Experimental Determination of Apéry-like Identities for $\zeta(2n + 2)$

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CONTENTS

We document the discovery of two generating functions for $\zeta(2n+2)$, analogous to earlier work for $\zeta(2n+1)$ and $\zeta(4n+3)$, initiated by Koecher and pursued further by Borwein, Bradley, and others.

1. INTRODUCTION

Stimulated by recent work in the arena of Apéry-like sums [Borwein and Bradley 97, Borwein and Bradley 96, Borwein et al. 01b], we decided to methodically look for series acceleration formulas for the Riemann zeta function involving central binomial coefficients in the denominators. Using the PSLQ integer relation algorithm, as described below, we uncovered several new results. In particular, we document the discovery of two generating functions for $\zeta(2n + 2)$, analogous to earlier work for $\zeta(2n+1)$ and $\zeta(4n+3)$, initiated by Koecher and pursued further by Borwein, Bradley, and others. As a conclusion to a very satisfactory experiment, we have been able to use the Wilf–Zeilberger technique to prove our results.

An integer-relation-detection algorithm accepts a vector of real numbers $\vec{x}$ of length n and a bound A as input, and either outputs a length-n vector $\vec{a}$ of integers such that the dot product $a_1 x_1 + \cdots + a_n x_n$ equals 0 to within the available numerical precision, or else establishes that no such vector of integers of length less than A exists. Here the length is the Euclidean norm $(a_1^2 + a_2^2 + \cdots + a_n^2)^{1/2}$, derived from the usual Euclidean metric on $\mathbf{R}^n$. Helaman Ferguson's PSLQ [Ferguson et al. 99, Bailey and Broadhurst 00] is currently the most widely used integer-relation-detection algorithm [Borwein and Bailey 04, pp. 230–235], although variants of the so-called LLL algorithm [Lenstra et al. 82] are also commonly employed. Such algorithms underlie the "Recognize" and "identify" commands in the respective computer algebra packages Mathematica and Maple. They

also play a fundamental role in the investigations we discuss here.

The origins of this work lie in the existence of infinite-series formulas involving central binomial coefficients in the denominators for the constants $\zeta(2)$, $\zeta(3)$, and $\zeta(4)$. These formulas, as well as the role of the formula for $\zeta(3)$ in Apéry's proof of its irrationality, have prompted considerable effort during the past few decades to extend these results to larger integer arguments. The formulas in question are

$$\zeta(2) = 3 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}}, \qquad (1\text{--}1)$$

$$\zeta(3) = \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}}, \qquad (1\text{--}2)$$

$$\zeta(4) = \frac{36}{17} \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}}. \qquad (1\text{--}3)$$

Identity (1–1) has been known since the nineteenth century—it relates to $\arcsin^2(x)$—while (1–2) was variously discovered in the last century and (1–3) was noted by Comtet [Comtet 74, p. 89]; see [Borwein et al. 01b, van der Poorten 78]. Indeed, in [Borwein et al. 01b] a coherent proof of all three identities was provided in the course of a more general study of such central binomial series and so-called *multi-Clausen sums*.

These results led many to conjecture that the constant $\mathcal{Q}_5$ defined by the ratio

$$\mathcal{Q}_5 := \zeta(5) \Big/ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}}$$

is rational, or at least algebraic. However, integer-relation computations using PSLQ and 10,000-digit precision have established that if $\mathcal{Q}_5$ is a zero of a polynomial of degree at most 25 with integer coefficients, then the Euclidean norm of the vector of coefficients exceeds 1.24×10^{383}. Similar computations for $\zeta(5)$ have yielded a bound of 1.98×10^{380}.

These computations lend credence to the belief that $\mathcal{Q}_5$ and $\zeta(5)$ are transcendental. If algebraic, they almost certainly satisfy no simple polynomial of low degree. In particular, if there exist relatively prime integers p and q such that

$$\zeta(5) = \frac{p}{q} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}},$$

then p and q must be astronomically large. Moreover, a study of polylogarithmic ladders in the golden ratio

produced [Almkvist and Granville 99, Borwein et al. 01b]

$$2\zeta(5) - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}} = \frac{5}{2} \mathrm{Li}_5(\rho) - \frac{5}{2} \mathrm{Li}_4(\rho) \log \rho \quad (1\text{--}4)$$
$$+ \zeta(3) \log^2 \rho - \frac{1}{3}\zeta(2) \log^3 \rho$$
$$- \frac{1}{24} \log^5 \rho,$$

where $\rho = (3 - \sqrt{5})/2$ and $\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} z^k / k^n$ is the *polylogarithm* of order n. Since the terms on the right-hand side of (1–4) are almost certainly algebraically independent [Burger and Tubbs 04], we see how unlikely it is that $\mathcal{Q}_5$ is rational. Although the irrationality of $\zeta(5)$ has not yet been confirmed, it is known that one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\zeta(11)$ is irrational [Zudilin 01].

