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Estimation and prediction of Gaussian processes using generalized Cauchy covariance model under fixed domain asymptotics

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Abstract: We study estimation and prediction of Gaussian processes with covariance model belonging to the generalized Cauchy (GC) family, under fixed domain asymptotics. Gaussian processes with this kind of covariance function provide separate characterization of fractal dimension and long range dependence, an appealing feature in many physical, biological or geological systems. The results of the paper are classified into three parts.

In the first part, we characterize the equivalence of two Gaussian measures with GC covariance functions. Then we provide sufficient conditions for the equivalence of two Gaussian measures with Matérn (MT) and GC covariance functions and two Gaussian measures with Generalized Wendland (GW) and GC covariance functions.

In the second part, we establish strong consistency and asymptotic distribution of the maximum likelihood estimator of the microergodic parameter associated to GC covariance model, under fixed domain asymptotics. The last part focuses on optimal prediction with GC model and specifically, we give conditions for asymptotic efficiency prediction and asymptotically correct estimation of mean square error using a misspecified GC, MT or GW model.

Our findings are illustrated through a simulation study: the first compares the finite sample behavior of the maximum likelihood estimation of the microergodic parameter of the GC model with the given asymptotic distribution. We then compare the finite-sample behavior of the prediction and its associated mean square error when the true model is GC and the prediction is performed using the true model and a misspecified GW model.

Keywords and phrases: Infill asymptotics, long memory, microergodic parameter, maximum likelihood.

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1. Introduction

Two fundamental steps in geostatistical analysis are estimating the parameters of a Gaussian stochastic process and predicting the process at new locations. In both steps, the covariance function covers a central aspect. For instance, mean square error optimal prediction at an unobserved site depends on the knowledge of the covariance function. Since a covariance function must be positive definite, practical estimation generally requires the selection of some parametric families of covariances and the corresponding estimation of these parameters.

The maximum likelihood (ML) estimation method is generally considered the best method for estimating the parameters of covariance models. Nevertheless, the study of the asymptotics properties of ML estimation, is complicated by the fact that more than one asymptotic frameworks can be considered when observing a single realization [37]. The increasing domain asymptotic framework corresponds to the case where the sampling domain increases with the number of observed data and where the distance between any two sampling locations is bounded away from 0. The fixed domain asymptotic framework, sometimes called infill asymptotics [6], corresponds to the case where more and more data are observed in some fixed bounded sampling domain.

General results for the asymptotics properties of the ML estimator, under increasing domain asymptotic framework and some mild regularity conditions, are given in [21] and [3]. Specifically, they show that ML estimates are consistent and asymptotically Gaussian with asymptotic covariance matrix equal to the inverse of the Fisher information matrix.

Under fixed domain asymptotics, no general results are available for the asymptotic properties of ML estimation. Yet, some results have been obtained when assuming the covariance belongs to Matérn (MT) [22] or Generalized Wendland (GW) [12] models. Both families allow for a continuous parameterization of smoothess of the underlying Gaussian process, the GW family being additionally compactly supported [4]. Specifically, when the smoothness parameter is known and fixed, not all parameters can be estimated consistently, when

$d = 1, 2, 3$, with d the dimension of the Euclidean space. Instead, the ratio of variance and scale (to the power of a function of the smoothing parameter), sometimes called microergodic parameter is consistently estimable. This follows from results given in [36] for the MT model and [4] for the GW model.

Asymptotic results for ML estimation of the microergodic parameter of the MT model can be found in [36], [7], [32] when the scale parameter is assumed known and fixed. [19] give strong consistency and asymptotic distribution of the microergodic parameter when estimating jointly the scale and the variance parameters and by means of a simulation study they show that the asymptotic approximation is considerably improved in this case. Similar results for the microergodic parameter of the GW model can be found in [4].

In terms of prediction, under fixed domain asymptotic, [27, 28] provides conditions under which optimal predictions under a misspecified covariance function are asymptotically efficient, and mean square errors converge almost surely to their targets. Stein's conditions translates into the fact that the true and the misspecified covariances must be compatible, that is the induced Gaussian measures are equivalent [26, 17]. A weaker condition, based on ratio of spectral densities, is given in [29].

In this paper we study ML estimation and prediction of Gaussian processes, under fixed domain asymptotics, using Generalized Cauchy (GC) covariance model. GC family of covariance models has been proposed in [13] and deeply studied in [20]. It is particularly attractive because Gaussian processes with such covariance function allow for any combination of fractal dimension and Hurst coefficient, an appealing feature in many physical, biological or geological systems (see [14] and [13] and the references therein).

In particular, we offer the following results. First, we characterize the equivalence of two Gaussian measures with covariance functions belonging to the GC family and sharing the same smoothness parameter. A consequence of this result is that, as in MT and GW covariance models, when the smoothness parameter is known and fixed, not all parameters can be estimated consistently, under fixed domain asymptotics. Then we give sufficient conditions for the equivalence of two Gaussian measures where the state of truth is represented by a member of the MT or GC family and the other Gaussian measure has a GC covariance model.

We then assess the asymptotic properties of the ML estimator of the microergodic parameter associated with the GC family. Specifically, for a fixed smoothness parameter, we establish strong consistency and asymptotic distribution of the microergodic parameter assuming the scale parameter fixed and known. Then, we generalize these results when jointly estimating with ML the variance and the scale parameter.

Finally, using results in [27] and [29], we study the implications of our results on prediction, under fixed domain asymptotics. One remarkable implication is that when the true covariance belongs to the GC family, asymptotic efficiency prediction and asymptotically correct estimation of mean square error can be achieved using a compatible compactly supported GW covariance model.

The remainder of the paper is organized as follows. In Section 2 we review some results about MT, GW and GC covariance models. In Section 3 we first characterize the equivalence of Gaussian measure under the GC covariance model. Then we give sufficient conditions for the equivalence of two Gaussian measures with MT and GC and two Gaussian measures with GW and GC covariance models. In Section 4 we establish strong consistency and asymptotic distribution of the ML estimation of the microergodic parameter of the GC models, under fixed domain asymptotics. Section 5 discuss the consequences of our results in terms of prediction, under fixed domain asymptotics. Section 6 provides two simulation studies: the first show how well the given asymptotic distribution of the microergodic parameter apply to finite sample cases, when estimating with ML a GC covariance model under fixed domain asymptotics. The second compare the finite-sample behavior of the prediction when using two compatible GC and GW models, when the true model is GC. The final Section provides a discussion on the consequence of our results and open problems for future research.

2. Matérn, Generalized Wendland and Generalized Cauchy covariance models

This section depicts the main features of the three covariance models involved in the paper. We denote $\{Z(\boldsymbol{s}), \boldsymbol{s} \in D\}$ a zero mean Gaussian stochastic process on a bounded set D of $\mathbb{R}^d$, with stationary covariance function $C : \mathbb{R}^d \to \mathbb{R}$. We consider the family Φ_d of continuous mappings $\phi : [0, \infty) \to \mathbb{R}$ with $0 < \phi(0) < \infty$, such that

$$\operatorname{cov}\left(Z(\boldsymbol{s}), Z(\boldsymbol{s}')\right) = C(\boldsymbol{s}' - \boldsymbol{s}) = \phi(\|\boldsymbol{s}' - \boldsymbol{s}\|),$$

with $\boldsymbol{s}, \boldsymbol{s}' \in D$, and $\|\cdot\|$ denoting the Euclidean norm. Gaussian processes with such covariance functions are called weakly stationary and isotropic.

[25] characterized the family Φ_d as being scale mixtures of the characteristic functions of random vectors uniformly distributed on the spherical shell of $\mathbb{R}^d$, with any probability measure, F:

$$\phi(r) = \int_0^\infty \Omega_d(r\xi) F(\mathrm{d}\xi), \qquad r \geq 0,$$

with $\Omega_d(r) = r^{-(d-2)/2} J_{(d-2)/2}(r)$ and J_ν is Bessel function of the first kind of order ν. The family Φ_d is nested, with the inclusion relation $\Phi_1 \supset \Phi_2 \supset \ldots \supset \Phi_\infty$ being strict, and where $\Phi_\infty := \bigcap_{d \geq 1} \Phi_d$ is the family of mappings ϕ whose radial version is positive definite on any d-dimensional Euclidean space.

The MT function, defined as:

$$\mathcal{M}_{\nu,\alpha,\sigma^2}(r) = \sigma^2 \frac{2^{1-\nu}}{\Gamma(\nu)} \left(\frac{r}{\alpha}\right)^\nu \mathcal{K}_\nu\left(\frac{r}{\alpha}\right), \qquad r \geq 0,$$

is a member of the family Φ_∞ for any positive values of α and ν. Here, $\mathcal{K}_\nu$ is a modified Bessel function of the second kind of order ν, σ^2 is the variance and α a positive scaling parameter.

We also define Φ_d^b as the family that consists of members of Φ_d being additionally compactly supported on a given interval, $[0, b]$, $b > 0$. Clearly, their radial versions are compactly supported over balls of $\mathbb{R}^d$ with radius b. For a given $\kappa > 0$, the GW correlation function is defined as [4, 12]:

$$\varphi_{\mu,\kappa,\beta,\sigma^2}(r) = \begin{cases} \frac{\sigma^2}{B(2\kappa,\mu+1)} \int_{r/\beta}^{1} u(u^2 - (r/\beta)^2)^{\kappa-1}(1-u)^{\mu}\, \mathrm{d}u, & 0 \le r/\beta < 1, \\ 0, & r/\beta \ge 1, \end{cases} \tag{2.1}$$

where B denotes the beta function, σ^2 is the variance and $\beta > 0$ is the compact support. Equivalent representations of (2.1) in terms of Gauss hypergeometric function or Legendre polynomials are given in [16]. Closed form solutions of integral (2.1) can be obtained when $\kappa = k$ with $k \in \mathbb{N}$, the so called *original Wendland* functions [33], and, using some results in [24], when $\kappa = k + 0.5$, the so called *missing Wendland* functions.

Arguments in [12] and [35] show that, for a given $\kappa > 0$, $\varphi_{\mu,\kappa,\beta,\sigma^2} \in \Phi_d^\beta$ if and only if $\mu \ge (d+1)/2 + \kappa$.

In this special case $\kappa = 0$ the GW correlation function is defined as:

$$\varphi_{\mu,0,\beta,\sigma^2}(r) = (1 - r/\beta)_+^{\mu} = \begin{cases} (1-r)^{\mu}, & 0 \le r/\beta < 1, \\ 0, & r/\beta \ge 1, \end{cases}$$

and arguments in [15] show that $\varphi_{\mu,0,\beta,\sigma^2} \in \Phi_d^1$ if and only if $\mu \ge (d+1)/2$.

