Prehomogeneous Spaces Associated with Nilpotent Orbits in Simple Real Lie Algebras $\mathrm{E}_{6(6)}$ and $\mathrm{E}_{6(-26)}$ and Their Relative Invariants

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CONTENTS

We give an efficient and stable algorithm for computing highest weights in a large class of prehomogeneous spaces associated with the nilpotent orbits of the real Lie algebras $\mathrm{E}_{6(6)}$ and $\mathrm{E}_{6(-26)}$. This paper concludes our classification of such prehomogeneous spaces for all complex and real reductive Lie algebras. For classical algebras using the fact that the nilpotent orbits are parameterized by partitions of integers we have given general formulas in [Jackson and Noël 05a] and [Jackson and Noël 06]. For complex or inner-type real exceptional algebras we have given general algorithms and tables in [Jackson and Noël 05b] and [Jackson and Noël 05c]. The present paper considers the case of real exceptional algebras that are not of inner type.

1. INTRODUCTION

In this paper, we continue our program begun in [Jackson and Noël 05a], [Jackson and Noël 06], [Jackson and Noël 05b], and [Jackson and Noël 05c] by describing a fast and stable algorithm for decomposing modules of a Lie subgroup of the Levi factor of Jacobson–Morozov parabolic subgroups defined by nilpotent orbits in simple real Lie algebras $\mathrm{E}_{6(6)}$ and $\mathrm{E}_{6(-26)}$. We will solve the problem by working on the other side of the Kostant–Sekiguchi correspondence [Sekiguchi 87]. In order to continue we need some definitions.

Let $\mathfrak{g}$ be a real semisimple Lie algebra with adjoint group G and Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ relative to a Cartan involution θ. We will denote by $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$. Let σ be the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$, where $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ are obtained by complexifying $\mathfrak{k}$ and $\mathfrak{p}$ respectively. We let K be a maximal compact Lie subgroup of G with Lie algebra $\mathfrak{k}$, and $K_{\mathbb{C}}$ will be the connected subgroup of the adjoint group $G_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$, with Lie algebra $\mathfrak{k}_{\mathbb{C}}$. It is well known that $K_{\mathbb{C}}$ acts on $\mathfrak{p}_{\mathbb{C}}$ and that the number of nilpotent orbits of $K_{\mathbb{C}}$

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in $\mathfrak{p}_{\mathbb{C}}$ is finite. Furthermore, for a nilpotent $e \in \mathfrak{p}_{\mathbb{C}}$, $K_{\mathbb{C}} \cdot e$ is a connected component of $G_{\mathbb{C}} \cdot e \cap \mathfrak{p}_{\mathbb{C}}$.

2. THE KOSTANT–SEKIGUCHI CORRESPONDENCE

A triple (x, e, f) in $\mathfrak{g}_{\mathbb{C}}$ is called a *standard triple* if $[x, e] = 2e$, $[x, f] = -2f$, and $[e, f] = x$. If $x \in \mathfrak{k}_{\mathbb{C}}$ and $e, f \in \mathfrak{p}_{\mathbb{C}}$, then (x, e, f) is a *normal* triple. It is a result of Kostant and Rallis [Kostant and Rallis 71] that any nilpotent e of $\mathfrak{p}_{\mathbb{C}}$ can be embedded in a standard normal triple (x, e, f). Moreover, e is $K_{\mathbb{C}}$-conjugate to a nilpotent e' inside of a normal triple (x', e', f') with $\sigma(e') = f'$; see [Sekiguchi 87]. The triple (x', e', f') will be called a *Kostant–Sekiguchi* or *KS* triple, and we will refer to the element e' as its *nilpositive element*.

Every nilpotent E' in $\mathfrak{g}$ is G-conjugate to a nilpotent E embedded in a triple (H, E, F) in $\mathfrak{g}$ with the property that $\theta(H) = -H$ and $\theta(E) = -F$; see [Sekiguchi 87]. Such a triple will also be called a KS triple.

Define a map c from the set of KS triples of $\mathfrak{g}$ to the set of normal triples of $\mathfrak{g}_{\mathbb{C}}$ as follows:

$$\begin{aligned} x &= c(H) = \mathbf{i}(E - F), \\ e &= c(E) = \frac{1}{2}(H + \mathbf{i}(E + F)), \\ f &= c(F) = \frac{1}{2}(H - \mathbf{i}(E + F)) \end{aligned}$$

(where $\mathbf{i} = \sqrt{-1}$). The triple (x, e, f) is called the *Cayley transform* of (H, E, F). It is easy to verify that the triple (x, e, f) is a KS triple and that $x \in \mathbf{i}\mathfrak{k}$. The Kostant–Sekiguchi correspondence [Sekiguchi 87] gives a one-to-one map between the set of G-conjugacy classes of nilpotents in $\mathfrak{g}$ and the $K_{\mathbb{C}}$-conjugacy classes of nilpotents in $\mathfrak{p}_{\mathbb{C}}$. This correspondence sends the zero orbit to the zero orbit and the orbit through the nilpositive element of a KS triple to the one through the nilpositive element of its Cayley transform. Michèle Vergne [Vergne 95] has proved that there is in fact a diffeomorphism between a G-conjugacy class and the $\mathfrak{k}_{\mathbb{C}}$-conjugacy class associated with it by the Kostant–Sekiguchi correspondence.

3. THE MODULES

In light of the Kostant–Sekiguchi correspondence it is reasonable to study modules associated with $K_{\mathbb{C}}$-nilpotent orbits in the symmetric spaces $\mathfrak{p}_{\mathbb{C}}$ in order to understand real nilpotent orbits. Let e be a nilpotent element in $\mathfrak{p}_{\mathbb{C}}$. Without loss of generality we can embed e in a KS triple (x, e, f). The action of ad_x determines a grading

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\mathbb{C}}^i,$$

where $\mathfrak{g}_{\mathbb{C}}^i = \{Z \in \mathfrak{g}_{\mathbb{C}} : [x, Z] = iz\}$.

It is a fact that $\mathfrak{g}_{\mathbb{C}}^0$ is a reductive Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let $G_{\mathbb{C}}^0$ be the connected subgroup of $G_{\mathbb{C}}$ such that $\mathrm{Lie}(G_{\mathbb{C}}^0) = \mathfrak{g}_{\mathbb{C}}^0$. Then for $i \neq 0$ the vector spaces $\mathfrak{g}_{\mathbb{C}}^i \cap \mathfrak{p}_{\mathbb{C}}$ are $G_{\mathbb{C}}^0 \cap K_{\mathbb{C}}$-modules. Moreover, a theorem of Kostant and Rallis [Kostant and Rallis 71] asserts that $G_{\mathbb{C}}^0 \cap K_{\mathbb{C}}$ admits a Zariski-open and -dense orbit on $\mathfrak{g}_{\mathbb{C}}^2 \cap \mathfrak{p}_{\mathbb{C}}$; that is, the pair $(G_{\mathbb{C}}^0 \cap K_{\mathbb{C}},\ \mathfrak{g}_{\mathbb{C}}^2 \cap \mathfrak{p}_{\mathbb{C}})$ is a prehomogeneous space in the sense of Sato and Kimura [Sato and Kimura 77]. This prehomogeneous space plays an important role in our work, and our effort to better understand it led us to develop this project. See [Noël 98] for more information.

Let us denote $G_{\mathbb{C}}^0 \cap K_{\mathbb{C}}$, $\mathfrak{g}_{\mathbb{C}}^i \cap \mathfrak{k}_{\mathbb{C}}$, and $\mathfrak{g}_{\mathbb{C}}^i \cap \mathfrak{p}_{\mathbb{C}}$ by $K_{\mathbb{C}}^0$, $\mathfrak{k}_{\mathbb{C}}^i$, and $\mathfrak{p}_{\mathbb{C}}^i$ respectively. Then we shall show that for $i \neq 0$, $(G_{\mathbb{C}}^0, \mathfrak{g}_{\mathbb{C}}^i)$, $(K_{\mathbb{C}}^0, \mathfrak{k}_{\mathbb{C}}^i)$, and $(K_{\mathbb{C}}^0, \mathfrak{p}_{\mathbb{C}}^i)$ are prehomogeneous spaces. We shall need the following lemma from È. B. Vinberg.

Lemma 3.1. *Let $H \subseteq \mathrm{GL}(U)$ be a linear algebraic group, let L be a closed, connected subgroup of H, and let $V \subseteq U$ be a subspace invariant with respect to L. Suppose that for any vector $v \in V$,*

$$\mathfrak{h} \cdot v \cap V = \mathfrak{l} \cdot v.$$

Then the intersection of any orbit of H with the subspace V is a smooth manifold, each irreducible component of which is an orbit of L.