Given the negative result from PSLQ computations for $\mathcal{Q}_5$, the authors of [Borwein and Bradley 97] systematically investigated the possibility of a multiterm identity of this general form for $\zeta(2n + 1)$. The following were recovered early [Borwein and Bradley 97, Borwein and Bradley 96] in experimental searches using computer-based integer-relation tools:

$$\zeta(5) = 2\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}} - \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^2}, \quad (1\text{--}5)$$

$$\zeta(7) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^7 \binom{2k}{k}} + \frac{25}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^4}, (1\text{--}6)$$

$$\zeta(9) = \frac{9}{4}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^9 \binom{2k}{k}} - \frac{5}{4} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^7 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^2}$$
$$+ 5\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^5 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^4} + \frac{45}{4} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^6}$$
$$- \frac{25}{4}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^4} \sum_{j=1}^{k-1} \frac{1}{j^2}, \qquad (1\text{--}7)$$

$$\zeta(11) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^{11} \binom{2k}{k}} + \frac{25}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^7 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^4}$$
$$- \frac{75}{4}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^8}$$
$$+ \frac{125}{4}\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \sum_{j=1}^{k-1} \frac{1}{j^4} \sum_{i=1}^{k-1} \frac{1}{i^4}. \qquad (1\text{--}8)$$

The general formula

$$\sum_{k=1}^{\infty} \frac{1}{k(k^2 - x^2)}$$

$$= \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}} \frac{5k^2 - x^2}{k^2 - x^2} \prod_{m=1}^{k-1} \left(1 - \frac{x^2}{m^2}\right) \quad (1\text{--}9)$$

was obtained by Koecher [Koecher 79] following techniques of Knopp and Schur. It gives (1–2) as its first term and (1–5) as its second term but more complicated expressions for $\zeta(7)$, $\zeta(9)$, and $\zeta(11)$ than (1–6), (1–7), and (1–8). The corresponding result that for its first three terms gives (1–2), (1–6), and (1–8) was worked out by Borwein and Bradley [Borwein and Bradley 97].

Using bootstrapping and an application of the "Pade" function (which in both Mathematica and Maple produces Padé approximations to a rational function satisfied by a truncated power series) produced the following remarkable and unanticipated result [Borwein and Bradley 97]:

$$\sum_{k=1}^{\infty} \frac{1}{k^3(1 - x^4/k^4)} \quad (1\text{--}10)$$

$$= \frac{5}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^3 \binom{2k}{k}(1 - x^4/k^4)} \prod_{m=1}^{k-1} \left(\frac{1 + 4x^4/m^4}{1 - x^4/m^4}\right).$$

The equivalent hypergeometric formulation of (1–10) is

$$_5F_4 \left(\begin{array}{c} 2, 1+x, 1-x, 1+ix, 1-ix \\ 2+x, 2-x, 2+ix, 2-ix \end{array} \middle| 1 \right)$$

$$= \left(\frac{5}{4}\right) {}_6F_5 \left(\begin{array}{c} 2, 2, x_{++}, x_{+-}, x_{-+}, x_{--} \\ 3/2, 2+x, 2-x, 2+ix, 2-ix \end{array} \middle| -\frac{1}{4} \right)$$

(where because of the narrow columns we have used the following abbreviations: $x_{++} = 1 + x + ix$, $x_{+-} = 1 + x - ix$, $x_{-+} = 1 - x + ix$, $x_{--} = 1 - x - ix$).

The identity (1–10) generates (1–2), (1–6), and (1–8) above, and more generally gives a formula for $\zeta(4n+3)$, which for $n > 1$ contains fewer summations than the corresponding formula generated by (1–9). The task of proving (1–10) was reduced in [Borwein and Bradley 97] to that of establishing any one of a number of equivalent finite combinatorial identities. One of these latter identities is

$$\sum_{k=1}^{n} \frac{2n^2}{k^2} \prod_{i=1}^{n-1} (4k^4 + i^4) \bigg/ \prod_{\substack{i=1 \\ i \neq k}}^{n} (k^4 - i^4) = \binom{2n}{n}. \quad (1\text{--}11)$$

This was proved in [Almkvist and Granville 99], so (1–10) is an established theorem. It is now known to be the

$x = 0$ case of the even more general formula

$$\sum_{k=1}^{\infty} \frac{k}{k^4 - x^2 k^2 - y^4} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k \binom{2k}{k}} \frac{5k^2 - x^2}{k^4 - x^2 k^2 - y^4}$$

$$\times \prod_{m=1}^{k-1} \frac{(m^2 - x^2)^2 + 4y^4}{m^4 - x^2 m^2 - y^4}, \quad (1\text{--}12)$$

in which setting $y = 0$ recovers (1–9). The bivariate generating function identity (1–12) was conjectured by Henri Cohen and proved by Bradley [Bradley 02]. It was subsequently and independently proved by Rivoal [Rivoal 04].

