Scaling in a Map of the Two-Torus

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CONTENTS

We discuss scaling in the parameter space of a family of maps arising from the iteration of a map of the two-torus defined in terms of a Jacobian elliptic function. This map appears to show a complex analog of the Feigenbaum–Kadanoff–Shenker scaling found in bifurcation sequences of circle maps.

1. INTRODUCTION

We first review the Feigenbaum–Kadanoff–Shenker scaling [Feigenbaum et al. 1982; Jensen et al. 1983; Cvitanović et al. 1985] in the cubic critical family of circle maps on $\mathbb{R}$:

$$\theta \rightarrowtail f_\Omega(\theta) = \Omega + \theta - \frac{1}{2\pi}\sin(2\pi\theta). \tag{1–1}$$

We thus have the fundamental property $f_\Omega(\theta+n) = f_\Omega(\theta) + n$ for all integer n. The average advance in θ per iteration is called the winding number w_f. Denoting the n-th iterate of f as $f^{<n>}$, we define w_f as the limit:

$$w_f(\Omega) = \lim_{n\to\infty} \frac{f_\Omega^{<n>}(\theta_0) - \theta_0}{n} \tag{1–2}$$

for some initial θ_0. If $w_f(\Omega) = p/q$ is rational, (here p and q are positive integers), the map represents mode-locked behaviour with period q.

If f is invertible, then $w(\Omega)$ is a monotonic increasing function of Ω and is independent of the initial value θ_0. It is, however, constant on an interval surrounding each rational, and its graph forms a so-called 'devil's staircase'.

To introduce the concept of *scaling*, recall that every irrational number w can be represented by an infinite regular continued fraction

$$w = [a_1, a_2, a_3, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots + \cfrac{1}{a_n + \cdots}}},$$

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and the sequence of best rational approximations p_n/q_n is found by truncating the continued fraction: $p_n/q_n = [a_1, a_2, \ldots, a_n]$. Considering now the case of the sine map defined in equation (1–1), we may define a sequence $\{\Omega_n\}$ by means of $f_{\Omega_n}^{<q_n>}(0) = p_n$, so that there is a q_n-cycle containing zero with winding number p_n/q_n.

We first consider the golden mean winding number, which has a continued fraction

$$w_{gm} = [1, 1, 1, \ldots].$$

By successively truncating this infinite fraction, one generates a sequence of best rational approximations $\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \ldots$ to w_{gm}. The n-th term in this sequence is F_{n-1}/F_n where the F_n are the Fibonacci numbers. These are defined recursively by $F_0 = 0$, $F_1 = 1$, and

$$F_{n+1} = F_n + F_{n-1},$$

for $n = 1, 2, 3, \ldots$. From this it follows that

$$\lim_{n\to\infty} \frac{F_{n-1}}{F_n} = w_{gm}.$$

Using the corresponding sequence $\{\Omega_n\}$, we find that the limit

$$\delta \equiv \lim_{j\to\infty} \frac{\Delta\Omega_j}{\Delta\Omega_{j+1}} \tag{1–3}$$

exists, and so defines the Feigenbaum–Kadanoff–Shenker δ, which has the approximate value -2.834. There is also an orbit scaling: if φ_k is the value of the nearest cycle element to zero in the q_k cycle, then

$$\alpha \equiv \lim_{k\to\infty} \frac{\varphi_k}{\varphi_{k+1}}$$

exists, and is about -1.289. These are examples of scaling laws.

The case of degree of inflection point equal to three has been much studied (e.g. [Cvitanović et al. 1985]), and generalizations to different degrees (including infinity) of the point of inflection, and a discussion of higher-order scaling laws (corrections to scaling) have been made in [Dixon et al. 1997; Briggs et al. 1998].

