Estimation of the Hurst and the stability indices of a H-self-similar stable process

Thi To Nhu Dang

The University of Danang, University of Economics,
71 Ngu Hanh Son, Danang, Vietnam
e-mail: nhudtt@due.edu.vn

and

Jacques Istas

Laboratoire Jean Kuntzmann,
Université de Grenoble Alpes et CNRS, F-38000 Grenoble, France
e-mail: jacques.istas@univ-grenoble-alpes.fr

Abstract: In this paper we estimate both the Hurst and the stability indices of a H-self-similar stable process. More precisely, let X be a H-ssi (self-similar stationary increments) symmetric α-stable process. The process X is observed at points $\frac{k}{n}$, $k = 0, \ldots, n$. Our estimate is based on β-negative power variations with $-\frac{1}{2} < \beta < 0$. We obtain consistent estimators, with rate of convergence, for several classical H-ssi α-stable processes (fractional Brownian motion, well-balanced linear fractional stable motion, Takenaka's process, Lévy motion). Moreover, we obtain asymptotic normality of our estimators for fractional Brownian motion and Lévy motion.

Keywords and phrases: H-sssi processes, stable processes, self-similarity parameter estimator, stability parameter estimator.

Contents

1. Introduction

Self-similar processes play an important role in probability because of their connection to limit theorems and they are widely used to model natural phenomena. For instance, persistent phenomena in internet traffic, hydrology, geophysics or financial markets, e.g., [9], [17], [21], are known to be self-similar. Stable processes have attracted growing interest in recent years: data with "heavy tails" have been collected in fields as diverse as economics, telecommunications, hydrology and physics of condensed matter, which suggests using non-Gaussian stable processes as possible models, e.g., [21]. Self-similar α-stable processes have been proposed to model some natural phenomena with heavy tails, as in [21] and references therein.

    The estimation of various indices of $H-$sssi $\alpha-$stable processes has been a problem studied since several decades ago and, even nowadays, it continues to be a challenge. In the case of fractional Brownian motion, the estimation of the self-similarity index H has attracted attention to many authors and many methods have been proposed for solving this problem. Among these, one can mention the quadratic variation method (see e.g. [6], [7], [9], [13]), the p-variation method (see e.g. [8], [18]), the wavelet coefficients method (see e.g. [1], [5], [14]), the log-variation method (see e.g. [9], [12]). Other references, like the works of J. Istas, recommend the use of complex variations for estimating the self-similarity index H of $H-$sssi processes, but not for estimating α, (see e.g. [11]). For linear fractional stable motions, strongly consistent estimators of the self-similarity index H, based on the discrete wavelet transform of the processes, have been proposed without requirement that α to be known, as in [2], [20], [23], [24]. Thus, regarding the estimation of the stability index α, in [3], the authors presented a wavelet estimator for linear fractional stable motions assuming that H is known. Recently, the corresponding estimation problem of the stability function and the localisability function for a class of multistable processes was considered in the discussion paper of R. Le Guével, see [15], based on some conditions that involve the consistency of the estimators. For linear multifractional stable motions, in [4], the authors presented strongly consistent estimators of the localisability

function $H(.)$ and the stability index α using wavelet coefficients when $\alpha \in (1, 2)$ and $H(.)$ is a Hölder function smooth enough, with values in a compact subinterval $[\underline{H}, \overline{H}]$ of $(1/\alpha, 1)$.

The aim of this work is to construct consistent estimators of the self-similar index H and the stable index α of H-sssi, $S\alpha S$-stable processes using a new framework. In the view of the fact that a stable random variable has a density function, β- negative power variations have expectations and covariances for $-1/2 < \beta < 0$. Our estimates are thus based on these variations. This new approach provides estimators of H and α without assumptions on the existence moments of the underlying processes. It also allows us to give an estimator for the self-similarity parameter H without assumption on α and vice versa, we can estimate the stability index α without assumption on H. In other words, using β- negative power variations $(-1/2 < \beta < 0)$, one can obtain the estimators of H and α separately. We prove the consistency and rates of convergence of the proposed estimators for H and α for the underlying processes under an assumption on the series of covariances of β-negative power variations $(-1/2 < \beta < 0)$. Then obtained results were illustrated by some classical examples: fractional Brownian motions, $S\alpha S$-stable Lévy motions, well-balanced linear fractional stable motions and Takenaka's processes. We then show that the asymptotic normality of our estimates can be ascertained for the proposed estimators when the underlying process is a fractional Brownian motion or an $S\alpha S$-stable Lévy motion.

The remainder part of this article is organized as follows: in the next section, we present the setting, the assumption and main results to construct the estimators of H and α. In Section 3, some classical examples for the obtained results in Section 2 are given: fractional Brownian motions, $S\alpha S-$stable Lévy motions, well-balanced linear fractional stable motions, Takenaka's processes. In this Section, we also show the central limit theorem for the cases of the fractional Brownian motion and the $S\alpha S-$stable Lévy motion. Finally, in Section 4, we gather all the proofs of the main results and of the illustrated examples: Subsection 4.1 contains auxiliary results on negative power variations which play an important role in the proofs in Subsection 4.2 of the main results and in the proofs in Subsection 4.3 of the results of four examples.

2. Main results

Let us recall the definition of a H-sssi process and an α- stable process (see e.g., [21]): A real-valued process X

- is *H-self-similar (H-ss)* if for all $a > 0$, $\{X(at), t \in \mathbb{R}\} \stackrel{(d)}{=} a^H \{X(t), t \in \mathbb{R}\}$,
- has *stationary increments (si)* if, for all $s \in \mathbb{R}$,

$$\{X(t + s) - X(s), t \in \mathbb{R}\} \stackrel{(d)}{=} \{X(t) - X(0), t \in \mathbb{R}\},$$

where $\stackrel{(d)}{=}$ stands for equality of finite dimensional distributions. A random variable X is said to have a symmetric α-*stable distribution* $(S\alpha S)$ if there are

parameters $\alpha \in (0,2]$ and $\sigma > 0$ such that its characteristic function has the form:

$$\mathbb{E}e^{i\theta X} = \exp\left(-\sigma^{\alpha} \mid \theta \mid^{\alpha}\right).$$

When $\sigma = 1$, a $S\alpha S$ is said to be standard. Let X be a H-sssi, $S\alpha S$ random process with $0 < \alpha \leq 2$.

Let $L \geq 1, K \geq 1$ be fixed integers, $a = (a_0, \ldots, a_K)$ be a finite sequence with exactly L vanishing first moments, that is for all $q \in \{0, \ldots, L\}$, one has

$$\sum_{k=0}^{K} k^q a_k = 0, \sum_{k=0}^{K} k^{L+1} a_k \neq 0 \tag{1}$$

with convention $0^0 = 1$. For example, here we can choose $K = L + 1$ and

$$a_k = (-1)^{L+1-k} \frac{(L+1)!}{k!(L+1-k)!}. \tag{2}$$

The increments of X with respect to the sequence a are defined by

$$\triangle_{p,n} X = \sum_{k=0}^{K} a_k X\left(\frac{k+p}{n}\right). \tag{3}$$

We define now an estimator of H. Let $\beta \in \mathbb{R}, -\frac{1}{2} < \beta < 0$, we set

$$V_n(\beta) = \frac{1}{n-K+1} \sum_{p=0}^{n-K} |\triangle_{p,n} X|^{\beta}, \tag{4}$$

$$W_n(\beta) = n^{\beta H} V_n(\beta). \tag{5}$$

Notice that $V_n(\beta)$ is the empirical mean of order β and $W_n(\beta)$ is expected to converge to its mean. The estimator of H is defined by

$$\widehat{H}_n = \frac{1}{\beta} \cdot \log_2 \frac{V_{n/2}(\beta)}{V_n(\beta)}. \tag{6}$$

We are now in position to define an estimator of α. We define first auxiliary functions $\psi_{u,v}, h_{u,v}, \varphi_{u,v}$ before introducing the estimator of α, where $u > v > 0$.

Let $\psi_{u,v} \colon \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be the function defined by

$$\psi_{u,v}(x,y) = -v \ln x + u \ln y + C(u,v), \tag{7}$$

where

$$C(u,v) = \frac{u-v}{2} \ln \pi + u \ln\left(\Gamma(1+\frac{v}{2})\right) + v \ln\left(\Gamma(\frac{1-u}{2})\right)$$
$$- v \ln\left(\Gamma(1+\frac{u}{2})\right) - u \ln\left(\Gamma(\frac{1-v}{2})\right).$$

Let $h_{u,v} \colon (0, +\infty) \to (-\infty, 0)$ be the function defined by

$$h_{u,v}(x) = u \ln\left(\Gamma(1+\frac{v}{x})\right) - v \ln\left(\Gamma(1+\frac{u}{x})\right) \tag{8}$$

We will prove later that $h_{u,v}$ is bijective.

Let $\varphi_{u,v} : \mathbb{R} \to [0, +\infty)$ be the function defined by

$$\varphi_{u,v}(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ h_{u,v}^{-1}(x) & \text{if } x < 0 \end{cases} \tag{9}$$

where $h_{u,v}$ is defined as in (8).

Let β_1, β_2 be in $\mathbb{R}$ such that $-1/2 < \beta_1 < \beta_2 < 0$. The estimator of α is defined by

$$\widehat{\alpha}_n = \varphi_{-\beta_1, -\beta_2} \left(\psi_{-\beta_1, -\beta_2}(W_n(\beta_1), W_n(\beta_2)) \right), \tag{10}$$

where $\psi_{u,v}, \varphi_{u,v}$ are defined as in (7) and (9), respectively.

With $\beta \in (-\frac{1}{2}, 0)$ fixed, we will make the following assumption: There exist a sequence $\{b_n, n \in \mathbb{N}\}$ and a constant C such that $\lim_{n \to +\infty} b_n = 0, b_{n/2} = O(b_n)$ and

$$\limsup_{n \to +\infty} \frac{1}{nb_n^2} \sum_{k \in \mathbb{Z}, |k| \leq n} |cov(|\triangle_{k,1}X|^\beta, |\triangle_{0,1}X|^\beta)| \leq C^2. \tag{11}$$

Remark 2.1. *The assumption (11) is important to prove the consistency of the estimators of the self-similarity and the stability indices. We will see its role in the main theorem below.*

Now we are in position to present our main results for the estimation of H and α, based on the assumption (11).

Theorem 2.1. *Let X be a H-sssi, $S\alpha S$ random process that satisfies assumption (11). Also, let $\beta, \beta_1, \beta_2 \in \mathbb{R}, -\frac{1}{2} < \beta < 0, -\frac{1}{2} < \beta_1 < \beta_2 < 0$ and $\widehat{H}_n, \widehat{\alpha}_n$ be defined as in (6) and (10), respectively. Then as $n \to +\infty$, one has*

$$\widehat{H}_n \xrightarrow{\mathbb{P}} H, \widehat{\alpha}_n \xrightarrow{\mathbb{P}} \alpha,$$

moreover $\widehat{H}_n - H = O_{\mathbb{P}}(b_n), \widehat{\alpha}_n - \alpha = O_{\mathbb{P}}(b_n)$, where $O_{\mathbb{P}}$ is defined by:

   • $X_n = O_{\mathbb{P}}(1)$ *iff for all $\epsilon > 0$, there exists $M > 0$ such that $\sup_n \mathbb{P}(|X_n| > M) < \epsilon$,*

   • $Y_n = O_{\mathbb{P}}(a_n)$ *means $Y_n = a_n X_n$ with $X_n = O_{\mathbb{P}}(1)$.*

See Subsection 4.2 for the proof of Theorem 2.1.

3. Examples

In this section, we study four classical examples: fractional Brownian motion, $S\alpha S$-stable Lévy motion, well-balanced linear fractional stable motion, Takenaka's process. For these, we will show in Section 4 that (11) is valid, so that the conclusion of Theorem 2.1 holds. We precise this theorem by providing the rate of convergence defined in (11) and a central limit theorem for the first two cases.

3.1. Fractional Brownian motion

Definition 3.1. *Fractional Brownian motion*
Fractional Brownian motion is a centered Gaussian process with covariance given by

$$\mathbb{E}X(t)X(s) = \frac{\mathbb{E}X(1)^2}{2}\{|s|^{2H} + |t|^{2H} - |s-t|^{2H}\}.$$

Fractional Brownian motion is a H-sssi 2-stable process (see, e.g., [9], p. 59). We will prove that the condition (11) is satisfied with $b_n = n^{-1/2}$, then the results in Theorem 2.1 are obtained. Moreover, we can obtain the asymptotic normality of the estimators of the self-similarity index H and the stability index $\alpha = 2$.

Let X be a H fractional Brownian motion with $H \in (0,1)$. We first present the variances Ξ_1, Σ_1 for the limit distributions of the central limit theorems for the estimators of H and α.

We will mimic the Breuer-Major's theorem (see e.g., Theorem 7.2.4 in [19]) to define these variances. For $\beta \in \mathbb{R}, -1/2 < \beta < 0$, let us introduce the following function

$$f_\beta(x) = \sqrt{var\triangle_{0,1}X}^\beta(|x|^\beta - \mathbb{E}|Z_0|^\beta), \tag{12}$$

where $Z_0 = \frac{\triangle_{0,1}X}{\sqrt{var\triangle_{0,1}X}}$.

Following Proposition A.1 in Appendix, we can write f_β in terms of Hermite polynomials in a unique way

$$f_\beta(x) = \sum_{q \geq d} f_{\beta,q} H_q(x), \tag{13}$$

where d is the Hermite rank of f_β and $d \geq 2$, $\sum_{q \geq d} q! f_{\beta,q}^2 < +\infty$. Let

$$\rho(r) = \frac{\sum_{p,p'=0}^{K} a_p a_{p'}|r+p-p'|^{2H}}{\sum_{p,p'=0}^{K} a_p a_{p'}|p-p'|^{2H}}, \tag{14}$$

$$\rho_1(r) = \frac{\sum_{p,p'=0}^{K} a_p a_{p'}|r+p-2p'|^{2H}}{2^H \sum_{p,p'=0}^{K} a_p a_{p'}|p-p'|^{2H}}, \tag{15}$$

$$\Gamma_1 = \begin{pmatrix} \sum_{q \geq d} q! f_{\beta,q}^2 \sum_{r \in \mathbb{Z}} \rho^q(r) & \sum_{q \geq d} q! f_{\beta,q}^2 \sum_{r \in \mathbb{Z}} \rho_1^q(r) \\ \sum_{q \geq d} q! f_{\beta,q}^2 \sum_{r \in \mathbb{Z}} \rho_1^q(r) & 2 \sum_{q \geq d} q! f_{\beta,q}^2 \sum_{r \in \mathbb{Z}} \rho^q(r) \end{pmatrix} \tag{16}$$

and $\phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$\phi(x, y) = \frac{1}{\beta} \log_2 \frac{x}{y}. \tag{17}$$

Then Ξ_1 is defined by

$$\Xi_1 = \phi'(x_0, y_0) \Gamma_1 \phi'(x_0, y_0)^t, \tag{18}$$

where

$$(x_0, y_0) = (\mathbb{E}|\triangle_{0,1} X|^\beta, \mathbb{E}|\triangle_{0,1} X|^\beta). \tag{19}$$

To define Σ_1, let $-1/2 < \beta_1 < \beta_2 < 0$, following Proposition A.1 in Appendix, we can write f_{β_1}, f_{β_2} in terms of Hermite polynomials in a unique way

$$f_{\beta_1}(x) = \sum_{q \geq d_1} f_{\beta_1, q} H_q(x), f_{\beta_2}(x) = \sum_{q \geq d_1} f_{\beta_2, q} H_q(x) \tag{20}$$

where d_1 is the minimum of the Hermite ranks of f_{β_1} and f_{β_2}, $d_1 \geq 2$ and

$$\sum_{q \geq d} q! f_{\beta_1, q}^2 < +\infty, \sum_{q \geq d} q! f_{\beta_2, q}^2 < +\infty.$$

Let

$$\Sigma_1 = \nabla_{\varphi_{-\beta_1, -\beta_2} \circ \psi_{-\beta_1, -\beta_2}}(x_1, y_1) \Gamma_2 \nabla_{\varphi_{-\beta_1, -\beta_2} \circ \psi_{-\beta_1, -\beta_2}}(x_1, y_1)^t \tag{21}$$

where $\psi_{u,v}, \varphi_{u,v}$ are defined by (7), (9) respectively, ∇ is the differential operator and

$$(x_1, y_1) = (\mathbb{E}|\triangle_{0,1} X|^{\beta_1}, \mathbb{E}|\triangle_{0,1} X|^{\beta_2}), \tag{22}$$

$$\Gamma_2 = \begin{pmatrix} \sigma_{\beta_1}^2 & \sigma_{\beta_1, \beta_2} \\ \sigma_{\beta_1, \beta_2} & \sigma_{\beta_2}^2 \end{pmatrix}, \tag{23}$$

$$\sigma_{\beta_1}^2 = \sum_{q=d_1}^{+\infty} q! f_{\beta_1, q}^2 \sum_{k \in \mathbb{Z}} \rho(k)^q, \sigma_{\beta_2}^2 = \sum_{q=d_1}^{+\infty} q! f_{\beta_2, q}^2 \sum_{k \in \mathbb{Z}} \rho(k)^q, \tag{24}$$

$$\sigma_{\beta_1, \beta_2} = \sum_{q=d_1}^{+\infty} q! f_{\beta_1, q} f_{\beta_2, q} \sum_{k \in \mathbb{Z}} \rho(k)^q. \tag{25}$$

We can now state the following theorem, which precises the results for the estimation of H and α in the case of fractional Brownian motion.

Theorem 3.1. *Let X be a fractional Brownian motion. Then*
a)

$$\widehat{H}_n - H = O_\mathbb{P}(n^{-1/2}), \widehat{\alpha}_n - 2 = O_\mathbb{P}(n^{-1/2}),$$

b)

$$\sqrt{n}(\widehat{H}_n - H) \xrightarrow{(d)} \mathcal{N}_1(0, \Xi_1)), \sqrt{n}(\widehat{\alpha}_n - 2) \xrightarrow{(d)} \mathcal{N}_1(0, \Sigma_1)$$

as $n \to +\infty$, where Ξ_1, Σ_1 are defined by (18) and (21), respectively.

See Subsection 4.3 for the proof of Theorem 3.1.

3.2. $S\alpha S$-stable Lévy motion

Definition 3.2. *$S\alpha S$-stable Lévy motion*
A stochastic process $\{X(t), t \geq 0\}$ is called (standard) $S\alpha S$-stable Lévy motion if $X(0) = 0$ (a.s.), X has independent increments and, for all $0 \leq s < t < \infty$ and for some $0 < \alpha \leq 2$, $X(t) - X(s)$ is a $S\alpha S$ random variable with characteristic function given by

$$\mathbb{E}e^{i\theta(X(t)-X(s))} = \exp\left(-(t-s)|\theta|^{\alpha}\right).$$

The condition (11) is proved to be satisfied with $b_n = n^{-1/2}$, then the results in Theorem 2.1 are ascertained. Similar to the case of fractional Brownian motion, we obtain the asymptotic normality of H and α.