Following an analogous—but more deliberate—experimental procedure, as detailed below, we provide a similar general formula for $\zeta(2n+2)$ that is pleasingly parallel to (1–10). It is given in the following theorem:

Theorem 1.1. *Let x be a complex number not equal to a nonzero integer. Then*

$$\sum_{k=1}^{\infty} \frac{1}{k^2 - x^2} = 3 \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}(1 - x^2/k^2)} \quad (1\text{--}13)$$

$$\times \prod_{m=1}^{k-1} \left(\frac{1 - 4x^2/m^2}{1 - x^2/m^2}\right).$$

Note that the left-hand side of (1–13) is trivially equal to

$$\sum_{n=0}^{\infty} \zeta(2n+2)x^{2n} = \frac{1 - \pi x \cot(\pi x)}{2x^2}. \quad (1\text{--}14)$$

Thus, (1–13) generates an Apéry-like formula for $\zeta(2n)$ for every positive integer n.

In Section 2 we shall outline the discovery path, and then in Section 3 we prove (1–13)—or rather the equivalent finite form

$$_3F_2 \left(\begin{array}{c} 3n, n+1, -n \\ 2n+1, n+1/2 \end{array} \middle| \frac{1}{4} \right) = \frac{\binom{2n}{n}}{\binom{3n}{n}}, \qquad 0 \leq n \in \mathbf{Z}. \quad (1\text{--}15)$$

In Section 4 we provide another generating function for which the leading term, or "seed," is Comtet's formula (1–3) for $\zeta(4)$, while the prior generating functions have seeds (1–1) and (1–2).

The paper concludes with some remarks concerning our lack of success in obtaining formulas analogous to equations (1–10) and (1–13) that would generate the simplest known Apéry-like formulas for $\zeta(4n+2)$ and

$\zeta(4n + 1)$, respectively. In this light we record

$$\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}} = \frac{2\pi\sqrt{3} + 9}{27}, \qquad (1\text{--}16)$$

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\binom{2k}{k}} = \frac{2}{\sqrt{5}} \log\left(\frac{\sqrt{5} + 1}{2}\right), \qquad (1\text{--}17)$$

which are perhaps more appropriate seeds in these cases; see [Borwein and Borwein 87, pp. 384–386].

2.   DISCOVERING THEOREM 1.1

As indicated, we have applied a more disciplined experimental approach to produce an analogous generating function for $\zeta(2n + 2)$. We describe this process of discovery in some detail here, since the general technique appears to be quite fruitful and may well yield results in other settings.

We first conjectured that $\zeta(2n + 2)$ is a rational combination of terms of the form

$$\sigma(2r; [2a_1, \ldots, 2a_N]) := \sum_{k=1}^{\infty} \frac{1}{k^{2r}\binom{2k}{k}} \prod_{i=1}^{N} \sum_{n_i=1}^{k-1} \frac{1}{n_i^{2a_i}}, \tag{2--1}$$

where $r + \sum_{i=1}^{N} a_i = n + 1$, and the a_i are listed in nonincreasing order (note that the right-hand-side value is independent of the order of the a_i). This dramatically reduces the size of the search space, while in addition the sums (2–1) are relatively easy to compute.

One can then write

$$\sum_{n=0}^{\infty} \zeta(2n + 2)\, x^{2n} \tag{2--2}$$

$$\stackrel{?}{=} \sum_{n=0}^{\infty} \sum_{r=1}^{n+1} \sum_{\pi \in \Pi(n+1-r)} \alpha(\pi)\, \sigma(2r; 2\pi)\, x^{2n},$$

where $\Pi(m)$ denotes the set of all additive partitions of m if $m > 0$, $\Pi(0)$ is the singleton set whose sole element is the null partition $[\,]$, and the coefficients $\alpha(\pi)$ are complex numbers. In principle, $\alpha(\pi)$ in (2–2) could depend not only on the partition π but also on n. However, since the first few coefficients appeared to be independent of n, we found it convenient to assume that the generating function could be expressed in the form given above.