The parameters $\nu > 0$ and $\kappa \ge 0$ are crucial for the differentiability at the origin and, as a consequence, for the degree of the differentiability of the associated sample paths in the MT and GW models. In particular for a positive integer k, the sample paths of a Gaussian process are k times differentiable if and only if $\nu > k$ in the MT case and if and only if $\kappa > k - 1/2$ in the GW case.

The smoothness of a Gaussian process can also be described via the Hausdorff or fractal dimension of a sample path. The fractal dimension $D \in [d, d+1)$ is a measure of the roughness for non-differentiable Gaussian processes and higher values indicating rougher surfaces. For a given covariance function $\phi \in \Phi_d$ if $1 - \phi(r) \sim r^{\chi}$ as $r \to 0$ for some $\chi \in (0, 2]$ then the sample paths of the associated random process have fractal dimension $D = d + 1 - \chi/2$. Here χ is the so called fractal index that governs the roughness of sample paths of a non-differentiable Gaussian process.

In the case of a MT model $\chi = 2\nu$ so $D = d + 1 - \nu$ if $0 < \nu < 1$ and d otherwise [2, 14]. Thus the MT model permits the full range of allowable values for the fractal dimension. In the case of GW family $\chi = 2\kappa + 1$, so that in this case $D = d + 0.5 - \kappa$ if $0 \le \kappa < 0.5$ and d otherwise. Thus the GW model does not allow to cover the full range of allowable values for the fractal dimension.

Long-memory dependence can be defined trough the asymptotic behavior of the covariance function at infinity. Specifically, for a given covariance function $\phi \in \Phi_d$, if the power-law $\phi(r) \sim r^{-\varepsilon+d}$ as $r \to \infty$ holds for some $\varepsilon \in (0, d]$ the stochastic process is said to have long memory with Hurst coefficient $H = \varepsilon/2$. MT and GW covariance models do not posses this feature.

A celebrated family of members of Φ_∞ is the GC class [13], defined as:

$$\mathcal{C}_{\delta,\lambda,\gamma,\sigma^2}(r) = \sigma^2 \left(1 + (r/\gamma)^\delta\right)^{-\lambda/\delta}, \qquad r \geq 0, \tag{2.2}$$

where the conditions $\delta \in (0, 2]$ and $\lambda > 0, \gamma > 0, \sigma^2 > 0$ are necessary and sufficient for $\mathcal{C}_{\delta,\lambda,\gamma,\sigma^2} \in \Phi_\infty$. The parameter δ is crucial for the differentiability at the origin and, as a consequence, for the degree of the differentiability of the associated sample paths. Specifically, for $\delta = 2$, they are infinitely times differentiable and they are not differentiable for $\delta \in (0, 2)$.

The GC family represents a breaking point with respect to earlier literature based on the assumption of self similarity, since it decouples the fractal dimension and the Hurst effect. Specifically, the sample paths of the associated stochastic process have fractal dimension $D = d + 1 - \delta/2$ for $\delta \in (0, 2)$ and if $\lambda \in (0, d]$ it has long memory with Hurst coefficient $H = \lambda/2$. Thus, D and H may vary independently of each other [13, 20].

Fourier transforms of radial versions of members of Φ_d, for a given d, have a simple expression, as reported in [30] and [34]. For a member ϕ of the family Φ_d, we define its isotropic spectral density as

$$\widehat{\phi}(z) = \frac{z^{1-d/2}}{(2\pi)^d} \int_0^\infty u^{d/2} J_{d/2-1}(uz)\phi(u)\mathrm{d}u, \qquad z \geq 0, \tag{2.3}$$

and through the paper we use the notation $\widehat{\mathcal{C}}_{\delta,\lambda,\gamma,\sigma^2}$, $\widehat{\mathcal{M}}_{\nu,\alpha,\sigma^2}$ and $\widehat{\varphi}_{\mu,\kappa,\beta,\sigma^2}$ for the spectral density associated with $\mathcal{C}_{\delta,\lambda,\gamma,\sigma^2}$, $\mathcal{M}_{\nu,\alpha,\sigma^2}$ and $\varphi_{\mu,\kappa,\beta,\sigma^2}$. A well-known result about the spectral density of the Matérn model is the following:

$$\widehat{\mathcal{M}}_{\nu,\alpha,\sigma^2}(z) = \frac{\Gamma(\nu + d/2)}{\pi^{d/2}\Gamma(\nu)} \frac{\sigma^2\alpha^d}{(1 + \alpha^2 z^2)^{\nu+d/2}}, \qquad z \geq 0. \tag{2.4}$$

Define the function ${}_1F_2$ as:

$${}_1F_2(a; b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k z^k}{(b)_k (c)_k k!}, \qquad z \in \mathbb{R},$$

which is a special case of the generalized hypergeometric functions ${}_qF_p$ [1], with $(q)_k = \Gamma(q + k)/\Gamma(q)$ for $k \in \mathbb{N} \cup \{0\}$, being the Pochhammer symbol. The spectral density of $\varphi_{\mu,\kappa,\beta,\sigma^2}$ for $\kappa \geq 0$ is given by [4]:

$$\widehat{\varphi}_{\mu,\kappa,\beta,\sigma^2}(z) = \sigma^2 L \beta^d {}_1F_2\Big(\lambda; \lambda + \frac{\mu}{2}, \lambda + \frac{\mu}{2} + \frac{1}{2}; -\frac{(z\beta)^2}{4}\Big), \quad z \geq 0 \tag{2.5}$$

where $\lambda = (d + 1)/2 + \kappa$, and $L = (\Gamma(2\kappa + \mu + 1)\Gamma(\kappa)\Gamma(2\kappa + d))(2^d\pi^{\frac{d}{2}}\Gamma(\kappa + \frac{d}{2})\Gamma(\mu + 2\lambda)\Gamma(2\kappa))^{-1}$.

For two given functions $g_1(x)$ and $g_2(x)$, with $g_1(x) \asymp g_2(x)$ we mean that there exist two constants c and C such that $0 < c < C < \infty$ and $c|g_2(x)| \leq |g_1(x)| \leq C|g_2(x)|$ for each x. The next result follows from [20] and describe the spectral density of the GC covariance function and its asymptotic behaviour.

Theorem 1. *Let $\mathcal{C}_{\delta,\lambda,\gamma,\sigma^2}$ be the function defined at Equation (2.2). Then, for $\gamma > 0, \sigma^2 > 0, \lambda > 0$, $\delta \in (0,2)$ and $z > 0$:*

1.
$$\widehat{\mathcal{C}}_{\delta,\lambda,\gamma,\sigma^2}(z) = -\frac{\sigma^2\gamma^{d/2+1}z^{-d}}{2^{d/2-1}\pi^{d/2+1}} Im \int_0^\infty \frac{\mathcal{K}_{(d-2)/2}(\gamma t)}{(1+\exp(i\frac{\pi\delta}{2})(t/z)^\delta)^{\lambda/\delta}} t^{d/2}\, dt,$$

2.
$$\widehat{\mathcal{C}}_{\delta,\lambda,\gamma,\sigma^2}(z) = \varrho z^{-(d+\delta)} - \mathcal{O}(z^{-(d+2\delta)}) \quad \text{for } z \to \infty,$$

3.
$$\widehat{\mathcal{C}}_{\delta,\lambda,\gamma,\sigma^2}(z) \asymp z^{-(d+\delta)} \quad \text{for } z \to \infty,$$

where $\varrho = \frac{2^\delta \sigma^2 \lambda \Gamma(\frac{\delta+d}{2})\Gamma(\frac{\delta+2}{2})\sin(\frac{\pi\delta}{2})}{\delta\gamma^\delta \pi^{\frac{d}{2}+1}}$.

The existence of the spectral density (2.3) is guaranteed if the integral on the right part of (2.3) is convergent. If the integral does not converge, a generalized covariance function should be considered and the spectral density must be defined as the Fourier transform of a covariance function in the Schwartz space of test functions [11]. [20] show that if $\lambda \in (0,d]$, *i.e.* under long range depeendence, $\widehat{\mathcal{C}}_{\delta,\lambda,\gamma,\sigma^2}(z)$ diverge when $z \to 0^+$.

3. Equivalence of Gaussian measures with Generalized Cauchy, Matérn and Generalized Wendland covariance models

Equivalence and orthogonality of probability measures are useful tools when assessing the asymptotic properties of both prediction and estimation for stochastic processes. Denote with P_i, $i = 0,1$, two probability measures defined on the same measurable space $\{\Omega, \mathcal{F}\}$. P_0 and P_1 are called equivalent (denoted $P_0 \equiv P_1$) if $P_1(A) = 1$ for any $A \in \mathcal{F}$ implies $P_0(A) = 1$ and vice versa. On the other hand, P_0 and P_1 are orthogonal (denoted $P_0 \perp P_1$) if there exists an event A such that $P_1(A) = 1$ but $P_0(A) = 0$. For a stochastic process $\{Z(\boldsymbol{s}), \boldsymbol{s} \in \mathbb{R}^d\}$, to define previous concepts, we restrict the event A to the σ-algebra generated by $\{Z(\boldsymbol{s}), \boldsymbol{s} \in D\}$ where $D \subset \mathbb{R}^d$. We emphasize this restriction by saying that the two measures are equivalent on the paths of $\{Z(\boldsymbol{s}), \boldsymbol{s} \in D\}$.

Gaussian measures are completely characterized by their mean and covariance function. We write $P(\rho)$ for a Gaussian measure with zero mean and covariance function ρ. It is well known that two Gaussian measures are either equivalent or orthogonal on the paths of $\{Z(\boldsymbol{s}), \boldsymbol{s} \in D\}$ [17].

Let $P(\rho_i)$, $i = 0,1$ be two zero mean Gaussian measures with isotropic covariance function ρ_i and associated spectral density $\widehat{\rho}_i$, $i = 0,1$, as defined through (2.3). Using results in [26] and [17], [31] has shown that, if for some $a > 0$, $\widehat{\rho}_0(z)z^a$ is bounded away from 0 and ∞ as $z \to \infty$, and for some finite and positive c,

$$\int_c^\infty z^{d-1} \left\{\frac{\widehat{\rho}_1(z) - \widehat{\rho}_0(z)}{\widehat{\rho}_0(z)}\right\}^2 \mathrm{d}z < \infty, \tag{3.1}$$

then for any bounded subset $D \subset \mathbb{R}^d$, $P(\rho_0) \equiv P(\rho_1)$ on the paths of $Z(\boldsymbol{s}), \boldsymbol{s} \in D$. For the rest of the paper, we denote with $P(\mathcal{M}_{\nu,\alpha,\sigma^2})$, $P(\varphi_{\mu,\kappa,\beta,\sigma^2})$, $P(\mathcal{C}_{\delta,\lambda,\gamma,\sigma^2})$ a zero mean Gaussian measure induced by a MT, GW and GC covariance function respectively. The following Theorem is due to [36]. It characterizes the compatibility of two MT covariance models sharing a common smoothness parameter ν.