Proof: See [Vinberg 76, p. 469]. □

Using the previous lemma we prove the following proposition (the case of $(G_{\mathbb{C}}^0, \mathfrak{g}_{\mathbb{C}}^i)$ appears in [Vinberg 75] and [Rubenthaler 92], and the case of $(K_{\mathbb{C}}^0, \mathfrak{p}_{\mathbb{C}}^2)$ appears in [Ohta 91]):

Proposition 3.2. *For $i \neq 0$, the modules $(G_{\mathbb{C}}^0, \mathfrak{g}_{\mathbb{C}}^i)$, $(K_{\mathbb{C}}^0, \mathfrak{k}_{\mathbb{C}}^i)$, and $(K_{\mathbb{C}}^0, \mathfrak{p}_{\mathbb{C}}^i)$ have only finitely many orbits; hence they are prehomogeneous spaces. Moreover, $(G_{\mathbb{C}}^0, \mathfrak{g}_{\mathbb{C}}^2)$ and $(K_{\mathbb{C}}^0, \mathfrak{p}_{\mathbb{C}}^2)$ are regular; that is, the complements of the open dense orbits in $\mathfrak{g}_{\mathbb{C}}^2$ and $\mathfrak{p}_{\mathbb{C}}^2$ (the* singular loci*) are hypersurfaces.*

Proof: To prove that $(G_{\mathbb{C}}^0, \mathfrak{g}_{\mathbb{C}}^i)$ is a prehomogeneous space we identify $G_{\mathbb{C}}$, $G_{\mathbb{C}}^0$, $\mathfrak{g}_{\mathbb{C}}$, and $\mathfrak{g}_{\mathbb{C}}^i$ with H, L, U, and V respectively in the preceding lemma. Hence we need to show only that for any $v \in \mathfrak{g}_{\mathbb{C}}^i$,

$$\mathfrak{g}_{\mathbb{C}} \cdot v \cap \mathfrak{g}_{\mathbb{C}}^i = \mathfrak{g}_{\mathbb{C}}^0 \cdot v.$$

Clearly, $\mathfrak{g}_{\mathbb{C}}^0 \cdot v \subseteq \mathfrak{g}_{\mathbb{C}} \cdot v \cap \mathfrak{g}_{\mathbb{C}}^i$. Let $u \in \mathfrak{g}_{\mathbb{C}}$ be such that $[u,v] \in \mathfrak{g}_{\mathbb{C}}^i$. Since $u = \sum_j u_j$ with $u_j \in \mathfrak{g}_{\mathbb{C}}^j$, it follows that $[u,v] = \sum_j [u_j, v]$ and $[u_j, v] \subseteq \mathfrak{g}_{\mathbb{C}}^{i+j}$. Hence $[u_j, v] = 0$ for $j \neq 0$, and we must have $[u,v] = [u_0, v]$ and $\mathfrak{g}_{\mathbb{C}} \cdot v \cap \mathfrak{g}_{\mathbb{C}}^i \subseteq \mathfrak{g}_{\mathbb{C}}^0 \cdot v$. The result follows.

To prove that $(K_{\mathbb{C}}^0, \mathfrak{p}_{\mathbb{C}}^i)$ is a prehomogeneous space we identify $G_{\mathbb{C}}^0$, $K_{\mathbb{C}}^0$, $\mathfrak{g}_{\mathbb{C}}^i$, and $\mathfrak{p}_{\mathbb{C}}^i$ with H, L, U, and V respectively in the preceding lemma. We need to show only that for any $v \in \mathfrak{p}_{\mathbb{C}}^i$,

$$\mathfrak{g}_{\mathbb{C}}^0 \cdot v \cap \mathfrak{p}_{\mathbb{C}}^i = \mathfrak{k}_{\mathbb{C}}^0 \cdot v.$$

Clearly $\mathfrak{k}_{\mathbb{C}}^0 \cdot v \subseteq \mathfrak{g}_{\mathbb{C}}^0 \cdot v \cap \mathfrak{p}_{\mathbb{C}}^i$. Let $u \in \mathfrak{g}_{\mathbb{C}}^0$ be such that $[u,v] \in \mathfrak{p}_{\mathbb{C}}^i$. Since $x = x_k + x_p$ with $x_k \in \mathfrak{k}_{\mathbb{C}}$ and $x_p \in \mathfrak{p}_{\mathbb{C}}$, it follows that $[x,v] = [x_k, v] + [x_p, v]$ with $[x_k, v] \in \mathfrak{p}_{\mathbb{C}}^i$ and $[x_p, v] \in \mathfrak{k}_{\mathbb{C}}^i$. Since $[x,v] \in \mathfrak{p}_{\mathbb{C}}^i$, we must have $[x_p, v] = 0$. Hence $\mathfrak{g}_{\mathbb{C}}^0 \cdot v \cap \mathfrak{p}_{\mathbb{C}}^i \subseteq \mathfrak{k}_{\mathbb{C}}^0 \cdot v$.

To prove that $(K_{\mathbb{C}}^0, \mathfrak{k}_{\mathbb{C}}^i)$ is a prehomogeneous space we have only to repeat the previous argument, replacing $\mathfrak{p}_{\mathbb{C}}^i$ by $\mathfrak{k}_{\mathbb{C}}^i$.

From [Rubenthaler 92, Theorem 1.4.4], in order to show that $(G_{\mathbb{C}}^0, \mathfrak{g}_{\mathbb{C}}^2)$ and $(K_{\mathbb{C}}^0, \mathfrak{p}_{\mathbb{C}}^2)$ are regular, we need to show only that the centralizers $(G_{\mathbb{C}}^0)^e$ and $(K_{\mathbb{C}}^0)^e$ of e in $G_{\mathbb{C}}^0$ and $K_{\mathbb{C}}^0$ are reductive Lie subgroups. For $G_{\mathbb{C}}^0$ this was done by Springer and Steinberg in [Springer and Steinberg 70]. The $K_{\mathbb{C}}^0$ case was settled by Ohta [Ohta 91]. $\square$

The reader may wonder whether $(K_{\mathbb{C}}^0, \mathfrak{k}_{\mathbb{C}}^2)$ is regular in general. Here is a counterexample. Let $\mathfrak{g} = FI$, the split real form of F_4. Consider Orbit 20, labeled (204 4) in [Đoković 88a]. Then $\mathfrak{k}_{\mathbb{C}}^2$ is a two-dimensional representation of $K_{\mathbb{C}}^0$. The singular locus is $\{0\}$ and therefore is not a hypersurface.

Our goal is to describe the irreducible components of the $K_{\mathbb{C}}^0$-modules $\mathfrak{p}_{\mathbb{C}}^i$ and $\mathfrak{k}_{\mathbb{C}}^i$ with $i \neq 0$ for all nilpotent orbits of the Lie group $K_{\mathbb{C}}$ in the symmetric space $\mathfrak{p}_{\mathbb{C}}$.

4. ROOT DECOMPOSITION

Let $\mathfrak{h}$ be a fundamental Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{s}$, where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{s} \subseteq \mathfrak{p}$. Let I_c be the set of compact imaginary roots, let I_n be the set of noncompact imaginary roots, and let C be the set of complex pairs of roots. We have the following decompositions in the root spaces of $\mathfrak{g}_{\mathbb{C}}$ generated by the roots of $\mathfrak{h}_{\mathbb{C}}$:

$$\mathfrak{k}_{\mathbb{C}} = \mathfrak{t}_{\mathbb{C}} \oplus \sum_{\alpha \in I_c} \mathbb{C} X_\alpha \oplus \sum_{(\alpha, \theta\alpha) \in C} \mathbb{C}(X_\alpha + \theta(X_\alpha)),$$
$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{s}_{\mathbb{C}} \oplus \sum_{\alpha \in I_n} \mathbb{C} X_\alpha \oplus \sum_{(\alpha, \theta\alpha) \in C} \mathbb{C}(X_\alpha - \theta(X_\alpha)).$$

Here X_α is a nonzero vector of the root space $\mathfrak{g}_{\mathbb{C}}^\alpha$. An imaginary root α is compact (noncompact) if its root space $\mathfrak{g}_{\mathbb{C}}^\alpha$ lies in $\mathfrak{k}_{\mathbb{C}}$ ($\mathfrak{p}_{\mathbb{C}}$). See [Knapp 02] for more details.

