Determinants of Latin Squares of Order 8

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A *latin square* is an $n \times n$ array of n symbols in which each symbol appears exactly once in each row and column. Regarding each symbol as a variable and taking the determinant, we get a degree-n polynomial in n variables. Can two latin squares L, M have the same determinant, up to a renaming of the variables, apart from the obvious cases when L is obtained from M by a sequence of row interchanges, column interchanges, renaming of variables, and transposition? The answer was known to be no if $n \leq 7$; we show that it is yes for $n = 8$. The latin squares for which this situation occurs have interesting special characteristics.

1. INTRODUCTION

A *latin square of order* n is an $n \times n$ array of n symbols, usually denoted by $\{1, \dots, n\}$, in which each symbol appears exactly once in each row and in each column. Standard references are [Dénes and Keedwell 1974; 1991]. The unbordered multiplication table of any group forms a latin square. For example, the cyclic group C_4 of order 4, with elements in the order a^0, a^1, a^3, a^2, yields the square

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

An arbitrary latin square is the unbordered multiplication table of a quasigroup (a set with a binary operation with left and right cancellation).

The *matrix* X_L of a latin square L is obtained by replacing each element i in L by the variable x_i. The *determinant* of L is $\Theta_L = \det X_L$; if L has order n, its determinant is a homogeneous polynomial of degree n in n variables.

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Two polynomials φ, ψ in $\{x_1, \dots, x_n\}$ are *similar* if there exists a permutation σ in S_n such that $\varphi(x_1, \dots, x_n) = \pm\psi(x_{\sigma(1)}, \dots, x_{\sigma(n)})$.

Latin squares L and M are *isotopic* if M can be obtained from L by a sequence of row interchanges, column interchanges, and renaming of elements. More precisely, squares L and M are isotopic if there exist permutations π_1, π_2, π_3 such that

$$M(i,j) = \big(L(i\pi_1, j\pi_2)\big)\pi_3$$

for all i, j. It is clear that isotopic latin squares have similar determinants, as do L and its transpose L^t.

If G is a group, the *group matrix* X_G is the matrix corresponding to the latin square whose (g,h)-entry is gh^{-1}. Thus, the group matrix for C_4 is

$$\begin{bmatrix} x_1 & x_3 & x_2 & x_4 \\ x_2 & x_1 & x_4 & x_3 \\ x_3 & x_4 & x_1 & x_2 \\ x_4 & x_2 & x_3 & x_1 \end{bmatrix}.$$

The group matrix is interesting because X_G^2 has exactly the same pattern as X_G, with x_i replaced by $\sum_{jk=i} x_j x_k$. In fact, X_G^n has the same symmetry for all n. This follows from the fact, proved by Frobenius, that $X_G Y_G = Z_G$, where $Y_G = \{y_{gh^{-1}}\}$, $Z_G = \{z_{gh^{-1}}\}$ and $z_k = \sum_{gh=k} x_g y_h$. It follows that if G is commutative then X_G and Y_G commute, a result we apply below. The *group determinant* Θ_G of G is $\det(X_G)$.

The investigation of Θ_G led Frobenius to the character theory of nonabelian groups [Frobenius 1896; Hawkins 1971; 1974; Johnson 1991; 1993]. A latin square determinant Θ_L may be written as $\det\left(\sum_{i=1}^n \pi_i x_i\right)$, where the π_i are permutation matrices.

Results in the theory of invariants of a finite set of $n \times n$ matrices suggest that Θ_L alone will not characterise L up to isotopy and transposition; but in the case of group latin squares, the determinant Θ_G does determine G [Formanek and Sibley 1991]. More surprisingly, the sequence of coefficients of monomials in Θ_G of the form $x_1^{n-3} x_g x_h x_k$ (or equivalently the "regular 3-character") determines G [Hoehnke and Johnson 1992].

A character theory is available for quasigroups or latin squares, developed by one of the authors and J. D. H. Smith. As in the group case, the theory can be developed from Θ_L, but as might be expected the characters give less information in the quasigroup case. We refer to [Johnson 1988; 1992] and the references given there for the details. Before embarking on an investigation of "higher characters" that arise from the determinant in an analogous manner to the k-characters of groups [Hoehnke and Johnson 1992], it appears appropriate to determine the extent of the information on L that is contained in Θ_L.