2. TWO-DIMENSIONAL GENERALIZATIONS

There has been much speculation as to whether any of this behaviour exists in higher dimensions [Hu and Mao 1987], and several $\mathbb{R}^2$-analytic maps of the two-dimensional torus $\mathbb{T}^2$ (that is, the unit square with opposite sides identified), have been studied numerically, but no scaling has been found [Kim and Ostlund 1986; Chen and Wang 1991].

Our intention here is to see if an analytic map of $\mathbb{C}$ possesses different behaviour to the $\mathbb{R}^2$-analytic case. A preliminary study was described in [Briggs 1995]. We consider the torus map defined by the complex analytic family

$$f_\Omega(z) \equiv \Omega + z - \operatorname{sn}(\gamma z, m)/\gamma, \tag{2–1}$$

where sn is Jacobi's elliptic function (see, for example, [Lawden 1989], chapter 2), Ω is a complex constant, and m and γ are real constants. sn is doubly periodic and we choose for convenience to make the unit square $[0, 1]^2$ the period parallelogram. Since the periods of $\operatorname{sn}(z, m)$ are given by the complete elliptic integral of the first kind [Lawden 1989] as

$$4K(m) = 4\int_0^{\pi/2} dt/\sqrt{1 - m\sin^2 t}$$

and $2iK(1-m)$, the choice $m = (3-2\sqrt{2})^2$ uniquely enforces $4K(m) = 2iK(1-m)$, so that with $\gamma = 4K(m)$, we have the desired relation $f_\Omega(z+k) = f_\Omega(z)+k$ for all Gaussian integers k (that is, complex numbers with integer real and imaginary parts), and for all z, Ω. This map is thus a natural generalization of the sine circle map family, and being meromorphic has a large amount of relevant theory [Bergweiler 1993]. Note also that f_Ω is real whenever both z and Ω are real. Thus this family can be considered as a complexification of the real sine map, as a nonzero imaginary part is introduced to Ω.

However, although we have maintained the desirable property of meromorphicity, several potential disadvantages should be kept in mind:

(1) the presence of poles, in our case at $\frac{i}{2}$ and $\frac{1+i}{2}$;

(2) sn maps the unit square twice over the entire complex plane, so the map f_Ω is thus not invertible; and

(3) an arbitrary perturbation of our map will destroy the double-periodicity, so that we cannot claim that any observed scaling relations are universal. A reasonable hypothesis would be that the family $f_\Omega : z \mapsto z + \Omega + sn(4K(m)z, m)/(4K(m))$ for different real m defined on a appropriate rectangle also have an analogous scaling behaviour to

that to be described, but we have not yet tested this hypothesis numerically.

A consequence of (2) is that it is not possible to define a winding number independently of the initial point of the orbit. Thus, we will only study here orbits starting at the origin.

3. THE PARAMETER SPACE

We first study the parameter space of the family f_Ω. Due to the symmetries of the function ($f_{1-\Omega}(0) = 1 - f_\Omega(0)$ and $f_{i-\Omega}(0) = i - f_\Omega(0)$), it is sufficient to study the region $0 < \operatorname{Re}\Omega, \operatorname{Im}\Omega < 1/2$. For a large number of values of Ω in this region, we computed the orbit of zero under the map f_Ω, that is, the sequence $\{0, f_\Omega(0), f_\Omega(f_\Omega(0)) \equiv f_\Omega^{<2>}(0), \dots\}$. If convergence to a periodic orbit modulo the integer lattice $\mathbb{Z}^2$ was detected (that is, convergence to a point z_0 satisfying $f_\Omega^{<q>}(z_0) = z_0 + p$ for some integer q and Gaussian integer p), then the point Ω was coloured. The result is shown in Figure 1. This can be considered the Mandelbrot set for the family f_Ω.

Each individual component appears to be just a Mandelbrot set for the cubic family z^3+c, and these Mandelbrot sets are nonintersecting. This can be understood from the fact that the family f_Ω has a cubic critical point at the origin [Briggs et al. 1991]. Each such component has constant 'period' q, and varying p, depending on the bifurcation path followed from the central region. That is, each complete component, including 'ears', has constant q.