For positive integer k and partition $\pi = (a_,a_2, \ldots, a_N)$ of the positive integer m, let

$$\widehat{\sigma}_k(\pi) := \prod_{i=1}^{N} \sum_{n_i=1}^{k-1} \frac{1}{n_i^{2a_i}}.$$

Then

$$\sigma(2r; 2\pi) = \sum_{k=1}^{\infty} \frac{\widehat{\sigma}_k(\pi)}{k^{2r}\binom{2k}{k}},$$

and from (2–2), we deduce that

$$\sum_{n=0}^{\infty} \zeta(2n + 2)\, x^{2n}$$

$$= \sum_{n=0}^{\infty} \sum_{r=1}^{n+1} \sum_{\pi \in \Pi(n+1-r)} \alpha(\pi)\sigma(2r; 2\pi)x^{2n}$$

$$= \sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}} \sum_{r=1}^{\infty} \frac{x^{2r-2}}{k^{2r}}$$

$$\times \sum_{n=r-1}^{\infty} \sum_{\pi \in \Pi(n+1-r)} \alpha(\pi)\, \widehat{\sigma}_k(\pi)x^{2(n+1-r)}$$

$$= \sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2 - x^2)} \sum_{m=0}^{\infty} x^{2m} \sum_{\pi \in \Pi(m)} \alpha(\pi)\, \widehat{\sigma}_k(\pi)$$

$$= \sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2 - x^2)}\, P_k(x), \tag{2--3}$$

where

$$P_k(x) := \sum_{m=0}^{\infty} x^{2m} \sum_{\pi \in \Pi(m)} \alpha(\pi)\, \widehat{\sigma}_k(\pi), \tag{2--4}$$

whose closed form is yet to be determined. Our strategy, as in the case of (1–10) [Borwein and Bradley 96], was to compute $P_k(x)$ explicitly for a few small values of k in the hope that these would suggest a closed form for general k.

Some examples we produced are shown below. At each step we "bootstrapped" by assuming that the first few coefficients of the current result are the coefficients of the previous result. Then we found the remaining coefficients (which are in each case unique) by means of integer-relation computations.

In particular, we computed high-precision (200-digit) numerical values of the assumed terms and the left-hand-side zeta value, and then applied PSLQ to find the rational coefficients. In each case we "hard-wired" the first few coefficients to agree with the coefficients of the preceding formula. Note below that in the sigma notation, the first few coefficients of each expression are simply the previous step's terms, where the first argument of σ (corresponding to r) has been increased by two.

These initial terms (with coefficients in bold) are then followed by terms with the other partitions as arguments, with all terms ordered lexicographically by partition (shorter partitions are listed before longer partitions,

and within a partition of a given length, larger entries are listed before smaller entries in the first position where they differ; the integers in brackets are nonincreasing):

$$\zeta(2) = 3\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}k^2} = 3\sigma(2,[0]), \tag{2--5}$$

$$\zeta(4) = \mathbf{3}\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}k^4} - \mathbf{9}\sum_{k=1}^{\infty} \frac{\sum_{j=1}^{k-1}j^{-2}}{\binom{2k}{k}k^2}$$
$$= \mathbf{3}\sigma(4,[0]) - \mathbf{9}\sigma(2,[2]), \tag{2--6}$$

$$\zeta(6) = \mathbf{3}\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}k^6} - \mathbf{9}\sum_{k=1}^{\infty} \frac{\sum_{j=1}^{k-1}j^{-2}}{\binom{2k}{k}k^4}$$
$$- \frac{45}{2}\sum_{k=1}^{\infty} \frac{\sum_{j=1}^{k-1}j^{-4}}{\binom{2k}{k}k^2} + \frac{27}{2}\sum_{k=1}^{\infty}\sum_{j=1}^{k-1} \frac{\sum_{i=1}^{k-1}i^{-2}}{j^2\binom{2k}{k}k^2}$$
$$= \mathbf{3}\sigma(6,[\,]) - \mathbf{9}\sigma(4,[2])$$
$$- \frac{45}{2}\sigma(2,[4]) + \frac{27}{2}\sigma(2,[2,2]),$$

$$\zeta(8) = \mathbf{3}\sigma(8,[\,]) - \mathbf{9}\sigma(6,[2]) - \frac{\mathbf{45}}{\mathbf{2}}\sigma(4,[4]) \tag{2--7}$$
$$+ \frac{\mathbf{27}}{\mathbf{2}}\sigma(4,[2,2]) - 63\sigma(2,[6]) + \frac{135}{2}\sigma(2,[4,2])$$
$$- \frac{27}{2}\sigma(2,[2,2,2]),$$

$$\zeta(10) = \mathbf{3}\sigma(10,[\,]) - \mathbf{9}\sigma(8,[2]) - \frac{\mathbf{45}}{\mathbf{2}}\sigma(6,[4]) \tag{2--8}$$
$$+ \frac{\mathbf{27}}{\mathbf{2}}\sigma(6,[2,2]) - \mathbf{63}\sigma(4,[6]) + \frac{\mathbf{135}}{\mathbf{2}}\sigma(4,[4,2])$$
$$- \frac{\mathbf{27}}{\mathbf{2}}\sigma(4,[2,2,2]) - \frac{765}{4}\sigma(2,[8])$$
$$+ 189\sigma(2,[6,2]) + \frac{675}{8}\sigma(2,[4,4])\sigma(2,[4,2,2])$$
$$- \frac{405}{4} + \frac{81}{8}\sigma(2,[2,2,2,2]).$$

Next, from the above results, one can immediately read that $\alpha([\,]) = 3$, $\alpha([1]) = -9$, $\alpha([2]) = -45/2$, $\alpha([1,1]) = 27/2$, and so forth. Table 1 presents the values of α that we obtained in this manner.