Let E be the equivalence relation on the set of latin squares of order n in which L is related to M if L is isotopic to M or M^t. A basic question for latin square determinants is this: Are there squares L, M that are not E-equivalent but have similar determinants? Note that squares arising from groups are E-equivalent if and only if the groups are isomorphic.

Previously it has been shown that for $n \le 7$ any two latin squares with similar determinants are E-equivalent. The cases $n \le 1, 2, 3, 4$ are easy. Cases 5 and 6 can be checked by direct calculation using symbolic manipulation packages, and the case $n = 7$ is handled in [Ferguson 1989]. Here we describe calculations that settle the case $n = 8$. We have found that of the 842227 E-classes all but 37 have dissimilar determinants. The 37 exceptional E-classes merge into 12 classes of squares with similar determinants, each containing between 2 and 7 distinct E-classes. Moreover, the exceptional squares are all of a special type, in that they are isotopic to squares of the form $\left[\begin{smallmatrix} Q & R \\ S & T \end{smallmatrix}\right]$, where Q and T are latin squares on $\{1,2,3,4\}$ and R and S are latin squares on $\{5,6,7,8\}$. Since any latin square of order 4 is isotopic to a square arising from a group, the determinants may be described in terms of the group matrices of the two groups of order 4. Using the properties of group matrices given above it is reasonably easy to find the symbolic determinants of the 37 exceptional squares by hand and to write the results in a compact form. This contrasts with a typical latin square determinant of order 8, which cannot in practice be calculated by hand, since it occupies several pages when expressed in monomials.

From the point of view of invariants of matrices, our calculations show that a pair of latin squares of order 8 have identical determinants if and only if the coefficients of all monomials $x_i^{n_i} x_j^{n_j} x_k^{n_k} x_m^{n_m}$, where $n_i + n_j + n_k + n_m = 8$, coincide.

Section 2 describes the details of the computation. Section 3 contains a representative list of exceptional squares with their determinants, and examples of how hand calculation of the determinants can be carried out. Other invariants of the exceptional E-classes are listed. In Section 4 we indicate how to use pairs of E-inequivalent squares of order n with similar determinants to construct squares of order kn that are E-inequivalent but have similar determinants. We conclude with some remarks and questions.

2. OVERVIEW OF THE COMPUTATION

The computations were performed in Pascal and Maple on a DIGITAL VAXstation 4000-90 in the Computer Science Department at Concordia University. We started with representatives for the 283657 main classes of 8×8 latin squares, provided by Kolesova [Kolesova et al. 1990]. For each main class representative, we computed row and column conjugates, thus obtaining a set of 850971 squares that represent (with some redundancy) all the E-classes.

1. For each representative L, we calculated $\Theta_L(v)$ for the 56 distinct permutations v of
$$(1, 1, 1, 0, 0, 0, 0, 0),$$
and computed n_m, n_z, and n_p, the number of choices of v for which $\Theta_L(v)$ was negative, zero, and positive, respectively. The set $\{n_m, n_p\}$ is invariant for latin squares with similar determinants. CPU time: 4h 40min.
2. For each representative L, we calculated $\Theta_L(v)$ for the 168 distinct permutations v of
$$(1, 1, 2, 0, 0, 0, 0, 0),$$
and computed s_m, s_p, and s_q, respectively the sums of the negative values, positive values, and squares of values of $\Theta_L(v)$. The sum s_q and the set $\{|s_m|, s_p\}$ are invariant for latin squares with similar determinants. By sorting, we determined which triples $t = (\{n_m, n_p\}, \{|s_m|, s_p\}, s_q)$ appeared only once. Discarding the corresponding representatives, we eliminated all but 17596 of the representatives. CPU time: 15h 25min.
3. We tested representatives with same values of the invariant triple t for isotopy and transposed isotopy, eliminating redundant E-class representatives. After this step, 529 representatives remained. Incidentally, we confirmed at this point that there are exactly 842227 E-classes, in agreement with [Kolesova et al. 1990, Table IV]. CPU time: 23h 43min.
4. For each surviving representative L, we computed $\Theta_L(v)$ for the 1680 distinct permutations v of $(1, 2, 3, 5, 0, 0, 0, 0)$. This sequence of values, when sorted and paired with its negative, is invariant for latin squares with similar determinants. For all but 37 representatives this invariant was unique; altogether these 37 exceptional representatives had 12 distinct invariants. CPU time: 28min.
5. For representatives with coincident $(1, 2, 3, 5)$-sequence values, we searched the isotopy classes to produce representatives with same $(1, 1, 1)$-sequence and $(1, 1, 2)$-sequence determinant values. CPU time: 36min.
6. We computed the symbolic determinant for each of the 37 representative squares, obtaining 12 distinct determinants, all dissimilar. CPU time: 10min.