Theorem 2. *For a given $\nu > 0$, let $P(\mathcal{M}_{\nu,\alpha_i,\sigma_i^2})$, $i = 0, 1$, be two zero mean Gaussian measures. For any bounded infinite set $D \subset \mathbb{R}^d$, $d = 1, 2, 3$, $P(\mathcal{M}_{\nu,\alpha_0,\sigma_0^2}) \equiv P(\mathcal{M}_{\nu,\alpha_1,\sigma_1^2})$ on the paths of $Z(\boldsymbol{s}), \boldsymbol{s} \in D$, if and only if*

$$\frac{\sigma_0^2}{\alpha_0^{2\nu}} = \frac{\sigma_1^2}{\alpha_1^{2\nu}}. \tag{3.2}$$

The following Theorem is a generalization of Theorem 4 in [4] and it characterizes the compatibility of two GW covariance models sharing a common smoothness parameter κ. We omit the proof since the result can be obtained using the same arguments.

Theorem 3. *For a given $\kappa \geq 0$, let $P(\varphi_{\mu_i,\kappa,\beta_i,\sigma_i^2})$, $i = 0, 1$, be two zero mean Gaussian measures and let $\mu_i > d+\kappa+1/2$. For any bounded infinite set $D \subset \mathbb{R}^d$, $d = 1, 2, 3$, $P(\varphi_{\mu_0,\kappa,\beta_0,\sigma_0^2}) \equiv P(\varphi_{\mu_1,\kappa,\beta_1,\sigma_1^2})$ on the paths of $Z(\boldsymbol{s}), \boldsymbol{s} \in D$ if and only if*

$$\frac{\sigma_0^2}{\beta_0^{2\kappa+1}}\mu_0 = \frac{\sigma_1^2}{\beta_1^{2\kappa+1}}\mu_1. \tag{3.3}$$

The first relevant result of this paper concerns the characterization of the compatibility of two GC functions sharing a common smoothness parameter.

Theorem 4. *For a given $\delta \in (d/2, 2)$, let $P(\mathcal{C}_{\delta,\lambda_i,\gamma_i,\sigma_i^2})$ $i = 0, 1$ be two zero mean Gaussian measures. For any bounded infinite set $D \subset \mathbb{R}^d$, $d = 1, 2, 3$, $P(\mathcal{C}_{\delta,\lambda_0,\gamma_0,\sigma_0^2}) \equiv P(\mathcal{C}_{\delta,\lambda_1,\gamma_1,\sigma_1^2})$ on the paths of $Z(\boldsymbol{s}), \boldsymbol{s} \in D$ if and only if*

$$\frac{\sigma_0^2}{\gamma_0^{\delta}}\lambda_0 = \frac{\sigma_1^2}{\gamma_1^{\delta}}\lambda_1. \tag{3.4}$$

Proof. Let us start with the sufficient part of the assertion. From Theorem 1 point 3, $k < z^{d+\delta}\widehat{C}_{\delta,\lambda_0,\gamma_0,\sigma_0^2}(z) < K$ as $z \to \infty$. Let

$$B(z) := \frac{\widehat{C}_{\delta,\lambda_1,\gamma_1,\sigma_1^2}(z) - \widehat{C}_{\delta,\lambda_0,\gamma_0,\sigma_0^2}(z)}{\widehat{C}_{\delta,\lambda_0,\gamma_0,\sigma_0^2}(z)}.$$

In order to prove the sufficient part, we need to find conditions such that for some positive and finite c,

$$\int_c^{\infty} z^{d-1}B^2(z)dz < \infty \tag{3.5}$$

We proceed by direct construction, and, using Theorem 1 Point 2 we find that as $z \to \infty$,

$$|B(z)| \leq \frac{z^{d+\delta}}{k} |\varrho_1 z^{-(d+\delta)} - \mathcal{O}(z^{-(d+2\delta)}) - \varrho_0 z^{-(d+\delta)} + \mathcal{O}(z^{-(d+2\delta)})|$$
$$\leq \frac{1}{k} |\varrho_1 - \varrho_0 + \mathcal{O}(z^{-\delta})|$$

where $\varrho_i = \frac{2^\delta \sigma_i^2 \lambda_i \Gamma(\frac{\delta+d}{2}) \Gamma(\frac{\delta+2}{2}) \sin(\frac{\pi\delta}{2})}{\delta \gamma_i^\delta \pi^{\frac{d}{2}+1}}$, with $i = 0, 1$.

Then we obtain,

$$\int_c^\infty z^{d-1} B^2(z) dz \leq \frac{z^{d+\delta}}{k^2} \int_c^\infty z^{d-1} \left(\varrho_1 - \varrho_0 + \mathcal{O}(z^{-\delta})\right)^2 \mathrm{d}z$$

We conclude that (3.5) is true if $\delta > d/2$ and $\varrho_0 = \varrho_1$. This last condition implies (3.4). Moreover since $\delta < 2$, the condition $\delta > d/2$ can be satisfied only for $d = 1, 2, 3$. The sufficient part of our claim is thus proved. The necessary part follows the arguments in the proof of [36]. □

An immediate consequence of Theorem 4 is that, for a fixed $\delta \in (d/2, 2)$, the parameters λ, γ and σ^2 cannot be estimated consistently. Nevertheless the microergodic parameter $\sigma^2\lambda/\gamma^\delta$ is consistently estimable. In Section 4, we establish the asymptotic properties of ML estimation associated with the microergodic parameter of the GC model.

The second relevant result of this paper give sufficient conditions for the compatibility of a GC and a MT covariance model.

Theorem 5. *For given $\delta \in (d/2, 2)$, let $P(\mathcal{C}_{\delta,\lambda_1,\gamma_1,\sigma_1^2})$ and $P(\mathcal{M}_{\nu,\alpha,\sigma_0^2})$ be two zero mean Gaussian measures. If $\nu = \delta/2$ and*

$$\frac{\sigma_0^2}{\alpha^{2\nu}} = \left(\frac{\Gamma^2(\delta/2)\sin(\pi\delta/2)}{2^{1-\delta}\pi}\right) \frac{\sigma_1^2}{\gamma_1^\delta} \lambda_1, \tag{3.6}$$

then for any bounded infinite set $D \subset \mathbb{R}^d$, $d{=}1, 2, 3$, $P(\mathcal{M}_{\nu,\alpha,\sigma_0^2}) \equiv P(\mathcal{C}_{\delta,\lambda_1,\gamma_1,\sigma_1^2})$ on the paths of $Z(\boldsymbol{s}), \boldsymbol{s} \in D$,

Proof. The spectral density of the MT model is given by:

$$\widehat{\mathcal{M}}_{\nu,\alpha,\sigma_0^2}(z) = \frac{\Gamma(\nu + d/2)}{\pi^{d/2}\Gamma(\nu)} \frac{\sigma^2 \alpha^d}{(1 + \alpha^2 z^2)^{\nu+d/2}}, \qquad z \geq 0. \tag{3.7}$$

It is known that $\widehat{\mathcal{M}}_{\nu,\alpha,\sigma_0^2}(z) z^a$ is bounded away from 0 and ∞ as $z \to \infty$ for some $a > 0$ [36]. Let

$$K(z) := \frac{\widehat{C}_{\delta,\lambda_1,\gamma_1,\sigma_1^2}(z) - \widehat{\mathcal{M}}_{\nu,\alpha,\sigma_0^2}(z)}{\widehat{\mathcal{M}}_{\nu,\alpha,\sigma_0^2}(z)}.$$

In order to prove the sufficient part we need to find conditions such that for some positive and finite c,

$$\int_c^\infty z^{d-1} K^2(z) \mathrm{d}z < \infty. \tag{3.8}$$

Let $\varrho_2^{-1} = \frac{\Gamma(\nu+d/2)\sigma_0^2\alpha^{-2\nu}}{\pi^{d/2}\Gamma(\nu)}$. Using asymptotic expansion of (3.7) and Theorem 1, point 2, as $z \to \infty$:

$$\begin{aligned}
K(z) &= \left|\varrho_2^{-1}\left[\varrho_1 z^{-(d+\delta)} - \mathcal{O}(z^{-(d+2\delta)})\right](\alpha^{-2}+z^2)^{\nu+\frac{d}{2}} - 1\right| \\
&= \left|\varrho_2^{-1}\left[\varrho_1 z^{-(d+\delta)} - \mathcal{O}(z^{-(d+2\delta)})\right] z^{2\nu+d}((\alpha z)^{-2}+1)^{\nu+\frac{d}{2}} - 1\right| \\
&= \Big|\varrho_2^{-1}\left[\varrho_1 z^{-(d+\delta)} - \mathcal{O}(z^{-(d+2\delta)})\right] z^{2\nu+d}\big[1+(\nu+d/2)(\alpha z)^{-2} \\
&\quad + \mathcal{O}(z^{-2})\big] - 1\Big| \\
&= \Big|\varrho_2^{-1}\varrho_1 z^{2\nu-\delta} - 1 + \varrho_2^{-1}\varrho_1(\nu+d/2)\alpha^{-2}z^{2\nu-\delta-2} + \mathcal{O}(z^{2\nu-\delta-2}) \\
&\quad - \mathcal{O}(z^{2\nu-2\delta}) - \mathcal{O}(z^{2\nu-2\delta-2})\Big| \\
&\le \left|\varrho_2^{-1}\varrho_1 z^{2\nu-\delta} - 1\right| + \varrho_2^{-1}\varrho_1(\nu+d/2)\alpha^{-2}z^{2\nu-\delta-2} + \mathcal{O}(z^{2\nu-2\delta}) \\
&\quad + \mathcal{O}(z^{2\nu-2\delta-2}) + \mathcal{O}(z^{2\nu-\delta-2}).
\end{aligned}$$

Then, if $2\nu = \delta$ and $\varrho_2^{-1}\varrho_1 = 1$ we obtain,

$$\int_c^\infty z^{d-1} K^2(z) \mathrm{d}z \le \int_c^\infty z^{d-1}\left((\nu+d/2)\alpha^{-2}z^{-2} + \mathcal{O}(z^{-\delta})\right)^2 \mathrm{d}z$$

and the second term of the inequality is finite for $\delta > d/2$. Moreover since $\delta < 2$, the condition $\delta > d/2$ can be satisfied only for $d = 1, 2, 3$. Then for a given $\delta \in (d/2, 2)$ and $d = 1, 2, 3$, inequality (3.8) is true if $\nu = \delta/2$ and $\varrho_2^{-1}\varrho_1 = 1$. This last two conditions implies (3.6). □

Remark I. As expected, compatibility between GC and MT covariance models is achieved only for a subset of the parametric space of ν that leads to non differentiable sample paths and in particular for $d/4 < \nu < 1$, $d = 1, 2, 3$.

The following are sufficient conditions given in [4] concerning the compatibility of a MT and a GW covariance models.