We have considered the real classical algebras in [Jackson and Noël 06] and the real exceptional algebras of inner type in [Jackson and Noël 05c]. It remains to consider real exceptional algebras that are not of inner type. The only such algebras are $\mathrm{E}_{6(-26)}$ and $\mathrm{E}_{6(6)}$, which we now consider in turn. We begin by summarizing some results of Đoković [Đoković 88b]:

4.1 The Algebra $\mathrm{E}_{6(-26)}$

Let $\mathfrak{g}_{\mathbb{C}} = \mathrm{E}_6$ and let $\Delta = \{\alpha_1, \alpha_2, \ldots, \alpha_6\}$ be the Bourbaki simple roots of $\mathfrak{g}_{\mathbb{C}}$. Define an involution θ on Δ as follows:

$$\theta(\alpha_1) = \alpha_6, \quad \theta(\alpha_2) = \alpha_2, \quad \theta(\alpha_3) = \alpha_5,$$
$$\theta(\alpha_4) = \alpha_4, \quad \theta(\alpha_5) = \alpha_3, \quad \theta(\alpha_6) = \alpha_1.$$

Furthermore, we require that

$$\theta(X_{\alpha_1}) = X_{\alpha_6}, \quad \theta(X_{\alpha_2}) = X_{\alpha_2}, \quad \theta(X_{\alpha_3}) = X_{\alpha_5},$$
$$\theta(X_{\alpha_4}) = X_{\alpha_4}, \quad \theta(X_{\alpha_5}) = X_{\alpha_3}, \quad \theta(X_{\alpha_6}) = X_{\alpha_1}.$$

It is well known [Knapp 02, Đoković 88b] that $\mathfrak{k}_{\mathbb{C}}$ is of type F_4 and $\mathfrak{p}_{\mathbb{C}}$ is the symmetric space associated with the real form $\mathrm{E}_{6(-26)}$ of E_6. Let $\mathfrak{h}_{\mathbb{C}}$ be the Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$ associated with the root system generated by Δ. Then $\mathfrak{t}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$. The simple roots of $(\mathfrak{k}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ are

$$\beta_1 = \alpha_2, \quad \beta_2 = \alpha_4, \quad \beta_3 = \frac{\alpha_3 + \alpha_5}{2}, \quad \beta_4 = \frac{\alpha_1 + \alpha_6}{2}.$$

We should also point out that $\mathrm{E}_{6(-26)}$ has no noncompact imaginary roots. The compact imaginary roots are given in Table 1.

4.2 The Algebra $\mathrm{E}_{6(6)}$

Let $\mathfrak{g}_{\mathbb{C}}, \{\alpha_1, \ldots, \alpha_6\}$, and $\{\beta_1, \ldots, \beta_4\}$ be as above. Define

$$\beta_0 = -\beta_1 - 2\beta_2 - 3\beta_3 - 2\beta_4.$$

By computing the entries $\{\langle \beta_i, \beta_j \rangle\}_{i,j \in \{0,2,3,4\}}$ of the Cartan matrix, one can verify that

$$\Delta' = \{\beta_0, \beta_4, \beta_3, \beta_2\}$$

1.	$\pm\alpha_2$
2.	$\pm\alpha_4$
3.	$\pm(\alpha_2+\alpha_4)$
4.	$\pm(\alpha_3+\alpha_4+\alpha_5)$
5.	$\pm(\alpha_2+\alpha_3+\alpha_4+\alpha_5)$
6.	$\pm(\alpha_2+\alpha_3+2\alpha_4+\alpha_5)$
7.	$\pm(\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6)$
8.	$\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)$
9.	$\pm(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6)$
10.	$\pm(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6)$
11.	$\pm(\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6)$
12.	$\pm(\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6)$

TABLE 1. Compact imaginary roots of $\mathrm{E}_{6(-26)}$.

1.	$\pm\alpha_4$
2.	$\pm(\alpha_3+\alpha_4+\alpha_5)$
3.	$\pm(\alpha_1+\alpha_3+\alpha_4+\alpha_5+\alpha_6)$
4.	$\pm(\alpha_1+2\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6)$

TABLE 2. Compact imaginary roots of $\mathrm{E}_{6(6)}$.

is a system of simple roots in a root system of type C_4. There is a unique involution θ of $\mathfrak{g}_{\mathbb{C}}$ that coincides with the Cartan involution of $\mathrm{E}_{6(-26)}$ on simple roots except that $\theta(X_{\alpha_2}) = -X_{\alpha_2}$, and with respect to this new involution the system Δ' is a system of simple roots for $\mathfrak{k}_{\mathbb{C}}$ [Đoković 88b, pp. 197–199]. In this new decomposition $\mathfrak{k}_{\mathbb{C}}$ is of type C_4, and $\mathfrak{p}_{\mathbb{C}}$ is the symmetric space associated with the split real form $\mathrm{E}_{6(6)}$ of E_6. Furthermore, the new involution retains $\mathfrak{t}_{\mathbb{C}}$ as Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$.

Tables 2 and 3 contain the compact and noncompact imaginary roots of $\mathrm{E}_{6(6)}$. Observe that θ defines the same Vogan diagram as that given in [Knapp 02, p. 361], where Proposition 6.104 allows us to decide which imaginary roots are compact or noncompact.

1.	$\pm\alpha_2$
2.	$\pm(\alpha_2+\alpha_4)$
3.	$\pm(\alpha_2+\alpha_3+\alpha_4+\alpha_5)$
4.	$\pm(\alpha_2+\alpha_3+2\alpha_4+\alpha_5)$
5.	$\pm(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6)$
6.	$\pm(\alpha_1+\alpha_2+\alpha_3+2\alpha_4+\alpha_5+\alpha_6)$
7.	$\pm(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6)$
8.	$\pm(\alpha_1+\alpha_2+2\alpha_3+3\alpha_4+2\alpha_5+\alpha_6)$

TABLE 3. Noncompact imaginary roots of $\mathrm{E}_{6(6)}$.

5. THE ALGORITHM

We now describe an algorithm for computing the highest weights of the prehomogeneous spaces $(K^0_{\mathbb{C}}, \mathfrak{p}^d_{\mathbb{C}})$ and $(K^0_{\mathbb{C}}, \mathfrak{k}^d_{\mathbb{C}})$ associated with a KS triple (h, e, f). We need the following notation:

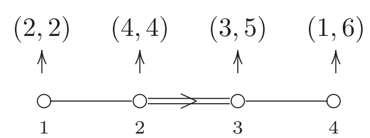
1. $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$ are simple roots of E_6 in the usual Bourbaki system.

2. $\alpha_0 = \alpha_1 - \alpha_2 - 2\alpha_3 - 2\alpha_4 - \alpha_5 - \alpha_6$ and $\alpha_7 = \theta(\alpha_0)$.

3. Relative to $\mathfrak{t}_{\mathbb{C}}$, the Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$, we have the following root restrictions:
$$\begin{aligned}\beta_0 &= \alpha_0|_{\mathfrak{t}_{\mathbb{C}}} = \alpha_7|_{\mathfrak{t}_{\mathbb{C}}},\\ \beta_1 &= \alpha_2|_{\mathfrak{t}_{\mathbb{C}}},\\ \beta_2 &= \alpha_4|_{\mathfrak{t}_{\mathbb{C}}},\\ \beta_3 &= \alpha_3|_{\mathfrak{t}_{\mathbb{C}}} = \alpha_5|_{\mathfrak{t}_{\mathbb{C}}},\\ \beta_4 &= \alpha_1|_{\mathfrak{t}_{\mathbb{C}}} = \alpha_6|_{\mathfrak{t}_{\mathbb{C}}}.\end{aligned}$$

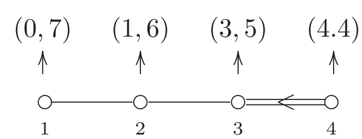
4. Let
$$(\gamma_1,\gamma_2,\gamma_3,\gamma_4) = \begin{cases}(\beta_1,\beta_2,\beta_3,\beta_4) & \text{if } \mathfrak{g} = \mathrm{E}_{6(-26)},\\ (\beta_0,\beta_4,\beta_3,\beta_2) & \text{if } \mathfrak{g} = \mathrm{E}_{6(6)}\,.\end{cases}$$

5. Define a map $\phi = (\phi_1, \phi_2)$ on the simple roots of $\mathfrak{k}_{\mathbb{C}}$ as follows:



if $\mathfrak{g} = \mathrm{E}_{6(-26)}$

and



if $\mathfrak{g} = \mathrm{E}_{6(6)}$.

We will usually write $\phi_j(i)$ in place of $\phi_j(\gamma_i)$; for example, $\phi_2(3) = 5$ when $\mathfrak{g} = \mathfrak{e}_{6(-26)}$.

6. Let
$$X_{\gamma_i} = X_{\alpha_{\phi_1(i)}} + \theta X_{\alpha_{\phi_1(i)}} = X_{\alpha_{\phi_1(i)}} + \epsilon X_{\alpha_{\phi_2(i)}},$$
where
$$\epsilon = \begin{cases}-1 & \text{if } \mathfrak{g} = \mathrm{E}_{6(6)} \text{ and } i = 1,\\ 1 & \text{otherwise.}\end{cases}$$

5.1 Description of the Algorithm

In this section we describe an algorithm that calculates $K^0_{\mathbb{C}}$-highest weights of $\mathfrak{p}^d_{\mathbb{C}}$ (respectively $\mathfrak{k}^d_{\mathbb{C}}$).