For example, the largest region, centered on the origin, has winding number $(0+0i)/1$, and moving up the imaginary axis we have a bifurcation to a winding number $(0+0i)/2$. The other regions are (in decreasing order of size) $(0+0i)/3$, $(0+0i)/4$, etc. Secondly, starting at $\Omega = 1/2$, we have regions of winding number $(1+0i)/3$, $(1+0i)/4$, $(1+0i)/5$ and so on in decreasing order of size along the real axis. At approximately $\Omega = 1/2 + 0.27i$, the region has winding number $(1+i)/4$.

4. SCALING

Our aim is to look for possible scaling laws in the family f_Ω. Firstly, within each connected region of Figure 1, we have the well-understood scaling behaviour of complex cubics described in [Briggs et al. 1991]. To investigate more general scaling laws that link different components, we consider third-order recurrences [Kim and Ostlund 1986; Chen and Wang 1991; Hu and Mao 1987]. As part of a study of $\mathbb{R}^2$ maps, Hu and Mao [Hu and Mao 1987] generalized the Fibonacci recurrence by using $F_0 = F_1 = 0$, $F_2 = 1$, and

$$F_{n+3} = F_{n+2} + F_{n+1} + F_n, \qquad (4\text{–}1)$$

for $n = 0, 1, 2, \dots$, on the grounds that it is 'the simplest possible ternary continued fraction expansion'. On the other hand, Kim and Ostlund [1986] used the same initial conditions but

$$F_{n+3} = F_{n+1} + F_n. \qquad (4\text{–}2)$$

To unify these concepts it is useful to recall some elementary number field theory: a *cubic number field* is a set

$$\{\, a_0 + a_1\zeta + a_2\zeta^2 \mid a_0, a_1, a_2 \in \mathbb{Q} \,\},$$

where ζ is a root of a monic irreducible cubic polynomial with integer coefficients. Every cubic number field K has a *discriminant* denoted $d(K)$, and the discriminant of the defining cubic (that is,

$$\big((z_1 - z_2)(z_1 - z_3)(z_2 - z_3)\big)^2,$$

where z_1, z_2, z_3 are the roots of the cubic) is always a squared-integer multiple of $d(K)$. The cubic number field is called *cyclic* if the discriminant of its defining cubic polynomial is a square. Thus the linear recurrence (4–1) has characteristic polynomial

$$z^3 - z^2 - z - 1,$$

which has discriminant -44 and generates a field with the same discriminant, whereas the recurrence (4–2) has characteristic polynomial

$$z^3 - z - 1$$

which has discriminant -23, again the discriminant of the generated cubic number field. The number 23 is in fact the smallest possible absolute value of the discriminant of any cubic number field, which is an intriguing analog of the fact that the golden mean in one dimension is associated with the quadratic number field $\mathbb{Q}(\sqrt{5})$ of smallest discriminant, namely 5. In fact, all these authors neglected to consider another cubic of small discriminant, namely