Theorem 6. *For given $\nu \ge 1/2$ and $\kappa \ge 0$, let $P(\mathcal{M}_{\nu,\alpha,\sigma_0^2})$ and $P(\varphi_{\mu,\kappa,\beta,\sigma_1^2})$ be two zero mean Gaussian measures. If $\nu = \kappa + 1/2$, $\mu > d + \kappa + 1/2$, and*

$$\frac{\sigma_0^2}{\alpha^{2\nu}} = \mu\left(\frac{\Gamma(2\kappa+\mu+1)}{\Gamma(\mu+1)}\right)\frac{\sigma_1^2}{\beta^{2\kappa+1}}, \tag{3.9}$$

then for any bounded infinite set $D \subset \mathbb{R}^d$, $d = 1, 2, 3$, $P(\mathcal{M}_{\nu,\alpha,\sigma_0^2}) \equiv P(\varphi_{\mu,\kappa,\beta,\sigma_1^2})$ on the paths of $Z(\boldsymbol{s}), \boldsymbol{s} \in D$.

Putting together Theorem 5 and Theorem 6 we obtain the next new result that establish sufficient conditions for the compatibility of a GW and GC covariance function:

Theorem 7. *For given* $\delta \in (d/2, 2) \cap [1, 2)$ *let* $P(\mathcal{C}_{\delta,\lambda,\gamma,\sigma_0^2})$ *and* $P(\varphi_{\mu,\kappa,\beta,\sigma_1^2})$ *be two zero mean Gaussian measures. If* $\kappa + 1/2 = \delta/2$, $\mu > d + \kappa + 1/2$ *and*

$$\left(\frac{\Gamma(2\kappa+\mu+1)}{\Gamma(\mu+1)}\right)\frac{\sigma_1^2}{\beta^{2\kappa+1}}\mu = \left(\frac{\Gamma^2(\delta/2)\sin(\pi\delta/2)}{2^{1-\delta}\pi}\right)\frac{\sigma_0^2}{\gamma^\delta}\lambda, \tag{3.10}$$

then for any bounded infinite set $D \subset \mathbb{R}^d$, $d = 1, 2, 3$, $P(\mathcal{C}_{\delta,\lambda,\gamma,\sigma_0^2}) \equiv P(\varphi_{\mu,\kappa,\beta,\sigma_1^2})$ *on the paths of* $Z(\boldsymbol{s}), \boldsymbol{s} \in D$.

Remark II. As expected, compatibility between GC and GW covariance models is achieved only for a subset of the parametric space of κ that leads to non differentiable sample paths and in particular $0 \le \kappa < 1/2$, $d = 1, 2$ and $1/4 \le \kappa < 1/2$, $d = 3$.

4. Asymptotic properties of the maximum likelihood estimation for the Generalized Cauchy model

We now focus on the microergodic parameter $\sigma^2\lambda/\gamma^\delta$ associated with the GC family. The following results fix the asymptotic properties of its ML estimator. In particular, we shall show that the microergodic parameter can be estimated consistently, and then assess the asymptotic distribution of the ML estimator.

Let $D \subset \mathbb{R}^d$ be a bounded subset of $\mathbb{R}^d$ and $S_n = \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_n \in D \subset \mathbb{R}^d\}$ denote any set of distinct locations. Let $\boldsymbol{Z}_n = (Z(\boldsymbol{s}_1), \ldots, Z(\boldsymbol{s}_n))'$ be a finite realization of $Z(\boldsymbol{s})$, $\boldsymbol{s} \in D$, a zero mean stationary Gaussian process with a given parametric covariance function $\sigma^2\phi(\cdot;\boldsymbol{\tau})$, with $\sigma^2 > 0$, $\boldsymbol{\tau}$ a parameter vector and ϕ a member of the family Φ_d, with $\phi(0;\boldsymbol{\tau}) = 1$.

We then write $R_n(\boldsymbol{\tau}) = [\phi(\|\boldsymbol{s}_i - \boldsymbol{s}_j\|;\boldsymbol{\tau})]_{i,j=1}^n$ for the associated correlation matrix. The Gaussian log-likelihood function is defined as:

$$\mathcal{L}_n(\sigma^2, \boldsymbol{\tau}) = -\frac{1}{2}\left(n\log(2\pi\sigma^2) + \log(|R_n(\boldsymbol{\tau})|) + \frac{1}{\sigma^2}\boldsymbol{Z}_n' R_n(\boldsymbol{\tau})^{-1}\boldsymbol{Z}_n\right). \tag{4.1}$$

Under the GC model, the Gaussian log-likelihood is obtained with $\phi(\cdot;\boldsymbol{\tau}) \equiv \mathcal{C}_{1,\lambda,\delta,\gamma}$ and $\boldsymbol{\tau} = (\lambda, \delta, \gamma)'$. Since in what follows δ and λ are assumed known and fixed, for notation convenience, we write $\tau = \gamma$. Let $\hat{\sigma}_n^2$ and $\hat{\gamma}_n$ be the maximum likelihood estimator obtained maximizing $\mathcal{L}_n(\sigma^2, \gamma)$ for fixed δ and λ.

In order to prove consistency and asymptotic Gaussianity of the microergodic parameter, we first consider an estimator that maximizes (4.1) with respect to σ^2 for a fixed arbitrary scale parameter $\gamma > 0$, obtaining the following estimator

$$\hat{\sigma}_n^2(\gamma) = \arg\max_{\sigma^2} \mathcal{L}_n(\sigma^2, \gamma) = \boldsymbol{Z}_n' R_n(\gamma)^{-1}\boldsymbol{Z}_n/n. \tag{4.2}$$

Here $R_n(\gamma)$ is the correlation matrix coming from the GC family $\mathcal{C}_{1,\lambda,\delta,\gamma}$. The following result offers some asymptotic properties of ML estimator of the migroergodic parameter $\hat{\sigma}_n^2(\gamma)\lambda/\gamma^{2\delta}$ both in terms of consistency and asymptotic distribution. The proof is omitted since it follows the same steps in [4] and [32].

Theorem 8. *Let $Z(\boldsymbol{s})$, $\boldsymbol{s} \in D$, be a zero mean Gaussian process with covariance function belonging to the GC family, i.e.* $\mathcal{C}_{\sigma_0^2,\lambda,\delta,\gamma_0}$, *with* $\delta \in (d/2, 2)$, $d = 1, 2, 3$ *and* $\lambda > d$. *Suppose* $(\sigma_0^2, \gamma_0) \in (0, \infty) \times (0, \infty)$. *For a fixed* $\gamma > 0$, *let* $\hat{\sigma}_n^2(\gamma)$ *as defined through Equation* (4.2). *Then, as* $n \to \infty$,

1. $\hat{\sigma}_n^2(\gamma)\lambda/\gamma^\delta \xrightarrow{a.s} \sigma_0^2\lambda/\gamma_0^\delta$ *and*
2. $n^{\frac{1}{2}}(\hat{\sigma}_n^2(\gamma)\lambda/\gamma^{2\delta} - \sigma_0^2\lambda/\gamma_0^\delta) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 2(\sigma_0^2\lambda/\gamma_0^\delta)^2)$.

The second type of estimation considers the joint maximization of (4.1) with respect to $(\sigma^2, \gamma) \in (0, \infty) \times I$ where $I = [\gamma_L, \gamma_U]$ and $0 < \gamma_L < \gamma_U < \infty$. The solution of this optimization problem is given by $(\hat{\sigma}_n^2(\hat{\gamma}_n), \hat{\gamma}_n)$ where

$$\hat{\sigma}_n^2(\hat{\gamma}_n) = \boldsymbol{Z}_n' R_n(\hat{\gamma}_n)^{-1}\boldsymbol{Z}_n/n$$

and $\hat{\gamma}_n = \arg\max_{\gamma \in I} \mathcal{PL}_n(\gamma)$. Here $\mathcal{PL}_n(\gamma)$ is the profile log-likelihood:

$$\mathcal{PL}_n(\gamma) = -\frac{1}{2}\left(\log(2\pi) + n\log(\hat{\sigma}_n^2(\gamma)) + \log|R_n(\gamma)| + n\right). \tag{4.3}$$

We now establish the asymptotic properties of the sequence of random variables $\hat{\sigma}_n^2(\hat{\gamma}_n)\lambda/\hat{\gamma}_n^\delta$ in a special case. The following Lemma is needed in order to establish consistency and asymptotic distribution.

Lemma 1. *For any* $\gamma_1 < \gamma_2$, $\gamma_i \in I = [\gamma_L, \gamma_U]$, $i = 1, 2$ *and* $\delta \in (0, 1]$ *and* $\lambda > d$ *then* $\hat{\sigma}_n^2(\gamma_1)/\gamma_1^\delta \leq \hat{\sigma}_n^2(\gamma_2)/\gamma_2^\delta$ *for each* n.

Proof. The proof follows [19] and [4] which use the same arguments in the MT and GW cases. Let $0 < \gamma_1 < \gamma_2$, with $\gamma_1, \gamma_2 \in I$. Then, for any $\boldsymbol{Z}_n$,

$$\hat{\sigma}_n^2(\gamma_1)/\gamma_1^\delta - \hat{\sigma}_n^2(\gamma_2)/\gamma_2^\delta = \frac{1}{n}\boldsymbol{Z}_n'(R_n(\gamma_1)^{-1}\gamma_1^{-\delta} - R_n(\gamma_2)^{-1}\gamma_2^{-\delta})\boldsymbol{Z}_n$$

is nonnegative if the matrix $R_n(\gamma_1)^{-1}\gamma_1^{-\delta} - R_n(\gamma_2)^{-1}\gamma_2^{-\delta}$ is positive semi-definite and this happens if and only if the matrix $B = R_n(\gamma_2)\gamma_2^\delta - R_n(\gamma_1)\gamma_1^\delta$ with generic element

$$B_{ij} = \gamma_2^\delta \mathcal{C}_{\delta,\lambda,\gamma_2,1}(\|\boldsymbol{s}_i - \boldsymbol{s}_j\|) - \gamma_1^\delta \mathcal{C}_{\delta,\lambda,\gamma_1,1}(\|\boldsymbol{s}_i - \boldsymbol{s}_j\|),$$

is positive semi-definite. From Theorem *3.3* of [8], this happens if $\delta \in (0, 1]$ and $\lambda > d$. □

We now establish strong consistency and asymptotic distribution of the sequence of random variables $\hat{\sigma}_n^2(\hat{\gamma}_n)\lambda/\hat{\gamma}_n^\delta$.