3. EXPERIMENTAL RESULTS

Consider the equivalence classes induced on the set of 8×8 latin squares by similarity of determinant.

$d_1 = 2f_v^2 - 2f_w^2$				
1:1	A	B	B	$A_{(234)}$
1:2	A	B	${}_{(234)}B_{(234)}$	${}_{(234)}A_{(243)}$
1:3	A	B	B	${}_{(243)}A_{(234)}$
1:4	A	B	${}_{(34)}B_{(34)}$	${}_{(34)}A_{(24)}$
1:5	A	B	${}_{(34)}B_{(34)}$	${}_{(23)}A_{(24)}$
1:6	A	B	${}_{(243)}B_{(243)}$	${}_{(24)}A_{(34)}$
1:7	A	B	${}_{(24)}B_{(24)}$	${}_{(234)}A_{(243)}$
$d_2 = 2f_v^2 + 2f_w^2 - 2(v_4^2 + w_4^2)v_2v_3w_2w_3$				
2:1	A	B	${}_{(243)}B_{(243)}$	${}_{(23)}A_{(243)}$
2:2	A	B	${}_{(23)}B_{(23)}$	${}_{(243)}A_{(23)}$
$d_3 = 2f_vg_v + 2f_w^2 - (v_2 + v_3)(v_2v_3w_4^2 + w_2^2w_3^2)v_4$				
3:1	A	B	B	${}_{(24)}D_{(243)}$
3:2	A	B	${}_{(23)}B_{(23)}$	${}_{(243)}D_{(24)}$
$d_4 = 2f_vg_v + 2f_w^2 - (v_2 + v_3)(v_2w_4^2 + v_3v_4^2)w_2w_3$				
4:1	A	B	${}_{(243)}B_{(243)}$	${}_{(243)}D_{(23)}$
4:2	A	B	${}_{(243)}B_{(243)}$	${}_{(23)}D_{(243)}$
4:3	A	B	${}_{(23)}B_{(23)}$	${}_{(23)}D_{(243)}$
4:4	A	${}_{(34)}B_{(34)}$	${}_{(243)}B_{(243)}$	${}_{(243)}D_{(23)}$
$d_5 = 2f_vg_v + 2f_w^2 - (v_2 + v_3)(v_2v_3 + w_2^2)v_4w_3w_4$				
5:1	A	B	${}_{(34)}B_{(34)}$	${}_{(24)}D_{(243)}$
5:2	A	B	${}_{(234)}B_{(234)}$	${}_{(243)}D_{(24)}$
5:3	A	B	${}_{(234)}B_{(234)}$	${}_{(24)}D_{(243)}$
$d_6 = 2f_vg_v + 2f_w^2 - (v_2 + v_3)(v_2w_3^2w_4^2 + v_3v_4^2w_2^2)$				
6:1	A	B	B	${}_{(23)}D_{(243)}$
6:2	A	B	${}_{(34)}B_{(34)}$	${}_{(243)}D_{(23)}$
$d_7 = 2f_vg_v + 2f_w^2 + (v_2 + v_3)(v_2w_3^2w_4^2 + v_3v_4^2w_2^2)$				
7:1	A	B	B	$D_{(1423)}$
7:2	A	B	${}_{(34)}B_{(34)}$	${}_{(34)}D_{(13)(24)}$
$d_8 = 2f_vg_v + 2f_wg_w - (v_2w_2 + v_3w_3)(v_2v_3w_3 + w_2w_4^2)v_4$				
8:1	A	${}_{(34)}B_{(34)}$	${}_{(243)}C_{(243)}$	${}_{(243)}D_{(24)}$
8:2	A	${}_{(34)}B_{(34)}$	${}_{(24)}C_{(24)}$	${}_{(24)}D_{(243)}$
8:3	A	${}_{(234)}B_{(234)}$	${}_{(243)}C_{(243)}$	${}_{(243)}D_{(24)}$
8:4	A	${}_{(234)}B_{(234)}$	${}_{(24)}C_{(24)}$	${}_{(24)}D_{(243)}$
$d_9 = 2f_vg_v + 2f_wg_w - (v_2w_2 + v_3w_3)(v_2v_3 + w_2w_3)v_4w_4$				
9:1	A	${}_{(23)}B_{(23)}$	${}_{(24)}C_{(24)}$	${}_{(24)}D_{(243)}$
9:2	A	B	${}_{(24)}C_{(24)}$	${}_{(24)}D_{(243)}$
$d_{10} = 2f_vg_v + 2f_wg_w - (v_2w_2 + v_3w_3)(v_2v_4^2w_2 + v_3w_3w_4^2)$				
10:1	A	${}_{(234)}B_{(234)}$	${}_{(23)}C_{(23)}$	${}_{(23)}D_{(234)}$
10:2	A	${}_{(23)}B_{(23)}$	${}_{(23)}C_{(23)}$	${}_{(23)}D_{(234)}$
10:3	A	${}_{(234)}B_{(234)}$	${}_{(24)}C_{(24)}$	${}_{(24)}D$
$d_{11} = 2f_vg_v + 2f_wg_w - (v_2w_2 + v_3w_3)(v_2w_3w_4^2 + v_3v_4^2w_2)$				
11:1	A	${}_{(34)}B_{(34)}$	${}_{(23)}C_{(23)}$	${}_{(23)}D_{(243)}$
11:2	A	${}_{(34)}B_{(34)}$	${}_{(243)}C_{(243)}$	${}_{(243)}D_{(23)}$
11:3	A	B	${}_{(243)}C_{(243)}$	${}_{(243)}D_{(23)}$
$d_{12} = 2f_vg_v + 2f_wg_w + (v_2w_2 + v_3w_3)(v_2w_3w_4^2 + v_3v_4^2w_2)$				
12:1	A	${}_{(34)}B_{(34)}$	${}_{(23)}C_{(23)}$	${}_{(23)}D_{(142)}$
12:2	A	${}_{(34)}B_{(34)}$	${}_{(243)}C_{(243)}$	${}_{(243)}D_{(1342)}$
12:3	A	B	${}_{(23)}C_{(23)}$	${}_{(23)}D_{(142)}$