The variances Ξ_2, Σ_2 for the limit distributions of the central limit theorems for the estimators of H and α are defined as follows.

Let X be a $S\alpha S-$stable Lévy motion, we define the variance for the limit distribution of the central limit theorem for the estimator of H by

$$\Xi_2 = \phi'(x_0, y_0)\Gamma_3\phi'(x_0, y_0)^t, \tag{26}$$

where $\phi(x, y), (x_0, y_0)$ are defined by (17), (19), respectively and

$$\Gamma_3 = \begin{pmatrix} \sigma_1^2 & \sigma_{1,2} \\ \sigma_{1,2} & \sigma_2^2 \end{pmatrix}, \tag{27}$$

$$\begin{aligned}
\sigma_1^2 &= var|\triangle_{0,1}X|^{\beta} + cov(|\triangle_{0,1}X|^{\beta}, |\triangle_{1,1}X|^{\beta}) \\
&\quad + 2\sum_{p=1}^{K-1}\left(cov(|\triangle_{0,1}X|^{\beta}, |\triangle_{2p,1}X|^{\beta}) + cov(|\triangle_{0,1}X|^{\beta}, |\triangle_{2p+1,1}X|^{\beta})\right) \\
&\quad + 2\sum_{p=1}^{K-1}\left(cov(|\triangle_{1,1}X|^{\beta}, |\triangle_{2p,1}X|^{\beta}) + cov(|\triangle_{1,1}X|^{\beta}, |\triangle_{2p+1,1}X|^{\beta})\right),
\end{aligned} \tag{28}$$

$$\sigma_2^2 = 2\left(var|\triangle_{0,1}X|^{\beta} + 2\sum_{p=1}^{K-1}cov(|\triangle_{0,1}X|^{\beta}, |\triangle_{p,1}X|^{\beta})\right) \tag{29}$$

$$\begin{aligned}
\sigma_{1,2} &= 2^{\beta H}\left(cov(|\triangle_{0,2}X|^{\beta}, |\triangle_{0,1}X|^{\beta}) + cov(|\triangle_{1,2}X|^{\beta}, |\triangle_{0,1}X|^{\beta})\right) \\
&\quad + 2^{\beta H}\sum_{p=1}^{K-1}\left(cov(|\triangle_{0,2}X|^{\beta}, |\triangle_{p,1}X|^{\beta}) + cov(|\triangle_{1,2}X|^{\beta}, |\triangle_{p,1}X|^{\beta})\right) \\
&\quad + 2^{\beta H}\sum_{p=1}^{K-1}\left(cov(|\triangle_{0,1}X|^{\beta}, |\triangle_{2p,2}X|^{\beta}) + cov(|\triangle_{0,1}X|^{\beta}, |\triangle_{2p+1,2}X|^{\beta})\right)
\end{aligned}$$
$$\tag{30}$$

The variance for the limit distribution of the central limit theorem for the estimator of α is defined by

$$\Sigma_2 = \nabla_{\varphi_{-\beta_1,-\beta_2}\circ\psi_{-\beta_1,-\beta_2}}(x_1, y_1)\Gamma_4\nabla_{\varphi_{-\beta_1,-\beta_2}\circ\psi_{-\beta_1,-\beta_2}}(x_1, y_1)^t \tag{31}$$

where $\psi_{u,v}, \varphi_{u,v}, (x_1, y_1)$ are defined as in (7), (9) and (22), respectively,

$$\Gamma_4 = \begin{pmatrix} \sigma_1^2 & \sigma_{1,2} \\ \sigma_{1,2} & \sigma_2^2 \end{pmatrix}, \tag{32}$$

$$\sigma_1^2 = var|\triangle_{0,1}X|^{\beta_1} + 2\sum_{k=1}^{K-1} cov(|\triangle_{0,1}X|^{\beta_1}, |\triangle_{k,1}X|^{\beta_1}), \tag{33}$$

$$\sigma_2^2 = var|\triangle_{0,1}X|^{\beta_2} + 2\sum_{k=1}^{K-1} cov(|\triangle_{0,1}X|^{\beta_2}, |\triangle_{k,1}X|^{\beta_2}) \tag{34}$$

$$\sigma_{1,2} = cov(|\triangle_{0,1}X|^{\beta_1}, |\triangle_{0,1}X|^{\beta_2})$$
$$+ \frac{1}{2}\sum_{k=1}^{K-1}\left(cov(|\triangle_{0,1}X|^{\beta_1}, |\triangle_{k,1}X|^{\beta_2}) + cov(|\triangle_{0,1}X|^{\beta_2}, |\triangle_{k,1}X|^{\beta_1})\right). \tag{35}$$

We now present the results on the asymptotic normality for the case of $S\alpha S$-stable Lévy motion.

Theorem 3.2. *Let X be a $S\alpha S$-stable Lévy motion. Then*
a)

$$\widehat{H}_n - H = O_{\mathbb{P}}(n^{-1/2}), \widehat{\alpha}_n - \alpha = O_{\mathbb{P}}(n^{-1/2})$$

b)

$$\sqrt{n}(\widehat{H}_n - H) \xrightarrow{(d)} \mathcal{N}_1(0, \Xi_2)), \sqrt{n}(\widehat{\alpha}_n - \alpha) \xrightarrow{(d)} \mathcal{N}_1(0, \Sigma_2)$$

as $n \to +\infty$, where Ξ_2, Σ_2 are defined by (26) and (31), respectively.

The proof of Theorem 3.2 is given in Subsection 4.3.

3.3. *Well-balanced linear fractional stable motion*

Definition 3.3. *Well-balanced linear fractional stable motion*
Let M be a $S\alpha S$ random measure, $0 < \alpha \le 2$, with Lebesgue control measure and consider

$$X(t) = \int_{-\infty}^{+\infty} (\mid t - x \mid^{H-1/\alpha} - \mid x \mid^{H-1/\alpha})M(dx), -\infty < t < +\infty$$

where $0 < H < 1, H \ne 1/\alpha$. The process X is called the well-balanced linear fractional stable motion. Then X is a H-sssi process (Proposition 7.4.2, [21]).

Let

$$b_n = \begin{cases} n^{-1/2} & \text{if } H < L + 1 - \frac{2}{\alpha} \\ n^{\frac{\alpha H - (L+1)\alpha}{4}} & \text{if } H > L + 1 - \frac{2}{\alpha} \\ \sqrt{\frac{\ln n}{n}} & \text{if } H = L + 1 - \frac{2}{\alpha}. \end{cases} \tag{36}$$

It is clear that $\lim_{n\to+\infty} b_n = 0$ and $b_{n/2} = O(b_n)$. We get the following results for the estimation of H and α.

Theorem 3.3. *Let* $\{X(t)\}_{t\in\mathbb{R}}$ *be a well-balanced linear fractional stable motion with* $0 < H < 1, H \neq 1/\alpha$ *and* $0 < \alpha < 2$. *Then for every* $\beta \in (-1/2, 0)$, *Theorem 2.1 is true with* b_n *defined by (36).*

See Subsection 4.3 for the proof of Theorem 3.3.

3.4. Takenaka's processes

Definition 3.4. *Takenaka's process*
Let M be a symmetric $\alpha-$ stable random measure $(0 < \alpha < 2)$ with control measure
$$m(dx, dr) = r^{\nu-2}dxdr, (0 < \nu < 1).$$
Let $t \in \mathbb{R}$, set
$$C_t = \{(x, r) \in \mathbb{R} \times \mathbb{R}^+, |x - t| \leq r\}, S_t = C_t \Delta C_0$$
where Δ denotes the symmetric difference between two sets.

Takenaka's process is defined by

$$X(t) = \int_{\mathbb{R}\times\mathbb{R}^+} \mathbb{1}_{S_t}(x, r)M(dx, dr). \tag{37}$$

Following Theorem 4 in [25], the process X is $\nu/\alpha-$sssi. Let

$$b_n = n^{\frac{\nu-1}{2}}. \tag{38}$$

We can now ascertain the following.

Theorem 3.4. *Let* $\{X_t, t \in \mathbb{R}\}$ *be a Takenaka's process defined by (37). Then for every* $\beta, \beta \in (-1/2, 0)$, *Theorem 2.1 is true with* b_n *defined by (38).*

The proof of Theorem 3.4 is given in Subsection 4.3.

4. Proofs

First, we give results on expectation of negative power variations of H-sssi, $S\alpha S$ random processes in Subsection 4.1. Then we apply these results in Subsection 4.2 to the estimation of H and α, in order to prove Theorem 2.1. Finally, we prove that Theorem 2.1 is true for four classical examples presented in Section 3.

4.1. Negative power expectation and auxiliary results

Now we present some results on expectation of negative power variations of H-sssi, $S\alpha S$ random processes proved by using theory of distribution. These results are the tools to prove assumptions (11) for four examples in Section 3 and to prove the main result on the estimation for α.

4.1.1. Auxiliary results

We start with the following lemma which confirms the existence of the expectation of β-negative power variation of a symmetric stable random variable when $\beta \in \mathbb{C}, Re(\beta) \in (-1, 0)$.

Lemma 4.1. *Let X be a $S\alpha S$ random variable, $\beta \in \mathbb{C}, Re(\beta) \in (-1, 0)$, then $|\mathbb{E}|X|^{\beta}| < +\infty$.*

The proof of Lemma 4.1 is given in Subsection 4.1.2.

The next two important results will be used to prove the condition (11) for our examples in next section. Theorem 4.1 gives a way to determine the expectation of β-negative power variation of a symmetric stable random variable whereas Theorem 4.2 helps to establish the inequality of (11) for illustrated examples.

Let (S, μ) be a measure space, $h, g \in L^{\alpha}(S, \mu)$ and M be a symmetric α-stable random measure on S with control measure μ, $\alpha \in (0, 2)$. Set

$$U = \int_S h(s)M(ds), V = \int_S g(s)M(ds). \tag{39}$$

Let

$$C_{\beta} = \frac{2^{\beta+1/2}\Gamma(\frac{\beta+1}{2})}{\Gamma(-\frac{\beta}{2})} \tag{40}$$

where $\beta \in \mathbb{C}$ such that $Re(\beta) \in (-1, 0)$.

Theorem 4.1. *For $\beta \in \mathbb{C}, Re(\beta) \in (-1, 0)$, we have*

$$\mathbb{E}|U|^{\beta} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathcal{F}T(y)\mathbb{E}e^{iUy}dy = \frac{C_{\beta}}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\mathbb{E}e^{iUy}}{|y|^{\beta+1}}dy \tag{41}$$

in the sense of distributions, where U, V are defined by (39), $T = |x|^{\beta}$ and $\mathcal{F}T$ is Fourier transform of T.

See Subsection 4.1.3 for the proof of Theorem 4.1.

Theorem 4.2. *Assume that*

$$||U||_{\alpha}^{\alpha} = \int_S |h(s)|^{\alpha}\mu(ds) = 1, ||V||_{\alpha}^{\alpha} = \int_S |g(s)|^{\alpha}\mu(ds) = 1$$

$$[U, V]_2 = \int_S |h(s)g(s)|^{\alpha/2} \le \eta < 1,$$

where U, V are defined as in (39). Then for $-1/2 < Re(\beta) < 0$, we have

$$\mathbb{E}|U|^{\beta}|V|^{\overline{\beta}} = \frac{C_{\beta}C_{\overline{\beta}}}{2\pi} \int_{\mathbb{R}^2} \frac{\mathbb{E}e^{ixU+iyV}}{|x|^{1+\beta}|y|^{1+\overline{\beta}}}dxdy. \tag{42}$$

Moreover, there exists a constant $C(\eta)$ such that

$$|cov(|U|^\beta, |V|^\beta)| \leq C(\eta) \int_S |h(s)g(s)|^{\alpha/2} ds. \tag{43}$$

The proof of Theorem 4.2 is given in Subsection 4.1.4.

The following two lemmas follow from Theorem 4.1 in which Lemma 4.2 provides an important formula to construct the estimator for α.

Lemma 4.2. *Let X be a standard $S\alpha S$ variable with $0 < \alpha \leq 2$ and $\beta \in \mathbb{C}, -1 < Re(\beta) < 0$, then*

$$\mathbb{E}|X|^\beta = \frac{2^\beta \Gamma(\frac{\beta+1}{2})\Gamma(1 - \frac{\beta}{\alpha})}{\sqrt{\pi}\Gamma(1 - \frac{\beta}{2})}. \tag{44}$$

See Subsection 4.1.5 for the proof of Lemma 4.2.

Lemma 4.3. *Let X be a $S\alpha S$ process where $0 < \alpha \leq 2$, $\beta \in \mathbb{C}, -\frac{1}{2} < Re(\beta) < 0$, then*

$$\mathbb{E}|\triangle_{0,1}X|^\beta \neq 0.$$

See Subsection 4.1.6 for the proof of Lemma 4.3. Now we will give the proofs for the latter results.

4.1.2. Proof of Lemma 4.1

Since X is a $S\alpha S$-stable random variable, X has a density function $f(x)$ that is even and continuous on $\mathbb{R}$. We first consider the case $\beta \in \mathbb{R}$ and $-1 < \beta < 0$.

For $-1 < \beta < 0$, we can write:

$$\mathbb{E}|X|^\beta = \int_\mathbb{R} |x|^\beta f(x)dx = \int_{|x|\leq 1} |x|^\beta f(x)dx + \int_{|x|\geq 1} |x|^\beta f(x)dx := A + B.$$

We have

$$A = \int_{|x|\leq 1} |x|^\beta f(x)dx \leq \sup_{|x|\leq 1} |f(x)| \int_{|x|\leq 1} |x|^\beta dx < +\infty, B = 2\int_1^\infty x^\beta f(x)dx \leq 2.$$

It follows that $\mathbb{E}|X|^\beta < +\infty$. For $\beta \in \mathbb{C}, -1 < Re(\beta) < 0$, we have

$$\left|\mathbb{E}|X|^\beta\right| = \left|\int_\mathbb{R} |x|^{a+ib} f(x)dx\right| \leq \int_\mathbb{R} |x|^{Re(\beta)} f(x)dx < +\infty.$$

Then we obtain the conclusion.

4.1.3. Proof of Theorem 4.1

To prove Theorem 4.1, we start with the following lemma.

Lemma 4.4. *For all $x \in \mathbb{R}, \beta \in \mathbb{C}, -1 < Re(\beta) < 0$, let $T(x) = |x|^\beta$. Then T has Fourier transform defined by*

$$\mathcal{F}T(y) = \frac{C_\beta}{|y|^{\beta+1}} \tag{45}$$

in the sense of distributions, where C_β is defined as in (40).

Proof. For $\beta \in \mathbb{C}, -1 < Re(\beta) < 0$, following example 5, chapter VII of [22], then T is a distribution and it has Fourier transform $\mathcal{F}T(y) = C|y|^{-(\beta+1)}$, where C is a constant. We will find C using function $k(x) = e^{-x^2/2}$. Since $T \in L^1_{loc}(\mathbb{R})$ and $k \in S(\mathbb{R})$, in the sense of distributions, we have $\langle \mathcal{F}T, k \rangle = \langle T, \mathcal{F}k \rangle$. On the other hand,

$$\mathcal{F}k(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} e^{-x^2/2} dx = e^{-y^2/2}$$

then

$$\int_{\mathbb{R}} |x|^\beta e^{-x^2/2} dx = \int_{\mathbb{R}} C|y|^{-(\beta+1)} e^{-y^2/2} dy.$$

By taking the change of variable, we obtain that

$$\int_{\mathbb{R}} |x|^\beta e^{-x^2/2} dx = 2^{\frac{\beta+1}{2}} \Gamma(\frac{\beta+1}{2}), \int_{\mathbb{R}} |y|^{-(\beta+1)} e^{-y^2/2} dy = 2^{-\beta/2} \Gamma(-\frac{\beta}{2}).$$

It follows that

$$C = \frac{2^{\beta+1/2} \Gamma(\frac{\beta+1}{2})}{\Gamma(-\frac{\beta}{2})} = C_\beta. \qquad \square$$

From Lemma 4.4, we have $\mathcal{F}T(y) = \frac{C_\beta}{|y|^{1+\beta}}$, where f is the density function of U and $C_u = 2^{u+1/2} \frac{\Gamma(\frac{u+1}{2})}{\Gamma(-\frac{u}{2})}$.

Let φ be a non-negative, even function such that

$$\varphi \in C_0^\infty(\mathbb{R}), supp\varphi \subset [-1, 1], \int_{\mathbb{R}} \varphi(y) dy = 1.$$

Set $\varphi_\epsilon(x) = \frac{\varphi(x/\epsilon)}{\epsilon}$, we will prove that $g_\epsilon = \mathcal{F}^{-1}f * \varphi_\epsilon \in S(\mathbb{R})$.

Indeed, let $\chi(x)$ be a function in $C_0^\infty(\mathbb{R})$ such that $\chi(x) = 1$ for $|x| \le 1$ and $\chi(x) = 0$ for $|x| \ge 2$.

We can write the characteristic function corresponding with the density function f as

$$e^{-\sigma^\alpha |x|^\alpha} = \sqrt{2\pi} \mathcal{F}^{-1} f(x) := g(x) = \chi(x)g(x) + (1 - \chi(x))g(x) := g_1(x) + g_2(x)$$

and

$$g * \varphi_\epsilon(x) = g_1 * \varphi_\epsilon(x) + g_2 * \varphi_\epsilon(x).$$

It is clearly that $g_1 \in L^1(\mathbb{R})$, g_1 has compact support, $\varphi_\epsilon \in C_0^\infty(R)$, so $g_1 * \varphi_\epsilon \in S(\mathbb{R})$.

We also have $g_2 * \varphi \in S(\mathbb{R})$ since g_2 and $\varphi_\epsilon(x)$ are in $S(\mathbb{R})$.

Then we get $g_\epsilon \in S(\mathbb{R})$.

We have

$$\mathcal{F}g_\epsilon(x) = \sqrt{2\pi} f(x) \mathcal{F}\varphi_\epsilon(x).$$

Since

$$\langle T, \mathcal{F}g_\epsilon \rangle = \langle \mathcal{F}T, g_\epsilon \rangle,$$

we obtain

$$\int_{\mathbb{R}} \sqrt{2\pi} |x|^\beta f(x) \mathcal{F}\varphi_\epsilon(x) dx = \int_{\mathbb{R}} \mathcal{F}T(y) \mathcal{F}^{-1} f * \varphi_\epsilon(y) dy \tag{46}$$

$$= \int_{\mathbb{R}} \mathcal{F}^{-1} f(y) \mathcal{F}T * \varphi_\epsilon(y) dy, \tag{47}$$

Here we used Fubini's theorem since $\mathcal{F}T, \mathcal{F}^{-1}f \in L^1_{loc}(\mathbb{R}), \varphi_\epsilon \in C_0^\infty(\mathbb{R})$ and φ_ϵ is an even function.