Input: $\gamma_i(h)$ for $(i = 1, 2, 3, 4)$

Step 1. Compute $\alpha_i(h)$ for $(i = 1, 2, 3, 4, 5, 6)$

Step 2. Make a list $\mathcal{L}$ of all roots δ of E_6 such that $\delta(h) = d$

Step 3. For each $\delta \in \mathcal{L}$ do

- if δ is complex set $Y_\delta = X_\delta - \theta(X_\delta)$ (respectively $Y_\delta = X_\delta + \theta(X_\delta)$) and delete $\theta(\delta)$ from $\mathcal{L}$
- if δ is noncompact (respectively compact) imaginary set $Y_\delta = X_\delta$
- if δ is compact (respectively noncompact) imaginary delete δ from $\mathcal{L}$

[Now $\{Y_\delta\}_{\delta\in\mathcal{L}}$ is a basis for $\mathfrak{p}^d_{\mathbb{C}}$ (respectively $\mathfrak{k}^d_{\mathbb{C}}$)]

Step 4. For each $\delta \in \mathcal{L}$ check whether $[X_{\gamma_i}, Y_\delta] = 0$ for all i such that $\gamma_i(h) = 0$
if not delete δ from $\mathcal{L}$

[Now $\mathcal{L}$ is the list of $K^0_{\mathbb{C}}$-highest weights in $\mathfrak{p}^d_{\mathbb{C}}$ expressed in the $(\alpha_1, \dots, \alpha_6)$ basis]

Step 5. Restrict to $\mathfrak{t}_{\mathbb{C}}$ expressing the highest weights in the basis $(\gamma_1, \dots, \gamma_4)$.

Step 6. Use the Cartan matrix of $\mathfrak{k}$ to express the highest weights in the fundamental basis $(\omega_1, \dots, \omega_4)$

End

5.2 Correctness of the Algorithm

Observe that Step 4 is the most important part of the algorithm. The next lemma gives us a proof of correctness.

Lemma 5.1. *Maintaining the above notation, $[X_{\gamma_i}, Y_\delta] = 0$ if and only if $\alpha_{\phi_{j(i)}} + \delta$ is not a root or is a compact (respectively noncompact) imaginary root for $j = 1, 2$.*

Proof: For simplicity we shall give a proof for the $\mathfrak{p}_{\mathbb{C}}$ case. The proof of the $\mathfrak{k}_{\mathbb{C}}$ case is similar. We consider two cases:

1. δ is a complex root. Then

$$\begin{aligned}
[X_{\gamma_i}, Y_\delta] &= [X_{\gamma_i}, X_\delta - \theta X_\delta] \\
&= [X_{\alpha_{\phi_1(i)}} + \theta X_{\alpha_{\phi_1(i)}}, X_\delta - \theta X_\delta] \\
&= [X_{\alpha_{\phi_1(i)}}, X_\delta] - [\theta X_{\alpha_{\phi_1(i)}}, \theta X_\delta] \\
&\quad + [\theta X_{\alpha_{\phi_1(i)}}, X_\delta] - [X_{\alpha_{\phi_1(i)}}, \theta X_\delta] \\
&= [X_{\alpha_{\phi_1(i)}}, X_\delta] - \theta[X_{\alpha_{\phi_1(i)}}, X_\delta] \\
&\quad + [\theta X_{\alpha_{\phi_1(i)}}, X_\delta] - \theta[\theta X_{\alpha_{\phi_1(i)}}, X_\delta] \\
&= [X_{\alpha_{\phi_1(i)}}, X_\delta] - \theta[X_{\alpha_{\phi_1(i)}}, X_\delta] \\
&\quad + \epsilon([X_{\alpha_{\phi_2(i)}}, X_\delta] - \theta[X_{\alpha_{\phi_2(i)}}, X_\delta]).
\end{aligned}$$

If $\phi_1(i) = \phi_2(i)$ then $\epsilon = 1$ and $[X_{\gamma_i}, Y_\delta]$ vanishes if and only if $[X_{\alpha_{\phi_1(i)}}, X_\delta] \in \mathfrak{k}_{\mathbb{C}}$.

If $\phi_1(i) \neq \phi_2(i)$, then $[X_{\alpha_{\phi_1(i)}}, X_\delta] - \theta[X_{\alpha_{\phi_1(i)}}, X_\delta]$ and $[X_{\alpha_{\phi_2(i)}}, X_\delta] - \theta[X_{\alpha_{\phi_2(i)}}, X_\delta]$ belong to independent subspaces of $\mathfrak{g}_{\mathbb{C}}$, so $[X_{\gamma_i}, Y_\delta]$ vanishes if and only if $[X_{\alpha_{\phi_j(i)}}, X_\delta] \in \mathfrak{k}_{\mathbb{C}}$ for $j = 1, 2$.

2. δ is noncompact imaginary. Then

$$[X_{\gamma_i}, Y_\delta] = [X_{\gamma_i}, X_\delta] = [X_{\alpha_{\phi_1(i)}}, X_\delta] + \epsilon[X_{\alpha_{\phi_2(i)}}, X_\delta].$$

If $\phi_1(i) = \phi_2(i)$, then $\epsilon = 1$ and $\alpha_{\phi_1(i)}$ is compact imaginary, so $\alpha_{\phi_1(i)} + \delta$ is not a compact imaginary root, and $[X_{\gamma_i}, Y_\delta]$ vanishes if and only if $\alpha_{\phi_1(i)} + \delta$ is not a root.

If $\phi_1(i) \neq \phi_2(i)$, then $\alpha_{\phi_j(i)} + \delta$ is not a compact imaginary root, and $[X_{\gamma_i}, Y_\delta]$ vanishes if and only if $\alpha_{\phi_j(i)} + \delta$ is not a root for $j = 1, 2$. □

6. RELATIVE INVARIANTS

Let (G, V) be a prehomogeneous space. A *relative invariant* of (G, V) is a polynomial $f \in \mathbb{C}[V]$ that transforms by a character of G:

$$f(g^{-1}\vec{v}) = \chi(g) f(\vec{v}).$$

Since the open orbit is dense, a relative invariant is determined up to scalar multiplication by its character. Evidently any relative invariant is fixed by the action of the commutator subgroup $[G, G]$. Conversely, if G is reductive, then any $[G, G]$-invariant polynomial is a sum of relative invariants:

$$*\mathbb{C}[V]^{[G,G]} = \bigoplus_{\chi} F^\chi,$$

where F^χ is the space of relative invariants of character χ. If $F^\chi \neq 0$, then $\dim_{\mathbb{C}} F^\chi = 1$.

Let M be the set of all characters χ with $F^\chi \neq 0$. Then M is a submonoid of the character group of G, and $\mathbb{C}[V]^{[G,G]}$ is M-graded. Indeed, the ring of invariants $\mathbb{C}[V]^{[G,G]}$ is isomorphic to the monoid ring $\mathbb{C}[M]$. Since $\mathbb{C}[V]$ has unique factorization and $[G,G]$ is connected, $\mathbb{C}[V]^{[G,G]}$ also has unique factorization and M is a free commutative monoid. We call the generators of M the *fundamental characters* of (G,V). Fundamental characters and their associated relative invariants can be calculated using methods from classical invariant theory; see [Jackson and Noël 05a] and [Jackson and Noël 06] for details.

7. PREHOMOGENEOUS SPACES ASSOCIATED WITH $\mathbf{E_{6(6)}}$ AND $\mathbf{E_{6(-26)}}$

In the tables below, we give for each nilpotent orbit $\mathcal{O}$ the "labeled Dynkin diagram" of $\mathcal{O}$, which is the Dynkin diagram of $\mathfrak{k}_{\mathbb{C}}$ with the integers $\gamma_i(x)$ attached to the corresponding nodes.

Since $\mathfrak{k}_{\mathbb{C}}^0$ contains $\mathfrak{t}$, it is a sum of root spaces. Moreover, because the $\gamma_i(x)$ are nonnegative, a positive root space X_α lies in $\mathfrak{k}_{\mathbb{C}}^0$ if and only if α is a sum of simple roots with label 0. Consequently, the Dynkin diagram of $\mathfrak{k}_{\mathbb{C}}^0$ is obtained from that of $\mathfrak{k}_{\mathbb{C}}$ by deleting the nodes with positive labels, and the dimension of the center $\mathfrak{z}(\mathfrak{k}_{\mathbb{C}}^0)$ is the number of deleted nodes.

For each nilpotent orbit $\mathcal{O}$, we list those $i > 0$ for which $\mathfrak{p}_{\mathbb{C}}^i \neq 0$ (respectively $\mathfrak{k}_{\mathbb{C}}^i \neq 0$), and give the $K_{\mathbb{C}}^0$-highest weights of these prehomogeneous spaces expressed on the basis of the fundamental weights of $K_{\mathbb{C}}$ in the Bourbaki order. When interpreting the results given in the table, one should keep in mind that the action of the semisimple part of $\mathfrak{k}_{\mathbb{C}}^0$ on $\mathfrak{p}_{\mathbb{C}}^i$ (respectively $\mathfrak{k}_{\mathbb{C}}^i$) is completely determined by those coefficients associated with the nodes of Dynkin–Kostant label 0; the other coefficients affect only the action of the center of $\mathfrak{k}_{\mathbb{C}}^0$, which in any case must act by scalars on each irreducible component of $\mathfrak{p}_{\mathbb{C}}^i$ (respectively $\mathfrak{k}_{\mathbb{C}}^i$).