TABLE 1. For each of the 12 determinant similarity classes comprising more than one E-class, we show one representative $\left[\begin{smallmatrix} Q & R \\ S & T \end{smallmatrix}\right]$ of each component E-class in the form (class identifier, Q, R, S, T). See page 321.

All but twelve of these similarity classes coincide with E-classes; the exceptional classes are unions of two or more E-classes.

For each exceptional class, Table 1 shows representatives of the distinct E-classes from which it is composed. These representatives all have the same determinant, which Table 1 shows in terms of the polynomials

$$\begin{aligned} v_1 &= x_1 + x_2 + x_3 + x_4, & w_1 &= x_5 + x_6 + x_7 + x_8, \\ v_2 &= x_1 + x_2 - x_3 - x_4, & w_2 &= x_5 + x_6 - x_7 - x_8, \\ v_3 &= x_1 - x_2 + x_3 - x_4, & w_3 &= x_5 - x_6 + x_7 - x_8, \\ v_4 &= x_1 - x_2 - x_3 + x_4, & w_4 &= x_5 - x_6 - x_7 + x_8, \end{aligned}$$

$$\begin{aligned} f_v &= v_2 v_3 v_4, & f_w &= w_2 w_3 w_4, \\ g_v &= \tfrac{1}{2}(v_2^2 + v_3^2) v_4, & g_w &= \tfrac{1}{2}(w_2^2 + w_3^2) w_4, \end{aligned}$$

and with a common factor of $\pm\frac{1}{2}(v_1^2 - w_1^2)$ suppressed. Also in the table, A, B, C, D stand for