Now we will find the limits of two sides of the equation (47) when $\epsilon \to 0$. We first consider the left hand side of (47). One has

$$\lim_{\epsilon \to 0} \mathcal{F}\varphi_\epsilon(x) = \lim_{\epsilon \to 0} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-itx} \varphi_\epsilon(t) dt = \lim_{\epsilon \to 0} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-itx} \frac{\varphi(t/\epsilon)}{\epsilon} dt$$

$$= \lim_{\epsilon \to 0} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} e^{-i\epsilon ux} \varphi(u) du.$$

For $x, u \in \mathbb{R}, e^{-i\epsilon ux} \varphi(u) \to \varphi(u)$ when $\epsilon \to 0$, and

$$|e^{-i\epsilon ux} \varphi(u)| = \varphi(u), \int_{\mathbb{R}} \varphi(u) du = 1.$$

Following Lebesgue dominated convergence theorem, one gets

$$\lim_{\epsilon \to 0} \mathcal{F}\varphi_\epsilon(x) = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \varphi(u) du = \frac{1}{\sqrt{2\pi}}.$$

Therefore, for $x \neq 0$,

$$\sqrt{2\pi} |x|^\beta f(x) \mathcal{F}\varphi_\epsilon(x) \to \sqrt{2\pi} |x|^\beta f(x)$$

pointwise when $\epsilon \to 0$. We have

$$\left| |x|^\beta f(x) \mathcal{F}\varphi_\epsilon(x) \right| = \frac{1}{\sqrt{2\pi}} |x|^{Re(\beta)} f(x) \left| \int_{\mathbb{R}} e^{-itx} \frac{\varphi(t/\epsilon)}{\epsilon} dt \right|$$

$$= \frac{1}{\sqrt{2\pi}}|x|^{e(\beta)}f(x)\left|\int_{\mathbb{R}} e^{-i\epsilon ux}\varphi(u)du\right|$$

$$\leq \frac{1}{\sqrt{2\pi}}|x|^{Re(\beta)}f(x)\left|\int_{\mathbb{R}} \varphi(u)du\right| = \frac{1}{\sqrt{2\pi}}|x|^{Re(\beta)}f(x).$$

Moreover, applying Lemma 4.1, it follows that $\int_{\mathbb{R}} |x|^{Re(\beta)}f(x)dx < \infty$. Thus applying Lebesgue dominated convergence theorem, the left hand side of (47) converges to $\int_{\mathbb{R}} |x|^{Re(\beta)}f(x)dx$.

Turning back to right hand side of (47), since $\mathcal{F}T$ is continuous at $y \neq 0$ and $\mathcal{F}T \in L^1_{loc}(\mathbb{R})$, we get

$$\lim_{\epsilon \to 0} \mathcal{F}T * \varphi_\epsilon(y) = \mathcal{F}T(y)$$

for $y \in \mathbb{R}^*$. It follows that

$$\lim_{\epsilon \to 0} \mathcal{F}^{-1}f(y)\mathcal{F}T * \varphi_\epsilon(y) = \mathcal{F}^{-1}f(y)\mathcal{F}T(y)$$

pointwise almost everywhere. We have the following inequality on $\mathcal{F}T$ and φ_ϵ.

Lemma 4.5. *There exists a constant $C > 0$ such that for all $\epsilon > 0, x \neq 0$, we have*

$$|\mathcal{F}T| * \varphi_\epsilon(x) \leq C|\mathcal{F}T|(x).$$

Proof. Since $\mathcal{F}T$ and φ_ϵ are even functions, we just need to prove this lemma for $x > 0$. From the fact that φ has compact support, then there exists a constant C such that for all $x > 0$, $\varphi(x) \leq C\mathbb{1}_{[-1,1]}(x)$.

We consider first the case $x > 2\epsilon$. One has

$$|\mathcal{F}T| * \varphi_\epsilon(x) = \int_{\mathbb{R}} |\mathcal{F}T|(y)\varphi_\epsilon(x-y)dy \leq \frac{C}{\epsilon}\int_{x-\epsilon}^{x+\epsilon} |\mathcal{F}T|(y)dy$$

$$\leq 2C|\mathcal{F}T|(x-\epsilon) \leq 2C|\mathcal{F}T|(x/2) = C_1|\mathcal{F}T|(x).$$

If $x \leq 2\epsilon$, then

$$|\mathcal{F}T| * \varphi_\epsilon(x) \leq \frac{C}{\epsilon}\int_{x-\epsilon}^{x+\epsilon} |\mathcal{F}T|(y)dy \leq \frac{C}{\epsilon}\int_{-3\epsilon}^{3\epsilon} |\mathcal{F}T|(y)dy$$

$$= \frac{2C}{\epsilon}\int_0^{3\epsilon} \frac{|F_\beta|}{|y|^{1+Re(\beta)}}dy = \frac{C_1}{\epsilon}\frac{1}{(3\epsilon)^{Re(\beta)}} \leq C_2|\mathcal{F}T|(x). \qquad \square$$

Applying Lemma 4.5, then we deduce that

$$|\mathcal{F}^{-1}f(y)\mathcal{F}T * \varphi_\epsilon(y)| \leq C|\mathcal{F}^{-1}f(y)||\mathcal{F}T|(y)$$

almost everywhere. But

$$\int_{\mathbb{R}} |\mathcal{F}^{-1}f(y)||\mathcal{F}T|(y)dy = \int_{\mathbb{R}} \frac{e^{-|y|^\alpha}}{\sqrt{2\pi}} \frac{|C_\beta|}{|y|^{1+Re(\beta)}}dy = \frac{|C_\beta|}{\sqrt{2\pi}}\int_{\mathbb{R}} \frac{e^{-|y|^\alpha}}{|y|^{1+Re(\beta)}}dy < \infty$$

since $Re(\beta) \in (-1,0)$. Applying Lebesgue dominated convergence theorem again, the right hand side of (47) converges to $\int_{\mathbb{R}} \mathcal{F}^{-1}f(y)\mathcal{F}T(y)dy$. So we get (41).

4.1.4. Proof of Theorem 4.2

Let χ be in $C_0^\infty(\mathbb{R}), \chi \geq 0, \chi(x) = 1$ if $x \in [-1,1], supp\chi \in [-2,2]$. For $\epsilon > 0$, we define

$$\phi_\epsilon(x) = (1 - \chi(x/\epsilon))\chi(\epsilon x).$$

Let μ be the distribution of random vector (U, V), then μ is a probability measure on $\mathbb{R}^2$.

Let $T_1(x) = |x|^\beta, T_2(x) = |y|^{\overline{\beta}}$. Following Lemma 4.4, T_1, T_2 are distributions and have Fourier transforms

$$\mathcal{F}T_1(y) = \frac{C_\beta}{|y|^{1+\beta}}, \mathcal{F}T_2(x) = \frac{C_{\overline{\beta}}}{|x|^{1+\overline{\beta}}},$$

respectively, in the sense of distributions, where $C_u = 2^{u+1/2}\frac{\Gamma(\frac{u+1}{2})}{\Gamma(\frac{-u}{2})}$.

Set $F_{1\epsilon}(x) = T_1(x)\phi_\epsilon(x), F_{2\epsilon}(y) = T_2(y)\phi_\epsilon(y)$.
It is clearly that $F_{1\epsilon}(x) \in S(\mathbb{R}), F_{2\epsilon}(y) \in S(\mathbb{R})$.
Then $F_{1\epsilon} \otimes F_{2\epsilon}(x,y) \in S(\mathbb{R}^2)$. It follows that

$$\int_{\mathbb{R}^2} F_{1\epsilon} \otimes F_{2\epsilon}(x,y)d\mu(x,y) = \int_{\mathbb{R}^2} \mathcal{F}^{-1}(d\mu)(x,y)\mathcal{F}(F_{1\epsilon} \otimes F_{2\epsilon})(x,y)dxdy. \quad (48)$$

Now we consider the right-hand side of (48). We have

$$\mathcal{F}(F_{1\epsilon} \otimes F_{2\epsilon})(x,y) = \mathcal{F}F_{1\epsilon} \otimes \mathcal{F}F_{2\epsilon}(x,y).$$

We can write

$$F_{1\epsilon}(x) = T_1(x)\chi(\epsilon x) - T_1(x)\chi(\epsilon x)\chi(\frac{x}{\epsilon}).$$

Set $\psi(x) = \mathcal{F}\chi(x)$. One has

$$\mathcal{F}F_{1\epsilon}(x) = \frac{1}{\sqrt{2\pi}}\mathcal{F}T_1 * \psi_\epsilon - \frac{1}{2\pi}\mathcal{F}T_1 * \psi_\epsilon * \psi_{1/\epsilon}$$

$$= \mathcal{F}T_1 * (\frac{1}{\sqrt{2\pi}}\psi_\epsilon) - \frac{1}{\sqrt{2\pi}}\mathcal{F}T_1 * (\frac{1}{\sqrt{2\pi}}\psi_\epsilon) * \psi_{1/\epsilon}.$$

We will use the following lemma.

Lemma 4.6. *Let ψ be a function in the Schwartz class, $T(t) = |t|^\beta$ where $Re(\beta) \in (-1, 0)$. Then for all $x \neq 0$, there exists a constant $C > 0$ such that*

$$|T * \psi(x)| \leq C|T(x)|.$$

Proof. We denote C a running constant which may change from an occurrence to another occurrence. For $\epsilon > 0$, set

$$\psi_\epsilon(x) = \frac{1}{\epsilon}\psi(\frac{x}{\epsilon}).$$

We first prove that there exists a constant $C > 0$ such that for all $\epsilon > 0$,

$$|T * \psi_\epsilon(x)| \leq C \sup_{a>0} \frac{1}{2a} \int_{x-a}^{x+a} |T(t)|dt.$$

Let $k(y) = |T(x - y)|$, $I_\epsilon = T * \psi_\epsilon(x)$.

By taking the change of variable $u = \frac{y}{\epsilon}$, we obtain that $I_\epsilon = \int_{\mathbb{R}} k(\epsilon u)\psi(u)du$.

Set $F(x) = \int_0^x k(\epsilon u)du$, one has

$$F(x) = \frac{1}{\epsilon} \int_0^{\epsilon x} k(t)dt, F'(x) = k(\epsilon x).$$

Combining with the fact that $\lim_{x \to \infty} F(x)\psi(x) = 0$ and $F(0) = 0$, we deduce that

$$\int_0^{+\infty} k(\epsilon u)\psi(u)du = \int_0^{+\infty} F'(u)\psi(u)du = -\int_0^{+\infty} F(u)\psi'(u)du$$

$$= -\int_0^{+\infty} \left\{ \frac{1}{\epsilon u} \int_0^{\epsilon u} k(t)dt \right\} u\psi'(u)du.$$

Since $k(u) \geq 0$, it follows that

$$\frac{1}{\epsilon u} \int_0^{\epsilon u} k(t)dt \leq \sup_{a>0} \frac{1}{a} \int_{-a}^{+a} k(t)dt.$$

We also have ψ is a function in the Schwartz class, then

$$|\int_0^{+\infty} k(\epsilon u)\psi(u)du| \leq |\int_0^{+\infty} u\psi'(u)du| \sup_{a>0} \frac{1}{a} \int_{-a}^{+a} k(t)dt = C \sup_{a>0} \frac{1}{2a} \int_{-a}^{+a} k(t)dt.$$

We can get a similar bound for the integral $|\int_{-\infty}^{0} k(\epsilon u)\psi(u)du|$. Therefore we obtain

$$|I_\epsilon| \leq C \sup_{a>0} \frac{1}{2a} \int_{x-a}^{x+a} |T(t)| \, dt.$$

Taking $\epsilon = 1$, it follows that

$$|T * \psi(x)| \le C \sup_{a>0} \frac{1}{2a} \int\limits_{x-a}^{x+a} |T(t)| dt.$$

Now we will prove that there exists a constant $C > 0$ such that for all $a > 0$, then

$$\frac{1}{2a} \int\limits_{x-a}^{x+a} |T(t)| dt \le C|T(x)|.$$

We first consider the case $x > 0$.

If $x > 2a$, then $0 < \frac{x}{2} < x - a < x + a$ and $T(t)$ decreases over $[x - a, x + a]$. We get

$$\frac{1}{2a} \int\limits_{x-a}^{x+a} |T(t)| dt \le \frac{1}{2a} (x + a - (x - a)) (x - a)^{Re(\beta)}$$

$$\le \frac{(x/2)^{Re(\beta)}}{2} = C|T(x)|.$$

If $0 < x \le 2a < 3a$, then

$$\frac{1}{2a} \int\limits_{x-a}^{x+a} |T(t)| \le \frac{1}{2a} \int\limits_{-3a}^{3a} |T(t)| dt$$

$$= \frac{1}{a} \int_0^{3a} t^{Re(\beta)} dt = \frac{(3a)^{1+Re(\beta)}}{a(1 + Re(\beta))}$$

$$\le C(3a)^{Re(\beta)} \le Cx^{Re(\beta)} = C|T(x)|.$$

For the case $x < 0$, if $x \le -2a$, then $x - a < x + a < x/2 < 0$, we obtain

$$\frac{1}{2a} \int\limits_{x-a}^{x+a} |T(t)| \le \frac{(x + a - (x - a))|x + a|^{Re(\beta)}}{2a}$$

$$\le |x/2|^{Re(\beta)} = C|T(x)|.$$

If $-2a < x < 0$, then $-3a < x - a < x + a < 3a$, one gets

$$\frac{1}{2a} \int\limits_{x-a}^{x+a} |T(t)| \le \frac{1}{2a} \int\limits_{-3a}^{3a} |T(t)| dt$$

$$= \frac{1}{a} \int\limits_0^{3a} t^{Re(\beta)} \le C|x|^{Re(\beta)} = C|T(x)|.$$

One can therefore obtain the conclusion. $\square$

Since $\psi_\epsilon * \psi_{1/\epsilon} \in S(\mathbb{R})$, following Lemma 4.6, we have

$$|\mathcal{F}T_1 * (\psi_\epsilon * \psi_{1/\epsilon})(x)| \leq C|\mathcal{F}T_1(x)|.$$

Then, there exists a constant $C > 0$ such that

$$|\mathcal{F}F_{1\epsilon}(x)| \leq C|\mathcal{F}T_1(x)|.$$

In a similar way, we also get $|\mathcal{F}F_{2\epsilon}(y)| \leq C|\mathcal{F}T_2(y)|$. It follows that

$$|\mathcal{F}^{-1}(d\mu)(x,y)\mathcal{F}(F_{1\epsilon} \otimes F_{2\epsilon})(x,y)| \leq C|\mathcal{F}^{-1}(d\mu)(x,y)||\mathcal{F}T_1(x)\mathcal{F}T_2(y)|.$$

Let us recall that $\int_{\mathbb{R}} \frac{\psi(t)}{\sqrt{2\pi}} dt = \chi(0) = 1$. We will use the two following lemmas to get

$$\lim_{\epsilon \to 0} \mathcal{F}F_{1\epsilon}(x) = \mathcal{F}T_1(x), \lim_{\epsilon \to 0} \mathcal{F}F_{2\epsilon}(y) = \mathcal{F}T_2(y). \tag{49}$$

Lemma 4.7. *Let* $T(x) = |x|^\beta$ *where* $Re(\beta) \in (-1, 0)$, ψ *be a function in Schwartz class. Then almost everywhere,*

$$\lim_{\epsilon \to 0} T * \psi_{1/\epsilon}(x) = 0.$$

where $\psi_{1/\epsilon}(x) = \epsilon\psi(\epsilon x)$.

Proof. Let $x \in \mathbb{R}, x \neq 0$, we have

$$T * \psi_{1/\epsilon}(x) = \int_{\mathbb{R}} T(y)\epsilon\psi(\epsilon(x - y))dy = \int_{\mathbb{R}} |y|^\beta \epsilon\psi(\epsilon(x - y))dy.$$

By taking the change of variable $t = \epsilon y$, one gets

$$|T * \psi_{1/\epsilon}(x)| \leq \int_{\mathbb{R}} |t/\epsilon|^{Re(\beta)}|\psi(\epsilon x - t)|dt = \epsilon^{-Re(\beta)} \int_{\mathbb{R}} |t|^{Re(\beta)}|\psi(\epsilon x - t)|dt.$$

We write

$$\int_{\mathbb{R}} |t|^{Re(\beta)}|\psi(\epsilon x - t)|dt = \int_{-1}^{1} |t|^{Re(\beta)}|\psi(\epsilon x - t)|dt + \int_{|t|\geq 1} |t|^{Re(\beta)}|\psi(\epsilon x - t)|dt$$

$$:= I_1 + I_2.$$

We consider I_1 and I_2. Since ψ is a function in Schwartz class, one gets $||\psi||_\infty < \infty, ||\psi||_1 < \infty$. Then

$$I_1 \leq 2||\psi||_\infty \int_0^1 t^{Re(\beta)}dt = C < +\infty.$$

$$I_2 = \int\limits_{|t|\geq 1} |t|^{Re(\beta)}|\psi(\epsilon x - t)|dt \leq \int\limits_{|t|\geq 1} |\psi(\epsilon x - t)|dt \leq ||\psi||_1 = C < +\infty.$$

Since then $|T * \psi_{1/\epsilon}(x)| \leq C\epsilon^{-Re(\beta)} \to 0$ as $\epsilon \to 0$. It follows that $T * \psi_{1/\epsilon}(x) \to 0$ almost everywhere as $\epsilon \to 0$. $\qquad\square$

Lemma 4.8. *Let ψ be a function in Schwartz class such that $\int\limits_{\mathbb{R}} \psi(t)dt = 1$, $T(t) = |t|^\beta$ where $Re(\beta) \in (-1, 0)$. Then we have $\lim\limits_{\epsilon\to 0} T * \psi_\epsilon(x) = T(x)$ almost everywhere, where $\psi_\epsilon(x) = \frac{\psi(x/\epsilon)}{\epsilon}$.*

Proof. Let $x \in \mathbb{R}, x \neq 0$, we consider $I = T * \psi_\epsilon(x) - T(x)$. Let us recall that

$$\int\limits_{\mathbb{R}} \psi_\epsilon(t)dt = \int\limits_{\mathbb{R}} \frac{\psi(t/\epsilon)}{\epsilon}dt = \int\limits_{\mathbb{R}} \psi(u)du = 1.$$

Therefore $I = \int\limits_{\mathbb{R}} \{T(x-y) - T(x)\} \frac{1}{\epsilon}\psi(y/\epsilon)dy$.