Using these values, we then calculated series approximations to the functions $P_k(x)$ using formula (2–4). We obtained

$$P_3(x) \approx 3 - \frac{45}{4}x^2 - \frac{45}{16}x^4 - \frac{45}{64}x^6 - \frac{45}{256}x^8 - \frac{45}{1024}x^{10}$$
$$- \frac{45}{4096}x^{12} - \frac{45}{16384}x^{14} - \frac{45}{65536}x^{16},$$
$$P_4(x) \approx 3 - \frac{49}{4}x^2 + \frac{119}{144}x^4 + \frac{3311}{5184}x^4 + \frac{38759}{186624}x^6$$
$$+ \frac{384671}{6718464}x^8 + \frac{3605399}{241864704}x^{10} + \frac{33022031}{8707129344}x^{12}$$
$$+ \frac{299492039}{313456656384}x^{14},$$

$$P_5(x) \approx 3 - \frac{205}{16}x^2 + \frac{7115}{2304}x^4 + \frac{207395}{331776}x^6 + \frac{4160315}{47775744}x^8$$
$$+ \frac{74142995}{6879707136}x^{10} + \frac{1254489515}{990677827584}x^{12}$$
$$+ \frac{20685646595}{142657607172096}x^{14} + \frac{336494674715}{20542695432781824}x^{16},$$
$$P_6(x) \approx 3 - \frac{5269}{400}x^2 + \frac{6640139}{1440000}x^4 + \frac{1635326891}{5184000000}x^6$$
$$- \frac{5944880821}{18662400000000}x^8 - \frac{212874252291349}{67184640000000000}x^{10}$$
$$- \frac{141436384956907381}{241864704000000000000}x^{12}$$
$$- \frac{70524260274859115989}{8707129344000000000000000}x^{14}$$
$$- \frac{31533457168819214655541}{313456656384000000000000000000}x^{16},$$
$$P_7(x) \approx 3 - \frac{5369}{400}x^2 + \frac{8210839}{1440000}x^4 - \frac{199644809}{5184000000}x^6$$
$$- \frac{680040118121}{18662400000000}x^8 - \frac{278500311775049}{67184640000000000}x^{10}$$
$$- \frac{84136715217872681}{241864704000000000000}x^{12}$$
$$- \frac{22363377813883431689}{8707129344000000000000000}x^{14}$$
$$- \frac{5560090840263911428841}{313456656384000000000000000000}x^{16}.$$

With these approximations in hand, we were then in a position to attempt to determine closed-form expressions for $P_k(x)$. This can be done using the "Pade" function in either Mathematica or Maple. We obtained the following:

$$P_1(x) \stackrel{?}{=} 3,$$
$$P_2(x) \stackrel{?}{=} \frac{3(4x^2-1)}{(x^2-1)},$$
$$P_3(x) \stackrel{?}{=} \frac{12(4x^2-1)}{(x^2-4)},$$
$$P_4(x) \stackrel{?}{=} \frac{12(4x^2-1)(4x^2-9)}{(x^2-4)(x^2-9)},$$
$$P_5(x) \stackrel{?}{=} \frac{48(4x^2-1)(4x^2-9)}{(x^2-9)(x^2-16)},$$
$$P_6(x) \stackrel{?}{=} \frac{48(4x^2-1)(4x^2-9)(4x^2-25)}{(x^2-9)(x^2-16)(x^2-25)},$$
$$P_7(x) \stackrel{?}{=} \frac{192(4x^2-1)(4x^2-9)(4x^2-25)}{(x^2-16)(x^2-25)(x^2-36)}.$$

These results immediately suggest that the general form of a generating-function identity is

$$\sum_{n=0}^{\infty} \zeta(2n+2)x^{2n} \stackrel{?}{=} 3\sum_{k=1}^{\infty} \frac{1}{\binom{2k}{k}(k^2-x^2)} \prod_{m=1}^{k-1} \frac{4x^2-m^2}{x^2-m^2}, \tag{2--9}$$

which is equivalent to (1–13).