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 6 & 5 & 8 & 7 \\ 7 & 8 & 5 & 6 \\ 8 & 7 & 6 & 5 \end{bmatrix},$$

$$C = \begin{bmatrix} 5 & 7 & 6 & 8 \\ 6 & 5 & 8 & 7 \\ 7 & 8 & 5 & 6 \\ 8 & 6 & 7 & 5 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 2 & 3 & 1 \end{bmatrix},$$

and the notation ${}_\rho S_\sigma$ indicates that permutation ρ is applied to the rows of S and permutation σ is applied to the columns of S; thus

$${}_{(234)}A_{(243)} = \begin{bmatrix} 1 & 3 & 4 & 2 \\ 4 & 2 & 1 & 3 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 2 & 4 \end{bmatrix}.$$

Each of the 8×8 latin squares in Table 1 is of the form $\left[\begin{smallmatrix} Q & R \\ S & T \end{smallmatrix}\right]$, with the 4×4 latin squares Q, R, S, T shown sequentially.

As a sample, we offer on page 322 the calculation of the determinants of the squares in classes 1:1 and 8:1. We use the following well-known result.

Proposition. *If Q, R, S, T are $n \times n$ matrices and $\det Q \neq 0$ then*

$$\det \begin{pmatrix} Q & R \\ S & T \end{pmatrix} = \det Q \cdot \det(T - SQ^{-1}R).$$

In particular, if Q commutes with R then

$$\det \begin{pmatrix} Q & R \\ S & T \end{pmatrix} = \det(TQ - SR),$$

and if Q commutes with S then

$$\det \begin{pmatrix} Q & R \\ S & T \end{pmatrix} = \det(QT - SR).$$

Proof.

$$\begin{pmatrix} Q & R \\ S & T \end{pmatrix} = \begin{pmatrix} Q & 0 \\ S & I \end{pmatrix} \begin{pmatrix} I & Q^{-1}R \\ 0 & T - SQ^{-1}R \end{pmatrix}. \qquad \square$$

Mapping groups and automorphism groups

The *left mapping group* $M_\lambda(L)$ is the group generated by the permutations given by the rows of L; the *right mapping group* $M_\rho(L)$ is generated by the permutations given by the columns. For example, the row 4 1 3 5 2 6 8 7 represents the permutation (1 4 5 2)(7 8). The full mapping group $M(L)$ is the group generated by $M_\lambda(L)$ and $M_\rho(L)$. If L arises from the Cayley table of a group G, then $M_\lambda(L)$ and $M_\rho(L)$ are isomorphic to G and correspond to the left and right regular representations of G, and $M(L)$ is isomorphic to $\mathrm{Inn}(G) \rtimes G$ where $\mathrm{Inn}(G)$ denotes the group of inner automorphisms of G. If L_1, L_2 are in reduced form and L_1 is isotopic to L_2 then $M_\lambda(L_1)$, $M_\rho(L_1)$, $M(L_1)$ are respectively isomorphic to $M_\lambda(L_2)$, $M_\rho(L_2)$, $M(L_2)$ [Albert 1943]. It is obvious that $M_\rho(L) = M_\lambda(L^t)$ and $M(L) \simeq M(L^t)$. Hence M and the set $\{M_\lambda, M_\rho\}$ are invariants of an E-class.

The automorphism group $\mathrm{Aut}(L)$ of L is the group of triples of permutations (π_1, π_2, π_3) which fix L in the sense that

$$(L(i\pi_1, j\pi_2))\,\pi_3 = L(i, j) \quad \text{for all } i, j.$$

$\mathrm{Aut}(L)$ is an invariant of the E-class of L.

