to the various strata of codimension one. Each stratum S receives a coorientation corresponding to the normal direction in which the index of S increases as one passes through S.

In order to produce a well-defined invariant (the Vassiliev cycle), the indices must satisfy a compatibility condition: the total increment of the indices along any closed path encircling a codimension-two stratum must be zero. The invariant associated with a given Vassiliev cycle is defined to be zero for an arbitrarily chosen stable map f_0, and its value on any other stable map f is then the total increment in the cocycle along a generic path from f_0 to f.

3. WINDING NUMBERS OF BRANCH SETS OF FOLD MAPS

The branch sets of any fold map from a closed surface to the plane consists of closed regular plane curves whose winding (or rotation) numbers (also known as Whitney indices [Whitney 37]) satisfy certain compatibility conditions. These are based on well-known properties of regions immersed in the plane [Chillingworth 72, Kauffman 79, Yamamoto 03, Whitney 37].

Proposition 3.1. *The sum of the winding numbers of the boundary curves of a surface immersed in the plane is equal to the Euler characteristic of the surface.*

Since each branch curve of f lies on the boundary of exactly one positive and one negative region, we have the following corollary:

Corollary 3.2. *The sum of the winding numbers of the branch curves of a fold map is equal to the Euler characteristic of each half (positive or negative) of the surface and hence equals half the Euler characteristic of the surface. Furthermore, the graph of any fold map of the sphere is balanced.*

A *balanced* graph is a bipartite graph having the same number of positive/negative vertices. We observe in passing that there exist fold maps of surfaces (of higher genus) whose graphs are not balanced.

Proof: The first part of the corollary follows immediately from the proposition. As for the second part, the Euler characteristic of the positive part of the surface is equal to the sum of the Euler characteristics of the positive regions and hence to twice the number of positive regions minus the total number of branch curves (since each region, being contained in the sphere, is a disk with holes, and therefore its Euler characteristic is two minus the number of boundary curves). The same goes for the negative half, whence the result. □

Not every immersed curve can occur as a branch curve of a fold map. In fact we have the following theorem.

Theorem 3.3. *Any branch curve of a fold map of the sphere has odd winding number (or equivalently, an even number of double points).*

Proof: Consider the graph of the map in which each edge is indexed by one plus the winding number of the corresponding branch curve (in particular, the index is even if and only if the winding number is odd). In the case of a sphere, the graph is a tree (as shown in [Hacon et al. 03]). By the proposition, at each vertex the local sum of the indices is even (in fact, equal to 2). Since the graph is a tree, there is a vertex that belongs to just one edge. The index of this edge is therefore even (in fact, equal to 2). Removing this edge, we obtain a subtree for which the local sums are also all even. By induction on the number of edges of the tree, starting with the case of one edge, the indices of the subtree are all even. In other words, the winding numbers are all odd. □

Figure 1 displays representatives of the different Whitney isotopy classes with odd winding number having min-

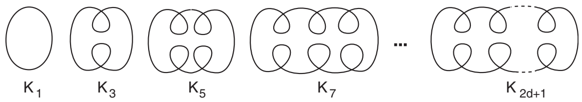

FIGURE 1. Arnol′d's basic curves having odd winding number.

imal numbers of double points. Such curves are called basic curves by Arnol′d [Arnol′d 94b].

A (finite) set of (possibly repeated) integers $\{x, y, z, \dots\}$ is called *compatible* if

$$(x + 1) + (y + 1) + (z + 1) + \cdots = 2.$$

Thus the winding numbers of the boundary curves of an immersed planar region are compatible in the above sense.

4. STABLE MAPS WITH PRESCRIBED BRANCH SET

Given these necessary conditions, we next consider the question of realizing a set of immersed curves as the branch set of a fold map of the sphere with prescribed graph. For the sphere, this question is answered in Theorem 4.3, which characterizes graphs of such maps, as well as the set of winding numbers of their branch sets.

Recall that in general, it is sufficient to construct the immersed regions one by one and then assemble them to produce a stable map. For the sphere these regions must necessarily be planar regions, and the existence of an immersed planar region with prescribed boundary depends on the Blank words [Malta et al. 93] corresponding to the immersed boundary curves. Thus it is sufficient to partition the (oriented) curves into disjoint classes, each of which is the boundary of an immersed planar region. It is usually not necessary to use the Blank word; one simply produces the desired immersed region as a union of immersed disks as in Figure 2.