$$z^3 - z^2 - 1,$$

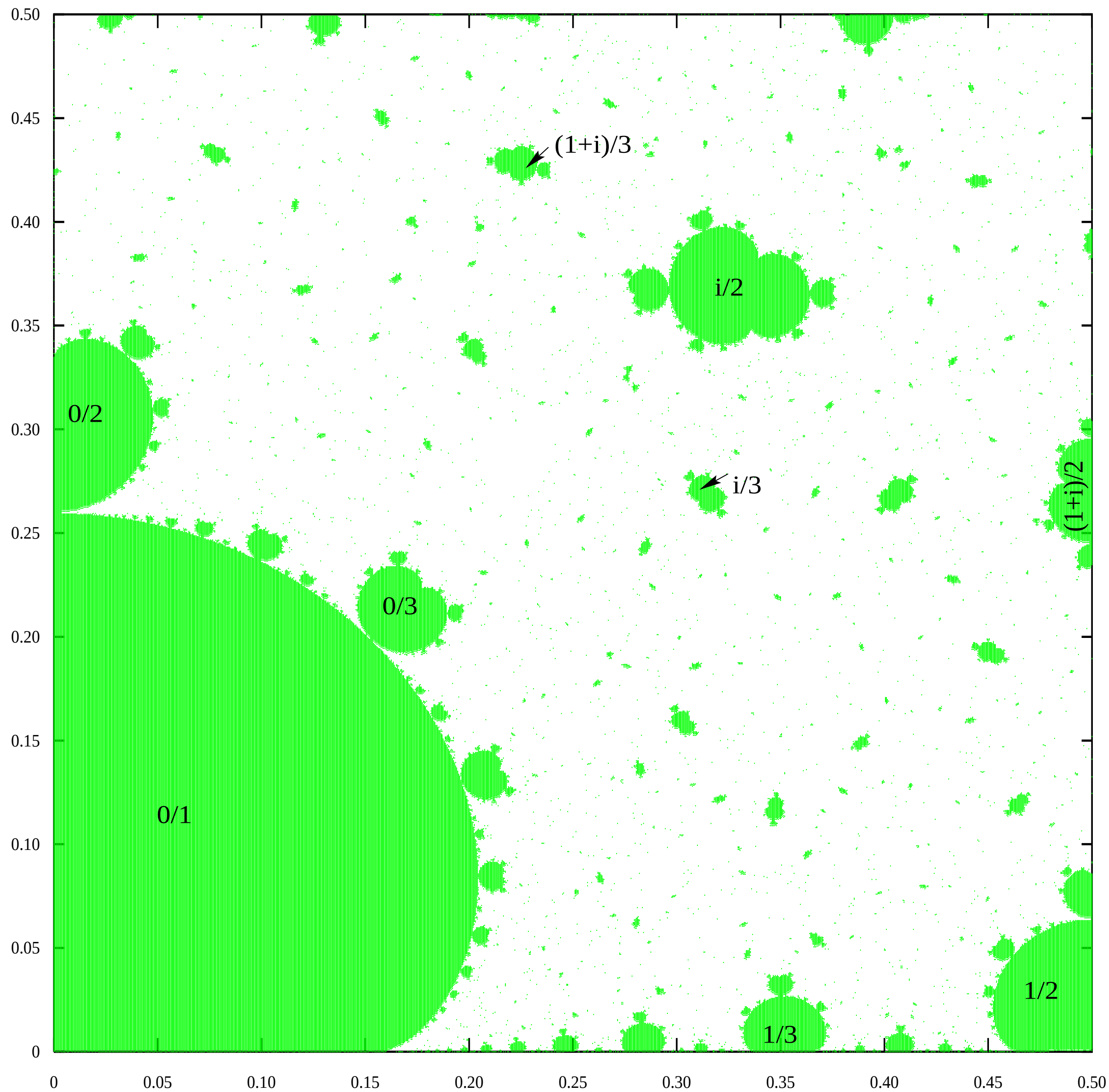

FIGURE 1. One-quarter of the Mandelset of $\Omega + z - \mathrm{sn}(\gamma z, m)/\gamma$ for $m = (3-2\sqrt{2})^2$ and $\gamma = 4K(m)$. The coloured regions are values of Ω for which the orbit of the origin is bounded. The full figure is obtained by reflecting at the edges of the square.

which generates a field with $d(K) = -31$. In any case, all these polynomials have negative discriminant, a property not shared by the quadratic number field $\mathbb{Q}(\sqrt{5})$, which suggests that the analogy mentioned above is inappropriate. In fact, the smallest possible *positive* discriminant of a cubic number field is 49, corresponding to the cyclic cubic field $\mathbb{Q}(\theta)$ of z^3+z^2-2z-1, which has the nicely symmetric roots $\{\theta_1 = 2\cos(2\pi/7),\ \theta_2 = 2\cos(4\pi/7),\ \theta_3 = 2\cos(6\pi/7)\}$. (Note that $2\cos(2\pi/5) = (\sqrt{5}-1)/2$, a very suggestive analogy!)