Sample determinant calculations. For simplicity we write A, B, C and D for X_A, X_B, X_C and X_D. Setting

$$P=\begin{pmatrix} \frac12 & \frac12 & \frac12 & \frac12 \\ \frac12 & \frac12 & -\frac12 & -\frac12 \\ \frac12 & -\frac12 & \frac12 & -\frac12 \\ \frac12 & -\frac12 & -\frac12 & \frac12 \end{pmatrix}, \qquad I=\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}, \qquad \text{and} \qquad \begin{cases} v_5=x_1-x_4=\frac12(v_2+v_3),\\ v_6=x_2-x_3=\frac12(v_2-v_3),\\ w_5=x_5-x_8=\frac12(w_2+w_3),\\ w_6=x_6-x_7=\frac12(w_2-w_3),\end{cases}$$

we have

$$PAP=\begin{bmatrix} v_1&0&0&0\\0&v_2&0&0\\0&0&v_3&0\\0&0&0&v_4\end{bmatrix}, \quad PBP=\begin{bmatrix} w_1&0&0&0\\0&w_2&0&0\\0&0&w_3&0\\0&0&0&w_4\end{bmatrix}, \quad PCB=\begin{bmatrix} w_1&0&0&0\\0&w_5&w_6&0\\0&-w_6&w_5&0\\0&0&0&w_4\end{bmatrix}, \quad PDB=\begin{bmatrix} v_1&0&0&0\\0&v_5&v_6&0\\0&-v_6&v_5&0\\0&0&0&v_4\end{bmatrix};$$

moreover $P^{-1}=P$ and $PI_\sigma=I_\sigma P$ for any permutation σ of $\{2,3,4\}$. The following sample derivations use these equalities and the proposition on page 321.

Class 1:1

$$\begin{aligned}
\begin{vmatrix} A & B \\ B & A_{(234)} \end{vmatrix} &= \left| A^2_{(234)}-B^2 \right| = \left| P\left(A^2_{(234)}-B^2\right)P \right| = \left| PA^2P_{(234)}-PB^2P \right| \\
&= \left| \begin{bmatrix} v_1^2&0&0&0\\0&v_2^2&0&0\\0&0&v_3^2&0\\0&0&0&v_4^2\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{bmatrix} - \begin{bmatrix} w_1^2&0&0&0\\0&w_2^2&0&0\\0&0&w_3^2&0\\0&0&0&w_4^2\end{bmatrix} \right| = \begin{vmatrix} v_1^2-w_1^2 & 0 & 0 & 0\\ 0 & -w_2^2 & v_2^2 & 0\\ 0&0&-w_3^2&v_3^2\\ 0&v_4^2&0&-w_4^2\end{vmatrix} \\
&= \left(v_1^2-w_1^2\right)\left(v_2^2v_3^2v_4^2-w_2^2w_3^2w_4^2\right).
\end{aligned}$$

