Electronic Journal of Statistics
Vol. 11 (2017) 3673–3702
ISSN: 1935-7524
DOI: 10.1214/17-EJS1319

A discontinuity adjustment for subdistribution function confidence bands applied to right-censored competing risks data

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Abstract: The wild bootstrap is the resampling method of choice in survival analytic applications. Theoretic justifications typically rely on the assumption of existing intensity functions which is equivalent to an exclusion of ties among the event times. However, such ties are omnipresent in practical studies. It turns out that the wild bootstrap should only be applied in a modified manner that corrects for altered limit variances and emerging dependencies. This again ensures the asymptotic exactness of inferential procedures. An analogous necessity is the use of the Greenwood-type variance estimator for Nelson-Aalen estimators which is particularly preferred in tied data regimes. All theoretic arguments are transferred to bootstrapping Aalen-Johansen estimators for cumulative incidence functions in competing risks. An extensive simulation study as well as an application to real competing risks data of male intensive care unit patients suffering from pneumonia illustrate the practicability of the proposed technique.

MSC 2010 subject classifications: 62N03, 62G09.
Keywords and phrases: Aalen-Johansen estimator, counting process, discontinuous cumulative hazard functions, discontinuous cumulative incidence functions, Greenwood-type variance estimator, Nelson-Aalen estimator, survival analysis, Tied event times, wild bootstrap.

Received February 2017.

Contents

1. Introduction

In non- and semiparametric survival analysis, one of the most frequent assumptions is imposed by the existence of hazard rates. Such a rate quantifies the instantaneous probability of an event occurrence given the survival up to the present time. In the case of its existence, it can be expressed with the help of the survival function and its first derivative. On the other hand, the survival function can then be written as an exponential function that involves an integral of the hazard rate. In this case, the occurrence of several individuals with exactly the same event time is theoretically excluded.

However, following [21], p. 65, there are two reasons for tied survival times, i.e. at least two individuals having exactly the same recorded event time: either due to rounding of continuous underlying data or because the event times are genuinely discrete, such as the "number of menstrual cycles until pregnancy" in a "study of effectiveness of birth control methods". In case of rounding, the recorded event times are usually rounded to the whole day, week, month, year, or some other fixed time unit. This rounding procedure almost always implicates the appearance of tied event times. Strictly speaking, this rounding of times alters the underlying survival function by inflicting discrete components. In this case, the product-integral representation of the survival function does not reduce to the above-mentioned exponential function formula; cf. [15].

There are many procedures which are able to cope with tied survival data. The perhaps most famous of those is the Kaplan-Meier estimator [17] which consistently estimates survival functions possibly having discrete components; cf. e.g. Section 3.9 in [24]. Furthermore, the classical bootstrap for survival data, as proposed by [13], is able to reproduce the limit distribution of the Kaplan-Meier process, even if ties are present; cf. [1]. For estimating its unknown limit variances (or those of the related Nelson-Aalen estimators) it is well-known (e.g. [3]) that Greenwood-type estimators should be utilized in the presence of ties: They result from predictable variation processes of martingales which are "sufficiently general to handle discrete, continuous, and mixed cases" [16, p. 164]. Furthermore, the same authors state on p. 171 that, in contrast to other variance estimators, the Greenwood estimate "has the advantage of providing a natural bridge between the discrete and continuous cases". [2] even found a general preference (also under left-truncation) for this kind of estimator. There is a vast number of further work with a focus on estimation or model fitting under tied survival data. We only briefly mention the following: In Section 7.6 of [9], it is suggested for the modeling of discrete-time survival models with covariates to either replace the proportional hazards model by a logistic model involving the discrete hazard function or to treat tied data as if they were generated from continuous-time models. [14] discusses the use of logistic regression techniques for fitting parametric survival curves while allowing for ties in the case of censored data. Smoothing techniques for discrete survival data and model checking methods are described in [23].

Due to the complicated limit behaviour of process-valued estimators in survival analysis, such as the general Aalen-Johansen estimator for state transition probabilities in Markovian multi-state models (e.g. Chapter IV.4 in [3]), resampling methods are mandatory for deducing inference methods. The perhaps most convenient resampling method in non- and semiparametric survival analysis is not the classical bootstrap but rather the wild bootstrap; cf. [18], [20], [5]. One of its advantages is the capability of creating martingale residuals with approximately the right variances in semiparametric applications, even in the case of various different covariates. Typically, however, large sample properties of the wild bootstrap are verified by using the assumption of existing hazard rates which is sometimes too restrictive as discussed above. See [6] and [7] for a wild bootstrap treatment of, respectively, the general Nelson-Aalen and Aalen-Johansen estimators under absolute continuity.