Let $\theta > 0$ be a constant. There exists $0 < \delta < |x|$ such that for $|y| \leq \delta$, we have $|T(x-y) - T(x)| \leq \frac{\theta}{2||\psi||_1}$. Then

$$|I| \leq \int\limits_{|y|\leq\delta} |T(x-y) - T(x)|\frac{|\psi(y/\epsilon)|}{\epsilon}dy + \int\limits_{|y|\geq\delta} |T(x-y) - T(x)|\frac{|\psi(y/\epsilon)|}{\epsilon}dy$$

$$:= I_1 + I_2.$$

We have

$$I_1 \leq \frac{\theta}{2\epsilon||\psi||_1} \int\limits_{|y|\leq\delta} |\psi(y/\epsilon)|dy = \frac{\theta}{2\epsilon||\psi||_1} \int\limits_{|u|\leq\epsilon\delta} \epsilon|\psi(u)|du$$

$$\leq \frac{\theta}{2\epsilon||\psi||_1} \int\limits_{\mathbb{R}} \epsilon|\psi(u)|du = \frac{\theta}{2}.$$

Now we consider I_2. Since ψ is a function in Schwartz class, there exists a constant $C > 0$ such that for $|t| \geq 1$ then $|\psi(t)| \leq \frac{C}{t^2}$.

We choose $\epsilon > 0$ such that $\frac{\delta}{\epsilon} \geq 1$. By taking the change of variable $t = y/\epsilon$, we get

$$I_2 \leq \int\limits_{|t|\geq\frac{\delta}{\epsilon}} |T(x - \epsilon t)||\psi(t)|dt + \int\limits_{|t|\geq\frac{\delta}{\epsilon}} |T(x)||\psi(t)|dt$$

$$\leq \int\limits_{\substack{|t|\geq\frac{\delta}{\epsilon}\\|x-\epsilon t|\leq 1}} |T(x - \epsilon t)||\psi(t)|dt + \int\limits_{\substack{|t|\geq\frac{\delta}{\epsilon}\\|x-\epsilon t|\geq 1}} |T(x - \epsilon t)||\psi(t)|dt + \int\limits_{|t|\geq\frac{\delta}{\epsilon}} |T(x)||\psi(t)|dt$$

$$:= J_1 + J_2 + J_3.$$

We have

$$J_1 = \int\limits_{|t| \geq \frac{\delta}{\epsilon}, |x - \epsilon t| \leq 1} |T(x - \epsilon t)||\psi(t)|dt \leq C \int\limits_{|x - \epsilon t| \leq 1} |T(x - \epsilon t)|(\epsilon/\delta)^2 dt.$$

By taking the change of variable $u = \epsilon t - x$, one gets

$$J_1 \leq \frac{C\epsilon^2}{\delta^2} \int\limits_{|u| \leq 1} \frac{|T(u)|}{\epsilon} du = C_1 \epsilon.$$

Here C_1 is a constant depending on δ.

Let us consider J_2. Since $|T(t)| = |t|^{Re(\beta)}$ and $Re(\beta) \in (-1, 0)$, if $|x - \epsilon t| \geq 1$ we get $|T(x - \epsilon t)| \leq 1$.

Moreover $\delta/\epsilon \geq 1$, it follows that

$$J_2 \leq \int\limits_{|t| \geq \delta/\epsilon} \frac{C}{t^2} dt = C_2 \frac{\epsilon}{\delta}$$

where C_2 is a constant depending on δ. Similarly, since $\delta/\epsilon \geq 1$, we get

$$J_3 \leq |T(x)| \int\limits_{|t| \geq \delta/\epsilon} \frac{C}{t^2} dt = C_3 \epsilon$$

where C_3 is a constant depending on x, δ. So we get $I_2 \leq C\epsilon$ where C is a constant depending on x, δ. We can choose ϵ small enough to get $I_2 \leq \frac{\theta}{2}$.

Then for all $\theta > 0$, there exists ϵ_0 such that for all $0 < \epsilon < \epsilon_0$, we have $|I| \leq \theta$. Therefore we get the conclusion. $\square$

From (49), one gets

$$\lim_{\epsilon \to 0} \mathcal{F}(F_{1\epsilon} \otimes F_{2\epsilon})(x, y) = \mathcal{F}T_1(x)\mathcal{F}T_2(y) = \frac{C_\beta C_{\overline{\beta}}}{|x|^{1+\beta}|y|^{1+\overline{\beta}}}.$$

Moreover $\mathcal{F}^{-1}(d\mu)(x, y) = \frac{\mathbb{E}e^{ixU + iyV}}{2\pi}$. We use Theorem 4.1, Lemma 4.1 and the following lemma to deduce that

$$\int\limits_{\mathbb{R}^2} \frac{|\mathcal{F}^{-1}(d\mu)(x, y)|}{|x|^{1+Re(\beta)}|y|^{1+Re(\overline{\beta})}} dxdy = \int\limits_{\mathbb{R}^2} \frac{\mathbb{E}e^{ixU + iyV}}{2\pi|x|^{1+Re(\beta)}|y|^{1+Re(\overline{\beta})}} dxdy < +\infty.$$

Lemma 4.9. *Set*

$$M_{U,V}(x, y) = \mathbb{E}e^{ixU + iyV} - \mathbb{E}e^{ixU}\mathbb{E}e^{iyV}, I = \int\limits_{\mathbb{R}^2} \frac{|M_{U,V}(x, y)|}{|xy|^{1+Re(\beta)}} dxdy.$$

$$[U, V]_2 = \int\limits_S |h(s)g(s)|^{\alpha/2} \leq \eta < 1,$$

where U, V are defined as in Theorem 4.2. Then $I \leq C(\eta)[U, V]_2 < \infty$, where the constant $C(\eta)$ depends on η.

Proof. We just need to consider the integral only over $(0, +\infty) \times (0, +\infty)$. We divide this domain into four regions $(0, 1) \times (0, 1), (0, 1) \times (1, +\infty), (1, +\infty) \times (0, 1), (1, +\infty) \times (1, +\infty)$ and let I_1, I_2, I_3, I_4 be the integrals over those domains, respectively.

Over $(0, 1) \times (0, 1)$, by using inequality (3.4) in [20] we get

$$I_1 \leq \int_0^1 \int_0^1 \frac{|M_{U,V}(x, y)|}{|xy|^{1+Re(\beta)}} dx dy \leq 2 \int_0^1 \int_0^1 (xy)^{\alpha/2-1-Re(\beta)} dx dy [U, V]_2 = C[U, V]_2.$$

Over $(1, +\infty) \times (1, +\infty)$, by using inequality (3.6) in [20] and assumptions

$$||U||_\alpha^\alpha = 1, ||V||_\alpha^\alpha = 1, [U, V]_2 = \int_S |f(s)g(s)|^{\alpha/2} \leq \eta < 1,$$

we can bound the integral over this domain by

$$I_2 \leq 2 \int_1^{+\infty} \int_1^{+\infty} (xy)^{\alpha/2-1-Re(\beta)} e^{-2(1-\eta)(xy)^{\alpha/2}} dx dy [U, V]_2.$$

Here we can bound $e^{-2(1-\eta)(xy)^{\alpha/2}}$ up to a constant depending on η by $(xy)^{-p}$ for arbitrarily large $p > 0$.

So $I_2 \leq C(\eta)[U, V]_2$. Over $(0, 1) \times (1, +\infty)$, by using inequality (3.5) in [20] we obtain a bound

$$I_3 \leq 2 \int_0^1 \int_1^{+\infty} (xy)^{\alpha/2-1-Re(\beta)} e^{-(x^{\alpha/2}-y^{\alpha/2})^2} dx dy [U, V]_2$$

$$\leq 2 \int_0^1 \int_1^{+\infty} (xy)^{\alpha/2-1-Re(\beta)} e^{-(y^{\alpha/2}-1)^2} dx dy [U, V]_2$$

$$\leq C \int_1^{+\infty} (xy)^{\alpha/2-1-Re(\beta)} e^{-(y^{\alpha/2}-1)^2} dx dy [U, V]_2$$

$$= C \int_1^{+\infty} y^{\alpha/2-1-Re(\beta)} e^{-(y^{\alpha/2}-1)^2} dy [U, V]_2.$$

Since $\int_1^{+\infty} y^{\alpha/2-1-Re(\beta)} e^{-(y^{\alpha/2}-1)^2} dy < +\infty$, we get $I_3 \leq C[U, V]_2$.

A similar bound holds for I_4. Then we have the conclusion. $\qquad \square$

By Lebesgue dominated convergence theorem, as $\epsilon \to 0$, the right-hand side of (48) converges to

$$\frac{C_\beta C_{\overline{\beta}}}{2\pi} \int_{\mathbb{R}^2} \frac{\mathbb{E}e^{ixU+iyV}}{|x|^{1+\beta}|y|^{1+\overline{\beta}}} dx dy.$$

Now we consider the left-hand side of (48). Since $\lim\limits_{\epsilon \to 0} \phi_\epsilon(x) = 1$ for all $x \in \mathbb{R}$, it follows

$$\lim_{\epsilon \to 0} F_{1\epsilon}(x) F_{2\epsilon}(y) = |x|^\beta |y|^{\overline{\beta}}$$

for all $x \neq 0, y \neq 0$.

It is clear that $|F_{1\epsilon}(x)| \leq C|x|^{Re(\beta)}, |F_{2\epsilon}(y)| \leq C|y|^{Re(\overline{\beta})}$. Moreover

$$\int\limits_{\mathbb{R}^2} |x|^{Re(\beta)} |y|^{Re(\overline{\beta})} d\mu(x,y) = \left| \mathbb{E}|U|^\beta |V|^{\overline{\beta}} \right| \leq \sqrt{\mathbb{E}|U|^{2\beta} \mathbb{E}|V|^{2\overline{\beta}}} < +\infty$$

since $Re(\beta) \in (-1/2, 0)$.

We can therefore apply Lebesgue dominated convergence theorem for the left-hand side of (48). As $\epsilon \to 0$, it converges to

$$\int\limits_{\mathbb{R}^2} |x|^\beta |y|^{\overline{\beta}} d\mu(x,y) = \mathbb{E}|U|^\beta |V|^{\overline{\beta}}.$$

This proves the result (42). Now we prove (43).

Following Theorem 4.1 and (42), for $-1/2 < Re(\beta) < 0$, we get

$$\mathbb{E}|U|^\beta = \frac{C_\beta}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \frac{\mathbb{E}e^{ixU}}{|x|^{1+\beta}} dx, \mathbb{E}|V|^{\overline{\beta}} = \frac{C_{\overline{\beta}}}{\sqrt{2\pi}} \int\limits_{\mathbb{R}} \frac{\mathbb{E}e^{iyV}}{|y|^{1+\overline{\beta}}} dy$$

$$\mathbb{E}|U|^\beta |V|^{\overline{\beta}} = \frac{C_\beta C_{\overline{\beta}}}{2\pi} \int\limits_{\mathbb{R}^2} \frac{\mathbb{E}e^{ixU+iyV}}{|x|^{1+\beta}|y|^{1+\overline{\beta}}} dx dy.$$

Then

$$\left| cov(|U|^\beta, |V|^\beta) \right| = \left| \mathbb{E}|U|^\beta |V|^{\overline{\beta}} - \mathbb{E}|U|^\beta \mathbb{E}|V|^{\overline{\beta}} \right|$$

$$= \left| C_\beta C_{\overline{\beta}} \int\limits_{\mathbb{R}^2} \frac{\mathbb{E}e^{ixU+iyV} - \mathbb{E}e^{ixU}\mathbb{E}e^{iyV}}{|x|^{1+\beta}|y|^{1+\overline{\beta}}} dx dy \right|$$

$$\leq |C_\beta C_{\overline{\beta}}| \int\limits_{\mathbb{R}^2} \frac{\left| \mathbb{E}e^{ixU+iyV} - \mathbb{E}e^{ixU}\mathbb{E}e^{iyV} \right|}{|xy|^{1+Re(\beta)}} dx dy.$$

Applying Lemma 4.9, we obtain (43).

4.1.5. *Proof of Lemma 4.2*

For the case $\alpha = 2$, let Y be a standard $S2S$ variable. Then for $-1 < Re(\beta) < 0$, we have

$$\mathbb{E}|Y|^\beta = \frac{1}{2\sqrt{\pi}} \int\limits_{-\infty}^{+\infty} |x|^\beta e^{-\frac{x^2}{4}} dx = \frac{1}{\sqrt{\pi}} \int\limits_0^{+\infty} x^\beta e^{-x^2/4} dx = \frac{2^\beta}{\sqrt{\pi}} \cdot \Gamma(\frac{\beta+1}{2}).$$

Let us now consider the case $\alpha \neq 2$. Following (45) and Theorem 4.1, we have

$$\mathbb{E}|X|^\beta = \frac{C_\beta}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\mathbb{E}e^{iXy}}{|y|^{\beta+1}} dy = \frac{C_\beta}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{e^{-|y|^\alpha}}{|y|^{\beta+1}} dy = \frac{2C_\beta}{\sqrt{2\pi}} \int_0^{+\infty} \frac{e^{-y^\alpha}}{y^{\beta+1}} dy.$$

By making the change of variable $y^\alpha = t$, then

$$\mathbb{E}|X|^\beta = \frac{2C_\beta}{\sqrt{2\pi}} \int_0^{+\infty} t^{-\beta/\alpha-1} e^{-t} dt = \frac{2C_\beta}{\sqrt{2\pi}} \int_0^{+\infty} t^{-\beta/\alpha-1} e^{-t} dt = \frac{\sqrt{2}C_\beta \Gamma(-\beta/\alpha)}{\alpha\sqrt{\pi}}.$$

Since $\Gamma(x+1) = x\Gamma(x)$, one gets

$$\mathbb{E}|X|^\beta = \frac{2^\beta \Gamma(\frac{\beta+1}{2})\Gamma(1-\frac{\beta}{\alpha})}{\sqrt{\pi}\Gamma(1-\frac{\beta}{2})}.$$

4.1.6. Proof of Lemma 4.3

From the fact that X is a $S\alpha S$ process, one can write

$$\triangle_{0,1}X = \sum_{k=0}^K a_k X(k) \stackrel{(d)}{=} \sigma Y$$

where $\sigma > 0$ and Y is a standard $S\alpha S$ random variable. Then $\mathbb{E}|\triangle_{0,1}X|^\beta = \sigma^\beta \mathbb{E}|Y|^\beta$. Following Theorem 4.1, since there doesn't exist any $x \in \mathbb{C}$ such that $\Gamma(x) = 0$, we deduce that $\mathbb{E}|Y|^\beta \neq 0$.
    Thus $\mathbb{E}|\triangle_{0,1}X|^\beta \neq 0$.

4.2. Proof of Theorem 2.1

Proof of Theorem 2.1. In this proof, we shall denote by C a generic constant which may change from occurrence to occurrence.
    We will prove that $W_n(\beta) - \mathbb{E}|\triangle_{0,1}X|^\beta = O_{\mathbb{P}}(b_n)$ where b_n is defined by (11).
    Indeed, from Lemma 4.1, it follows that $\mathbb{E}|\triangle_{0,1}X|^\beta < +\infty$.
    Because of H-self similarity and stationary increment properties of X, one has

$$\triangle_{p,n}X = \sum_{k=0}^K a_k X(\frac{k+p}{n}) \stackrel{(d)}{=} \sum_{k=0}^K \frac{a_k}{n^H} X(k+p) = \sum_{k=0}^K \frac{a_k}{n^H}(X(k+p) - X(p))$$

$$\stackrel{(d)}{=} \sum_{k=0}^K \frac{a_k}{n^H} X(k) = \frac{\triangle_{0,1}X}{n^H}.$$

We get $\mathbb{E}|\triangle_{p,n}X|^\beta = \frac{\mathbb{E}|\triangle_{0,1}X|^\beta}{n^{\beta H}}$ and $\mathbb{E}W_n(\beta) = \mathbb{E}|\triangle_{0,1}X|^\beta$. Now we will prove that $W_n(\beta) \xrightarrow{(\mathbb{P})} \mathbb{E}|\triangle_{0,1}X|^\beta$ when $n \to \infty$.

We have

$$\mathbb{E}|W_n(\beta)|^2 = \frac{n^{2\beta H}}{(n-K+1)^2} \sum_{p,p'=0}^{n-K} \mathbb{E}|\triangle_{p,n}X|^\beta |\triangle_{p',n}X|^\beta.$$

Moreover

$$|\triangle_{p,n}X|^\beta |\triangle_{p',n}X|^\beta$$

$$\stackrel{(d)}{=} |\sum_{k=0}^{K} \frac{a_k}{n^H} X(k+p)|^\beta |\sum_{k=0}^{K} \frac{a_k}{n^H} X(k+p')|^\beta$$

$$= \frac{1}{n^{2\beta H}} |\sum_{k=0}^{K} a_p[X(k+p) - X(p')]|^\beta |\sum_{k=0}^{K} a_k[X(k+p') - X(p')]|^\beta$$

$$\stackrel{(d)}{=} \frac{1}{n^{2\beta H}} |\sum_{k=0}^{K} a_k X(k+p-p')|^\beta |\sum_{k=0}^{K} a_k X(k)|^\beta$$

$$= \frac{1}{n^{2\beta H}} |\triangle_{p-p',1}X|^\beta |\triangle_{0,1}X|^\beta.$$

It follows that

$$\mathbb{E}|\triangle_{p,n}X|^\beta |\triangle_{p',n}X|^\beta = \frac{\mathbb{E}|\triangle_{p-p',1}X|^\beta |\triangle_{0,1}X|^\beta}{n^{2\beta H}} = \frac{\mathbb{E}|\triangle_{k,1}X|^\beta |\triangle_{0,1}X|^\beta}{n^{2\beta H}}$$

with $k = p - p'$. Thus

$$\mathbb{E}|W_n(\beta)|^2 = \frac{1}{n-K+1} \sum_{|k| \le n-K} (1 - \frac{|k|}{n-K+1}) \mathbb{E}|\triangle_{k,1}X|^\beta |\triangle_{0,1}X|^\beta.$$

One has

$$\mathbb{E}|W_n(\beta) - \mathbb{E}|\triangle_{0,1}X|^\beta|^2 = \mathbb{E}|W_n(\beta)|^2 - \mathbb{E}|\triangle_{0,1}X|^\beta \mathbb{E}|\triangle_{0,1}X|^\beta.$$

On the other hand, since $\mathbb{E}|\triangle_{k,1}X|^\beta = \mathbb{E}|\triangle_{0,1}X|^\beta$ and

$$\frac{1}{n-K+1} \sum_{|k| \le n-K} \left(1 - \frac{|k|}{n-K+1}\right) = 1,$$

it follows that

$$\mathbb{E}|W_n(\beta) - \mathbb{E}|\triangle_{0,1}X|^\beta|^2 = \frac{\sum_{|k| \le n-K} (1 - \frac{|k|}{n-K+1}) cov(|\triangle_{k,1}X|^\beta, |\triangle_{0,1}X|^\beta)}{n-K+1}. \quad (50)$$

Using (50) and the assumption (11), one obtains

$$\limsup_n \frac{1}{b_n^2} \mathbb{E}|W_n(\beta) - E|\triangle_{0,1}X|^\beta|^2 \le \Sigma^2.$$

For all $\epsilon > 0$, applying Markov's inequality and using (50), we get

$$\sup_n P(|W_n(\beta) - \mathbb{E}|\triangle_{0,1}X|^\beta| > b_n \frac{\Sigma}{\sqrt{\epsilon}}) \leq \limsup_n \frac{\mathbb{E}|W_n(\beta) - E|\triangle_{0,1}X|^\beta|^2}{b_n^2 \frac{\Sigma^2}{\epsilon}} \leq \epsilon.$$

It follows that

$$W_n(\beta) - \mathbb{E}|\triangle_{0,1}X|^\beta = O_{\mathbb{P}}(b_n). \tag{51}$$

In a similar way, combining with the fact that $b_{n/2} = O(b_n)$, one also has

$$W_{n/2}(\beta) - \mathbb{E}|\triangle_{0,1}X|^\beta = O_{\mathbb{P}}(b_n). \tag{52}$$

Now we will prove that $\widehat{H}_n - H = O_{\mathbb{P}}(b_n)$.