Class 8:1

$$\begin{aligned}
&\begin{vmatrix} A & {}_{(34)}B_{(34)} \\ {}_{(243)}C_{(243)} & {}_{(243)}D_{(24)} \end{vmatrix} \\
&= \left| {}_{(243)}D_{(24)}\,A - {}_{(243)}C_{(243)}\,{}_{(34)}B_{(34)} \right| = \left| P\left({}_{(243)}D_{(24)}\,A - {}_{(243)}C_{(243)}\,{}_{(34)}B_{(34)}\right)P \right| \\
&= \left| {}_{(243)}PDP_{(24)}\,PAP - {}_{(243)}PCP_{(243)}\,{}_{(34)}PBP_{(34)} \right| \\
&= \left| \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{bmatrix}\begin{bmatrix} v_1&0&0&0\\0&v_5&v_6&0\\0&-v_6&v_5&0\\0&0&0&v_4\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix} PAP - \begin{bmatrix}1&0&0&0\\0&0&1&0\\0&0&0&1\\0&1&0&0\end{bmatrix}\begin{bmatrix} w_1&0&0&0\\0&w_5&w_6&0\\0&-w_6&w_5&0\\0&0&0&w_4\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&1\\0&1&0&0\\0&0&1&0\end{bmatrix} {}_{(34)}PBP_{(34)} \right| \\
&= \left| \begin{bmatrix} v_1&0&0&0\\0&0&v_5&-v_6\\0&v_4&0&0\\0&0&v_6&v_5\end{bmatrix}\begin{bmatrix} v_1&0&0&0\\0&v_2&0&0\\0&0&v_3&0\\0&0&0&v_4\end{bmatrix} - \begin{bmatrix} w_1&0&0&0\\0&w_5&0&-w_6\\0&0&w_4&0\\0&w_6&0&w_5\end{bmatrix}\begin{bmatrix} w_1&0&0&0\\0&w_2&0&0\\0&0&w_4&0\\0&0&0&w_3\end{bmatrix} \right| \\
&= \left| \begin{bmatrix} v_1^2&0&0&0\\0&0&v_3v_5&-v_4v_6\\0&v_2v_4&0&0\\0&0&v_3v_6&v_4v_5\end{bmatrix} - \begin{bmatrix} w_1^2&0&0&0\\0&w_2w_5&0&-w_3w_6\\0&0&w_4^2&0\\0&w_2w_6&0&w_3w_5\end{bmatrix} \right| = \begin{vmatrix} v_1^2-w_1^2&0&0&0\\0&-w_2w_5&v_3v_5&-v_4v_6+w_3w_6\\0&v_2v_4&-w_4^2&0\\0&-w_2w_6&v_3v_6&v_4v_5-w_3w_5\end{vmatrix} \\
&= -\left(v_1^2-w_1^2\right)\left(v_2v_3v_4^2(v_5^2+v_6^2)+w_2w_3w_4^2(w_5^2+w_6^2)-v_4(v_2v_3w_3+w_2w_4^2)(v_5w_5+v_6w_6)\right).
\end{aligned}$$

E-cl	M_λ	M_ρ	M	a
1:1	96	32	288	96
1:2	32	32	96′	192
1:3	96	96	288	48
1:4	96	32	192	64
1:5	96	96	576	64
1:6	64	64	576′	16
1:7	64	64	576′	32
2:1	96	64	1152	32
2:2	32	64	1152	32

E-cl	M_λ	M_ρ	M	a
3:1	1152	1152	1152	4
3:2	1152	1152	1152	4
4:1	1152	128	1152	4
4:2	1152	1152	1152	4
4:3	1152	128	1152	4
4:4	1152	1152	1152	4
5:1	128	1152	1152	4
5:2	1152	1152	1152	4
5:3	128	128	1152	4

E-cl	M_λ	M_ρ	M	a
6:1	1152	128	1152	8
6:2	1152	128	1152	8
7:1	1152	128	1152	8
7:2	1152	128	1152	8
8:1	192	1152	1152	4
8:2	576	1152	1152	4
8:3	576′	1152	1152	4
8:4	576′	1152	1152	4
9:1	192	128	1152	8
9:2	576′	128	1152	8

E-cl	M_λ	M_ρ	M	a
10:1	64	1152	1152	8
10:2	576	128	1152	4
10:3	576′	1152	1152	8
11:1	64′	1152	1152	16
11:2	192	1152	1152	16
11:3	576′	128	1152	8
12:1	64′	1152	1152	16
12:2	192	1152	1152	16
12:3	576′	128	1152	8

TABLE 2. Mapping groups of representatives of the 37 exceptional E-classes, and size a of the automorphism group of the classes. Key for the groups, in GAP notation: $32 = E(8){:}E_4 = [2^2]D(4)$, $64 = \frac{1}{2}[2^4]dD(4) = E(4)^2{:}D_{12}$, $64' = E(8){:}D_8 = [2^3]D(4)$, $96 = E(8){:}A_4 = [\frac{1}{3}A(4)^2]2 = E(4){:}6$, $96' = \frac{1}{2}[E(4)^2{:}S_3]2 = E(4)^2{:}D_6$, $128 = [2^4]D(4)$, $192 = E(8){:}S_4 = [E(4)^2{:}S_3]2$, $288 = [A(4)^2]2$, $576 = [\frac{1}{2}S(4)^2]2$, $576' = \frac{1}{2}[S(4)^2]2$, $1152 = [S(4)^2]2$.