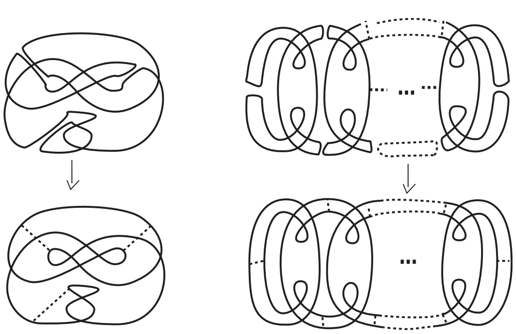

FIGURE 2. Unions of immersed disks.

As for the question of which sets of winding numbers can occur, it will be shown below that (for the sphere) the necessary conditions given above are, in fact, sufficient to realize any set of compatible winding numbers.

Recall that for a fold map of the sphere, the winding numbers of the branch curves are odd integers that satisfy the following compatibility conditions, one for each vertex v of the associated tree: With each edge vw (with vertices v and w) we associate a variable I_{vw}. We write C_v for the sum of the I_{uv} for all edges uv containing v. The compatibility condition is then $C_v = 2$. For a fold map of the sphere, we have a solution of the compatibility equations given by setting I_{vw} equal to one plus the winding number of the branch curve corresponding to the edge vw (all these integers being, in fact, even). We thus have a system of $2n$ equations in $2n - 1$ variables. Since the number, n of positive vertices is equal to the number of negative vertices, we have that

$$\sum (C_v - 2) = \sum I_{xy} - 2n = \sum (C_w - 2),$$

where v runs over all positive vertices, w over all negative vertices, and xy over all edges. Thus any equation is a consequence of the rest.

Now fix a vertex $\star$. For any vertex v define d_v to be the length of the (unique) path in the tree between v and $\star$. Thus $d_\star = 0$, $d_v = 1$ if $v\star$ is an edge, for any edge vw we have that d_v and d_w differ by one, and for any vertex $v \neq \star$ there is a unique edge vw such that $d_v = d_w + 1$. The equation $C_v = 2$ determines I_{vw} in terms of the other variables, i.e., in terms of the I_{uv} for which $d_u = d_v + 1$. For the largest value of d_v, C_v is just I_{vw}, for which $I_{vw} = 2$ is, of course, the unique solution. Thus the equations $C_v = 2$ may be solved uniquely for successively smaller values of d_v up to and including $d_v = 1$. The remaining equation $C_\star = 2$ is a consequence of the rest. We observe that the solution consists entirely of even integers, corresponding to the fact (already proved) that the winding numbers must all be odd. We summarize the above as the following lemma.

Lemma 4.1. *A tree is balanced if and only if the compatibility conditions $C_v = 2$ have a unique solution (which necessarily consists only of even integers).*

It remains to show that any compatible set of odd integers is the set of winding numbers of an immersed planar region. The region will be assembled from a set of basic immersed disks by a connected sum (see Figure 2 for a typical example). Here, we observe that a

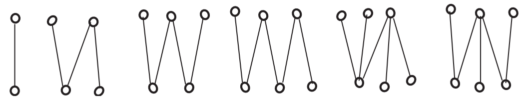

FIGURE 3. Balanced trees.

connected sum of immersions can be obtained in a natural way by means of connected sums of their boundary curves. The relations between connected sums of curves and their stable isotopy invariants (Arnol′d's invariants) have been discussed in [Mendes de Jesus and Romero Fuster 02]. From now on, all compatible sets will be assumed to consist entirely of odd integers. We define the sum $\{x, y, z, \dots\} \oplus \{u, v, w, \dots\}$ of two compatible sets to be the (compatible) set consisting of $x+u-1$, $y, z, \dots,$ and $v, w, \dots,$ provided that x and u have the same sign. The total number of elements of $X \oplus Y$ is the number of elements of X plus the number of elements of Y minus one.

By connecting (through a convenient bridge, as explained in [Mendes de Jesus and Romero Fuster 02]) one of the curves α_1 in the image of the boundary of an immersion $\mathcal{R}_1$ with a curve α_2 of the image of the boundary of an immersion $\mathcal{R}_2$, we obtain a new set of plane curves that can be seen as the images of the boundary of an immersed region $\mathcal{R}_3$. If X_1, X_2, and X_3 are the compatible sets of integers obtained from the winding numbers (plus one) of the images of the boundary curves of the immersed regions $\mathcal{R}_1$, $\mathcal{R}_2$, and $\mathcal{R}_3$ as explained above, we can easily see that $X_1 \oplus X_2 = X_3$, where the first integers in the sets X_1, X_2, and X_3 correspond to the curves α_1, α_2, and $\alpha_1 \oplus \alpha_2$ respectively. This relation between both sums is illustrated in Figure 4. We observe that the requirement on the signs of the integers x and u above is due to the fact that two plane curves can be summed only if they have compatible orientations.