Let $\phi : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$\phi(x, y) = \log_2 \frac{x}{y}. \tag{53}$$

Then $\widehat{H}_n - H = \phi(W_{n/2}(\beta), W_n(\beta))$.

We need the following lemma.

Lemma 4.10. *Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$, be differentiable at a constant vector $(a, b) \in D$. Let (X_n, Y_n) be random vectors whose ranges lie in D such that $X_n \xrightarrow{\mathbb{P}} a, Y_n \xrightarrow{\mathbb{P}} b$ and $X_n - a = O_{\mathbb{P}}(b_n), Y_n - b = O_{\mathbb{P}}(b_n)$ where $\{b_n\}_n$ is a non-negative sequence and $b_n \to 0$ as $n \to +\infty$.*

Then $f(X_n, Y_n) - f(a, b) = O_{\mathbb{P}}(b_n)$.

Proof. Since f is differentiable at (a, b), we can write

$$f(a + h_1, b + h_2) = f(a, b) + h_1 \frac{\partial f}{\partial x}(a, b) + h_2 \frac{\partial f}{\partial y}(a, b) + o(||(h_1, h_2)||).$$

as $||h|| = ||(h_1, h_2)|| \to 0$.

By applying Lemma 2.12 in [26] for

$$R(x, y) = f(a + x, b + y) - f(a, b) - x \frac{\partial f}{\partial x}(a, b) - y \frac{\partial f}{\partial y}(a, b)$$

and the sequence random vector $(X_n - a, Y_n - b)$, one gets

$$\begin{aligned}
&f(X_n, Y_n) - f(a, b) \\
&= (X_n - a)\frac{\partial f}{\partial x}(a, b) + (Y_n - a)\frac{\partial f}{\partial y}(a, b) + o_{\mathbb{P}}(||(X_n - a, Y_n - b)||) \\
&= (X_n - a)O_{\mathbb{P}}(1) + (Y_n - a)O_{\mathbb{P}}(1) + o_{\mathbb{P}}(||(X_n - a, Y_n - b)||) \\
&= b_n O_{\mathbb{P}}(1)O_{\mathbb{P}}(1) + b_n O_{\mathbb{P}}(1)O_{\mathbb{P}}(1) + b_n O_{\mathbb{P}}(1)o_{\mathbb{P}}(1) \\
&= b_n O_{\mathbb{P}}(1) + b_n O_{\mathbb{P}}(1) + b_n o_{\mathbb{P}}(1) \\
&= b_n O_{\mathbb{P}}(1). \qquad \square
\end{aligned}$$

Applying Lemma 4.10 with $f = \phi$ and vector $(\mathbb{E} \mid \triangle_{0,1}X \mid^{\beta}, \mathbb{E} \mid \triangle_{0,1}X \mid^{\beta})$, combining with (51), (52) and the fact that $\phi(\mathbb{E} \mid \triangle_{0,1}X \mid^{\beta}, \mathbb{E} \mid \triangle_{0,1}X \mid^{\beta}) = 0$, it follows that $\widehat{H}_n - H = O_{\mathbb{P}}(b_n)$.

Since $\lim_{n \to +\infty} b_n = 0$, it induces that $\lim_{n \to +\infty} \widehat{H}_n \overset{(\mathbb{P})}{=} H$.

To prove that $\widehat{\alpha}_n - \alpha = O_{\mathbb{P}}(b_n)$, we first prove that

$$h_{-\beta_1,-\beta_2}(\alpha) = \psi_{-\beta_1,-\beta_2}(\mathbb{E}|\triangle_{0,1}X|^{\beta_1}, \mathbb{E}|\triangle_{0,1}X|^{\beta_2}) \tag{54}$$

where $\psi_{u,v}, h_{u,v}$ are defined by (7), (8), respectively.

Indeed, since $\{X_t, t \in \mathbb{R}\}$ is a H-sssi $S\alpha S-$stable process, there exists a constant $\sigma > 0$ such that $\triangle_{0,1}X = \sigma Y$, where Y is the standard H-sssi, $S\alpha S$ random variable. For $\beta_1, \beta_2 \in \mathbb{R}, -1/2 < \beta_1, \beta_2 < 0$, from Lemma 4.2, we have

$$\mathbb{E}|\triangle_{0,1}X|^{\beta_1} = \sigma^{\beta_1}\mathbb{E}|Y|^{\beta_1} = \sigma^{\beta_1}\frac{2^{\beta_1}\Gamma(\frac{\beta_1+1}{2})\Gamma(1-\frac{\beta_1}{\alpha})}{\sqrt{\pi}\Gamma(1-\frac{\beta_1}{2})}.$$

Thus

$$\left(\mathbb{E}|\triangle_{0,1}X|^{\beta_1}\right)^{\beta_2} = \sigma^{\beta_1\beta_2}\left(\frac{2^{\beta_1}\Gamma(\frac{\beta_1+1}{2})\Gamma(1-\frac{\beta_1}{\alpha})}{\sqrt{\pi}\Gamma(1-\frac{\beta_1}{2})}\right)^{\beta_2}.$$

Similarly, we also get

$$\left(\mathbb{E}|\triangle_{0,1}X|^{\beta_2}\right)^{\beta_1} = \sigma^{\beta_1\beta_2}\left(\frac{2^{\beta_2}\Gamma(\frac{\beta_2+1}{2})\Gamma(1-\frac{\beta_2}{\alpha})}{\sqrt{\pi}\Gamma(1-\frac{\beta_2}{2})}\right)^{\beta_1}.$$

Moreover, from Lemma 4.3, $\mathbb{E}|\triangle_{0,1}X|^{\beta} \neq 0$ for all $-1/2 < \beta < 0$, then it induces

$$\frac{\left(\mathbb{E}|\triangle_{0,1}X|^{\beta_1}\right)^{\beta_2}}{\left(\mathbb{E}|\triangle_{0,1}X|^{\beta_2}\right)^{\beta_1}} = \frac{\pi^{\frac{\beta_1-\beta_2}{2}}\Gamma^{\beta_1}(1-\frac{\beta_2}{2})\Gamma^{\beta_2}(\frac{\beta_1+1}{2})\Gamma^{\beta_2}(1-\frac{\beta_1}{\alpha})}{\Gamma^{\beta_2}(1-\frac{\beta_1}{2})\Gamma^{\beta_1}(\frac{\beta_2+1}{2})\Gamma^{\beta_1}(1-\frac{\beta_2}{\alpha})}.$$

Taking the natural logarithm, we have

$$\beta_2\ln(\mathbb{E}|\triangle_{0,1}X|^{\beta_1}) - \beta_1\ln(\mathbb{E}|\triangle_{0,1}X|^{\beta_2})$$
$$= \frac{\beta_1-\beta_2}{2}\ln\pi + \beta_1\ln\left(\Gamma(1-\frac{\beta_2}{2})\right) + \beta_2\ln\left(\Gamma(\frac{\beta_1+1}{2})\right) + \beta_2\ln\left(\Gamma(1-\frac{\beta_1}{\alpha})\right)$$
$$- \beta_2\ln\left(\Gamma(1-\frac{\beta_1}{2})\right) - \beta_1\ln\left(\Gamma(\frac{\beta_2+1}{2})\right) - \beta_1\ln\left(\Gamma(1-\frac{\beta_2}{\alpha})\right).$$

It follows that

$$\beta_2\ln\left(\Gamma(1-\frac{\beta_1}{\alpha})\right) - \beta_1\ln\left(\Gamma(1-\frac{\beta_2}{\alpha})\right)$$
$$= \beta_2\ln(\mathbb{E}|\triangle_{0,1}X|^{\beta_1}) - \beta_1\ln(\mathbb{E}|\triangle_{0,1}X|^{\beta_2}) + \frac{\beta_2-\beta_1}{2}\ln\pi - \beta_1\ln\left(\Gamma(1-\frac{\beta_2}{2})\right)$$

$$- \beta_2 \ln \left(\Gamma(\frac{\beta_1 + 1}{2}) \right) + \beta_2 \ln \left(\Gamma(1 - \frac{\beta_1}{2}) \right) + \beta_1 \ln \left(\Gamma(\frac{\beta_2 + 1}{2}) \right)$$

or $h_{-\beta_1, -\beta_2}(\alpha) = \psi_{-\beta_1, -\beta_2}(\mathbb{E}|\triangle_{0,1}X|^{\beta_1}, \mathbb{E}|\triangle_{0,1}X|^{\beta_2})$.

From the following lemma, we can deduce that $h_{u,v}$ is a strictly increasing function on $(0, +\infty)$ and $\lim_{x \to +\infty} h_{u,v}(x) = 0$, $\lim_{x \to 0} h_{u,v}(x) = -\infty$. Moreover, there exists an inverse function

$$h_{u,v}^{-1} : (-\infty, 0) \to (0, +\infty)$$

which is continuous and on $(-\infty, 0)$.

Lemma 4.11. *Let* $0 < v < u$ *and* $g_{u,v} : (0, +\infty) \to \mathbb{R}$ *be a function defined by*

$$g_{u,v}(x) = u \ln \left(\Gamma(1 + vx) \right) - v \ln \left(\Gamma(1 + ux) \right).$$

Then $g_{u,v}$ *is a strictly decreasing function on* $(0, +\infty)$ *and*

$$\lim_{x \to 0} g_{u,v}(x) = 0, \lim_{x \to +\infty} g_{u,v}(x) = -\infty.$$

Proof. We have

$$g'_{u,v}(x) = uv \frac{\Gamma'(1 + vx)}{\Gamma(1 + vx)} - uv \frac{\Gamma'(1 + ux)}{\Gamma(1 + ux)} = uv \left(\frac{\Gamma'(1 + vx)}{\Gamma(1 + vx)} - \frac{\Gamma'(1 + ux)}{\Gamma(1 + ux)} \right).$$

Following Bohn-Mollerup's theorem, Γ is a log-convex function. Let $k(y)$ be defined by $k(y) = \ln \Gamma(y)$. Then $k''(y) \geq 0$ for all $y > 0$. It follows that $\psi(y) := k'(y) = \frac{\Gamma'(y)}{\Gamma(y)}$ is an increasing function.

Since $\Gamma(x + 1) = x\Gamma(x)$, we have $\Gamma'(x + 1) = \Gamma(x) + x\Gamma'(x)$. We obtain

$$\psi(x + 1) = \frac{\Gamma'(x + 1)}{\Gamma(x + 1)} = \frac{\Gamma(x) + x\Gamma'(x)}{x\Gamma(x)} = \frac{1}{x} + \psi(x).$$

We will prove that ψ increases strictly.

Assume that there exist x_0, y_0 such that $0 < x_0 < y_0$ and $\psi(x_0) = \psi(y_0)$, then

$$\psi(x_0 + 1) - \psi(y_0 + 1) = \frac{1}{x_0} - \frac{1}{y_0} = \frac{y_0 - x_0}{x_0 y_0} > 0.$$

However, $x_0 + 1 < y_0 + 1$, then $\psi(x_0 + 1) \leq \psi(y_0 + 1)$ but this could not happen. Thus ψ is a strictly increasing function.

We also have $1 < 1 + vx < 1 + ux$, so $\frac{\Gamma'(1+vx)}{\Gamma(1+vx)} - \frac{\Gamma'(1+ux)}{\Gamma(1+ux)} < 0$.

It induces that $g'_{u,v}(x) < 0$ for all $0 < v < u$ and $x > 0$. This proves that $g_{u,v}(x)$ is a strictly decreasing function.

It is clear that $\lim_{x \to 0} g_{u,v}(x) = 0$. Now we need to prove that $\lim_{x \to +\infty} g_{u,v}(x) = -\infty$.

Applying Stirling's formula, we have

$$\ln \Gamma(1 + z) = \ln(z\Gamma(z)) = \ln z + \ln \Gamma(z) = (z + \frac{1}{2}) \ln z - z + \frac{1}{2} \ln(2\pi) + O(z^{-1})$$

as $z \to +\infty$. Then

$$
\begin{aligned}
g_{u,v}(x) =& u\left((vx + \frac{1}{2})\ln(vx) - vx + \frac{1}{2}\ln(2\pi) + O((vx)^{-1})\right) \\
& - v\left((ux + \frac{1}{2})\ln(ux) - ux + \frac{1}{2}\ln(2\pi) + O((ux)^{-1})\right) \\
=& uv(\ln v - \ln u)x + \frac{u-v}{2}\ln x + \frac{u\ln v - v\ln u}{2} + \frac{(u-v)\ln(2\pi)}{2} \\
& + O(\frac{u}{vx}) - O(\frac{v}{ux})
\end{aligned}
$$

as $x \to +\infty$.

Since $0 < v < u$ we deduce $uv(\ln v - \ln u) < 0$. Moreover $\lim_{x\to+\infty} \frac{\ln x}{x} = 0$, it follows that

$$
\lim_{x\to+\infty} g_{u,v}(x) = -\infty. \qquad \square
$$

Then since $\alpha \in (0,2]$, we obtain that

$$
h_{-\beta_1,-\beta_2}(\alpha) = \psi_{-\beta_1,-\beta_2}(\mathbb{E}|\triangle_{0,1}X|^{\beta_1}, \mathbb{E}|\triangle_{0,1}X|^{\beta_2}) < 0.
$$

On the other hand

$$
\psi_{-\beta_1,-\beta_2}(W_n(\beta_1), W_n(\beta_2)) = \psi_{-\beta_1,-\beta_2}(V_n(\beta_1), V_n(\beta_2)).
$$

We deduce that

$$
\begin{aligned}
\widehat{\alpha}_n - \alpha =& \varphi_{-\beta_1,-\beta_2}\left(\psi_{-\beta_1,-\beta_2}(W_n(\beta_1), W_n(\beta_2))\right) - h_{-\beta_1,-\beta_2}^{-1}\left(h_{-\beta_1,-\beta_2}(\alpha)\right) \\
=& \varphi_{-\beta_1,-\beta_2}\left(\psi_{-\beta_1,-\beta_2}(W_n(\beta_1), W_n(\beta_2))\right) - \varphi_{-\beta_1,-\beta_2}\left(h_{-\beta_1,-\beta_2}(\alpha)\right) \\
=& \varphi_{-\beta_1,-\beta_2}\left(\psi_{-\beta_1,-\beta_2}(W_n(\beta_1), W_n(\beta_2))\right) \\
& - \varphi_{-\beta_1,-\beta_2}(\psi_{-\beta_1,-\beta_2}(\mathbb{E}|\triangle_{0,1}X|^{\beta_1}, \mathbb{E}|\triangle_{0,1}X|^{\beta_2})).
\end{aligned}
$$

Moreover, $\varphi_{-\beta_1,-\beta_2} \circ \psi_{-\beta_1,-\beta_2}$ is continuous and differentiable at

$$
(x_1, y_1) = (\mathbb{E}|\triangle_{0,1}X|^{\beta_1}, \mathbb{E}|\triangle_{0,1}X|^{\beta_2}). \tag{55}
$$

Combining with (51), (52), we apply Lemma 4.10, and get

$$
\widehat{\alpha}_n - \alpha = O_{\mathbb{P}}(b_n).
$$

It also induces that $\lim_{n\to+\infty} \widehat{\alpha}_n \overset{(\mathbb{P})}{=} \alpha$. $\qquad \square$

4.3. Proofs related to Section 3

Now we are in position to prove Theorems related to examples presented in Section 3.