Thus, our computation was as follows: for each polynomial $x^3 + a_2x^2 + a_1x + a_0$ listed in Table 2, we generated a sequence of rational winding numbers $(p_1(n)+ip_2(n))/q(n)$ by setting $q(n) = r(n)$, $p_1(n) = r(n-1)$, $p_2(n) = r(n-2)$, and

$$r(n+3) = -a_2r(n+2) - a_1r(n+1) - a_0r(n)$$

for $n = 0, 1, 2, \ldots$, with arbitrarily chosen initial values $r(0) = 1$, $r(1) = 1$, $r(2) = 2$. Such a sequence of winding numbers always converges to a complex limit with irrational real and imaginary parts; this

Ω	p_1	p_2	q
$0.500000000 + 0.27327863i$	1	1	2
$0.510040518 + 0.66814844i$	1	2	3
$0.333373483 + 0.50000000i$	2	3	6
$0.483926512 + 0.49638946i$	3	6	10
$0.419720394 + 0.56365287i$	6	10	19
$0.418427241 + 0.50625041i$	10	19	33
$0.444036793 + 0.53076717i$	19	33	61
$0.422150578 + 0.53122839i$	33	61	108
$0.431512126 + 0.52147797i$	61	108	197

TABLE 1. Ω values for the case of discriminant 49. Compare Figure 2, right middle.

polynomial	d	$q_{\max}$	δ
x^3-x-1	-23	351	$-1.05257-0.701933i$
x^3-x^2-1	-31	277	$-0.41029-1.34655i$
x^3-x^2-x-1	-44	274	$-0.32905+3.41170i$
x^3-2x^2-1	-59	258	$3.50307+4.27158i$
x^3-2x-2	-76	200	$0.11997+1.34173i$
x^3-x^2-2x+1	$+49$	197	$-1.14601-1.14434i$
x^3-3x-1	$+81$	172	$-0.32413+3.42384i$

TABLE 2. Characteristic polynomials of the order-3 recurrences used in this study, the discriminant of the corresponding cubic fields, the largest value of q used, and the estimated scaling constant δ.

limit may easily be computed from the roots of the polynomial. Table 2 includes polynomials generating cubic fields of all discriminants having absolute value less than 82. Note, however, unlike the one-dimensional case, these rational approximants do not in general give the sequence of best approximants to the irrational limit.

Then for each rational winding numbers we found a corresponding Ω_n by solving

$$f_{\Omega_n}^{<q(n)>}(0) = p_1 + ip_2$$

numerically. We estimated δ both from (1–3) from a second-order correction to scaling method [Briggs 1994]. The latter is a extrapolation method which will be valid if the convergence of the estimates δ_k to δ is itself geometric. A typical set of Ω values is shown in Table 1. The ratio estimates are shown in Table 2, and in each case the convergence of the limit in equation (1–3) to δ was rapid. The corresponding paths in the Ω plane are shown in Figure 2.

This is our main result, since such δ scaling has not been observed before in families of maps of $\mathbb{R}^2$.

A natural question is whether there is any geometric scaling in the size of the orbits, analogous to the α scaling in circle maps. To investigate this question, we need some characteristic quantity to measure the size of orbits, which conceivably might be the distance of the closest orbit point to the origin, or a typical orbit point obtained from the recurrence relations, perhaps the orbit point corresponding the the q value of the previous orbit. We tried both ideas, but found no evidence at all of systematic orbit scaling.