As a consequence, according to their relative positions, we have two types of bridges: outer and inner. They lead respectively to external and internal sums of curves (see [Mendes de Jesus and Romero Fuster 02] for details). As illustrated by the different examples of Figure 4, external (respectively internal) sums of curves correspond to sums of positive (respectively negative) integers.

Clearly, any compatible set consisting of just two integers must already be of the form $\{a, -a\}$. Now consider a compatible set $\{x, y, z, \dots\}$. By compatibility, there must be at least two numbers (x and y, say) of opposite sign. The number $x+y+1$ is also odd, and $\{x, y, z, \dots\}$ is equal to $\{-x, x\} \oplus \{x+y+1, z, \dots\}$ if $-x$ and $x+y+1$ have the same sign, and is equal to $\{-y, y\} \oplus \{x+y+1, z, \dots\}$

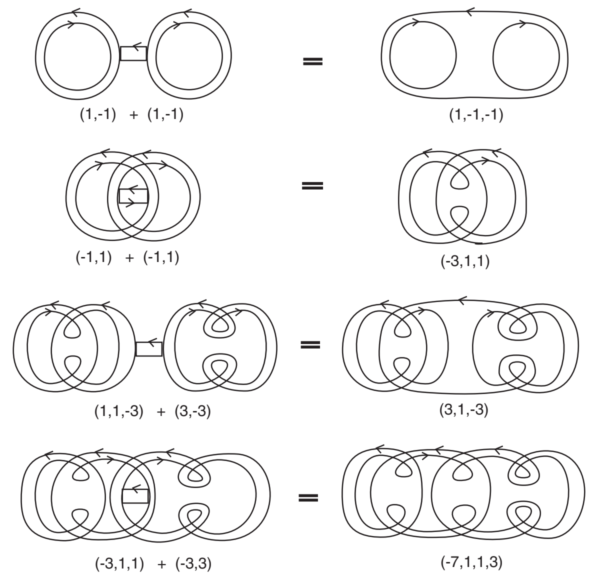

FIGURE 4. Sums of basic curves.

if $-y$ and $x + y + 1$ have the same sign. By induction on the number of elements of the set (starting with two-element sets), we may write $\{x + y + 1, z, \dots\}$ as a sum $\{a, -a\} \oplus \{b, -b\} \oplus \cdots \oplus \{k, -k\}$. Thus the original set $\{x, y, z, \dots\}$ can be written as a sum of two-element sets, as desired.

As illustrated in Figure 4, any such connected sum gives rise to an immersed planar region whose boundary consists of basic curves with the required winding numbers. We thus have the following result.

Lemma 4.2. *Every compatible set of odd integers occurs as the set of winding numbers of the boundary of an immersed planar region. Furthermore, we may insist that all the boundary curves be sums of basic curves, as illustrated in Figure 1.*

Figure 5 describes an inductive method for constructing the image of the boundary of immersed regions having k boundary components with total winding number i, for all possible compatible integer sets $(i_1+1, \dots, i_k+1)$, such that $i = i_1 + \cdots + i_k$. The following theorem summarizes the above discussion.

Theorem 4.3. *The graph of a fold map of the sphere is a balanced tree. Conversely, any balanced tree is the graph of a fold map of the sphere whose branch set consists of unions of basic curves (as in Figure 5). A tree is balanced if and only if its edges can be labeled with compatible even integers.*

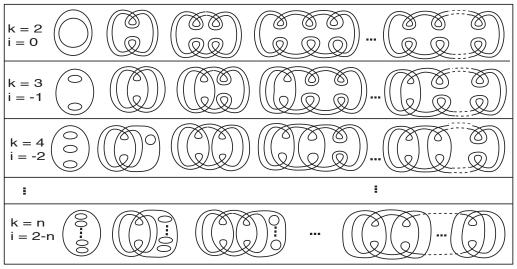

FIGURE 5. Basic curves in the boundary of immersed planar regions.

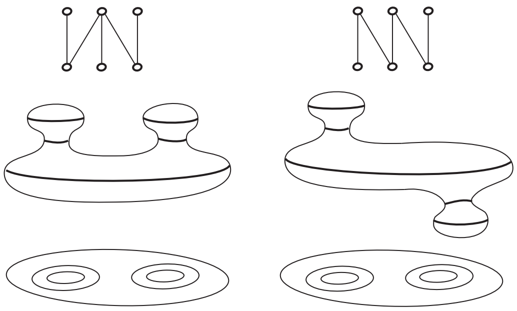

FIGURE 6. Different graphs for a branch set.