Proof of Theorem 3.1. a) From Lemma A.2 in Appendix, the assumption (11) is satisfied. Then following Theorem 2.1, we have

$$
\widehat{H}_n - H = O_{\mathbb{P}}(n^{-1/2}), \widehat{\alpha}_n - 2 = O_{\mathbb{P}}(n^{-1/2}).
$$

b) We now prove the asymptotic normality for the estimators of H and α. To prove $\sqrt{n}(\widehat{H}_n - H)$ converges to a normal distribution as $n \to +\infty$, we will first prove that

$$\sqrt{n}\left((W_n(\beta), W_{n/2}(\beta)) - (\mathbb{E}|\triangle_{0,1}X|^\beta, \mathbb{E}|\triangle_{0,1}X|^\beta)\right) \xrightarrow{(d)} \mathcal{N}_2(0, \Gamma_1) \qquad (56)$$

as $n \to +\infty$, where Γ_1 is defined by (16). Then, we need to prove that for all $a, b \in \mathbb{R}, ab \neq 0$,

$$V_n := a\sqrt{n}(W_n(\beta) - \mathbb{E}|\triangle_{0,1}X|^\beta) + b\sqrt{n}(W_{n/2}(\beta) - \mathbb{E}|\triangle_{0,1}X|^\beta) \qquad (57)$$

converges to $G \sim \mathcal{N}_1(0, \sigma^2)$ as $n \to +\infty$, where

$$\sigma^2 = (a^2 + 2b^2) \sum_{q \geq d} q! f_{\beta,q}^2 \sum_{r \in \mathbb{Z}} \rho^q(r) + 2ab \sum_{q \geq d} q! f_{\beta,q}^2 \sum_{r \in \mathbb{Z}} \rho_1^q(r), \qquad (58)$$

$f_{\beta,q}$s, ρ, ρ_1 are defined by (13), (14), (15), respectively. Since $\{X_t\}_{t \geq 0}$ is a H-sssi process, for all $n \in \mathbb{N}^*$, we get

$$\left(\triangle_{0,n}X, \triangle_{1,n}X, \ldots, \triangle_{n-K,n}X, \triangle_{0,n/2}X, \triangle_{1,n/2}X, \ldots, \triangle_{n/2-K,n/2}X\right)$$

$$\overset{(d)}{=} \frac{1}{(n/2)^H}\left(\triangle_{0,2}X, \triangle_{1,2}X, \ldots, \triangle_{n-K,2}X, \triangle_{0,1}X, \triangle_{1,1}X, \ldots, \triangle_{n/2-K,1}X\right).$$

Moreover $var\triangle_{k,2} = \frac{var\triangle_{0,1}X}{2^{2H}}, var\triangle_{k,1}X = var\triangle_{0,1}X$. It follows that

$$\sqrt{n}\left((W_n(\beta), W_{n/2}(\beta)) - (\mathbb{E}|\triangle_{0,1}X|^\beta, \mathbb{E}|\triangle_{0,1}X|^\beta)\right)$$

$$\overset{(d)}{=} \sqrt{n}\left(\frac{2^{\beta H}\sum_{k=0}^{n-K}|\triangle_{k,2}X|^\beta}{n - K + 1} - \mathbb{E}|\triangle_{0,1}X|^\beta, \frac{\sum_{k=0}^{n/2-K}|\triangle_{k,1}X|^\beta}{n/2 - K + 1} - \mathbb{E}|\triangle_{0,1}X|^\beta\right)$$

$$= (var\triangle_{0,1}X)^{\beta/2}\left(\frac{\sqrt{n}\sum_{k=0}^{n-K}(|Y_k|^\beta - \mathbb{E}|Z_0|^\beta)}{n - K + 1}, \frac{\sqrt{n}\sum_{l=0}^{n/2-K}(|Z_l|^\beta - \mathbb{E}|Z_0|^\beta)}{n/2 - K + 1}\right)$$

$$= \left(\frac{\sqrt{n}}{n - K + 1}\sum_{k=0}^{n-K}f_\beta(Y_k), \frac{\sqrt{n}}{n/2 - K + 1}\sum_{l=0}^{n/2-K}f_\beta(Z_l)\right)$$

where $Y_k = \frac{\triangle_{k,2}X}{\sqrt{var\triangle_{k,2}X}}, Z_l = \frac{\triangle_{l,1}X}{\sqrt{var\triangle_{l,1}X}}$ and f_β is defined by (12).

We obtain that $Y_k \sim \mathcal{N}_1(0, 1), Z_l \sim \mathcal{N}_1(0, 1)$, and

$$\mathbb{E}Y_k Y_{k'} = \frac{\mathbb{E}(\triangle_{k,2}X \triangle_{k',2}X)}{\frac{var\triangle_{0,1}X}{2^{2H}}} = \frac{\sum_{p,p'=0}^{K}a_p a_{p'}|k - k' + p - p'|^{2H}}{\sum_{p,p'=0}^{K}a_p a_{p'}|p - p'|^{2H}}.$$

Then $\mathbb{E}Y_k Y_{k'} = \rho(k - k'), \mathbb{E}Z_l Z_{l'} = \rho(l - l')$ and $\mathbb{E}Y_k Z_l = \rho_1(k - 2l)$ where ρ, ρ_1 are defined by (14), (15), respectively. As in the proof of Lemma A.2 in Appendix, we can prove that for r big enough

$$\rho(r)| \leq C|r|^{2H-3}, |\rho_1(r)| \leq C|r|^{2H-3}. \tag{59}$$

We then mimic the proof of Theorem 7.2.4 in [19] to get $V_n \xrightarrow{(d)} \mathcal{N}_1(0, \sigma^2)$ as $n \to +\infty$, it follows (56). On the other hand, we have

$$\sqrt{n}(\widehat{H}_n - H) = \sqrt{n}\frac{1}{\beta}\log_2 \frac{W_{n/2}(\beta)}{W_n(\beta)}$$
$$= \sqrt{n}\left(\phi(W_n(\beta), W_{n/2}(\beta)) - \phi(\mathbb{E}|\triangle_{0,1}X|^\beta, \mathbb{E}|\triangle_{0,1}X|^\beta)\right),$$

where ϕ is defined as in (17).

Since ϕ is differentiable at $(x_0, y_0) = (\mathbb{E}|\triangle_{0,1}X|^\beta, \mathbb{E}|\triangle_{0,1}X|^\beta)$, we can apply Theorem 3.1 in [10] to get

$$\sqrt{n}(\hat{H}_n - H) \xrightarrow{(d)} \mathcal{N}_1(0, \Xi_1)$$

as $n \to +\infty$, where Ξ_1 is defined by (18).

Now we prove central limit theorem for the estimation of α. We will prove that

$$\left(\sqrt{n}(W_n(\beta_1) - \mathbb{E}|\triangle_{0,1}X|^{\beta_1}), \sqrt{n}(W_n(\beta_2) - \mathbb{E}|\triangle_{0,1}X|^{\beta_2}))\right) \xrightarrow{(d)} \mathcal{N}_2(0, \Gamma_2) \tag{60}$$

as $n \to +\infty$, with Γ_2 defined by (23).

Since $\{X_t\}_{t \in \mathbb{R}}$ is a H-sssi process, we have

$$(\triangle_{0,n}X, \ldots, \triangle_{n-K,n}X) \stackrel{(d)}{=} \frac{1}{n^H}(\triangle_{0,1}X, \ldots, \triangle_{n-K,1}X). \tag{61}$$

On the other hand, $var\triangle_{k,1}X = var\triangle_{0,1}X$. Then we can write

$$\left(\sqrt{n}(W_n(\beta_1) - \mathbb{E}|\triangle_{0,1}X|^{\beta_1}), \sqrt{n}(W_n(\beta_2) - \mathbb{E}|\triangle_{0,1}X|^{\beta_2}))\right)$$
$$\stackrel{(d)}{=} \sqrt{n}\left(\frac{\sum_{k=0}^{n-K}|\triangle_{k,1}X|^{\beta_1}}{n - K + 1} - \mathbb{E}|\triangle_{0,1}X|^{\beta_1}, \frac{\sum_{k=0}^{n-K}|\triangle_{k,1}X|^{\beta_2}}{n - K + 1} - \mathbb{E}|\triangle_{0,1}X|^{\beta_2}\right)$$
$$= \sqrt{\frac{n}{n - K + 1}}\left(\frac{1}{\sqrt{n - K + 1}}\sum_{k=0}^{n-K} f_{\beta_1}(Z_k), \frac{1}{\sqrt{n - K + 1}}\sum_{k=0}^{n-K} f_{\beta_2}(Z_k)\right).$$

where f_{β_1} and f_{β_2} are defined as in (12) and $Z_k = \frac{\triangle_{k,1}X}{\sqrt{var\triangle_{k,1}X}}, Z_k \sim \mathcal{N}_1(0, 1)$.

We have $\mathbb{E}f_{\beta_1}(Z_0) = \mathbb{E}f_{\beta_2}(Z_0) = 0, \mathbb{E}f_{\beta_1}^2(Z_0) < +\infty, \mathbb{E}f_{\beta_2}^2(Z_0) < +\infty$ and $\mathbb{E}Z_k Z_l = \rho(k - l)$ where ρ is defined by (14).

We mimic the proof of Theorem 7.2.4 of [19] to obtain that

$$a\sqrt{n}\left(W_n(\beta_1) - \mathbb{E}|\triangle_{0,1}|^{\beta_1}\right) + b\sqrt{n}\left(W_n(\beta_2) - \mathbb{E}|\triangle_{0,1}|^{\beta_2}\right)$$

converges to $\mathcal{N}_1(0, \sigma^2)$ as $n \to +\infty$ for all $a, b \in \mathbb{R}, ab \neq 0$, where

$$\sigma^2 = \sum_{q=d}^{+\infty} q!(af_{\beta_1,q} + bf_{\beta_2,q})^2 \sum_{r\in\mathbb{Z}} \rho(r)^q.$$

Here $\rho, f_{\beta_1,q}, f_{\beta_2,q}$ are defined by (14), (20) respectively. This proves (60).

The function $\varphi_{-\beta_1,-\beta_2} \circ \psi_{-\beta_1,-\beta_2} : \mathbb{R}^+ \times \mathbb{R}^+ \to [0, +\infty)$ is differentiable at

$$(x_1, y_1) = (\mathbb{E}|\triangle_{0,1}X|^{\beta_1}, \mathbb{E}|\triangle_{0,1}X|^{\beta_2}),$$

where $\psi_{u,v}, \varphi_{u,v}$ are defined by (7), (9) respectively.

We can therefore apply Theorem 3.1 of [10] to get the conclusion. $\qquad\square$

Proof of Theorem 3.2. a) We will check the assumption (11).

Let $0 \leq l < k, k - l \geq K$, then

$$\triangle_{k,1}X = \sum_{p=0}^{K} a_p\left[X(k+p) - X(k)\right], \triangle_{l,1}X = \sum_{p'=0}^{K} a_{p'}\left[X(l+p') - X(l)\right].$$

By the fact that $\{X_t\}_{t\geq 0}$ has independent increments, we obtain that $X(l + p') - X(l), X(k+p) - X(k)$ are independent for all $p, p' = 0, \ldots, K$ since $0 \leq l \leq l + p' \leq k \leq k + p$.

It follows that $\triangle_{k,1}X$ and $\triangle_{0,1}X$ are independent for $|k| \geq K$. Thus

$$cov(|\triangle_{k,1}X|^{\beta}, |\triangle_{0,1}X|^{\beta}) = 0.$$

We deduce that

$$\frac{1}{n}\sum_{k\in\mathbb{Z},|k|\leq n} |cov(|\triangle_{k,1}X|^{\beta}, |\triangle_{0,1}X|^{\beta})| = \frac{1}{n}\sum_{k\in\mathbb{Z},|k|\leq K} |cov(|\triangle_{k,1}X|^{\beta}, |\triangle_{0,1}X|^{\beta})|$$
$$= \frac{C}{n}$$

where C is a positive constant. We thus get (11) with $b_n = n^{-1/2}$ and it follows that $\widehat{H}_n - H = O_{\mathbb{P}}(n^{-1/2}), \widehat{\alpha}_n - \alpha = O_{\mathbb{P}}(n^{-1/2})$.

b) To prove the asymptotic normality for the estimator of H, we first prove that for all $n \in \mathbb{N}, n > 2K$,

$$\sqrt{n}\left((W_n, W_{n/2}) - (\mathbb{E}|\triangle_{0,1}X|^{\beta}, \mathbb{E}|\triangle_{0,1}X|^{\beta})\right)$$

converges in distribution to a normal distribution as $n \to +\infty$.

Since $\{X_t\}_{t\geq 0}$ is a H-sssi process, one has

$$\left(\triangle_{0,n}X, \triangle_{1,n}X, \ldots, \triangle_{n-K,n}X, \triangle_{0,n/2}X, \triangle_{1,n/2}X, \ldots, \triangle_{n/2-K,n/2}X\right)$$

$$\overset{(d)}{=} \frac{1}{(n/2)^H} \left(\triangle_{0,2}X, \triangle_{1,2}X, \ldots, \triangle_{n-K,2}X, \triangle_{0,1}X, \triangle_{1,1}X, \ldots, \triangle_{n/2-K,1}X \right).$$

Moreover

$$\mathbb{E}|\triangle_{k,2}|^\beta = \frac{\mathbb{E}|\triangle_{0,1}X|^\beta}{2^{\beta H}}, \mathbb{E}|\triangle_{k,1}X|^\beta = \mathbb{E}|\triangle_{0,1}X|^\beta,$$

$$var|\triangle_{k,2}|^\beta = \frac{var|\triangle_{0,1}X|^\beta}{2^{2\beta H}}, var|\triangle_{k,1}X|^\beta = var|\triangle_{0,1}X|^\beta.$$

It follows that

$$\sqrt{n}\left(\left(W_n, W_{n/2} \right) - \left(\mathbb{E}|\triangle_{0,1}X|^\beta, \mathbb{E}|\triangle_{0,1}X|^\beta \right) \right)$$

$$\overset{(d}{=} \sqrt{n} \left(\left(\frac{2^{\beta H} \sum_{p=0}^{n-K} |\triangle_{p,2}X|^\beta}{n-K+1}, \frac{\sum_{p=0}^{n/2-K} |\triangle_{p,1}X|^\beta}{n/2-K+1} \right) - \left(\mathbb{E}|\triangle_{0,1}X|^\beta, \mathbb{E}|\triangle_{0,1}X|^\beta \right) \right)$$

$$= \sqrt{n} \left(\frac{2^{\beta H} \sum_{p=0}^{n-K} \left(|\triangle_{p,2}X|^\beta - \mathbb{E}|\frac{\triangle_{0,1}X}{2^H}|^\beta \right)}{n-K+1}, \frac{\sum_{p=0}^{n/2-K} \left(|\triangle_{p,1}X|^\beta - \mathbb{E}|\triangle_{0,1}X|^\beta \right)}{n/2-K+1} \right).$$

Now we need to prove that for all $a, b \in \mathbb{R}, ab \neq 0$,

$$S_n := a\sqrt{n} \left(\frac{2^{\beta H}}{n-K+1} \sum_{p=0}^{n-K} \left(|\triangle_{p,2}X)|^\beta - \mathbb{E}|\frac{\triangle_{0,1}X}{2^H}|^\beta \right) \right)$$

$$+ b\sqrt{n} \left(\frac{1}{n/2-K+1} \sum_{p=0}^{n/2-K} \left(|\triangle_{p,1}X)|^\beta - \mathbb{E}|\triangle_{0,1}X|^\beta \right) \right)$$

converges to a normal distribution when $n \to +\infty$. Let

$$Z_p := \frac{2^{\beta H}a}{2} \left(|\triangle_{2p,2}X|^\beta + |\triangle_{2p+1,2}X|^\beta \right) + b|\triangle_{p,1}X|^\beta. \tag{62}$$

It follows that

$$S_n = \sqrt{\frac{n}{n/2-K+1}} \left(\frac{1}{\sqrt{n/2-K+1}} \sum_{p=0}^{n/2-K} \left(Z_p - \mathbb{E}Z_p \right) \right) + U_n,$$

where

$$U_n = \frac{2^{\beta H}a\sqrt{n}(1-K)}{(n-K+1)(n-2K+2)} \sum_{p=0}^{n-2K+1} \left(|\triangle_{p,2}X|^\beta - \mathbb{E}\frac{\triangle_{0,1}X}{2^H}|^\beta \right)$$

$$+ \frac{2^{\beta H}a\sqrt{n}}{n-K+1} \sum_{p=n-2K+2}^{n-K} \left(|\triangle_{p,2}X|^\beta - \mathbb{E}\frac{\triangle_{0,1}X}{2^H}|^\beta \right)$$

$$= \frac{2^{\beta H} a \sqrt{n}}{n - K + 1} \left(\frac{1 - K}{n - 2K + 2} \sum_{p=0}^{n-2K+1} Y_p + \sum_{p=n-2K+2}^{n-K} Y_p \right),$$

and $Y_p = |\triangle_{p,2} X|^\beta - \mathbb{E} |\frac{\triangle_{0,1} X}{2^H}|^\beta$.

Since $\sum_{k=0}^{K} a_k = 0$, one can write

$$Z_{p+l} = \frac{2^{\beta H} a}{2} \left| \sum_{k=0}^{K} a_k \left(X(\frac{k + 2(p + l)}{2}) - X(l) \right) \right|^\beta$$

$$+ \frac{2^{\beta H} a}{2} \left| \sum_{k=0}^{K} a_k \left(X(\frac{k + 2(p + l) + 1}{2}) - X(l) \right) \right|^\beta$$

$$+ b \left| \sum_{k=0}^{K} a_k \left(X(k + p + l) - X(l) \right) \right|^\beta .$$

If $p - p' > K - 1$, since X has independent increments and

$$0 \le p' \le \frac{k + 2p'}{2} \le p \le \min\{\frac{k + 2p}{2}, \frac{k + 2p + 1}{2}, k + p\},$$

$$0 \le p' \le \frac{k + 2p' + 1}{2} \le p \le \min\{\frac{k + 2p}{2}, \frac{k + 2p + 1}{2}, k + p\},$$

$$0 \le p' \le k + p' \le p \le \min\{\frac{k + 2p}{2}, \frac{k + 2p + 1}{2}, k + p\},$$

for all $k = 0, \ldots, K$, it follows that $Z_p, Z_{p'}$ are independent. It induces that $\{Z_p\}_{p \in \mathbb{N}}$ is a $(K - 1)$-dependent sequence of random variables.

On the other hand, since X has stationary increments and $X(0) = 0$ almost surely, we have

$$(X(t + l) - X(l))_{t \in \mathbb{R}} \stackrel{(d)}{=} (X(t))_{t \in \mathbb{R}} .$$

Then for $l \in \mathbb{R}$ fixed, $(Z_{p+l}, p \in \mathbb{R}) \stackrel{(d)}{=} (Z_p, p \in \mathbb{R})$ or in another way, $(Z_p, p \in \mathbb{R})$ is stationary.

It follows that $\{Z_p\}_{p \in \mathbb{N}}$ is a stationary $(K - 1)$-dependent sequence of random variables. From Theorem 2.8.1 in [16], we get

$$\sqrt{\frac{n}{n/2 - K + 1}} \left(\frac{1}{\sqrt{n/2 - K + 1}} \sum_{p=0}^{n/2-K} (Z_p - \mathbb{E} Z_p) \right)$$

converges in distribution to a centered normal distribution with variance

$$\sigma^2 = 2(var Z_0 + 2 \sum_{k=1}^{K-1} cov(Z_0, Z_k)) = a^2 \sigma_1^2 + b^2 \sigma_2^2 + 2ab\sigma_{1,2} \qquad (63)$$

where $\sigma_1^2, \sigma_2^2, \sigma_{1,2}$ are defined as in (28), (29), (30). We also have $\mathbb{E}Y_p = 0, \mathbb{E}U_n = 0, Y_p \overset{(d)}{=} Y_0$ and $\mathbb{E}Y_p^2 = \mathbb{E}Y_0^2$ for all p. Thus

$$
\begin{aligned}
\mathbb{E}U_n^2 &= \frac{2^{2\beta H}a^2 n}{(n-K+1)^2}\mathbb{E}\left(\frac{1-K}{n-2K+2}\sum_{p=0}^{n-2K+1}Y_p + \sum_{p=n-2K+2}^{n-K}Y_p\right)^2 \\
&\leq \frac{2^{2\beta H+1}a^2 n}{(n-K+1)^2}\left(\frac{(1-K)^2}{(n-2K+2)^2}\mathbb{E}(\sum_{p=0}^{n-2K+1}Y_p)^2 + \mathbb{E}(\sum_{p=n-2K+2}^{n-K}Y_p)^2\right) \\
&\leq \frac{2^{2\beta H+1}a^2 n}{(n-K+1)^2}\left(\frac{(1-K)^2}{(n-2K+2)^2}(n-2K+2)^2\mathbb{E}Y_0^2 + (K-1)^2\mathbb{E}Y_0^2\right) \\
&= \frac{2^{2\beta H+2}a^2(K-1)^2 n}{(n-K+1)^2}\mathbb{E}Y_0^2 = \frac{nC}{(n-K+1)^2}.
\end{aligned}
$$

It follows that $\mathbb{E}U_n^2$ converges to 0 as $n \to +\infty$. Moreover $\mathbb{E}U_n = 0$, using Chebyshev's inequality, we obtain that $U_n \xrightarrow{(\mathbb{P})} 0$ as $n \to +\infty$.