APPENDIX

The accurate computation of the function sn presents some challenges. We used a representation in terms of theta functions, which have rapidly convergent series [Lawden 1989]. Setting $q := \exp(-2\pi)$ and $\gamma := 4K\big((3-2\sqrt{2})^2\big)$, the formula is

$$\operatorname{sn}\big(\gamma z,\, (3-2\sqrt{2})^2\big) = \frac{\theta_3(0)}{\theta_2(0)}\frac{\theta_1(2\pi z)}{\theta_4(2\pi z)}$$

with

$$\theta_1(z) = 2\sum_{n=0}^{\infty}(-1)^n q^{(n+1/2)^2}\sin{(2n+1)z},$$

$$\theta_2(0) = 2\sum_{n=0}^{\infty} q^{(n+1/2)^2},$$

$$\theta_3(0) = 1 + 2\sum_{n=0}^{\infty} q^{n^2},$$

$$\theta_4(z) = 1 + 2\sum_{n=0}^{\infty}(-1)^n q^{n^2}\cos 2nz.$$

These formulas were implemented in the C++ software CLN [Haible 1998], which supports arbitrary precision. We used a Newton–Raphson iteration for computing Ω values corresponding to periodic orbits; computation with up to 1000 decimal places was used to confirm the existence of these orbits.

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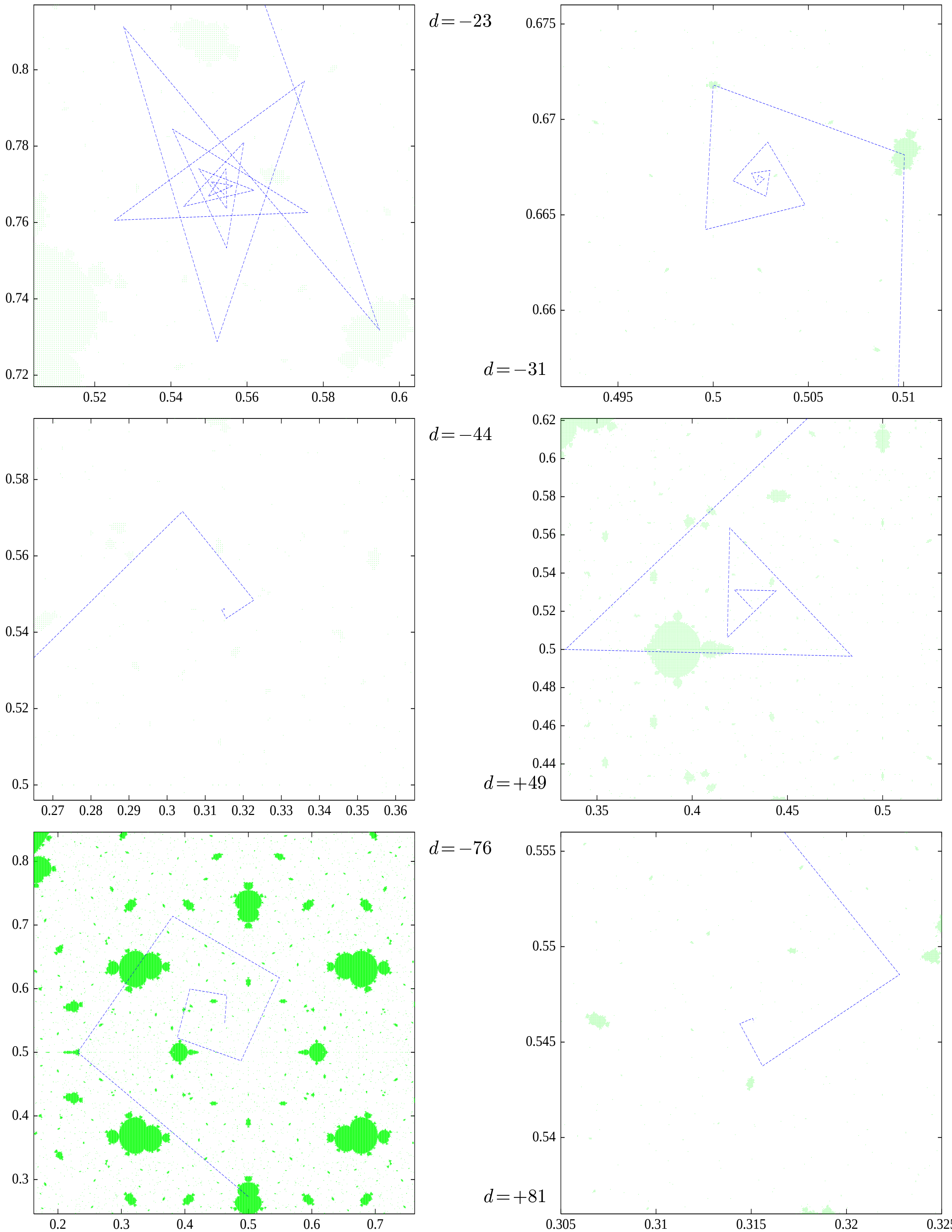