Theorem 9. *Let* $Z(\boldsymbol{s})$, $\boldsymbol{s} \in D \subset \mathbb{R}^d$, *be a zero mean Gaussian process with a Cauchy covariance model* $\mathcal{C}_{\sigma_0^2,\lambda,\delta,\gamma_0}$ *with* $d = 1$ *and* $\delta \in (1/2, 1]$, $\lambda > 1$ *or* $d = 2$ *and* $\delta = 1$, $\lambda > 2$ *Suppose* $(\sigma_0^2, \gamma_0) \in (0, \infty) \times I$ *where* $I = [\gamma_L, \gamma_U]$ *with* $0 < \gamma_L < \gamma_U < \infty$. *Let* $(\hat{\sigma}_n^2, \hat{\gamma}_n)'$ *maximize* (4.1) *over* $(0, \infty) \times I$. *Then as* $n \to \infty$,

1. $\hat{\sigma}_n^2(\hat{\gamma}_n)\lambda/\hat{\gamma}_n^\delta \xrightarrow{a.s} \sigma_0^2(\gamma_0)\lambda/\gamma_0^\delta$ *and*
2. $\sqrt{n}(\hat{\sigma}_n^2(\hat{\gamma}_n)\lambda/\hat{\gamma}_n^\delta - \sigma_0^2(\gamma_0)\lambda/\gamma_0^\delta) \xrightarrow{\mathcal{D}} \mathcal{N}(0, 2(\sigma_0^2(\gamma_0)\lambda/\gamma_0^\delta)^2)$.

Proof. The proof follows [19] and [4] which use the same arguments in the MT and GW cases. Let $\mathcal{G}_n(x) = \hat{\sigma}_n^2(x)/x^{\delta}$ and define the sequences $\mathcal{G}_n(\gamma_L)$ and $\mathcal{G}_n(\gamma_U)$. Since $\gamma_L \leq \hat{\gamma}_n \leq \gamma_U$ for every n, then, using Lemma 1, $\mathcal{G}_n(\gamma_L) \leq \mathcal{G}_n(\hat{\gamma}_n) \leq \mathcal{G}_n(\gamma_U)$ for all n with probability one. Combining this with Theorem 8 implies the result. □

5. Prediction using Generalized Cauchy model

We now consider prediction of a Gaussian process at a new location $\boldsymbol{s}_0$, using GC model, under fixed domain asymptotic. Specifically, we focus on two properties: asymptotic efficiency prediction and asymptotically correct estimation of prediction variance. [27] shows that both asymptotic properties hold when the Gaussian measures are equivalent. Let $P(\mathcal{C}_{\sigma_i^2,\lambda_i,\delta,\gamma_i})$, $i = 0, 1$ be two zero mean Gaussian measures. Under $P(\mathcal{C}_{\sigma_0^2,\lambda_0,\delta,\gamma_0})$, and using Theorem 4, both properties hold when $\sigma_0^2\lambda_0\gamma_0^{-\delta} = \sigma_1^2\lambda_1\gamma_1^{-\delta}$, $\delta \in (d/2, 2)$ and $d = 1, 2, 3$.

Similarly, let $P(\mathcal{M}_{\nu,\alpha,\sigma_2^2})$ and $P(\mathcal{C}_{\sigma^2,\lambda,\delta,\gamma})$ be two Gaussian measures with MT and Cauchy model. Using Theorem 5, under $P(\mathcal{M}_{\nu,\alpha,\sigma_2^2})$ both properties hold when (3.6) is true, $\delta \in (d/2, 2)$, $d = 1, 2, 3$. In addition, let $P(\varphi_{\mu,\kappa,\beta,\sigma_3^2})$ a Gaussian measure with GW model. Using Theorem 7, under $P(\varphi_{\mu,\kappa,\beta,\sigma_3^2})$ both properties hold when (3.10) is true, $\mu > d + \kappa + 1/2$, $\delta \in (d/2, 2) \cap [1, 2)$ and $d = 1, 2, 3$.

Actually, [29] gives a substantially weaker condition, based on the ratio of spectral densities, for asymptotic efficiency prediction based on the asymptotic behaviour of the ratio of the isotropic spectral densities. Now, let

$$\widehat{Z}_n(\delta,\lambda,\gamma) = \boldsymbol{c}_n(\delta,\lambda,\gamma)' R_n(\delta,\lambda,\gamma)^{-1}\boldsymbol{Z}_n \tag{5.1}$$

be the best linear unbiased predictor at an unknown location $\boldsymbol{s}_0 \in D \subset \mathbb{R}^d$, under the misspecified model $\mathcal{C}_{\delta,\lambda,\gamma,\sigma^2}$, where $\boldsymbol{c}_n(\delta,\lambda,\gamma) = [\mathcal{C}_{\delta,\lambda,\gamma,1}(\|\boldsymbol{s}_0 - \boldsymbol{s}_i)\|]_{i=1}^n$ and $R_n(\delta,\lambda,\gamma) = [\mathcal{C}_{\delta,\lambda,\gamma,1}(\|\boldsymbol{s}_i - \boldsymbol{s}_j)\|]_{i,j=1}^n$ is the correlation matrix.

If the correct model is $P(\mathcal{C}_{\delta,\lambda_0,\gamma_0,\sigma_0^2})$, then the mean squared error of the predictor is given by:

$$\begin{aligned}
&\mathrm{Var}_{\delta,\lambda_0,\gamma_0,\sigma_0^2}\left[\widehat{Z}_n(\delta,\lambda,\gamma) - Z(\boldsymbol{s}_0)\right] \\
&\quad= \sigma_0^2\Big(1 - 2\boldsymbol{c}_n(\delta,\lambda,\gamma,)' R_n(\delta,\lambda,\gamma)^{-1}\boldsymbol{c}_n(\delta,\lambda_0,\gamma_0) \\
&\qquad+ \boldsymbol{c}_n(\delta,\lambda,\gamma)' R_n(\delta,\lambda,\gamma)^{-1} R_n(\delta,\lambda_0,\gamma_0) R_n(\delta,\lambda,\gamma)^{-1}\boldsymbol{c}_n(\delta,\lambda,\gamma)\Big).
\end{aligned} \tag{5.2}$$

If $\gamma_0 = \gamma$ and $\lambda_0 = \lambda$, i.e., true and wrong models coincide, this expression simplifies to

$$\begin{aligned}
&\mathrm{Var}_{\delta,\lambda_0,\gamma_0,\sigma_0^2}\left[\widehat{Z}_n(\delta,\lambda_0,\gamma_0) - Z(\boldsymbol{s}_0)\right] \\
&\qquad= \sigma_0^2\big(1 - \boldsymbol{c}_n(\delta,\lambda_0,\gamma_0)' R_n(\delta,\lambda_0,\gamma_0)^{-1}\boldsymbol{c}_n(\delta,\lambda_0,\gamma_0)\big).
\end{aligned} \tag{5.3}$$

$\mathrm{Var}_{\nu,\alpha,\sigma_2^2}\big[\widehat{Z}_n(\delta,\lambda,\gamma) - Z(\boldsymbol{s}_0)\big]$, $\mathrm{Var}_{\nu,\alpha,\sigma_2^2}\big[\widehat{Z}_n(\nu,\alpha) - Z(\boldsymbol{s}_0)\big]$, $\mathrm{Var}_{\mu,\kappa,\beta,\sigma_3^2}\big[\widehat{Z}_n(\delta,\lambda,\gamma) - Z(\boldsymbol{s}_0)\big]$ and $\mathrm{Var}_{\mu,\kappa,\beta,\sigma_3^2}\big[\widehat{Z}_n(\mu,\kappa,\beta) - Z(\boldsymbol{s}_0)\big]$ can be similarly defined under, respectively, $P(\mathcal{M}_{\nu,\alpha,\sigma_2^2})$ and $P(\varphi_{\mu,\kappa,\beta,\sigma_3^2})$, where $\widehat{Z}_n(\nu,\alpha)$ and $\widehat{Z}_n(\mu,\kappa,\beta)$ are the best linear unbiased predictor using respectively the MT and GW models. The following results are an application of Theorems 1 and 2 of [29].

Theorem 10. *Let $P(\mathcal{C}_{\delta,\lambda_0,\gamma_0,\sigma_0^2})$, $P(\mathcal{C}_{\delta,\lambda_1,\gamma_1,\sigma_1^2})$, $P(\varphi_{\mu,\kappa,\beta,\sigma_3^2})$, $P(\mathcal{M}_{\nu,\alpha,\sigma_2^2})$ be four Gaussian probability measures on $D \subset \mathbb{R}^d$ with $\delta \in (d/2,2)$ and $d=1,2,3$. Then, for all $\boldsymbol{s}_0 \in D$:*

1. *Under $P(\mathcal{C}_{\delta,\lambda_0,\gamma_0,\sigma_0^2})$, as $n \to \infty$,*
$$\frac{\mathrm{Var}_{\delta,\lambda_0,\gamma_0,\sigma_0^2}\left[\widehat{Z}_n(\delta,\lambda_1,\gamma_1) - Z(\boldsymbol{s}_0)\right]}{\mathrm{Var}_{\delta,\lambda_0,\gamma_0,\sigma_0^2}\left[\widehat{Z}_n(\delta,\lambda_0,\gamma_0) - Z(\boldsymbol{s}_0)\right]} \longrightarrow 1, \tag{5.4}$$
for any fixed $\gamma_1 > 0$ and if $\sigma_0^2\lambda_0\gamma_0^{-\delta} = \sigma_1^2\lambda_1\gamma_1^{-\delta}$, then as $n \to \infty$,
$$\frac{\mathrm{Var}_{\delta,\lambda_1,\gamma_1,\sigma_1^2}\left[\widehat{Z}_n(\delta,\lambda_1,\gamma_1) - Z(\boldsymbol{s}_0)\right]}{\mathrm{Var}_{\delta,\lambda_0,\gamma_0,\sigma_0^2}\left[\widehat{Z}_n(\delta,\lambda_1,\gamma_1) - Z(\boldsymbol{s}_0)\right]} \longrightarrow 1. \tag{5.5}$$

2. *Under $P(\mathcal{M}_{\nu,\alpha,\sigma_2^2})$, if $\nu = \frac{\delta}{2}$ as $n \to \infty$,*
$$\frac{\mathrm{Var}_{\nu,\alpha,\sigma_2^2}\left[\widehat{Z}_n(\delta,\lambda_1,\gamma_1) - Z(\boldsymbol{s}_0)\right]}{\mathrm{Var}_{\nu,\alpha,\sigma_2^2}\left[\widehat{Z}_n(\nu,\alpha) - Z(\boldsymbol{s}_0)\right]} \longrightarrow 1, \tag{5.6}$$
for any fixed $\gamma_1 > 0$ and if $\left(\pi^{-1}2^{\delta-1}\Gamma^2(\delta/2)\sin(\pi\delta/2)\right)\sigma_1^2\lambda\gamma_1^{-\delta} = \sigma_2^2\alpha^{-2\nu}$, then as $n \to \infty$
$$\frac{\mathrm{Var}_{\delta,\lambda_1,\gamma_1,\sigma_1^2}\left[\widehat{Z}_n(\delta,\lambda_1,\gamma_1) - Z(\boldsymbol{s}_0)\right]}{\mathrm{Var}_{\nu,\alpha,\sigma_2^2}\left[\widehat{Z}_n(\delta,\lambda_1,\gamma_1) - Z(\boldsymbol{s}_0)\right]} \longrightarrow 1. \tag{5.7}$$