Following Slutsky's theorem, as $n \to +\infty$, S_n converges in distribution to a centered normal distribution with variance σ as in (63).

We deduce that

$$
\sqrt{n}\left((W_n, W_{n/2}) - (\mathbb{E}|\triangle_{0,1}X|^\beta, \mathbb{E}|\triangle_{0,1}X|^\beta)\right) \overset{(d)}{\to} \mathcal{N}_2(0, \Gamma_3),
$$

where Γ_3 is defined by (27). Since

$$
\begin{aligned}
\sqrt{n}(\widehat{H}_n - H) &= \sqrt{n}\frac{1}{\beta}\log_2\frac{W_{n/2}}{W_n} \\
&= \sqrt{n}\left(\phi(W_n(\beta), W_{n/2}(\beta)) - \phi(\mathbb{E}|\triangle_{0,1}X|^\beta, \mathbb{E}|\triangle_{0,1}X|^\beta)\right) \quad (64)
\end{aligned}
$$

where ϕ is defined by (17). Applying Theorem 3.1 of [10], we get $\sqrt{n}(\widehat{H} - H) \overset{(d)}{\to} \mathcal{N}_1(0, \Xi_2)$ with Ξ_2 defined by (26).

We now prove the central limit theorem for the estimation of α in the case of $S\alpha S-$stable Lévy motion.

We need to prove that for all $n \in \mathbb{N}, n > K$, then

$$
\sqrt{n}\left((W_n(\beta_1), W_n(\beta_2)) - (\mathbb{E}|\triangle_{0,1}X|^{\beta_1}, \mathbb{E}|\triangle_{0,1}X|^{\beta_2})\right)
$$

converges in distribution to a normal distribution as $n \to +\infty$.

We consider

$$
S_n = a\sqrt{n}\left(W_n(\beta_1) - \mathbb{E}|\triangle_{0,1}X|^{\beta_1}\right) + b\sqrt{n}\left(W_n(\beta_2) - \mathbb{E}|\triangle_{0,1}X|^{\beta_2}\right)
$$

for all $a, b \in \mathbb{R}, ab \neq 0$. Since $\{X_t, t \in \mathbb{R}\}$ is a H self-similar process, we have

$$
S_n \overset{(d)}{=} \frac{a\sqrt{n}}{n-K+1}\sum_{k=0}^{n-K}(|\triangle_{k,1}X|^{\beta_1} - \mathbb{E}|\triangle_{k,1}X|^{\beta_1})
$$

$$+ \frac{b\sqrt{n}}{n-K+1} \sum_{k=0}^{n-K} (|\triangle_{k,1}X|^{\beta_2} - \mathbb{E}|\triangle_{k,1}X|^{\beta_2})$$

$$= \frac{\sqrt{n}}{n-K+1} \sum_{k=0}^{n-K} (Z_k - \mathbb{E}Z_k)$$

where

$$Z_k = a|\triangle_{k,1}X|^{\beta_1} + b|\triangle_{k,1}X|^{\beta_2}. \tag{65}$$

Since $\{X_t, t \in \mathbb{R}\}$ has stationary increments, $\{Z_k, k \in \mathbb{N}\}$ is stationary.

Moreover, if $k - k' > K - 1$, since $\{X_t, t \in \mathbb{R}\}$ has independent increments, then $Z_k, Z_{k'}$ are independent. We obtain that $\{Z_k, k \in \mathbb{N}\}$ is a stationary $(K-1)$-dependent sequence of random variables. Then applying Theorem 2.8.1 of [16], as $n \to +\infty$, S_n converges to a centered normal distribution with variance:

$$\sigma^2 = var Z_0 + 2 \sum_{k=0}^{K-1} cov(Z_0, Z_k). \tag{66}$$

We can write σ^2 in details

$$\sigma^2 = a^2 \left(var|\triangle_{0,1}X|^{\beta_1} + 2\sum_{k=1}^{K-1} cov(|\triangle_{0,1}X|^{\beta_1}, |\triangle_{k,1}X|^{\beta_1}) \right)$$

$$+ b^2 \left(var|\triangle_{0,1}X|^{\beta_2} + 2\sum_{k=1}^{K-1} cov(|\triangle_{0,1}X|^{\beta_2}, |\triangle_{k,1}X|^{\beta_2}) \right)$$

$$+ ab \left(cov(|\triangle_{0,1}X|^{\beta_1}, |\triangle_{0,1}X|^{\beta_2}) + cov(|\triangle_{0,1}X|^{\beta_2}, |\triangle_{k,1}X|^{\beta_1}) \right)$$

$$+ 2ab \sum_{k=1}^{K-1} \left(cov(|\triangle_{0,1}X|^{\beta_1}, |\triangle_{k,1}X|^{\beta_2}) \right)$$

$$= a^2 \sigma_1^2 + b^2 \sigma_2^2 + 2ab\sigma_{1,2},$$

where $\sigma_1^2, \sigma_2^2, \sigma_{1,2}$ are defined by (33), (34), (35) respectively.

It follows that

$$\sqrt{n} \left((W_n(\beta_1), W_n(\beta_2)) - (\mathbb{E}|\triangle_{0,1}X|^{\beta_1}, \mathbb{E}|\triangle_{0,1}X|^{\beta_2}) \right) \overset{(d)}{\to} \mathcal{N}_2(0, \Gamma_4),$$

where Γ_4 defined by (32).

The function $\varphi_{-\beta_1, -\beta_2} \circ \psi_{-\beta_1, -\beta_2} : \mathbb{R}^+ \times \mathbb{R}^+ \to [0, +\infty)$ is differentiable at

$$(x_1, y_1) = (\mathbb{E}|\triangle_{0,1}X|^{\beta_1}, \mathbb{E}|\triangle_{0,1}X|^{\beta_2}),$$

where $\psi_{u,v}, \varphi_{u,v}$ are defined by (7), (9) respectively. Then we apply Theorem 3.1 of [10] to get the conclusion. $\quad\square$

Proof of Theorem 3.3. Set $f(t) = \sum_{k=0}^{K} a_k |k - t|^{H-1/\alpha}$. For all $k \in \mathbb{Z}$, one has

$$\triangle_{k,1}X = \int_{\mathbb{R}} \sum_{j=0}^{K} a_j (|k+j-s|^{H-1/\alpha} - |s|^{H-1/\alpha})M(ds) = \int_{\mathbb{R}} f(s-k)M(ds)$$

and $||\triangle_{k,1}X||_\alpha^\alpha = \int\limits_{\mathbb{R}} |f(s-k)|^\alpha ds$. By taking the change of variable $u = s - k$, we get

$$||\triangle_{k,1}X||_\alpha^\alpha = \int\limits_{\mathbb{R}} |f(u)|^\alpha du = ||\triangle_{0,1}X||_\alpha^\alpha.$$

Let $U_k = \frac{\triangle_{k,1}X}{||\triangle_{k,1}X||_\alpha}$, then $||U_k||_\alpha^\alpha = 1$ and $U_k = \int\limits_{\mathbb{R}} \frac{f(s-k)}{||\triangle_{k,1}X||_\alpha} M(ds)$. We now prove that the assumption (11) is satisfied. Therefore, we consider

$$S_n = \frac{1}{n} \sum_{k \in \mathbb{Z}, |k| \leq n} |cov(|\triangle_{k,1}X|^\beta, |\triangle_{0,1}X|^\beta)|. \tag{67}$$

Since $||\triangle_{k,1}X||_\alpha = ||\triangle_{0,1}X||_\alpha$, it follows that

$$\sum_{k \in \mathbb{Z}, |k| \leq n} |cov(|\triangle_{k,1}X|^\beta, |\triangle_{0,1}X|^\beta)| = ||\triangle_{0,1}X||_\alpha^{2\beta} \sum_{k \in \mathbb{Z}, |k| \leq n} |cov(|U_k|^\beta, |U_0|^\beta)|.$$

Moreover

$$[U_k, U_0]_2 = \int\limits_{\mathbb{R}} \left| \frac{f(s-k)f(s)}{||\triangle_{0,1}X||_\alpha^2} \right|^{\alpha/2} ds$$

Together with Lemma 3.6 in [11], there exist $k_0 > 4K$ and $0 < \eta < 1$ such that for all $k \in \mathbb{Z}, |k| > k_0$, one has

$$[U_k, U_0]_2 \leq \eta < 1.$$

Applying Theorem 4.2, there exists $C(\eta) > 0$ depending on η such that

$$|cov(|U_k|^\beta, |U_0|^\beta)| \leq C(\eta) \int\limits_{\mathbb{R}} |f(s-k)f(s)|^{\alpha/2} ds$$

for all $|k| > k_0$. Then for $n > k_0$, one obtains that

$$\sum_{\substack{k \in \mathbb{Z} \\ |k| \leq n}} |cov(|U_k|^\beta, |U_0|^\beta)| = \sum_{\substack{k \in \mathbb{Z} \\ |k| \leq k_0}} |cov(|U_k|^\beta, |U_0|^\beta)| + \sum_{\substack{k \in \mathbb{Z} \\ k_0 < |k| \leq n}} |cov(|U_k|^\beta, |U_0|^\beta)|$$

$$\leq C \sum_{\substack{k \in \mathbb{Z} \\ |k| \leq k_0}} \int\limits_{\mathbb{R}} |f(s-k)f(s)|^{\alpha/2} ds$$

$$+ C \sum_{\substack{k \in \mathbb{Z} \\ k_0 < |k| \leq n}} |k|^{\frac{\alpha H - (L+1)\alpha}{2}}.$$

Because $f(x) \in L^\alpha(\mathbb{R}, dx)$, one has $\sum\limits_{k \in \mathbb{Z}, |k| \leq k_0} \int\limits_{\mathbb{R}} | f(s-k)f(s) |^{\alpha/2} ds < +\infty$.

Then $S_n = \frac{C}{n} \sum\limits_{k \in \mathbb{Z}, k_0 < |k| \leq n} |k|^{\frac{\alpha H - (L+1)\alpha}{2}}.$

Since $\alpha H - (L+1)\alpha < 0$, using Lemma A.4 in Appendix, we also get

$$S_n = \begin{cases} O(n^{-1}) & \text{if } H < L+1-\frac{2}{\alpha} \\ O(n^{\frac{\alpha H - (L+1)\alpha}{2}}) & \text{if } H > L+1-\frac{2}{\alpha} \\ O(\frac{\ln n}{n}) & \text{if } H = L+1-\frac{2}{\alpha} \end{cases},$$

where S_n is defined by (67). Applying Theorem 2.1, we have

$$W_n(\beta) - \mathbb{E}|\triangle_{0,1}X|^\beta = O_\mathbb{P}(b_n), \widehat{H}_n - H = O_\mathbb{P}(b_n),$$

where b_n is defined by (36). $\qquad\square$

Proof of Theorem 3.4. We have

$$\triangle_{k,1}X = \int\limits_0^{+\infty}\int\limits_\mathbb{R} \left(\sum_{i=0}^K a_i \mathbb{1}_{S_{k+i}}(x,r)\right) M(dx,dr)$$

$$||\triangle_{k,1}X||_\alpha^\alpha = \int\limits_0^{+\infty}\int\limits_\mathbb{R} |\sum_{i=0}^K a_i \mathbb{1}_{S_{k+i}}(x,r)|^\alpha (r^{\nu-2})^\alpha dx dr$$

$$= \int\limits_0^{+\infty}\int\limits_\mathbb{R} |\sum_{i=0}^K a_i \mathbb{1}_{S_i}(x-k,r)|^\alpha (r^{\nu-2})^\alpha d(x-k)dr.$$

By taking the change of variable $u = x - k$, one obtains that

$$||\triangle_{k,1}X||_\alpha^\alpha = \int\limits_0^{+\infty}\int\limits_\mathbb{R} |\sum_{i=0}^K a_i \mathbb{1}_{S_i}(u,r)|^\alpha (r^{\nu-2})^\alpha du dr = ||\triangle_{0,1}X||_\alpha^\alpha.$$

Set $U_k = \frac{\triangle_{k,1}X}{||\triangle_{k,1}X||_\alpha} = \frac{\triangle_{k,1}X}{||\triangle_{0,1}X||_\alpha}$. Obviously, $||U_k||_\alpha^\alpha = 1$. We now prove that the condition (11) is satisfied. Set

$$I_n = \sum_{k\in\mathbb{Z},|k|\leq n} |cov(|\triangle_{k,1}X|^\beta, |\triangle_{0,1}X|^\beta)|$$

$$= ||\triangle_{0,1}X||_\alpha^\beta \sum_{k\in\mathbb{Z},|k|\leq n} |cov(|U_k|^\beta, |U_0|^\beta)|.$$

For $n > 2K$, applying Lemma A.3 in Appendix, one gets

$$I_n \leq C \left(\sum_{k\in\mathbb{Z},|k|\leq 2K} |cov(|U_k|^\beta, |U_0|^\beta)| + \sum_{k\in\mathbb{Z},2K<|k|\leq n} |cov(|U_k|^\beta, |U_0|^\beta)|\right)$$

$$\leq C \left(\sum_{k\in\mathbb{Z},|k|\leq 2K} |cov(|U_k|^\beta, |U_0|^\beta)| + \sum_{k\in\mathbb{Z},2K<|k|\leq n} |k|^{\nu-1}\right).$$

Since $0 < \nu < 1$, one gets $-1 < \nu - 1 < 0$. Following Lemma A.4 in Appendix, one obtains

$$\frac{1}{n} \sum_{k \in \mathbb{Z}, 2K < |k| \leq n} |k|^{\nu-1} = O(n^{\nu-1}).$$

Then we get the condition (11). Applying Theorem 2.1, we obtain that

$$W_n(X) - \mathbb{E}|\triangle_{0,1}X|^\beta = O_\mathbb{P}(b_n)$$

and

$$\widehat{H}_n - H = O_\mathbb{P}(b_n), \hat{\alpha}_n - \alpha = O_\mathbb{P}(b_n)$$

where b_n is defined as in (38). $\qquad\square$

Appendix A: Techincal results related to examples

We present here some technical results related to examples introduced in Section 3.

A.1. Auxiliary results related to Fractional Brownian motion

We are in position to provide and prove some technical results related to fractional Brownian motion. These results are used to present the variances for the limit distributions of the central limit theorems for the estimators of H and α and to prove Theorem 3.1 in Subsection 4.3.

Proposition A.1. *Let X be a H fractional Brownian motion with $H \in (0,1)$. For $\beta \in \mathbb{R}, -1/2 < \beta < 0$, let f_β be defined as in (12),*

$$f_\beta = \sqrt{var\triangle_{0,1}X}^\beta (|x|^\beta - \mathbb{E}|Z_0|^\beta)$$

where $Z_0 = \frac{\triangle_{0,1}X}{\sqrt{var\triangle_{0,1}X}}$. Then f_β can be expanded in a unique way into series of Hermite polynomials

$$f_\beta(x) = \sum_{q \geq d} f_{\beta,q} H_q(x)$$

and $\sum_{q \geq d} q! f_{\beta,q}^2 < +\infty$, where d is the Hermite rank of f_β, moreover $d \geq 2$.

Proof. Since $-1/2 < \beta < 0$, one has $\frac{1}{\sqrt{2\pi}} \int_\mathbb{R} f_\beta(x) e^{-x^2/2} dx = 0$ and

$$\frac{1}{\sqrt{2\pi}} \int_\mathbb{R} f_\beta^2(x) e^{-x^2/2} dx < +\infty.$$

Then following Proposition 1.4.2-(iv) in [19], we can write f_β in terms of Hermite polynomials in a unique way

$$f_\beta(x) = \sum_{q \geq d} f_{\beta,q} H_q(x),$$

where $d \geq 1$ is the Hermite rank of f_β and H_qs are the Hermite polynomials.

Moreover, it is clear that $Z_0 \sim \mathcal{N}(0,1)$. From Proposition 2.2.1 in [19], we get

$$\mathbb{E}[H_p(Z_0)H_q(Z_0)] = \begin{cases} 0 & \text{if } p \neq q \\ p! & \text{if } p = q \end{cases}$$

Then since $H_1(x) = x$, one has $\mathbb{E}H_1(Z_0)f_\beta(Z_0) = \mathbb{E}Z_0 f_\beta(Z_0) = f_{\beta,1}\mathbb{E}Z_0^2 = f_{\beta,1}$. Combining with the fact that

$$\mathbb{E}H_1(Z_0)f_\beta(Z_0) = \frac{(var\triangle_{0,1}X)^\beta}{\sqrt{2\pi}} \int_{\mathbb{R}} x(|x|^\beta - \mathbb{E}|Z_0|^\beta)e^{-x^2/2}dx = 0,$$

we deduce that $f_{\beta,1} = 0$. It follows that $d \geq 2$.