We next confirmed this result in several ways:

1. We symbolically computed the power-series coefficients of the left- and the right-hand sides of (2–9), and have verified that they agree up to the term with x^{100}.

Partition	Alpha	Partition	Alpha	Partition	Alpha
[empty]	3/1	1	−9/1	2	−45/2
1, 1	27/2	3	−63/1	2, 1	135/2
1, 1, 1	−27/2	4	−765/4	3, 1	189/1
2, 2	675/8	2, 1, 1	−405/4	1, 1, 1, 1	81/8
5	−3069/5	4, 1	2295/4	3, 2	945/2
3, 1, 1	−567/2	2, 2, 1	−2025/8	2, 1, 1, 1	405/4
1, 1, 1, 1, 1	−243/40	6	−4095/2	5, 1	9207/5
4, 2	11475/8	4, 1, 1	−6885/8	3, 3	1323/2
3, 2, 1	−2835/2	3, 1, 1, 1	567/2	2, 2, 2	−3375/16
2, 2, 1, 1	6075/16	2, 1, 1, 1, 1	−1215/16	1, 1, 1, 1, 1, 1	243/80
7	−49149/7	6, 1	49140/8	5, 2	36828/8
5, 1, 1	−27621/10	4, 3	32130/8	4, 2, 1	−34425/8
4, 1, 1, 1	6885/8	3, 3, 1	−15876/8	3, 2, 2	−14175/8
3, 2, 1, 1	17010/8	3, 1, 1, 1, 1	−1701/8	2, 2, 2, 1	10125/16
2, 2, 1, 1, 1	−6075/16	2, 1, 1, 1, 1, 1	729/16	1, 1, 1, 1, 1, 1, 1	−729/560
8	−1376235/56	7, 1	1179576/56	6, 2	859950/56
6, 1, 1	−515970/56	5, 3	902286/70	5, 2, 1	−773388/56
5, 1, 1, 1	193347/70	4, 4	390150/64	4, 3, 1	−674730/56
4, 2, 2	−344250/64	4, 2, 1, 1	413100/64	4, 1, 1, 1, 1	−41310/64
3, 3, 2	−277830/56	3, 3, 1, 1	166698/56	3, 2, 2, 1	297675/56
3, 2, 1, 1, 1	−119070/56	3, 1, 1, 1, 1, 1	10206/80	2, 2, 2, 2	50625/128
2, 2, 2, 1, 1	−60750/64	2, 2, 1, 1, 1, 1	18225/64	2, 1, 1, 1, 1, 1, 1	−1458/64
1, 1, 1, 1, 1, 1, 1, 1	2187/4480				

TABLE 1. Alpha coefficients found by PSLQ computations.

2. We verified that $\mathcal{Z}(1/6)$, where $\mathcal{Z}(x)$ is the right-hand side of (2–9), agrees with $18 - 3\sqrt{3}\pi$, computed using (1–14), to over 2,500-digit precision; likewise for $\mathcal{Z}(1/2) = 2$, $\mathcal{Z}(1/3) = 9/2 - 3\pi/(2\sqrt{3})$, $\mathcal{Z}(1/4) = 8 - 2\pi$, and $\mathcal{Z}(1/\sqrt{2}) = 1 - \pi/\sqrt{2} \cdot \cot(\pi/\sqrt{2})$.

3. We then affirmed that the formula (2–9) gives the same numerical value as (1–14) for the 100 pseudorandom values $\{m\pi\}$, for $1 \le m \le 100$, where $\{\cdot\}$ denotes the fractional part.

Thus, we were certain that (1–13) was correct and it remained only to find a proof of Theorem 1.1.

3. PROOF OF THEOREM 1.1

By partial fractions,

$$\frac{1}{1 - z^2/k^2} \prod_{m=1}^{k-1} \frac{1 - 4z^2/m^2}{1 - z^2/m^2} = \sum_{n=1}^{k} \frac{c_n(k)}{1 - z^2/n^2},$$

where

$$c_n(k) = \prod_{m=1}^{k-1} (1 - 4n^2/m^2) \Big/ \prod_{\substack{m=1 \\ m \ne n}}^{k} (1 - n^2/m^2)$$

if $1 \le n \le k$, and $c_n(k) = 0$ if $n > k$ or if $k \ge 2n + 1$. It follows that

$$\sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}(1 - z^2/k^2)} \prod_{m=1}^{k-1} \frac{1 - 4z^2/m^2}{1 - z^2/m^2}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}} \sum_{n=1}^{k} \frac{c_n(k)}{1 - z^2/n^2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{1 - z^2/n^2} \sum_{k=n}^{2n} \frac{c_n(k)}{k^2 \binom{2k}{k}}.$$