The above theorem characterizes those trees that can occur as graphs of fold maps of the sphere. We observe in passing that two maps can have the same branch set but different graphs, as Figure 6 illustrates. One may also ask which branch sets can occur. For maps of the sphere, equivalence is the same as isotopy. Thus we may make use of Arnol′d's isotopy classification of stable immersed plane curves (with at most five crossings) [Arnol′d 94a] in order to list the different possibilities for the branch sets.

If the branch set (of a stable map) consists of just one immersed curve, then the curve bounds an immersed disk. Arnol′d's list [Arnol′d 94a] contains exactly six such curves among the 110 representatives of stable isotopy classes of closed connected plane curves with $\mu \leq 3$ and $D \leq 4$, as can be verified by calculating Blank words [Malta et al. 93]. These curves are listed in Figure 7.

It seems probable that to each curve there corresponds just one class of stable maps. For six crossings or more, this is no longer true. Consider, for example, the curve of Figure 8, due to J. Milnor, which bounds two different immersed disks. According to [Eliashberg and Mishaev 02], this curve gives rise to two nonisotopic maps, depending on whether the two immersed disks used to construct the map are the same or not.

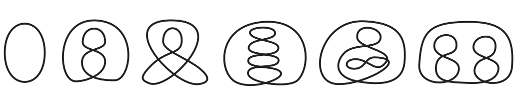

FIGURE 7. Branch sets with $\mu = 1$ and $D \leq 4$ bounding immersed disks.

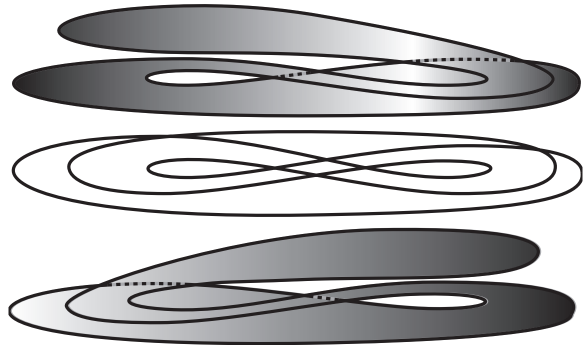

FIGURE 8. Two different extensions to the disk.

Taking into account Theorem 3.3 and the above considerations, inspection of Arnol′d's list of isotopy classes of immersed plane curves leads to the following theorem.

Theorem 4.4. *Figure 9 lists all possible branch sets of fold maps from the sphere to the plane with $\mu \leq 3$ and $D \leq 4$.*

Proof: As we have pointed out, there are only six curves (with up to four double points) in Arnol′d's list bounding immersed disks. If we allow more than one component in the branch set, the next case to be considered is that of three components (since by Corollary 3.2 there are as many positive regions as negative ones and hence an odd number of components). For three components only one tree occurs, namely the letter N. The other tree (the letter Y) is not balanced. In this case, the possible branch sets are given by isotopy classes of immersed curves X, Y, Z such that $D \leq 4$, X, and Z each bound an immersed disk and each pair X, Y and Y, Z bounds

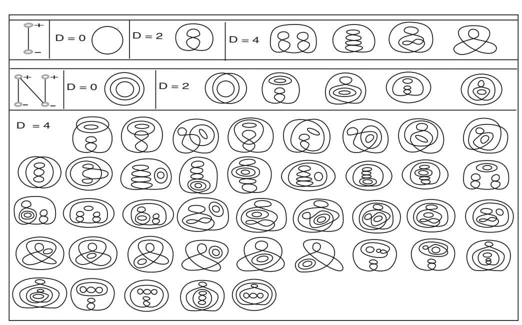

FIGURE 9. Graphs and branch sets for fold maps of the sphere with $\mu \leq 3$ and $D \leq 4$.

an immersed annulus. Figure 9 lists a number of (non-isotopic) branch sets with three components and at most four crossings.

These were determined as follows. Since two of the curves bound immersed disks, they must belong to the set of six curves shown in Figure 7. The third curve must be placed so as to lie on the common boundary of two annuli. This means that its winding number must be -1, which leaves us with 52 curves in Arnol′d's list. The restriction that $D = 0, 2, 4$, together with an application of Blank's criterion extended to a disk with one hole (i.e., Troyer's criterion [Troyer 73]; see also [Malta et al. 93]), leads to the complete list. $\square$

Not surprisingly, beyond this, listing all possibilities rapidly becomes a daunting task. A further complication is that there will be various inequivalent maps with the same branch set, as in Figure 6.

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