Moreover,

$$\mathbb{E}f_\beta^2(Z_0) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f_\beta^2(x)e^{-x^2/2}dx < +\infty.$$

On the other hand,

$$\mathbb{E}f_\beta^2(Z_0) = \sum_{p,q \geq d} f_{\beta,p}f_{\beta,q}\mathbb{E}[H_p(Z_0)H_q(Z_0)] = \sum_{q \geq d} q!f_{\beta,q}^2.$$

It follows that $\sum_{q \geq d} q!f_{\beta,q}^2 < +\infty$. $\qquad\qquad\square$

Lemma A.1. *Let* $(U,V) \overset{(d)}{=} \mathcal{N}_2\left((0,0), \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right), |\rho| \leq 1$. *Then for each* $\beta \in \mathbb{C}, Re(\beta) \in (-1/2, 0)$, *there exists a constant* $C > 0$ *such that* $\forall |\rho| \leq 1$, *we have:*

$$|cov(|U|^\beta, |V|^\beta)| \leq C\rho^2$$

Proof. Let $\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. We have $det(\Sigma) = 1 - \rho^2$ and $\Sigma^{-1} = \frac{1}{1-\rho^2}\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$. The density function of (U,V):

$$f(x,y) = |2\pi\Sigma|^{-1/2}exp\left[-\frac{1}{2}(xy)\Sigma^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right]$$

$$= \left((2\pi)^2 det\Sigma\right)^{-1/2} exp\left[-\frac{1}{2(1-\rho^2)}(x^2 + y^2 - 2\rho xy)\right].$$

We get

$$\mathbb{E}\left(|U|^\beta|V|^{\overline{\beta}}\right) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{\mathbb{R}^2} |x|^\beta|y|^{\overline{\beta}}exp\left[-\frac{1}{2(1-\rho^2)}(x^2 + y^2 - 2\rho xy)\right]dxdy$$

$$\mathbb{E}|U|^\beta\mathbb{E}|V|^{\overline{\beta}} = \frac{1}{2\pi} \int_{\mathbb{R}^2} |x|^\beta|y|^{\overline{\beta}}exp(-\frac{x^2 + y^2}{2})dxdy$$

and

$$cov(|U|^{\beta}, |V|^{\beta}) = \mathbb{E}(|U|^{\beta}|V|^{\overline{\beta}}) - \mathbb{E}|U|^{\beta}\mathbb{E}|V|^{\overline{\beta}}$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}^2} |x|^{\beta}|y|^{\overline{\beta}} exp(-\frac{x^2+y^2}{2}) A_{\rho}(x,y) dx dy$$

where

$$A_{\rho}(x,y) = \frac{1}{\sqrt{1-\rho^2}} exp\left(-\frac{\rho^2}{1-\rho^2}(x^2+y^2)\right) exp\left(\frac{\rho xy}{1-\rho^2}\right) - 1.$$

Since $\int_{\mathbb{R}} |x|^{\beta} x e^{-x^2/2} dx = 0$ we obtain that

$$cov(|U|^{\beta}, |V|^{\beta}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} |x|^{\beta}|y|^{\overline{\beta}} exp(-\frac{x^2+y^2}{2}) B_{\rho}(x,y) dx dy$$

with

$$B_{\rho}(x,y) = A_{\rho}(x,y) - \rho xy$$

$$= \frac{1}{\sqrt{1-\rho^2}} exp\left(-\frac{\rho^2}{1-\rho^2}(x^2+y^2)\right) exp\left(\frac{\rho xy}{1-\rho^2}\right) - 1 - \rho xy.$$

Using L'Hôpital rule, we get:

$$\lim_{\rho \to 0} \frac{B_{\rho}}{\rho^2} = \lim_{\rho \to 0} \frac{B_{\rho}'}{2\rho}$$

$$= \lim_{\rho \to 0} \frac{A_{\rho}'(x,y) - xy}{2\rho}$$

$$= xy \cdot \lim_{\rho \to 0} \frac{\left[exp\left(\frac{-\rho^2}{1-\rho^2}(x^2+y^2) + \frac{\rho xy}{1-\rho^2}\right)(1-\rho^2)^{-1} - 1\right]}{2\rho} + \frac{1}{2} - x^2 - y^2$$

$$:= A + \frac{1}{2} - x^2 - y^2.$$

Then we continue using L'Hôpital rule for the remaining limit:

$$A = \frac{xy}{2} \lim_{\rho \to 0} exp\left(\frac{-\rho^2}{1-\rho^2}(x^2+y^2) + \frac{\rho xy}{1-\rho^2}\right) \times$$

$$\left[\frac{2\rho}{(1-\rho^2)^2} - (x^2+y^2)\left(\frac{2\rho}{1-\rho^2} + \frac{2\rho^3}{(1-\rho^2)^2}\right) + xy\left(\frac{1}{1-\rho^2} + \frac{2\rho^2}{(1-\rho)^2}\right)\right]$$

$$= \frac{x^2 y^2}{2}.$$

One has

$$\lim_{\rho \to 0} \frac{B_{\rho}}{\rho^2} = \frac{x^2 y^2 + 1}{2} - (x^2+y^2)$$

$$\frac{\partial^2 B_\rho(x,y)}{\partial \rho^2} = P_\rho(x,y)exp\left(-\frac{\rho^2}{1-\rho^2}(x^2+y^2)\right)exp\left(\frac{\rho xy}{1-\rho^2}\right).$$

where $P_\rho(x,y)$ is a fourth degree polynomial that depends continuously on ρ. We also have

$$B_0(x,y) = 0, B'_\rho(x,y)\mid_{\rho=0} = 0.$$

A Taylor expansion up to order 2 leads to

$$B_\rho(x,y) = \rho^2 P_{\tilde{\rho}}(x,y)exp\left(-\frac{\tilde{\rho}^2}{1-\tilde{\rho}^2}(x^2+y^2)\right)exp\left(\frac{\tilde{\rho}xy}{1-\tilde{\rho}^2}\right)$$

with $\tilde{\rho} \in (0,\rho)$. On the compact set $|\rho| \leq 1/2$, the polynomial $P_\rho(x,y)$ can be bounded by a fourth degree polynomial $P(x,y)$, for all $x,y \in \mathbb{R}, |P_\rho(x,y)| \leq |P(|x|,|y|)|$. Moreover

$$exp\left(-\frac{\tilde{\rho}^2}{1-\tilde{\rho}^2}(x^2+y^2)\right)exp\left(\frac{\tilde{\rho}xy}{1-\tilde{\rho}^2}\right) \leq exp\left(\frac{\tilde{\rho}xy}{1-\tilde{\rho}^2}\right).$$

But with $|\tilde{\rho}| \leq 1/2$, we get $|\frac{\tilde{\rho}}{1-\tilde{\rho}^2}| \leq 2/3$. So $exp\left(\frac{\tilde{\rho}xy}{1-\tilde{\rho}^2}\right) \leq exp\left(\frac{2|xy|}{3}\right)$. Thus

$$|B_\rho(x,y)| \leq \rho^2 |P(|x|,|y|)|exp\left(\frac{2|xy|}{3}\right).$$

Because the power function grows faster than the polynomial function, we have

$$\int_{\mathbb{R}^2} |xy|^{Re(\beta)} exp(-\frac{x^2+y^2}{2})P(|x|,|y|)exp\left(\frac{2|xy|}{3}\right)dxdy < \infty.$$

So we have the conclusion. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma A.2. *Let X be a fractional Brownian motion, $\beta \in \mathbb{C}, -1/2 < Re(\beta) < 0$. Then*

$$\sum_{k\in\mathbb{Z}} |cov(|\triangle_{k,1}X|^\beta, |\triangle_{0,1}X|^\beta)| < +\infty.$$

Proof. We have

$$cov(\triangle_{k,1}X, \triangle_{0,1}X) = -\frac{1}{2}\sum_{p,p'=0}^{K} a_p a_{p'}|k+p-p'|^{2H}$$

$$= -\frac{k^{2H}}{2}\sum_{p,p'=0}^{K} a_p a_{p'}|1+\frac{p-p'}{k}|^{2H}.$$

We just need to consider $cov(\triangle_{k,1}X, \triangle_{0,1}X)$ when $|k| \geq K$. Since $1+\frac{p-p'}{k} \geq 0$, we get

$$cov(\triangle_{k,1}X, \triangle_{0,1}X) = -\frac{k^{2H}}{2}\sum_{p,p'=0}^{K} a_p a_{p'}(1+\frac{p-p'}{k})^{2H}.$$

Set

$$g(x) = -\frac{1}{2} \sum_{p,p'=0}^{K} a_p a_{p'} (1 + (p - p')x)^{2H}.$$

If $H = \frac{1}{2}$ then $g(x) = 0$. If $H \neq \frac{1}{2}$, using Taylor expansion as $x \to 0$, we get

$$g'(x) = -H \sum_{p,p'=0}^{K} a_p a_{p'} (p - p')(1 + (p - p')x)^{2H-1}$$

$$g''(x) = -H(2H-1) \sum_{p,p'=0}^{K} a_p a_{p'} (p - p')^2 (1 + (p - p')x)^{2H-2}$$

$$g^{(3)}(x) = -H(2H-1)(2H-2) \sum_{p,p'=0}^{K} a_p a_{p'} (p - p')^3 (1 + (p - p')x)^{2H-3}.$$

Thus $g(0) = 0, g'(0) = 0, g''(0) = 0, g^{(3)}(0) = 0$ and we obtain that $g(x) = o(x^3)$ as $x \to 0$. It follows that

$$cov(\triangle_{k,1}X, \triangle_{0,1}X) \sim k^{2H} \cdot o(\frac{1}{k^3}) \sim o(k^{2H-3})$$

as $k \to +\infty$. We can apply similarly as $k \to -\infty$. Then there exists a constant C such that for all $k, |k| \geq K$ and for all $H \in (0,1)$,

$$|cov(\triangle_{k,1}X, \triangle_{0,1}X)| \leq C|k|^{2H-3}. \tag{68}$$

For all $k \in \mathbb{Z}$, we have

$$var\triangle_{k,1}X = \mathbb{E}\left[\sum_{p=0}^{K} a_p X(k+p) \sum_{p'=0}^{K} a_{p'} X(k+p') \right]$$

$$= -\frac{1}{2} \sum_{p,p'=0}^{K} a_p a_{p'} |p - p'|^{2H}.$$

We now apply the Lemma A.1 with $U = \frac{\triangle_{k,1}X}{\sqrt{var(\triangle_{k,1}X)}}, V = \frac{\triangle_{0,1}X}{\sqrt{var(\triangle_{0,1}X)}}$. Then

$$\left| cov\left(|\frac{\triangle_{k,1}X}{\sqrt{var(\triangle_{k,1}X)}}|^\beta, |\frac{\triangle_{0,1}X}{\sqrt{var(\triangle_{0,1}X)}}|^\beta \right) \right| \leq C \cdot \frac{cov^2(\triangle_{k,1}X, \triangle_{0,1}X)}{var^2 \triangle_{0,1}X}.$$

It follows that

$$\left| cov(|\triangle_{k,1}X|^\beta, |\triangle_{0,1}X|^\beta) \right| \leq C cov^2(\triangle_{k,1}X, \triangle_{0,1}X), \forall k, k \in \mathbb{Z}.$$

Since $H \in (0,1)$, we get $\sum_{k \in \mathbb{Z}} |k|^{4H-6} < +\infty$. Applying inequality (68), we obtain

$$\sum_{k \in \mathbb{Z}} \left| cov(|\triangle_{k,1}X|^\beta, |\triangle_{0,1}X|^\beta) \right| = \sum_{k \in \mathbb{Z}, |k| < K} \left| cov(|\triangle_{k,1}X|^\beta, |\triangle_{0,1}X|^\beta) \right|$$

$$+ \sum_{k \in \mathbb{Z}, |k| \geq K} \left| cov(|\triangle_{k,1}X|^\beta, |\triangle_{0,1}X|^\beta) \right|$$

$$\leq C \sum_{k \in \mathbb{Z}, |k| < K} cov^2(\triangle_{k,1}X, \triangle_{0,1}X)$$

$$+ C \sum_{k \in \mathbb{Z}, |k| \geq K} |k|^{4H-6} < +\infty. \qquad \square$$

A.2. Auxiliary results related to Takenaka's process

Lemma A.3. *Let $\{X_t, t \in \mathbb{R}\}$ be a Takenaka's process defined by (37). Then for $\beta \in \mathbb{R}, \beta \in (-1/2, 0)$ and $|k| > 2K$, we have*

$$\left| cov(|\triangle_{k,1}X|^\beta, |\triangle_{0,1}X|^\beta) \right| \leq Ck^{\nu-1}.$$

Proof. One has

$$\triangle_{k,1}X = \sum_{i=0}^{K} a_i X(k+i) = \int_{\mathbb{R} \times \mathbb{R}^+} \sum_{i=0}^{K} a_i \mathbb{1}_{S_{k+i}}(x,r) M(dx,dr)$$

where $f_k = \sum_{i=0}^{K} a_i \mathbb{1}_{S_{k+i}} = \sum_{i=0}^{K} a_i (\mathbb{1}_{C_{k+i}} - \mathbb{1}_{C_0})^2$. From the fact that $\sum_{i=0}^{K} a_i = 0$ and $| 1 - 2\mathbb{1}_{C_0} |= 1$, it induces

$$|f_k| = |1 - 2\mathbb{1}_{C_0}|| \sum_{i=0}^{K} a_i \mathbb{1}_{C_{k+i}}| = | \sum_{i=0}^{K} a_i \mathbb{1}_{C_{k+i}}|.$$

Therefore we have to estimate, as $|k| \to +\infty$,

$$I_k = [\triangle_{k,1}X, \triangle_{0,1}X]_2 = \int_0^{+\infty} r^{\nu-2} \int_{\mathbb{R}} |f_k(x,r)f_0(x,r)|dxdr.$$

We will find an upper bound for I_k when $|k| \geq 2K$.

If $x > K + r$ then $\mathbb{1}_{C_i}(x,r) = 0$ for all $i = 0, \ldots, K$, thus $f_0(x,r) = 0$. If $x < k - r$, $\mathbb{1}_{C_{k+i}}(x,r) = 0$ for all $i = 0, \ldots, K$, it follows that $f_k(x,r) = 0$.

As a result, $f_k(x,r)f_0(x,r) = 0$ for all $x \in (-\infty, k - r) \cup (K + r, +\infty)$.

Let $k > 2K$. If $r < \frac{k-K}{2} \Leftrightarrow k - r > K + r$ then $f_k(x,r)f_0(x,r) = 0$ for all x. Thus one gets

$$I_k = \int_{\mathbb{R}} \left(\int_0^{\frac{k-K}{2}} r^{\nu-2}|f_k(x,r)f_0(x,r)|dr + \int_{\frac{k-K}{2}}^{+\infty} r^{\nu-2}|f_k(x,r)f_0(x,r)|dr \right) dx$$

$$= \int_{\mathbb{R}} \int_{\frac{k-K}{2}}^{+\infty} r^{\nu-2}|f_k(x,r)f_0(x,r)|drdx = \int_{\frac{k-K}{2}}^{+\infty} r^{\nu-2} \int_{\mathbb{R}} |f_k(x,r)f_0(x,r)|dxdr.$$

Here we consider $f_k(x,r)f_0(x,r)$.

Since $k > 2K$, then $k+K-r \leq k+r$. For $k+K-r \leq x \leq k+r$, $|x-k-i| \leq r$ for all $i = 0, \ldots, K$, it follows that $\mathbb{1}_{C_{k+i}}(x,r) = 1$ and $f_k(x,r) = \sum\limits_{i=0}^{K} a_i = 0$.

Therefore $f_k(x,r)f_0(x,r) = 0$ if $x \in (-\infty, k-r) \cup (k+K-r, +\infty)$. We also have

$$|f_k(x,r)| = |\sum_{i=0}^{K} a_i \mathbb{1}_{C_{k+i}}(x,r)| \leq \sum_{i=0}^{K} |a_i|$$

for all $k \in \mathbb{N}$. Thus

$$I_k \leq \int\limits_{\frac{k-K}{2}}^{+\infty} r^{\nu-2} \int\limits_{k-r}^{k+K-r} |f_k(x,r)f_0(x,r)| dx dr$$

$$\leq C \int\limits_{\frac{k-K}{2}}^{+\infty} r^{\nu-2} \int\limits_{k-r}^{k+K-r} dx dr = C(\frac{k-K}{2})^{\nu-1} \leq Ck^{\nu-1}$$

since $\frac{k-K}{2} \geq \frac{k}{4}$ and $0 < \nu < 1$.

Let $k < -2K$. If $r < -\frac{k+K}{2} \Leftrightarrow k+K+r < -r$, then for all $i = 1, \ldots, K$,

$$\mathbb{1}_{C_{k+i}}(x,r) = 0, \forall x \in (k+K+r, +\infty), \mathbb{1}_{C_i}(x,r) = 0, \forall x \in (-\infty, -r).$$

It follows that $f_k(x,r)f_0(x,r) = 0$, for all $x \in (-\infty, -r) \cup (k+K+r, +\infty) = \mathbb{R}$. Therefore

$$I_k = \int\limits_{-\frac{k+K}{2}}^{+\infty} r^{\nu-2} \int\limits_{\mathbb{R}} |f_k(x,r)f_0(x,r)| dx dr$$

For $r > -\frac{k+K}{2}$, $r > K - K/2 = K/2$. We have $f_k(x,r)f_0(x,r) = 0$ for all $x \in (-\infty, k-r) \cup (k-r+K, k+r) \cup (k+r+K, K+r) \cup (K+r, +\infty)$. It induces that

$$I_k = \int\limits_{-\frac{k+K}{2}}^{+\infty} r^{\nu-2} \int\limits_{\mathbb{R}} |f_k(x,r)f_0(x,r)| dx dr$$

$$= \int\limits_{-\frac{k+K}{2}}^{+\infty} r^{\nu-2} \left(\int\limits_{k-r}^{k-r+K} |f_k(x,r)f_0(x,r)| dx dr + \int\limits_{k+r}^{k+r+K} |f_k(x,r)f_0(x,r)| dx dr \right)$$

$$\leq C \int\limits_{-\frac{k+K}{2}}^{+\infty} r^{\nu-2} dr \leq C|k|^{\nu-1}.$$

Putting together with Theorem 4.2, for $|k| > 2K$ we obtain that

$$\left| cov(|\triangle_{k,1}X|^\beta, |\triangle_{0,1}X|^\beta) \right| \leq Ck^{\nu-1}. \qquad \square$$

A.3. Auxiliary results related to rate of convergence

We present here a lemma used to determine rate of convergence in the proofs of Theorems 3.3 and 3.4.

Lemma A.4. *For $p < 0$, let $S_n = \frac{1}{n} \sum_{|k| \leq n} |k|^p$, then $\lim_{n \to +\infty} S_n = 0$. Moreover*

$$
S_n = \begin{cases}
O(n^{-1}) & \text{if } p < -1 \\
O(n^p) & \text{if } -1 < p < 0 \\
O(\frac{\ln n}{n}) & \text{if } p = -1.
\end{cases}
$$

Proof. Set

$$
S_n = \frac{1}{n} \sum_{k=1}^{n} k^p.
$$

If $p < -1$, since $\int_1^\infty x^p dx < +\infty$, following the integral test for convergence, we get $\sum_{k=1}^\infty k^p < +\infty$. Then

$$
S_n = O(n^{-1}).
$$

If $-1 < p < 0$, we take a constant ϵ such that $0 < \epsilon < -p$, then

$$
S_n = \frac{1}{n} \sum_{k=1}^{n} \frac{k^{1+\epsilon+p}}{k^{1+\epsilon}} \leq \frac{1}{n} \sum_{k=1}^{n} \frac{n^{1+\epsilon+p}}{k^{1+\epsilon}} = n^{p+\epsilon} \sum_{k=1}^{n} \frac{1}{k^{1+\epsilon}}.
$$

Since $p + \epsilon < 0$, we get $\sum_{k=1}^{n} \frac{1}{k^{1+\epsilon}} < +\infty$. Then $S_n = O(n^{p+\epsilon})$ for all $0 < \epsilon < -p$. Thus $S_n = O(n^p)$.

If $p = -1$, then $S_n = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{k} = O(\frac{\ln n}{n})$.

In all cases, we have $\lim_{n \to +\infty} S_n = 0$. □

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