The interchange of summation order is justified by absolute convergence. To prove (1–13), it obviously suffices to show that

$$3 \sum_{k=n}^{2n} \frac{c_n(k)}{k^2 \binom{2k}{k}} = \frac{1}{n^2}.$$

Equivalently, we need only show that

$$S_n := \sum_{k=n}^{2n} \frac{3}{\binom{2k}{k}} \prod_{m=1}^{k-1} (4n^2 - m^2) \Big/ \prod_{\substack{m=1 \\ m \ne n}}^{k} (n^2 - m^2) = 1$$

for each positive integer n. But

$$S_n = \frac{6n^2}{(2n)!} \sum_{k=n}^{2n} \frac{1}{\binom{2k}{k}} \prod_{m=1}^{k-1}(4n^2 - m^2) \Big/ \prod_{m=n+1}^{k}(n^2 - m^2)$$

$$= \frac{(3n)!\,n!}{(2n)!\,(2n)!}\, {}_3F_2\left(\begin{array}{c}3n, n+1, -n \\ 2n+1, n+1/2\end{array}\middle|\frac{1}{4}\right).$$

Thus, we have reduced the problem of proving (1–13) to that of establishing the finite identity

$$T(n) := \frac{(3n)!\,n!}{(2n)!\,(2n)!}\, {}_3F_2\left(\begin{array}{c}3n, n+1, -n \\ 2n+1, n+1/2\end{array}\middle|\frac{1}{4}\right) = 1,\tag{3-1}$$

$n \in \mathbf{Z}^+$. But Maple readily simplifies $T(n+1)/T(n) = 1$, and since $T(0) = 1$, the identity (3–1) and hence (1–15) and (1–13) are established. If a certificate is desired, we can employ the Wilf–Zeilberger algorithm. In Maple 9.5 we set

$$r := \frac{\binom{2n}{n}}{\binom{3n}{n}},\tag{3-2}$$

$$f := \frac{(3n+k-1)!(n+k)!(-n-1+k)!(2n)!(n-\frac{1}{2})!(\frac{1}{4})^k}{(3n-1)!n!(-n-1)!(2n+k)!(n-\frac{1}{2}+k)!k!}.$$

Maple interprets the latter in terms of the Pochhammer symbol

$$(a)_k := \prod_{j=1}^{k}(a+j-1)$$

as

$$f = \frac{(3n)_k\,(n+1)_k\,(-n)_k}{(2n+1)_k\,(n+1/2)_k} \cdot \frac{(1/4)^k}{k!},$$

so despite the appearance of (3–2) the issue of factorials at negative integers does not arise for nonnegative integers k and n. Now execute

```
>  with(SumTools[Hypergeometric]):
>  WZMethod(f,r,n,k,'certify'): certify;
```

which returns the certificate

```
/     2                     \
\11 n  + 1 + 6 n + k + 5 k n/ k
- -----------------------------
  3 (n - k + 1) (2 n + k + 1) n
```

This proves that summing $f(n,k)$ over k produces $r(n)$, as asserted.

Indeed, the (suppressed) output of "WZMethod" is the *WZ-pair* (F, G) such that

$$F(n+1, k) - F(n, k) = G(n, k+1) - G(n, k),$$

where $F(n, k) := f(n, k)/r(n)$ for $r(n) \neq 0$ and is $f(n, k)$ otherwise. Sum both sides over $k \in \mathbf{Z}$ and use the fact that by construction, $G(n, k) \to 0$ as $k \to \pm\infty$. The certificate is

$$R(n, k) := \frac{G(n, k)}{F(n, k)}.$$

QED

4. AN IDENTITY FOR $\zeta(2n+4)$

We compare (1–13) to a result due to Leshchiner [Leshchiner 81], which is stated incorrectly in [Almkvist and Granville 99], and which, as the authors say, has a different flavor: for complex x not an integer,

$$\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}\frac{3k^2+x^2}{k^2-x^2}\prod_{m=1}^{k-1}\left(1-\frac{x^2}{m^2}\right)\tag{4-1}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-x^2} = \frac{\pi}{2x\sin(\pi x)} - \frac{1}{2x^2}.$$