3. *Under $P(\varphi_{\mu,\kappa,\beta,\sigma_3^2})$, if $\kappa+1/2 = \delta/2$, $\mu > d+\kappa+1/2$ and $\delta \in (d/2,2)\cap[1,2)$ as $n \to \infty$,*
$$U_1(\beta) = \frac{\mathrm{Var}_{\mu,\kappa,\beta,\sigma_3^2}\left[\widehat{Z}_n(\delta,\lambda_1,\gamma_1) - Z(\boldsymbol{s}_0)\right]}{\mathrm{Var}_{\mu,\kappa,\beta,\sigma_3^2}\left[\widehat{Z}_n(\mu,\kappa,\beta) - Z(\boldsymbol{s}_0)\right]} \longrightarrow 1, \tag{5.8}$$
for any fixed $\gamma_1 > 0$ and if $\left(\Gamma(2\kappa+\mu+1)\Gamma^{-1}(\mu+1)\right)\sigma_3^2\mu\beta^{-(2\kappa+1)} = \left(\pi^{-1}2^{\delta-1}\Gamma^2(\delta/2)\sin(\pi\delta/2)\right)\sigma_1^2\lambda\gamma_1^{-\delta}$, then as $n \to \infty$
$$U_2 = \frac{\mathrm{Var}_{\delta,\lambda_1,\gamma_1,\sigma_1^2}\left[\widehat{Z}_n(\delta,\lambda_1,\gamma_1) - Z(\boldsymbol{s}_0)\right]}{\mathrm{Var}_{\mu,\kappa,\beta,\sigma_3^2}\left[\widehat{Z}_n(\delta,\lambda_1,\gamma_1) - Z(\boldsymbol{s}_0)\right]} \longrightarrow 1. \tag{5.9}$$

Proof. Since $\widehat{\mathcal{C}}_{\sigma^2,\lambda,\delta,\gamma}(z)$ is bounded away from zero and infinity and as $z \to \infty$,
$$\frac{\widehat{\mathcal{C}}_{\delta,\lambda_1,\gamma_1,\sigma_1^2}(z)}{\widehat{\mathcal{C}}_{\delta,\lambda_0,\gamma_0,\sigma_0^2}(z)} = \frac{\varrho_1 z^{-(d+\delta)} - \mathcal{O}(z^{-(d+2\delta)})}{\varrho_0 z^{-(d+\delta)} - \mathcal{O}(z^{-(d+2\delta)})}$$

where ϱ_i, $i = 0, 1$ are defined in the Proof of Theorem 4, then, if $\delta \in (d/2, 2)$ and $d = 1, 2, 3$

$$\lim_{z\to\infty} \frac{\widehat{\mathcal{C}}_{\delta,\lambda_1,\gamma_1,\sigma_1^2}(z)}{\widehat{\mathcal{C}}_{\delta,\lambda_0,\gamma_0,\sigma_0^2}(z)} = \frac{\varrho_1}{\varrho_0} = \frac{\sigma_1^2\lambda_1\gamma_1^{-\delta}}{\sigma_0^2\lambda_0\gamma_0^{-\delta}}, \tag{5.10}$$

and using Theorem 1 of [29], we obtain (5.4). If $\sigma_1^2\lambda_0\gamma_1^{-\delta} = \sigma_0^2\lambda_1\gamma_0^{-\delta}$, using Theorem 2 of [29], we obtain (5.5).

Similarly, since $\widehat{\mathcal{M}}_{\nu,\alpha,\sigma_2^2}(z)$ is bounded away from zero and infinity and as $z \to \infty$

$$\begin{aligned}\frac{\widehat{\mathcal{C}}_{\delta,\lambda_1,\gamma_1,\sigma_1^2}(z)}{\widehat{\mathcal{M}}_{\nu,\alpha,\sigma_2^2}(z)} &= \varrho_2^{-1}\big[\varrho_1 z^{-(d+\delta)} - \mathcal{O}(z^{-(d+2\delta)})\big](\alpha^{-2}+z^2)^{\nu+\frac{d}{2}}\\ &= \varrho_2^{-1}\varrho_1 z^{2\nu-\delta} + \varrho_2^{-1}\varrho_1(\nu+d/2)\alpha^{-2}z^{2\nu-\delta-2} + \mathcal{O}(z^{2\nu-\delta-2})\\ &\quad - \mathcal{O}(z^{2\nu-2\delta}) - \mathcal{O}(z^{2\nu-2\delta-2})\end{aligned} \tag{5.11}$$

where ϱ_2^{-1} is defined in the Proof of Theorem 5, then if $2\nu = \delta$, $\delta \in (d/2, 2)$ and $d = 1, 2, 3$

$$\lim_{z\to\infty} \frac{\widehat{\mathcal{C}}_{\delta,\lambda_1,\gamma_1,\sigma_1^2}(z)}{\widehat{\mathcal{M}}_{\nu,\alpha,\sigma_2^2}(z)} = \varrho_2^{-1}\varrho_1 = \frac{\Gamma^2(\delta/2)\sin(\pi\delta/2)\sigma_1^2\lambda_1\gamma_1^{-\delta}}{2^{1-\delta}\pi\sigma_2^2\alpha^{-2\nu}}.$$

Using Theorem 1 of [29], we obtain (5.6). If

$$\left(2^{\delta-1}\Gamma^2(\delta/2)\sin(\pi\delta/2)\pi^{-1}\right)\sigma_1^2\lambda_1\gamma_1^{-\delta} = \sigma_2^2\alpha^{-2\nu}, \tag{5.12}$$

using Theorem 2 of [29], we obtain (5.7).

Similarly, since $\widehat{\varphi}_{\mu,\kappa,\beta,\sigma_3^2}(z)$ is bounded away from zero and infinity, if $2\kappa+1 = \delta$, $\mu > d + \delta/2$, $\delta \in (d/2, 2) \cap [1, 2)$, $d = 1, 2, 3$ and using the asymptotic results on the spectral density of the GW model in [4], we have that for $F(z) := \frac{\widehat{\mathcal{C}}_{\delta,\lambda_1,\gamma_1,\sigma_1^2}(z)}{\widehat{\varphi}_{\mu,\kappa,\beta,\sigma_3^2}(z)}$ as $z \to \infty$:

$$\begin{aligned}F(z) &= \frac{(\sigma^2 L\beta^d)^{-1}(\varrho z^{-(d+\delta)} - \mathcal{O}(z^{-(d+2\delta)}))}{c_3(z\beta)^{-(d+1)-2\kappa}\{1+\mathcal{O}(z^{-2})\}+c_4(z\beta)^{-(\mu+\frac{d+1}{2}+\kappa)}\{\cos(z\beta-c_5)+\mathcal{O}(z^{-1})\}}\\ &= \frac{\varrho}{\sigma^2 L\beta^{-(2\kappa+1)}c_3} = \frac{2^{\delta-1}\Gamma^2(\delta/2)\sin(\pi\delta/2)\pi^{-1}\sigma_1^2\lambda_1\gamma_1^{-\delta}}{\Gamma(2\kappa+\mu+1)\Gamma^{-1}(\mu)\sigma_3^2\beta^{-(2\kappa+1)}}\end{aligned}$$

with ϱ defined in Theorem 1, $c_3 = \Gamma(\mu+2\lambda)\Gamma^{-1}(\mu)$ and c_4, c_5 positive constants and L defined in (2.5). Then, using Theorem 1 of [29], we obtain (5.8). If

$$\left(\frac{\Gamma(2\kappa+\mu+1)}{\Gamma(\mu+1)}\right)\sigma_3^2\mu\beta^{-(2\kappa+1)} = \left(\frac{\Gamma^2(\delta/2)\sin(\pi\delta/2)}{2^{1-\delta}\pi}\right)\sigma_1^2\lambda_1\gamma_1^{-\delta} \tag{5.13}$$

and using Theorem 2 of [29], we obtain (5.9). ◻

The implication of Point 1 is that under $P(\mathcal{C}_{\delta,\lambda_0,\gamma_0,\sigma_0^2})$, performing prediction with $P(\mathcal{C}_{\delta,\lambda_1,\gamma_1,\sigma_1^2})$ with an arbitrary $\gamma_1 > 0$ gives asymptotic prediction efficiency, if $\delta \in (d/2, 2)$, $d = 1, 2, 3$. Moreover, if $\sigma_0^2\gamma_0^{-\delta} = \sigma_1^2\gamma_1^{-\delta}$ then asymptotic prediction efficiency and asymptotically correct estimates of error variance are achieved. By virtue of Point 2, under $P(\mathcal{M}_{\nu,\alpha,\sigma_2^2})$, prediction with $\mathcal{C}_{\delta,\lambda_1,\gamma_1,\sigma_0^2}$, with an arbitrary $\gamma_1 > 0$, gives asymptotic prediction efficiency, if $\nu = \delta/2$, $\delta \in (d/2, 2)$, $d = 1, 2, 3$. Moreover if (5.12) is true then asymptotic prediction efficiency and asymptotically correct estimates of error variance are achieved. Finally, Point 3 implies that under $P(\varphi_{\mu,\kappa,\beta,\sigma_3^2})$, prediction with $P(\mathcal{C}_{\delta,\lambda_1,\gamma_1,\sigma_0^2})$, with an arbitrary $\gamma_1 > 0$, gives asymptotic prediction efficiency, if $\kappa + 1/2 = \delta/2$, $\delta \in (d/2, 2) \cap [1, 2)$, $d = 1, 2, 3$. Moreover, if (5.13) is true, then asymptotic prediction efficiency and asymptotically correct estimates of error variance are achieved.

Theorem 10 is still valid interchanging the role of the correct model with the wrong model. For instance point 3 can be rewritten as follows.