We also give the irreducible decomposition of $\mathfrak{p}_{\mathbb{C}}^i$ (respectively $\mathfrak{k}_{\mathbb{C}}^i$) as a $[K_{\mathbb{C}}^0, K_{\mathbb{C}}^0]$-module in the notation of [Kac 80]; we use the name of a classical group to denote its standard representation. Usually, context makes it clear whether the group or the module is intended. When $[K_{\mathbb{C}}^0, K_{\mathbb{C}}^0]$ contains more than one factor isomorphic to a given classical group, we number the factors with superscripts. The symbol $\mathbb{C}$ denotes the trivial module; Spin_7 means the spin representation of the twofold cover of SO_7.

The last column of each table contains the degrees of the fundamental relative invariants corresponding to the prehomogeneous spaces. Details about computing such degrees are found in [Jackson and Noël 05a].

Nilpotent orbits in $\mathbf{E_{6(6)}}$ (type EI)

Orbit | $K_{\mathbb{C}}$ diagram | i | $\dim \mathfrak{g}_{\mathbb{C}}^i \cap \mathfrak{p}_{\mathbb{C}}$ | Highest weights of $\mathfrak{g}_{\mathbb{C}}^i \cap \mathfrak{p}_{\mathbb{C}}$ | Prehomogeneous space | Fundamental characters

Orbit	$K_{\mathbb{C}}$ diagram	i	$\dim \mathfrak{g}_{\mathbb{C}}^i \cap \mathfrak{p}_{\mathbb{C}}$	Highest weights of $\mathfrak{g}_{\mathbb{C}}^i \cap \mathfrak{p}_{\mathbb{C}}$	Prehomogeneous space	Fundamental characters
1.	○—○—○⇐○ 0 0 0 1	1	10	$(0,0,2,-1)$	$\mathrm{S}^2(\mathrm{SL}_4)$	(4)
		2	1	$(0,0,0,1)$	$\mathbb{C}$	(1)
2.	○—○—○⇐○ 0 1 0 0	1	8	$(1,-1,1,0)$	$\mathrm{SL}_2 \otimes \mathrm{Sp}_4$	(2)
		2	5	$(0,0,0,1)$	SO_5	(2)
3.	○—○—○⇐○ 1 0 0 1	1	9	$(-1,0,1,0)$ $(0,2,0,-1)$	$\mathrm{SL}_3^* \oplus \mathrm{S}^2(\mathrm{SL}_3)$	$(2,1)$ $(0,3)$
		2	6	$(0,0,2,-1)$	$\mathrm{S}^2(\mathrm{SL}_3^*)$	(3)
		3	1	$(0,0,0,1)$	$\mathbb{C}$	(1)
4.	○—○—○⇐○ 0 0 0 2	2	10	$(0,0,2,-1)$	$\mathrm{S}^2(\mathrm{SL}_4)$	(4)
		4	1	$(0,0,0,1)$	$\mathbb{C}$	(1)
5.	○—○—○⇐○ 2 0 0 0	2	14	$(0,0,0,1)$	$\wedge^3(\mathrm{Sp}_6)/\,\mathrm{Sp}_6$	(4)

(continued on next page)

Nilpotent orbits in $\mathbf{E_{6(6)}}$ (type EI) (continued)

Orbit	$K_{\mathbb{C}}$ diagram	i	$\dim \mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}$	Highest weights of $\mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}$	Prehomogeneous space	Fundamental characters
6.	○—○—○⇐○ 0 2 0 0	2	8	$(1,-1,1,0)$	$SL_2 \otimes Sp_4$	(2)
		4	5	$(0,0,0,1)$	SO_5	(2)
7.	○—○—○⇐○ 0 1 0 2	1	4	$(1,0,1,-1)$	$SL_2^1 \otimes SL_2^2$	(2)
		2	4	$(0,2,0,-1)$ $(2,-2,0,1)$	$\mathbb{C} \oplus S^2(SL_2^1)$	$(1,0)$ $(0,2)$
		3	4	$(1,-1,1,0)$	$SL_2^1 \otimes SL_2^2$	(2)
		4	3	$(0,0,2,-1)$	$S^2(SL_2^2)$	(2)
		6	1	$(0,0,0,1)$	$\mathbb{C}$	(1)
8.	○—○—○⇐○ 0 1 0 1	1	7	$(1,0,1,-1)$ $(2,-2,0,1)$	$(SL_2^1 \otimes SL_2^2)$ $\oplus S^2(SL_2^1)$	$(2,0)$ $(0,2)$
		2	5	$(1,-1,1,0)$ $(0,2,0,-1)$	$(SL_2^1 \otimes SL_2^2) \oplus \mathbb{C}$	$(2,0)$ $(0,1)$
		3	3	$(0,0,2,-1)$	$S^2(SL_2^2)$	(2)
		4	1	$(0,0,0,1)$	$\mathbb{C}$	(1)
9.	○—○—○⇐○ 0 2 0 2	2	7	$(1,0,1,-1)$ $(2,-2,0,1)$	$(SL_2^1 \otimes SL_2^2)$ $\oplus S^2(SL_2^1)$	$(2,0)$ $(0,2)$
		4	5	$(1,-1,1,0)$ $(0,2,0,-1)$	$(SL_2^1 \otimes SL_2^2) \oplus \mathbb{C}$	$(2,0)$ $(0,1)$
		6	3	$(0,0,2,-1)$	$S^2(SL_2^2)$	(2)
		8	1	$(0,0,0,1)$	$\mathbb{C}$	(1)
10.	○—○—○⇐○ 1 0 1 0	1	6	$(-1,1,-1,1)$ $(1,1,-1,0)$	$(SL_2^1 \otimes SL_2^2)$ $\oplus SL_2^1$	$(2,0)$
		2	7	$(-1,0,1,0)$ $(0,2,-2,1)$	$\mathbb{C} \oplus (S^2(SL_2^1)$ $\otimes SL_2^2)$	$(1,0)$ $(0,4)$
		3	2	$(0,1,0,0)$	SL_2^1	$\emptyset$
		4	2	$(0,0,0,1)$	SL_2^2	$\emptyset$
11.	○—○—○⇐○ 1 1 0 1	1	6	$(-1,1,1,-1)$ $(0,-1,0,1)$ $(2,-2,2,-1)$	$SL_2 \oplus \mathbb{C}$ $\oplus S^2(SL_2)$	$(2,0,1)$ $(0,1,0)$ $(0,0,2)$
		2	5	$(-1,0,1,0)$ $(1,0,1,-1)$ $(2,-2,0,1)$	$SL_2 \oplus SL_2 \oplus \mathbb{C}$	$(1,1,0)$ $(0,0,1)$
		3	3	$(1,-1,1,0)$ $(0,2,0,-1)$	$SL_2 \oplus \mathbb{C}$	$(0,1)$
		4	3	$(0,0,2,-1)$	$S^2(SL_2)$	(2)
		5	1	$(0,0,0,1)$	$\mathbb{C}$	(1)
12.	○—○—○⇐○ 2 0 0 2	2	9	$(-1,0,1,0)$ $(0,2,0,-1)$	$SL_3^* \oplus S^2(SL_3)$	$(2,1)$ $(0,3)$
		4	6	$(0,0,2,-1)$	$S^2(SL_3^*)$	(3)
		6	1	$(0,0,0,1)$	$\mathbb{C}$	(1)
13.	○—○—○⇐○ 2 0 0 4	2	7	$(-2,0,0,1)$ $(0,2,0,-1)$	$\mathbb{C} \oplus S^2(SL_3)$	$(1,0)$ $(0,3)$
		4	3	$(-1,0,1,0)$	SL_3^*	$\emptyset$
		6	6	$(0,0,2,-1)$	$S^2(SL_3^*)$	(3)
		10	1	$(0,0,0,1)$	$\mathbb{C}$	(1)

(continued on next page)

Nilpotent orbits in $\mathbf{E_{6(6)}}$ (type EI) (continued)