FIGURE 2. Ω sequences for recurrences corresponding to various discriminants d, superimposed on the Mandelset of the family f_Ω.

REFERENCES

[Bergweiler 1993] W. Bergweiler, "Iteration of meromorphic functions", *Bull. Amer. Math. Soc.* (*N.S.*) **29**:2 (1993), 151–188.

[Briggs 1994] K. Briggs, "Corrections to universal scaling in real maps", *Phys. Lett. A* **191**:1-2 (1994), 108–112.

[Briggs 1995] K. M. Briggs, "A torus map based on Jacobi's sn", *Computers and graphics* **19** (1995), 451.

[Briggs et al. 1991] K. M. Briggs, G. R. W. Quispel, and C. J. Thompson, "Feigenvalues for Mandelsets", *J. Phys. A* **24**:14 (1991), 3363–3368.

[Briggs et al. 1998] K. M. Briggs, T. W. Dixon, and G. Szekeres, "Analytic solutions of the Cvitanović-Feigenbaum and Feigenbaum-Kadanoff-Shenker equations", *Internat. J. Bifur. Chaos Appl. Sci. Engrg.* **8**:2 (1998), 347–357.

[Chen and Wang 1991] S. G. Chen and Y. Q. Wang, "Cubic irrational number and critical scaling law for quasi-periodic motion", *Phys. Lett. A* **153** (1991), 113–116.

[Cvitanović et al. 1985] P. Cvitanović, B. Shraiman, and B. Söderberg, "Scaling laws for mode lockings in circle maps", *Phys. Scripta* **32**:4 (1985), 263–270.

[Dixon et al. 1997] T. W. Dixon, B. G. Kenny, and K. M. Briggs, "On the universality of singular circle maps", *Phys. Lett. A* **231**:5-6 (1997), 359–366.

[Feigenbaum et al. 1982] M. J. Feigenbaum, L. P. Kadanoff, and S. J. Shenker, "Quasiperiodicity in dissipative systems: a renormalization group analysis", *Phys. D* **5**:2-3 (1982), 370–386.

[Haible 1998] B. Haible, "CLN (C++ class library for numbers)", software, 1998. See http://clisp.cons.org/~haible/packages-cln-README.html.

[Hu and Mao 1987] B. Hu and J.-M. Mao, "Transitions to chaos in higher dimensions", pp. 206–271 in *Directions in chaos*, vol. 1, edited by H. Bai-lin, World Sci. Publishing, Singapore, 1987.

[Jensen et al. 1983] M. H. Jensen, P. Bak, and T. Bohr, "Complete devil's staircase: fractal dimension, and universality of mode-locking structure in the circle map", *Phys. Rev. Lett.* **50** (1983), 1637.

[Kim and Ostlund 1986] S.-H. Kim and S. Ostlund, "Simultaneous rational approximations in the study of dynamical systems", *Phys. Rev. A* (3) **34**:4 (1986), 3426–3434.

[Lawden 1989] D. F. Lawden, *Elliptic functions and applications*, Springer, New York, 1989.

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