Table 2 lists M_λ, M_ρ, and M for a representative of each of the 37 exceptional E-classes (in reduced form), and gives the size a of the automorphism group of the class. The mapping groups are identified by their orders, and the corresponding notation from GAP [Schönert et al. 1994] is shown.

4. CONCLUSION

It is intriguing that of the large number of E-classes of squares of order 8 so few lie in the exceptional classes, and that the squares in these classes all have the regular form described above. This form may be described algebraically as follows. A latin square L in reduced form defines a loop Q with binary operation . by

$$i.j = L(i,j)$$

(see [Johnson 1992], for example). All the loops Q arising from squares in the exceptional E-classes are (non-associative) extensions of the form

$$1 \longrightarrow C_2 \times C_2 \longrightarrow Q \longrightarrow C_2 \longrightarrow 1.$$

The squares of order 8 arising from loops Q that are extensions of the form

$$1 \longrightarrow C_2 \longrightarrow Q \longrightarrow C_2 \times C_2 \longrightarrow 1$$

were investigated in [Johnson 1988; 1992], and it was found that distinct E-classes had dissimilar determinants. Our expectation was that if distinct E-classes had similar determinants, this would be most likely to occur among squares which had little regularity.

Once we have squares L_1, L_2 of order n in distinct E-classes with similar determinants, we can produce squares of higher orders with similar determinants as follows. Let $L^{(j)}$ be the square obtained from L by replacing the element k by $k+nj$, $j = 1, \dots, r$, and let rL be defined as

$$\begin{bmatrix} L^{(1)} & L^{(2)} & \cdots & L^{(r-1)} & L^{(r)} \\ L^{(2)} & L^{(3)} & \cdots & L^{(r)} & L^{(1)} \\ \vdots & \vdots & & \vdots & \vdots \\ L^{(r)} & L^{(1)} & \cdots & L^{(r-2)} & L^{(r-1)} \end{bmatrix}.$$

It is readily seen using [Muir and Matzler 1960, p. 487] that Θ_{rL} is given by

$$(-1)^{(r-1)(r-2)/2} \times \prod_{j=0}^{r-1} \det\bigl(X_{L^{(1)}} + \rho^j X_{L^{(2)}} + \rho^{2j} X_{L^{(3)}} + \cdots + \rho^{(r-1)j} X_{L^{(r)}}\bigr),$$

where $\rho = e^{2\pi i/r}$. Hence Θ_{rL} is obtained from Θ_L by replacing x_k with

$$x_k + \rho^h x_{k+n} + \cdots + \rho^{(r-1)h} x_{k+(r-1)n}.$$

It follows that Θ_{rL_1} and Θ_{rL_2} are similar.

We remark that if L has left and right mapping groups M_λ and M_ρ then the corresponding groups for rL are

$$rM_\lambda = M_\lambda \wr C_r \quad \text{and} \quad rM_\rho = M_\rho \wr C_r,$$

where $\wr$ indicates the wreath product. Thus, if the squares L_1 and L_2 are chosen so that the sets

$$\{|M_\lambda(L_1)|, |M_\rho(L_1)|\} \quad \text{and} \quad \{|M_\lambda(L_2)|, |M_\rho(L_2)|\}$$

are distinct (for example, if $L_1 = 1{:}1$ and $L_2 = 1{:}2$ in Table 1), it follows from considerations of orders that the sets $\{M_\lambda(rL_1), M_\rho(rL_1)\}$ and $\{M_\lambda(rL_2), M_\rho(rL_2)\}$ are also distinct, so that rL_1 and rL_2 are E-inequivalent, using the result in [Albert 1943] mentioned above. Thus we have E-inequivalent squares with similar determinants for all orders of the form $8k$, for $k = 1, 2, \ldots$.

Questions. 1. For which other orders are there E-inequivalent squares with similar determinants?
2. Are there E-inequivalent squares with trivial automorphism group with similar determinants?
3. Are there E-inequivalent squares with full mapping group S_n with similar determinants?

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