Orbit	$K_{\mathbb{C}}$ diagram	i	$\dim \mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}$	Highest weights of $\mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}$	Prehomogeneous space	Fundamental characters
14.	1 2 1 1	1	3	$(0,-1,2,-1)$ $(-2,0,0,1)$ $(2,-1,0,0)$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		2	3	$(-1,2,-1,0)$ $(0,-1,0,1)$ $(2,-2,2,-1)$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		3	3	$(-1,1,1,-1)$ $(2,-2,0,1)$ $(1,1,-1,0)$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		4	2	$(-1,1,-1,1)$ $(1,0,1,-1)$	$\mathbb{C}\oplus\mathbb{C}$	$(1,0)$ $(0,1)$
		5	2	$(-1,0,1,0)$ $(1,0,-1,1)$	$\mathbb{C}\oplus\mathbb{C}$	$(1,0)$ $(0,1)$
		6	2	$(1,-1,1,0)$ $(0,2,0,-1)$	$\mathbb{C}\oplus\mathbb{C}$	$(1,0)$ $(0,1)$
		7	1	$(0,2,-2,1)$	$\mathbb{C}$	(1)
		8	1	$(0,1,0,0)$	$\mathbb{C}$	(1)
		9	1	$(0,0,2,-1)$	$\mathbb{C}$	(1)
		10	1	$(0,0,0,1)$	$\mathbb{C}$	(1)
15.	1 0 1 1	1	5	$(-1,1,1,-1)$ $(-2,0,0,1)$ $(1,1,-1,0)$	$\mathrm{SL}_2\oplus\mathbb{C}\oplus\mathrm{SL}_2$	$(1,0,1)$ $(0,1,0)$
		2	5	$(-1,1,-1,1)$ $(0,2,0,-1)$	$\mathrm{SL}_2\oplus\mathrm{S}^2(\mathrm{SL}_2)$	$(2,1)$ $(0,2)$
		3	4	$(-1,0,1,0)$ $(0,2,-2,1)$	$\mathbb{C}\oplus\mathrm{S}^2(\mathrm{SL}_2)$	$(1,0)$ $(0,2)$
		4	2	$(0,1,0,0)$	SL_2	$\emptyset$
		5	1	$(0,0,2,-1)$	$\mathbb{C}$	(1)
		6	1	$(0,0,0,1)$	$\mathbb{C}$	(1)
16.	1 1 1 1	1	4	$(-1,2,-1,0)$ $(0,-1,2,-1)$ $(-2,0,0,1)$ $(2,-1,0,0)$	$\mathbb{C}\oplus\mathbb{C}$ $\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0,0)$ $(0,1,0,0)$ $(0,0,1,0)$ $(0,0,0,1)$
		2	4	$(-1,1,1,-1)$ $(0,-1,0,1)$ $(1,1,-1,0)$ $(2,-2,2,-1)$	$\mathbb{C}\oplus\mathbb{C}$ $\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0,0)$ $(0,1,0,0)$ $(0,0,1,0)$ $(0,0,0,1)$
		3	3	$(-1,1,-1,1)$ $(1,0,1,-1)$ $(2,-2,0,1)$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		4	3	$(-1,0,1,0)$ $(1,0,-1,1)$ $(0,2,0,-1)$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		5	2	$(1,-1,1,0)$ $(0,2,-2,1)$	$\mathbb{C}\oplus\mathbb{C}$	$(1,0)$ $(0,1)$
		6	1	$(0,1,0,0)$	$\mathbb{C}$	(1)
		7	1	$(0,0,2,-1)$	$\mathbb{C}$	(1)
		8	1	$(0,0,0,1)$	$\mathbb{C}$	(1)

(continued on next page)

Nilpotent orbits in $\mathbf{E_{6(6)}}$ (type EI) (continued)

Orbit	$K_{\mathbb{C}}$ diagram	i	$\dim \mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}$	Highest weights of $\mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}$	Prehomogeneous space	Fundamental characters
17.	1 1 1 2	1	3	$(-1,2,-1,0)$ $(0,-1,2,-1)$ $(2,-1,0,0)$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		2	4	$(-1,1,1,-1)$ $(-2,0,0,1)$ $(1,1,-1,0)$ $(2,-2,2,-1)$	$\mathbb{C}\oplus\mathbb{C}$ $\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0,0)$ $(0,1,0,0)$ $(0,0,1,0)$ $(0,0,0,1)$
		3	2	$(0,-1,0,1)$ $(1,0,1,-1)$	$\mathbb{C}\oplus\mathbb{C}$	$(1,0)$ $(0,1)$
		4	3	$(-1,1,-1,1)$ $(0,2,0,-1)$ $(2,-2,0,1)$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		5	2	$(-1,0,1,0)$ $(1,0,-1,1)$	$\mathbb{C}\oplus\mathbb{C}$	$(1,0)$ $(0,1)$
		6	2	$(1,-1,1,0)$ $(0,2,-2,1)$	$\mathbb{C}\oplus\mathbb{C}$	$(1,0)$ $(0,1)$
		7	1	$(0,1,0,0)$	$\mathbb{C}$	(1)
		8	1	$(0,0,2,-1)$	$\mathbb{C}$	(1)
		10	1	$(0,0,0,1)$	$\mathbb{C}$	(1)
18.	2 2 2 2	2	4	$(-1,2,-1,0)$ $(0,-1,2,-1)$ $(-2,0,0,1)$ $(2,-1,0,0)$	$\mathbb{C}\oplus\mathbb{C}$ $\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0,0)$ $(0,1,0,0)$ $(0,0,1,0)$ $(0,0,0,1)$
		4	4	$(-1,1,1,-1)$ $(0,-1,0,1)$ $(1,1,-1,0)$ $(2,-2,2,-1)$	$\mathbb{C}\oplus\mathbb{C}$ $\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0,0)$ $(0,1,0,0)$ $(0,0,1,0)$ $(0,0,0,1)$
		6	3	$(-1,1,-1,1)$ $(1,0,1,-1)$ $(2,-2,0,1)$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		8	3	$(-1,0,1,0)$ $(1,0,-1,1)$ $(0,2,0,-1)$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		10	2	$(1,-1,1,0)$ $(0,2,-2,1)$	$\mathbb{C}\oplus\mathbb{C}$	$(1,0)$ $(0,1)$
		12	1	$(0,1,0,0)$	$\mathbb{C}$	(1)
		14	1	$(0,0,2,-1)$	$\mathbb{C}$	(1)
		16	1	$(0,0,0,1)$	$\mathbb{C}$	(1)
19.	2 2 0 2	2	6	$(-1,1,1,-1)$ $(0,-1,0,1)$ $(2,-2,2,-1)$	$\mathrm{SL}_2\oplus\mathbb{C}$ $\oplus S^2(\mathrm{SL}_2)$	$(2,0,1)$ $(0,1,0)$ $(0,0,2)$
		4	5	$(-1,0,1,0)$ $(1,0,1,-1)$ $(2,-2,0,1)$	$\mathrm{SL}_2\oplus\mathrm{SL}_2\oplus\mathbb{C}$	$(1,1,0)$ $(0,0,1)$
		6	3	$(1,-1,1,0)$ $(0,2,0,-1)$	$\mathrm{SL}_2\oplus\mathbb{C}$	$(0,1)$
		8	3	$(0,0,2,-1)$	$S^2(\mathrm{SL}_2)$	(2)
		10	1	$(0,0,0,1)$	$\mathbb{C}$	(1)

(continued on next page)

Nilpotent orbits in $\mathbf{E_{6(6)}}$ (type EI) (continued)						
Orbit	$K_{\mathbb{C}}$ diagram	i	$\dim \mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}$	Highest weights of $\mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}$	Prehomogeneous space	Fundamental characters
20.	4 2 2 4	2	4	$(-1,2,-1,0)$ $(0,-1,2,-1)$ $(-2,0,0,1)$ $(2,0,-2,1)$	$\mathbb{C} \oplus \mathbb{C}$ $\oplus \mathbb{C} \oplus \mathbb{C}$	$(1,0,0,0)$ $(0,1,0,0)$ $(0,0,1,0)$ $(0,0,0,1)$
		4	2	$(-1,1,1,-1)$ $(2,-1,0,0)$	$\mathbb{C} \oplus \mathbb{C}$	$(1,0)$ $(0,1)$
		6	3	$(0,-1,0,1)$ $(1,1,-1,0)$ $(2,-2,2,-1)$	$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		8	2	$(-1,1,-1,1)$ $(1,0,1,-1)$	$\mathbb{C} \oplus \mathbb{C}$	$(1,0)$ $(0,1)$
		10	3	$(-1,0,1,0)$ $(0,2,0,-1)$ $(2,-2,0,1)$	$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		12	1	$(1,0,-1,1)$	$\mathbb{C}$	(1)
		14	2	$(1,-1,1,0)$ $(0,2,-2,1)$	$\mathbb{C} \oplus \mathbb{C}$	$(1,0)$ $(0,1)$
		16	1	$(0,1,0,0)$	$\mathbb{C}$	(1)
		18	1	$(0,0,2,-1)$	$\mathbb{C}$	(1)
		22	1	$(0,0,0,1)$	$\mathbb{C}$	(1)
21.	2 2 0 4	2	6	$(-1,1,1,-1)$ $(-2,0,0,1)$ $(2,-2,2,-1)$	$\mathrm{SL}_2 \oplus \mathbb{C}$ $\oplus \mathrm{S}^2(\mathrm{SL}_2)$	$(2,0,1)$ $(0,1,0)$ $(0,0,2)$
		4	3	$(0,-1,0,1)$ $(1,0,1,-1)$	$\mathbb{C} \oplus \mathrm{SL}_2$	$(1,0)$
		6	4	$(-1,0,1,0)$ $(0,2,0,-1)$ $(2,-2,0,1)$	$\mathrm{SL}_2 \oplus \mathbb{C} \oplus \mathbb{C}$	$(0,1,0)$ $(0,0,1)$
		8	2	$(1,-1,1,0)$	SL_2	$\emptyset$
		10	3	$(0,0,2,-1)$	$\mathrm{S}^2(\mathrm{SL}_2)$	(2)
		14	1	$(0,0,0,1)$	$\mathbb{C}$	(1)
22.	0 2 2 0	2	8	$(2,-2,0,1)$ $(1,1,-1,0)$	$(\mathrm{S}^2(\mathrm{SL}_2^1)$ $\otimes \mathrm{SL}_2^2) \oplus \mathrm{SL}_2^1$	$(4,0)$ $(2,2)$
		4	4	$(1,0,-1,1)$	$(\mathrm{SL}_2^1 \otimes \mathrm{SL}_2^2)$	(2)
		6	4	$(1,-1,1,0)$ $(0,2,-2,1)$	$\mathrm{SL}_2^1 \oplus \mathrm{SL}_2^2$	$(1,1)$
		8	1	$(0,1,0,0)$	$\mathbb{C}$	(1)
		10	2	$(0,0,0,1)$	SL_2^2	$\emptyset$
23.	0 0 2 0	2	12	$(0,2,-2,1)$	$(\mathrm{S}^2(\mathrm{SL}_3) \otimes \mathrm{SL}_2)$	(12)
		4	3	$(0,1,0,0)$	SL_3	$\emptyset$
		6	2	$(0,0,0,1)$	SL_2	$\emptyset$