Theorem 11. *Let $P(\mathcal{C}_{\delta,\lambda_1,\gamma_1,\sigma_1^2})$, $P(\varphi_{\mu,\kappa,\beta,\sigma_3^2})$ be two Gaussian probability measures on $D \subset \mathbb{R}^d$, $d = 1, 2, 3$. Then, for all $\boldsymbol{s}_0 \in D$ and under $P(\mathcal{C}_{\delta,\lambda_1,\gamma_1,\sigma_1^2})$, if $\kappa = \frac{\delta}{2} - \frac{1}{2}$, $\mu > d + \kappa + 1/2$ and $\delta \in (d/2, 2) \cap [1, 2)$ as $n \to \infty$,*

$$U(\beta) = \frac{\mathrm{Var}_{\delta,\lambda_1,\gamma_1,\sigma_1^2}\left[\widehat{Z}_n(\mu, \kappa, \beta) - Z(\boldsymbol{s}_0)\right]}{\mathrm{Var}_{\delta,\lambda_1,\gamma_1,\sigma_1^2}\left[\widehat{Z}_n(\delta, \lambda_1, \gamma) - Z(\boldsymbol{s}_0)\right]} \longrightarrow 1, \tag{5.14}$$

for any fixed $\beta > 0$ and if (5.13) is true, then as $n \to \infty$

$$U_2 = \frac{\mathrm{Var}_{\mu,\kappa,\beta,\sigma_3^2}\left[\widehat{Z}_n(\mu, \kappa, \beta) - Z(\boldsymbol{s}_0)\right]}{\mathrm{Var}_{\delta,\lambda_1,\gamma_1,\sigma_1^2}\left[\widehat{Z}_n(\mu, \kappa, \beta) - Z(\boldsymbol{s}_0)\right]} \longrightarrow 1. \tag{5.15}$$

One remarkable implication of Theorem 11 is that when the true covariance belongs to the GC family, asymptotic efficiency prediction and asymptotically correct estimation of mean square error can be achieved, under suitable conditions, using a compactly supported GW covariance model.

6. Simulations and illustrations

The main goals of this section are twofold: on the one hand, we compare the finite sample behavior of the ML estimation of the microergodic parameter of the GC model with the asymptotic distributions given in Theorems 8 and 9. On the other hand, we compare the finite sample behavior of MSE prediction of a zero mean Gaussian process with GC covariance model, using a compatible GW covariance model (Theorem 11).

For the first goal we have considered 4000 points uniformly distributed over $[0, 1]$ and then we randomnly select a sequence of $n = 250, 500, 1000$ points. For each n we simulate using Cholesky decomposition and then we estimate with ML, 500 realizations from a zero mean Gaussian process with GC model. For the GC covariance model, $\mathcal{C}_{\delta,\lambda,\gamma_0,\sigma_0^2}$ we fix $\sigma_0^2 = 1$ and in view of Theorem 9, we fix $\delta = 0.75$ and $\lambda = 1.5$. Then we fix γ_0 such that the practical range of the GC models is 0.3, 0.6 and 0.9. For a given correlation, with practical range x, we mean that the correlation is approximatively lower than 0.05 when $r > x$.

For each simulation, we consider δ and λ as known and fixed, and we estimate with ML the variance and scale parameters, obtaining $\hat{\sigma}_i^2$ and $\hat{\gamma}_i$, $i = 1, \dots, 1000$. In order to estimate, we first maximize the profile log-likelihood (4.3) to get $\hat{\gamma}_i$. Then, we obtain $\hat{\sigma}_i^2(\hat{\gamma}_i) = \boldsymbol{z}_i' R(\hat{\gamma}_i)^{-1}\boldsymbol{z}_i/n$, where $\boldsymbol{z}_i$ is the data vector of simulation i. Optimization was carried out using the R [23] function *optimize* where the parametric space was restricted to the interval $[\varepsilon, 10\gamma_0]$ and ε is slightly larger than machine precision, about 10^{-15} here.

Using the asymptotic distributions stated in Theorems 8 and 9, Table 1 compares the sample quantiles of order $0.05, 0.25, 0.5, 0.75, 0.95$, mean and variance of $\sqrt{n/2}\big(\widehat{\sigma}_i^2(x)\gamma_0^\delta/(\sigma_0^2 x^\delta) - 1\big)$ for $x = \widehat{\gamma}_i, \gamma_0, 0.75\gamma_0, 1.25\gamma_0$ with the associated theoretical values of the standard Gaussian distribution, for $n = 500, 1000, 2000$.

As expected, the best approximation is achieved overall when using the true scale parameter, i.e., $x = \gamma_0$. In the case of $x = \widehat{\gamma}_i$, the sample distribution converge to the the asymptotic distribution given in Theorem 9 when increasing n, even if the convergence seems to be slow. Note that, for a fixed n, when increasing the practical range the convergence to the standard Gaussian distribution is faster. In particular, for $n = 2000$ and practical range equal to 0.9 the asymptotic distribution given in Theorem 9 is a satisfactory approximation of the sample distribution. When using scale parameters that are too small or too large with respect to the true compact support ($x = 0.75\gamma_0, 1.25\gamma_0$), the convergence to the asymptotic distribution given in Theorem 8 is very slow. These results are consistent with [19] and [4] and when generating confidence intervals for the microergodic parameter we strongly recommend jointly estimating variance and compact support and using the asymptotic distribution given in Theorem 9.

As for the second goal, using the results given in Theorem 11, we now compare asymptotic prediction efficiency and asymptotically correct estimation of prediction variance using ratios $U(\beta)$ and U_2 defined in (5.14) and (5.15) respectively. Specifically, we consider as true model $\mathcal{C}_{\delta,\lambda_1,\gamma_1,\sigma_1^2}$ setting $\sigma_1^2 = 1$, $\delta = 1.2, 1.8$, $\lambda_1 = 5$ and γ_1 such that the practical range is $0.3, 0.6, 0.9$. As wrong model, following the conditions in Theorem 11, we consider $\varphi_{\mu,\kappa,\beta,\sigma_3^2}$ with $\sigma_3^2 = 1$, $\kappa = (\delta - 1)/2$, $\mu = 2 + \kappa$ and the "equivalent" compact support is obtained as:

$$\beta_1^* = \left[\gamma_1^{-\delta}\frac{\sigma_1^2\lambda_1}{\sigma_3^2\mu}\frac{2^{\delta-1}sin(\pi\delta/2)\Gamma^2(\delta/2)\Gamma(\mu+1)}{\Gamma(2\kappa+\mu+1)\pi}\right]^{-1/(2\kappa+1)}.$$

For instance if $\delta = 1.2$ and γ_1 is such that the practical range is equal to 0.3 then $\beta_1^* = 0.204$. Figure 1, top left part, compares the GW and GC covariance model in this case. The right part compares the GW and GC covariance model under the same setting but with $\delta = 1.8$. In Figure 1, bottom part, we also show two realizations from a Gaussian random process with the two compatible

TABLE 1
Sample quantiles, mean and variance of $\sqrt{n/2}(\widehat{\sigma}_i^2(x)\gamma_0^\delta/(\sigma_0^2 x^\delta)-1)$, $i=1,\dots,1000$, *for* $x=\widehat{\gamma},\gamma_0,(5/4)\gamma_0,(3/4)\gamma_0$ *when* $\delta=0.75$, $\lambda=1.5$ *and* $n=500,1000,2000$, *compared with the associated theoretical values of the standard Gaussian distribution when* $d=1$. *Here* γ_0 *is chosen such that the practical range (PR) is* $0.3, 0.6, 0.9$.

PR	x	n	5%	25%	50%	75%	95%	Mean	Var
0.3	$\widehat{\gamma}$	500	-1.806	-0.715	0.052	0.898	2.011	0.115	1.366
		1000	-1.756	-0.724	-0.006	0.880	2.071	0.067	1.377
		2000	-1.749	-0.757	0.075	0.751	1.779	0.022	1.205
	γ_0	500	-1.481	-0.622	-0.012	0.746	1.647	0.055	0.998
		1000	-1.613	-0.745	-0.092	0.696	1.532	-0.040	1.018
		2000	-1.615	-0.681	-0.015	0.658	1.567	-0.016	1.005
	$\frac{5}{4}\gamma_0$	500	-0.900	-0.025	0.640	1.399	2.348	0.687	1.086
		1000	-1.083	-0.178	0.453	1.281	2.202	0.518	1.068
		2000	-1.152	-0.216	0.483	1.143	2.088	0.479	1.038
	$\frac{3}{4}\gamma_0$	500	-1.809	-0.982	-0.383	0.351	1.247	-0.320	0.951
		1000	-1.923	-1.076	-0.418	0.346	1.182	-0.375	0.990
		2000	-1.884	-0.962	-0.306	0.363	1.237	-0.314	0.987
0.6	$\widehat{\gamma}$	500	-1.685	-0.667	0.055	0.868	2.026	0.124	1.274
		1000	-1.678	-0.728	0.009	0.849	2.007	0.060	1.280
		2000	-1.692	-0.688	0.043	0.723	1.771	0.028	1.154
	γ_0	500	-1.481	-0.622	-0.012	0.746	1.647	0.055	0.998
		1000	-1.613	-0.745	-0.092	0.696	1.532	-0.040	1.018
		2000	-1.616	-0.681	-0.015	0.658	1.567	-0.016	1.005
	$\frac{5}{4}\gamma_0$	500	-1.104	-0.249	0.426	1.165	2.105	0.460	1.053
		1000	-1.281	-0.388	0.252	1.058	1.949	0.308	1.049
		2000	-1.341	-0.391	0.296	0.954	1.875	0.289	1.025
	$\frac{3}{4}\gamma_0$	500	-1.693	-0.847	-0.247	0.487	1.391	-0.187	0.968
		1000	-1.810	-0.947	-0.293	0.472	1.312	-0.250	1.000
		2000	-1.778	-0.858	-0.197	0.474	1.355	-0.200	0.994
0.9	$\widehat{\gamma}$	500	-1.654	-0.651	0.076	0.848	1.979	0.130	1.232
		1000	-1.688	-0.730	0.026	0.835	1.975	0.067	1.230
		2000	-1.656	-0.712	0.051	0.708	1.751	0.032	1.131
	γ_0	500	-1.481	-0.622	-0.012	0.746	1.647	0.055	0.998
		1000	-1.613	-0.745	-0.092	0.696	1.532	-0.040	1.018
		2000	-1.613	-0.681	-0.015	0.658	1.567	-0.016	1.005
	$\frac{5}{4}\gamma_0$	500	-1.193	-0.339	0.311	1.069	1.988	0.364	1.039
		1000	-1.360	-0.475	0.170	0.973	1.845	0.222	1.041
		2000	-1.414	-0.465	0.219	0.882	1.797	0.212	1.020
	$\frac{3}{4}\gamma_0$	500	-1.644	-0.791	-0.189	0.547	1.452	-0.130	0.975
		1000	-1.761	-0.901	-0.245	0.532	1.364	-0.199	1.004
		2000	-1.736	-0.814	-0.154	0.518	1.402	-0.154	0.997
$N(0,1)$			-1.645	-0.674	0	0.674	1.645	0	1