Using the methods of the previous section—but using a basis of sums over simplices, not hypercubes— we have likewise now obtained for complex x not an integer,

$$\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}(k^2-x^2)}\prod_{m=1}^{k-1}\left(1-\frac{x^2}{m^2}\right)\tag{4-2}$$

$$= \frac{\pi}{4x^3\sin(\pi x)} - \frac{1}{x^4} + \frac{3\cos(\pi x/3)}{4x^4}.$$

To see this, let

$$W(x)$$
$$:= \left(\frac{1/2}{1-x^2}\right){}_5F_4\left(\begin{array}{c}1, 1+x, 1+x, 1-x, 1-x \\ 2, 3/2, 2+x, 2-x\end{array}\middle|\frac{1}{4}\right)$$

denote the left-hand side of (4–2), and let

$$V(x) := \left(\frac{1}{2}\right){}_3F_2\left(\begin{array}{c}1, 1+x, 1-x \\ 2, 3/2\end{array}\middle|\frac{1}{4}\right)$$

$$= \sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}\prod_{m=1}^{k-1}\left(1-\frac{x^2}{m^2}\right)$$

$$= \sum_{n=1}^{\infty}(-x^2)^{n-1}\frac{(2\arcsin(1/2))^{2n}}{(2n)!}$$

$$= \frac{1-\cos(\pi x/3)}{x^2}.$$

Expanding Leshchiner's series (4–1) now gives

$$\frac{3}{2}V(x) + 2x^2\,W(x) = \frac{\pi}{2x\sin(\pi x)} - \frac{1}{2x^2}.$$

Solving for $W(x)$ gives (4–2) as claimed.

To recapitulate, we have the following theorem.

Theorem 4.1. *Let x be a complex number, not an integer. Then*

$$\sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}(k^2 - x^2)} \prod_{m=1}^{k-1}\left(1 - \frac{x^2}{m^2}\right) \qquad (4\text{–}3)$$
$$= \frac{\pi x \csc(\pi x) + 3\cos(\pi x/3) - 4}{4x^4}.$$

If $0 \le |x| < 1$, then the Maclaurin series for the left-hand side of (4–3) is equal to

$$\sum_{n=2}^{\infty} \frac{(-1)^n \left\{3^{1-2n} - 2B_{2n}\left(2^{2n-1} - 1\right)\right\} \pi^{2n}}{4\,(2n)!} x^{2n-4}$$
$$= \frac{17}{36}\zeta(4) + \frac{313}{648}\zeta(6)\,x^2 + \frac{23147}{46656}\zeta(8)\,x^4$$
$$+ \frac{1047709}{2099520}\zeta(10)\,x^6 + \cdots,$$

where the rational coefficients B_{2n} refer to the even-indexed Bernoulli numbers generated by

$$x\coth(x) = \sum_{n=0}^{\infty} \frac{(2x)^{2n}}{(2n)!} B_{2n}.$$

Note that the constant term recaptures (1–3) as desired—as taking the limit on the right side of (4–3) confirms. Correspondingly, the constant term in (4–1) yields (1–1). The coefficient of x^2 is

$$\frac{313}{648}\zeta(6) = \sum_{k=1}^{\infty} \frac{1}{k^6\binom{2k}{k}} - \sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^2}.$$

This all suggests that there should be a unifying formula for our two identities (1–13) and (4–3) as there is for the odd cases; see (1–12).

5.   CONCLUSION

We believe that this general experimental procedure will ultimately yield results for many other classes of arguments, such as for $\zeta(4n + m)$, $m = 0, 1$, but our current experimental results are negative.

1. Considering $\zeta(4n + 1)$, for $n = 2$, the simplest evaluation we know is (1–7). This is one term shorter than that given by Rivoal [Rivoal 04], which comes from taking the coefficient of $x^2 y^4$ in (1–12).

2. For $\zeta(2n+4)$ (and $\zeta(4n)$) starting with (1–3), which we recall is

$$\zeta(4) = \frac{36 \cdot 1}{17}\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}},$$

the identity for $\zeta(6)$ most susceptible to bootstrapping is

$$\zeta(6) = \frac{36 \cdot 8}{163}\left[\sum_{k=1}^{\infty}\frac{1}{k^6\binom{2k}{k}} + \frac{3}{2}\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}\right].$$
$$(5\text{–}1)$$

For $\zeta(8)$ and $\zeta(10)$ we have enticingly found

$$\zeta(8) = \frac{36 \cdot 64}{1373}\left[\sum_{k=1}^{\infty}\frac{1}{k^8\binom{2k}{k}} + \frac{9}{4}\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}\right.$$
$$\left. + \frac{3}{2}\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^6}\right], \qquad (5\text{–}2)$$

$$\zeta(10) = \frac{36 \cdot 512}{11143}\left[\sum_{k=1}^{\infty}\frac{1}{k^{10}\binom{2k}{k}} + \frac{9}{4}\sum_{k=1}^{\infty}\frac{1}{k^6\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}\right.$$
$$+ \frac{3}{2}\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^8} + \frac{9}{4}\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^6}$$
$$\left. + \frac{27}{8}\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}\sum_{j=1}^{k-1}\frac{1}{j^4}\sum_{i=1}^{j-1}\frac{1}{i^4}\right]. \qquad (5\text{–}3)$$

But this pattern is not fruitful; the pattern stops at $n = 10$.

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