Nilpotent orbits in $\mathbf{E_{6(6)}}$ (type EI)						
Orbit	$K_{\mathbb{C}}$ diagram	i	$\dim \mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$	Highest weights of $\mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$	Prehomogeneous space	Fundamental characters
1.	0 0 0 1	1	10	$(2,0,0,0)$	$\mathrm{S}^2(\mathrm{SL}_4)$	(4)
2.	0 1 0 0	1	8	$(1,-1,1,0)$	$\mathrm{SL}_2 \otimes \mathrm{Sp}_4$	(2)
(continued on next page)						

Nilpotent orbits in $\mathbf{E_{6(6)}}$ (type EI) (continued)

Orbit	$K_{\mathbb{C}}$ diagram	i	$\dim \mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$	Highest weights of $\mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$	Prehomogeneous space	Fundamental characters
		2	3	$(2,0,0,0)$	$S^2(SL_2)$	(2)
3.	$\circ-\circ-\circ\Leftarrow\circ$ 1 0 0 1	1	9	$(-2,2,0,0)$ $(1,0,1,-1)$	$S^2(SL_3) \oplus SL_3^*$	$(1,2)$ $(3,0)$
		2	3	$(0,1,0,0)$	SL_3	$\emptyset$
		3	1	$(2,0,0,0)$	$\mathbb{C}$	(1)
4.	$\circ-\circ-\circ\Leftarrow\circ$ 0 0 0 2	2	10	$(2,0,0,0)$	$S^2(SL_4)$	(4)
5.	$\circ-\circ-\circ\Leftarrow\circ$ 2 0 0 0	2	6	$(0,1,0,0)$	Sp_6	$\emptyset$
		4	1	$(2,0,0,0)$	$\mathbb{C}$	(1)
6.	$\circ-\circ-\circ\Leftarrow\circ$ 0 2 0 0	2	8	$(1,-1,1,0)$	$SL_2 \otimes Sp_4$	(2)
		4	3	$(2,0,0,0)$	$S^2(SL_2)$	(2)
7.	$\circ-\circ-\circ\Leftarrow\circ$ 0 1 0 2	1	4	$(1,0,1,-1)$	$SL_2^1 \otimes SL_2^2$	(2)
		2	3	$(0,-2,2,0)$	$S^2(SL_2^2)$	(2)
		3	4	$(1,-1,1,0)$	$SL_2^1 \otimes SL_2^2$	(2)
		4	3	$(2,0,0,0)$	$S^2(SL_2^1)$	(2)
8.	$\circ-\circ-\circ\Leftarrow\circ$ 0 1 0 1	1	7	$(0,-2,2,0)$ $(1,0,1,-1)$	$S^2(SL_2^2)\oplus$ $(SL_2^1 \otimes SL_2^2)$	$(2,0)$ $(0,2)$
		2	4	$(1,-1,1,0)$	$SL_2^1 \otimes SL_2^2$	(2)
		3	3	$(2,0,0,0)$	$S^2(SL_2^1)$	(2)
9.	$\circ-\circ-\circ\Leftarrow\circ$ 0 2 0 2	2	7	$(0,-2,2,0)$ $(1,0,1,-1)$	$S^2(SL_2^2)\oplus$ $(SL_2^1 \otimes SL_2^2)$	$(2,0)$ $(0,2)$
		4	4	$(1,-1,1,0)$	$SL_2^1 \otimes SL_2^2$	(2)
		6	3	$(2,0,0,0)$	$S^2(SL_2^1)$	(2)
10.	$\circ-\circ-\circ\Leftarrow\circ$ 1 0 1 0	1	6	$(-1,1,-1,1)$ $(1,1,-1,0)$	$(SL_2^1 \otimes SL_2^2)$ $\oplus SL_2^1$	$(2,0)$
		2	5	$(-2,2,0,0)$ $(1,0,-1,1)$	$S^2(SL_2^1) \oplus SL_2^2$	$(2,0)$
		3	2	$(0,1,0,0)$	SL_2^1	$\emptyset$
		4	1	$(2,0,0,0)$	$\mathbb{C}$	(1)
11.	$\circ-\circ-\circ\Leftarrow\circ$ 1 1 0 1	1	6	$(-1,1,1,-1)$ $(0,-2,2,0)$ $(2,-1,0,0)$	$SL_2 \oplus$ $S^2(SL_2) \oplus \mathbb{C}$	$(2,1,0)$ $(0,2,0)$ $(0,0,1)$
		2	4	$(-1,0,1,0)$ $(1,0,1,-1)$	$SL_2 \oplus SL_2$	$(1,1)$
		3	3	$(-2,2,0,0)$ $(1,-1,1,0)$	$\mathbb{C} \oplus SL_2$	$(1,0)$
		4	1	$(0,1,0,0)$	$\mathbb{C}$	(1)
		5	1	$(2,0,0,0)$	$\mathbb{C}$	(1)
12.	$\circ-\circ-\circ\Leftarrow\circ$ 2 0 0 2	2	9	$(-2,2,0,0)$ $(1,0,1,-1)$	$S^2(SL_3) \oplus SL_3^*$	$(1,2)$ $(3,0)$
		4	3	$(0,1,0,0)$	SL_3	$\emptyset$
		6	1	$(2,0,0,0)$	$\mathbb{C}$	(1)
13.	$\circ-\circ-\circ\Leftarrow\circ$ 2 0 0 4	2	3	$(1,0,1,-1)$	SL_3^*	$\emptyset$
		4	6	$(-2,2,0,0)$	$S^2(SL_3)$	(3)
		6	3	$(0,1,0,0)$	SL_3	$\emptyset$
		8	1	$(2,0,0,0)$	$\mathbb{C}$	(1)

(continued on next page)

Nilpotent orbits in $\mathbf{E_{6(6)}}$ (type EI) (continued)

Orbit	$K_{\mathbb{C}}$ diagram	i	$\dim \mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$	Highest weights of $\mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$	Prehomogeneous space	Fundamental characters
14.	1 2 1 1	1	3	$(0,-1,2,-1)$ $(0,0,-2,2)$ $(2,-1,0,0)$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		2	2	$(-1,2,-1,0)$ $(0,-1,0,1)$	$\mathbb{C}\oplus\mathbb{C}$	$(1,0)$ $(0,1)$
		3	3	$(-1,1,1,-1)$ $(0,-2,2,0)$ $(1,1,-1,0)$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		4	2	$(-1,1,-1,1)$ $(1,0,1,-1)$	$\mathbb{C}\oplus\mathbb{C}$	$(1,0)$ $(0,1)$
		5	2	$(-1,0,1,0)$ $(1,0,-1,1)$	$\mathbb{C}\oplus\mathbb{C}$	$(1,0)$ $(0,1)$
		6	1	$(1,-1,1,0)$	$\mathbb{C}$	(1)
		7	1	$(-2,2,0,0)$	$\mathbb{C}$	(1)
		8	1	$(0,1,0,0)$	$\mathbb{C}$	(1)
		9	1	$(2,0,0,0)$	$\mathbb{C}$	(1)
15.	1 0 1 1	1	5	$(0,0,-2,2)$ $(-1,1,1,-1)$ $(1,1,-1,0)$	$\mathbb{C}\oplus\mathrm{SL}_2\oplus\mathrm{SL}_2$	$(1,0,0)$ $(0,1,1)$
		2	3	$(-1,1,-1,1)$ $(1,0,1,-1)$	$\mathrm{SL}_2\oplus\mathbb{C}$	$(0,1)$
		3	4	$(-2,2,0,0)$ $(1,0,-1,1)$	$\mathrm{S}^2(\mathrm{SL}_2)\oplus\mathbb{C}$	$(0,1)$ $(2,0)$
		4	2	$(0,1,0,0)$	SL_2	$\emptyset$
		5	1	$(2,0,0,0)$	$\mathbb{C}$	(1)
16.	1 1 1 1	1	4	$(-1,2,-1,0)$ $(0,-1,2,-1)$ $(0,0,-2,2)$ $(2,-1,0,0)$	$\mathbb{C}\oplus\mathbb{C}\oplus$ $\mathbb{C}\oplus\mathbb{C}$	$(1,0,0,0)$ $(0,1,0,0)$ $(0,0,1,0)$ $(0,0,0,1)$
		2	3	$(-1,1,1,-1)$ $(0,-1,0,1)$ $(1,1,-1,0)$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		3	3	$(-1,1,-1,1)$ $(0,-2,2,0)$ $(1,0,1,-1)$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		4	2	$(-1,0,1,0)$ $(1,0,-1,1)$	$\mathbb{C}\oplus\mathbb{C}$	$(1,0)$ $(0,1)$
		5	2	$(-2,2,0,0)$ $(1,-1,1,0)$	$\mathbb{C}\oplus\mathbb{C}$	$(1,0)$ $(0,1)$
		6	1	$(0,1,0,0)$	$\mathbb{C}$	(1)
		7	1	$(2,0,0,0)$	$\mathbb{C}$	(1)
17.	1 1 1 2	1	3	$(-1,2,-1,0)$ $(0,-1,2,-1)$ $(2,-1,0,0)$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		2	3	$(0,0,-2,2)$ $(-1,1,1,-1)$ $(1,1,-1,0)$	$\mathbb{C}\oplus\mathbb{C}\oplus\mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		3	2	$(0,-1,0,1)$ $(1,0,1,-1)$	$\mathbb{C}\oplus\mathbb{C}$	$(1,0)$ $(0,1)$