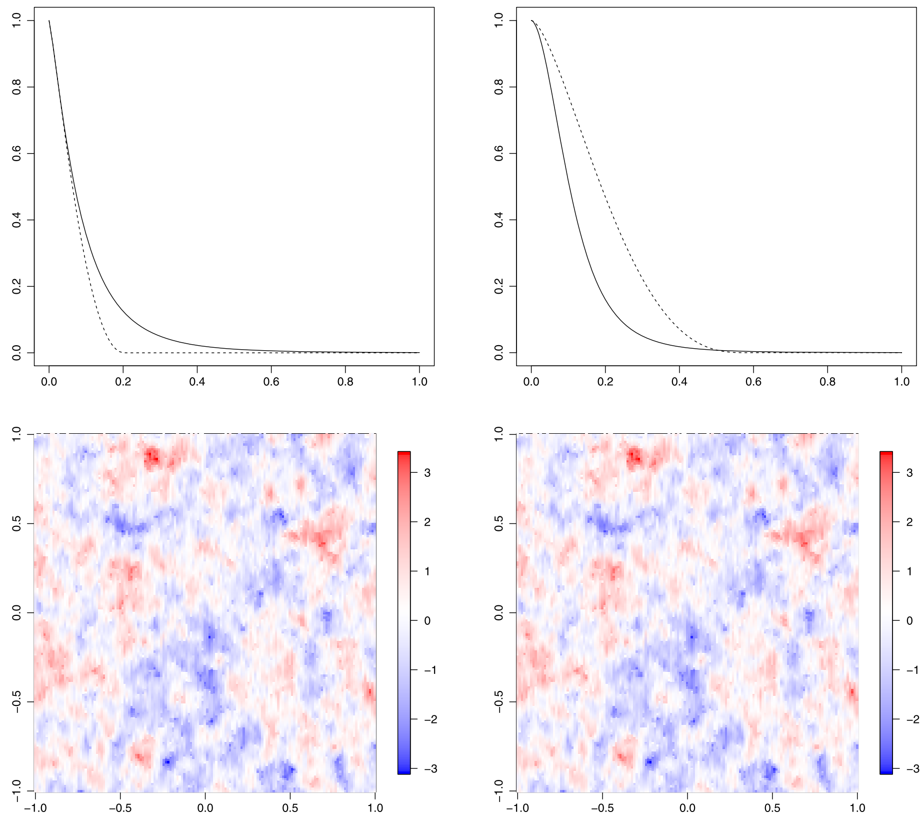

Fig 1. *Top left: a $\mathcal{C}_{1.2,5,\gamma_1,1}$ model (continous line) and a compatible $\varphi_{2.1,0.1,\beta_1^*,1}$ model (dotted line). Top right: A $\mathcal{C}_{1.8,5,\gamma_1,1}$ model (continous line) and a compatible $\varphi_{2.4,0.4,\beta_1^*,1}$ model (dotted line). In both cases γ_1 is chosen such that the practical range is 0.3 and β_1^* is computed using the equivalence condition. Bottom part: two realizations from two Gaussian random process with covariances as shown in top left part ($\mathcal{C}_{1.2,5,\gamma_1,1}$ on the left and $\varphi_{2.1,0.1,\beta_1^*,1}$ on the right).*

covariance functions shown in top left part. The two simulation are performed using cholesky decomposition and they share the same Gaussian simulation. It is apparent that the two realizations look very similar.

Then we randomly select $n_j = 50, 100, 500, 1000$, $j = 1, \dots, 100$ location sites without replacement from 5000 points uniformly distributed over $[0,1]^2$ and, for each j, we compute the ratio $U_{1j}(x\beta_1^*)$, $x = 1, 0.5, 2$ and the ratio U_{2j}, $j = 1, \dots, 500$, using closed form expressions in Equation (5.2) and (5.3) when predicting the location site $(0.26, 0.48)^T$. We consider $x = 1, 0.5, 2$ in order to investigate the effect of considering an arbitrary scale parameter on the convergence of ratio (5.14).

Table 2 shows the empirical means $\bar{U}_1(x\beta_1^*) = \sum_{j=1}^{100} U_{1j}(x\beta_1^*)/100$ for $x = 1, 0.5, 2$, and $\bar{U}_2 = \sum_{j=1}^{100} U_{2j}/100$ for n_j, $j = 1, \dots, 100$. Overall, the speed of

TABLE 2
$\bar{U}_1(x)$, $x = 0.5\beta_1^*, 2\beta_1^*, \beta_1^*$ and $\bar{U}_2$ as defined in (5.14) and (5.15), when considering a GC model with increasing practical range (PR) $(0.3, 0.6, 0.9)$, smoothness parameter $\delta = 1.2, 1.8$ and $n = 50, 100, 500, 1000$. Here β_1^* is the compact support parameter of the GW model computed using the equivalence condition.

δ		n	$PR = 0.3$	$PR = 0.6$	$PR = 0.9$
1.2	$\bar{U}_1(0.5\beta_1^*)$	50	1.164	1.256	1.155
		100	1.237	1.172	1.080
		500	1.126	1.043	1.027
		1000	1.068	1.029	1.018
	$\bar{U}_1(2\beta_1^*)$	50	1.002	1.035	1.050
		100	1.007	1.052	1.047
		500	1.055	1.036	1.024
		1000	1.047	1.025	1.016
	$\bar{U}_1(\beta_1^*)$	50	0.969	1.038	1.052
		100	0.999	1.056	1.048
		500	1.059	1.037	1.024
		1000	1.049	1.026	1.017
	$\bar{U}_2$	50	0.973	0.987	0.996
		100	0.979	0.993	0.998
		500	0.994	0.998	0.999
		1000	0.996	0.999	1.000
1.8	$\bar{U}_1(0.5\beta_1^*)$	50	2.696	2.055	1.512
		100	2.533	1.575	1.232
		500	1.423	1.084	1.035
		1000	1.215	1.040	1.018
	$\bar{U}_1(2\beta_1^*)$	50	2.812	2.045	1.508
		100	2.548	1.566	1.231
		500	1.411	1.083	1.035
		1000	1.209	1.039	1.018
	$\bar{U}_1(\beta_1^*)$	50	2.810	2.045	1.509
		100	2.550	1.566	1.231
		500	1.413	1.083	1.035
		1000	1.210	1.039	1.018
	$\bar{U}_2$	50	0.944	0.946	0.950
		100	0.958	0.947	0.973
		500	0.960	0.993	0.999
		1000	0.977	0.998	1.000

convergence for both $\bar{U}_1(x\beta_1^*)$, $x = 1, 0.5, 2$ and $\bar{U}_2$ is faster when increasing the dependence i.e. the practical range. Additionally, as expected, a too conservative choice of the arbitrary compact support ($0.5\beta_1^*$ in our simulations) deteriorates the convergence of the ratio $\bar{U}_1$. These results are consistent with the results in [4].

It is interesting to note that the speed of convergence is clearly affected by the magnitude of δ. In particular for $\delta = 1.8$ the convergence of both ratios is slower, in particular for $\bar{U}_1(x\beta_1^*)$, $x = 1, 0.5, 2$. For instance, when the practical range is equal to 0.3, $n = 1000$ is not sufficient to attain the convergence for $\bar{U}_1(x\beta_1^*)$, $x = 1, 0.5, 2$.

7. Concluding remarks

In this paper we studied estimation and prediction of Gaussian processes with covariance models belonging to the GC family, under fixed domain asymptotics. Specifically, we first characterize the equivalence of two Gaussian measures with CG models and then we establish strong consistency and asymptotic Gaussianity of the ML estimator of the associated microergodic parameter when considering both an arbitrary and an estimated scale parameter. Simulation results show that for a finite sample, the choice of an arbitrary scale parameter can result in a very poor approximation of the asymptotic distribution. These results are consistent with those in [19] in the MT case and [4] in the GW case.

We then give sufficient conditions for the equivalence of two Gaussian measures with GW and GC model and two Gaussian measures with MT and GC model and we study the consequence of these results on prediction under fixed domain asymptotics.

One remarkable consequence of our results on optimal prediction is that the mean square error prediction of a Gaussian process with a GC model can be achieved using a GW model under suitable conditions.

Then, under fixed domain asymptotics, a misspecified GW model can be used for optimal prediction when the true covariance model is GC or MT [4]. GW is an appealing model from computational point of view since the use of covariance functions with a compact support, leading to sparse matrices ([9], [18]), is a very accessible and scalable approach and well established and implemented algorithms for sparse matrices can be used when estimating the covariance parameters and/or predicting at unknown locations (e.g., [10]). An alternative strategy to produce sparse matrices is trough covariance tapering of the GC model but as outlined in [4], this kind of method is essentially an obsolete approach.

As highlighted in Section 1, the parameter δ is crucial for the differentiability at the origin and, as a consequence, for the degree of differentiability of the associated sample paths. Specifically, for $\delta = 2$, they are infinitely times differentiable and they are not differentiable for $\delta \in (0, 2)$. We do not offer results on equivalence of Gaussian measures when $\delta = 2$ and $0 < \lambda < \infty$ for the GC family. Nevertheless, it can be shown that $\mathcal{C}_{2,\lambda,\sqrt{\lambda\gamma/2},1}(r) \to e^{-r^2/\gamma}$ as $\lambda \to \infty$. This result is consistent with the MT and GW cases when considering the smoothness parameters going to infinity. Specifically, $\mathcal{M}_{\nu,\sqrt{\alpha}/(2\sqrt{\nu}),1}(r) \to e^{-r^2/\alpha}$ as $\nu \to \infty$ and $\varphi_{\mu,\kappa,g(\beta),1}(r) \to e^{-r^2/\beta}$ as $\kappa \to \infty$, where $g(\beta) = \sqrt{\beta}(\mu + 2\kappa + 1)\Gamma(\kappa + 1/2)(2\Gamma(k+1))^{-1}$ [5].

Thus, rescaled versions of GC, MT and GW converge to a squared exponential model when $\delta = 2$ and $\lambda \to \infty$, $\nu \to \infty$ and $\kappa \to \infty$ respectively. Now, let $P(\mathcal{G}_{\alpha_i,\sigma_i^2})$, $i = 0, 1$ two zero mean Gaussian measures with squared exponential covariance function. In this case $\widehat{\mathcal{G}}_{\alpha,\sigma^2}(z) = \sigma^2(\alpha/2)^{d/2}e^{-\alpha z^2/4}$ and using (3.1), it can be shown that the equivalence condition is given by $\sigma_0^2 = \sigma_1^2$, $\alpha_0 = \alpha_1$. Additionally,

$$\lim_{z\to\infty} \frac{\widehat{\mathcal{G}}_{\alpha_1,\sigma_1^2}(z)}{\widehat{\mathcal{G}}_{\alpha_0,\sigma_0^2}(z)} = \begin{cases} 0, & \text{if} \quad \alpha_1 > \alpha_0 \\ +\infty, & \text{if} \quad \alpha_1 < \alpha_0 \\ \sigma_1^2/\sigma_0^2, & \text{if} \quad \alpha_1 = \alpha_0 \end{cases}$$

and this implies that, under $P(\mathcal{G}_{\alpha_0,\sigma_0^2})$ and predicting with $P(\mathcal{G}_{\alpha_1,\sigma_1^2})$, asymptotic prediction efficiency is achieved only when $\alpha_0 = \alpha_1$ and asymptotically correct estimates of error variance are achieved under the trivial condition $\sigma_0^2 = \sigma_1^2$, $\alpha_0 = \alpha_1$.

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