(continued on next page)

Nilpotent orbits in $\mathbf{E_{6(6)}}$ (type EI) (continued)

Orbit	$K_\mathbb{C}$ diagram	i	$\dim \mathfrak{g}^i_\mathbb{C} \cap \mathfrak{k}_\mathbb{C}$	Highest weights of $\mathfrak{g}^i_\mathbb{C} \cap \mathfrak{k}_\mathbb{C}$	Prehomogeneous space	Fundamental characters
		4	2	$(-1,1,-1,1)$ $(0,-2,2,0)$	$\mathbb{C} \oplus \mathbb{C}$	$(1,0)$ $(0,1)$
		5	2	$(-1,0,1,0)$ $(1,0,-1,1)$	$\mathbb{C} \oplus \mathbb{C}$	$(1,0)$ $(0,1)$
		6	2	$(-2,2,0,0)$ $(1,-1,1,0)$	$\mathbb{C} \oplus \mathbb{C}$	$(1,0)$ $(0,1)$
		7	1	$(0,1,0,0)$	$\mathbb{C}$	(1)
		8	1	$(2,0,0,0)$	$\mathbb{C}$	(1)
18.	2 2 2 2	2	4	$(-1,2,-1,0)$ $(0,-1,2,-1)$ $(0,0,-2,2)$ $(2,-1,0,0)$	$\mathbb{C} \oplus \mathbb{C} \oplus$ $\mathbb{C} \oplus \mathbb{C}$	$(1,0,0,0)$ $(0,1,0,0)$ $(0,0,1,0)$ $(0,0,0,1)$
		4	3	$(-1,1,1,-1)$ $(0,-1,0,1)$ $(1,1,-1,0)$	$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		6	3	$(-1,1,-1,1)$ $(0,-2,2,0)$ $(1,0,1,-1)$	$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		8	2	$(-1,0,1,0)$ $(1,0,-1,1)$	$\mathbb{C} \oplus \mathbb{C}$	$(1,0)$ $(0,1)$
		10	2	$(-2,2,0,0)$ $(1,-1,1,0)$	$\mathbb{C} \oplus \mathbb{C}$	$(1,0)$ $(0,1)$
		12	1	$(0,1,0,0)$	$\mathbb{C}$	(1)
		14	1	$(2,0,0,0)$	$\mathbb{C}$	(1)
19.	2 2 0 2	2	6	$(-1,1,1,-1)$ $(0,-2,2,0)$ $(2,-1,0,0)$	$SL_2 \oplus$ $S^2(SL_2) \oplus \mathbb{C}$	$(2,1,0)$ $(0,2,0)$ $(0,0,1)$
		4	4	$(-1,0,1,0)$ $(1,0,1,-1)$	$SL_2 \oplus SL_2$	$(1,1)$
		6	3	$(-2,2,0,0)$ $(1,-1,1,0)$	$\mathbb{C} \oplus SL_2$	$(1,0)$
		8	1	$(0,1,0,0)$	$\mathbb{C}$	(1)
		10	1	$(2,0,0,0)$	$\mathbb{C}$	(1)
20.	4 2 2 4	2	2	$(-1,2,-1,0)$ $(0,-1,2,-1)$	$\mathbb{C} \oplus \mathbb{C}$	$(1,0)$ $(0,1)$
		4	3	$(0,0,-2,2)$ $(-1,1,1,-1)$ $(2,-1,0,0)$	$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		6	2	$(0,-1,0,1)$ $(1,1,-1,0)$	$\mathbb{C} \oplus \mathbb{C}$	$(1,0)$ $(0,1)$
		8	3	$(-1,1,-1,1)$ $(0,-2,2,0)$ $(1,0,1,-1)$	$\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$	$(1,0,0)$ $(0,1,0)$ $(0,0,1)$
		10	1	$(-1,0,1,0)$	$\mathbb{C}$	(1)
		12	2	$(-2,2,0,0)$ $(1,0,-1,1)$	$\mathbb{C} \oplus \mathbb{C}$	$(1,0)$ $(0,1)$
		14	1	$(1,-1,1,0)$	$\mathbb{C}$	(1)
		16	1	$(0,1,0,0)$	$\mathbb{C}$	(1)
		20	1	$(2,0,0,0)$	$\mathbb{C}$	(1)

(continued on next page)

Nilpotent orbits in $\mathbf{E_{6(6)}}$ (type EI) (continued)

Orbit	$K_{\mathbb{C}}$ diagram	i	$\dim \mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$	Highest weights of $\mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$	Prehomogeneous space	Fundamental characters
21.	○—○—○⇐○ 2 2 0 4	2	3	$(-1,1,1,-1)$ $(2,-1,0,0)$	$\mathrm{SL}_2 \oplus \mathbb{C}$	$(0,1)$
		4	5	$(0,-2,2,0)$ $(1,0,1,-1)$	$\mathrm{S}^2(\mathrm{SL}_2) \oplus \mathrm{SL}_2$	$(2,0)$ $(1,2)$
		6	2	$(-1,0,1,0)$	SL_2	$\emptyset$
		8	3	$(-2,2,0,0)$ $(1,-1,1,0)$	$\mathbb{C} \oplus \mathrm{SL}_2$	$(1,0)$
		10	1	$(0,1,0,0)$	$\mathbb{C}$	(1)
		12	1	$(2,0,0,0)$	$\mathbb{C}$	(1)
22.	○—○—○⇐○ 0 2 2 0	2	4	$(0,-1,0,1)$ $(1,1,-1,0)$	$\mathrm{SL}_2^2 \oplus \mathrm{SL}_2^1$	$(1,1)$
		4	5	$(0,-2,2,0)$ $(1,0,-1,1)$	$\mathbb{C} \oplus (\mathrm{SL}_2^1 \otimes \mathrm{SL}_2^2)$	$(1,0)$ $(0,2)$
		6	2	$(1,-1,1,0)$	SL_2^1	$\emptyset$
		8	3	$(2,0,0,0)$	$\mathrm{S}^2(\mathrm{SL}_2^1)$	(2)
23.	○—○—○⇐○ 0 0 2 0	2	6	$(1,0,-1,1)$	$\mathrm{SL}_3 \otimes \mathrm{SL}_2$	$\emptyset$
		4	6	$(2,0,0,0)$	$\mathrm{S}^2(\mathrm{SL}_3)$	(3)

Nilpotent orbits in $\mathbf{E_{6(-26)}}$ (type EIV)

Orbit	$K_{\mathbb{C}}$ diagram	i	$\dim \mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}$	Highest weights of $\mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{p}_{\mathbb{C}}$	Prehomogeneous space	Fundamental characters
1.	○—○⇒○—○ 0 0 0 1	1	8	$(0,0,1,-1)$	Spin_7	(2)
		2	1	$(0,0,0,1)$	$\mathbb{C}$	(1)
2.	○—○⇒○—○ 0 0 0 2	2	8	$(0,0,1,-1)$	Spin_7	(2)
		4	1	$(0,0,0,1)$	$\mathbb{C}$	(1)

Nilpotent orbits in $\mathbf{E_{6(-26)}}$ (type EIV)

Orbit	$K_{\mathbb{C}}$ diagram	i	$\dim \mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$	Highest weights of $\mathfrak{g}^i_{\mathbb{C}} \cap \mathfrak{k}_{\mathbb{C}}$	Prehomogeneous space	Fundamental characters
1.	○—○⇒○—○ 0 0 0 1	1	8	$(0,0,1,-1)$	Spin_7	(2)
		2	7	$(1,0,0,0)$	SO_7	(2)
2.	○—○⇒○—○ 0 0 0 2	2	8	$(0,0,1,-1)$	Spin_7	(2)
		4	7	$(1,0,0,0)$	SO_7	(2)

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