In this article, we will consider the nonparametric inference problem of confidence band construction for cause-specific cumulative hazard or cumulative incidence functions in the case of right-censored competing risks set-ups. At the same time, we allow these true, underlying functions to have both, discrete and absolutely continuous components. While this allows for theoretical generality, it practically also implies that data may be collected with different levels of precision. Even though the underlying unknown functions are altered in this process, the wild bootstrap adjustment as developed in this article shall be able to reproduce the asymptotic distribution of the estimators in any case.

The present article is organized as follows: In Section 2, we first discuss the consequences resulting from not using Greenwood-type estimators and from utilizing the usual wild bootstrap procedure. We propose a discontinuity adjustment for the wild bootstrap in Section 3, accompanied by our first main result: a conditional central limit theorem for this new technique. In the following Section 4, we show that the Nelson-Aalen estimators for different competing risks are in general asymptotically dependent. Therefore, we present in a next step an extension of the first proposal for the wild bootstrap adjustment that guarantees the correct limit dependence structure between the components for different risks. This technique has some direct implications on resampling the Aalen-Johansen estimator for cumulative incidence functions as these depend on all cause-specific hazard functions and, therefore, also on their dependencies.

We present conditional central limit theorems corresponding to this set-up in Section 5, where also variance estimators for these Aalen-Johansen estimators and time-simultaneous confidence bands for cumulative incidence functions are deduced. The performance of these bands in terms of coverage probabilities is analyzed in a simulation study in Section 6 and there it is compared to the behaviour of confidence bands based on the usual, unadjusted wild bootstrap. In this connection, we consider different variations of discretization coarseness and discretization probabilities. All considered resampling techniques are applied to a real data example with competing risks in Section 7, where confidence bands for the probability of an alive discharge of male patients with pneumonia from intensive care units are constructed. We conclude with a discussion in Section 8. All proofs and various detailed derivations are presented in Appendices A–E.

2. Implications of the unadjusted wild bootstrap

In general, we will assumme that there are discrete components in the event time distribution and that the event, if observed, can be classified to one out of $k \in \mathbb{N}, k \geqslant 2$ different causes, i.e. competing risks. If T denotes a random event time, this implies that the function

$$\alpha(t) = \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t} P(T \in [t, t + \Delta t] \mid T \geqslant t)$$

does not exist. Suppose that $n \in \mathbb{N}$ i.i.d. individuals participate in our study, but that their observability may be independently right-censored by i.i.d. censoring variables. Based on these observations, all available collected information is described by the counting and at risk processes,

$$\begin{aligned} & N_{ji}(t) = 1\{\text{“ individual } i \text{ is observed to fail in } [0,t] \text{ due to risk } j\text{” }\} \\ \text{and} \quad & Y_i(t) = 1\{\text{“ individual } i \text{ is under observation at time } t- \text{ ” }\} \end{aligned}$$

respectively, with $j = 1, \ldots, k$ and $i = 1, \ldots, n$. This notation may be used to extend the arguments below to also incorporate independent left-truncation in the sense of [3], Chapter III. For the rest of this section, we fix an arbitrary $j \in \{1, \ldots, k\}$ and define $H_j^{uc}(t) = E(N_{j1}(t))$ and $\bar{H}(t) = E(Y_1(t))$. Now, the sum $N_j(t) = \sum_{i=1}^n N_{ji}(t)$ has the compensator

$$\Lambda_j(t) = \sum_{i=1}^n \Lambda_{ji}(t) = \int_0^t Y(u) \mathrm{d}A_j(u) = \sum_{i=1}^n \int_0^t Y_i(u) \mathrm{d}A_j(u),$$

where $A_j(t) = \int_0^t \bar{H}(u)^{-1} \mathrm{d}H_j^{uc}(u)$ is the cumulative hazard function for a type j event. Therefore, $(M_j(t) = N_j(t) - \Lambda_j(t))_{t \geqslant 0}$ is a square-integrable martingale. The Nelson-Aalen estimator for $A_j(t)$ is defined as the stochastic integral $\widehat{A}_j(t) = \int_0^t Y^{-1} \mathrm{d}N_j$, where we let $0/0 = 0$. The limit covariance function of the normalized process $W_j = \sqrt{n}(\widehat{A}_j - A_j)$ is determined by the predictable and optional variation processes of the square-integrable martingales

$M_{1i} : t \mapsto N_{1i}(t) - \Lambda_{1i}(t)$ denoted by $\langle M_{1i}\rangle(t)$ and $[M_{1i}](t)$, respectively, which simplify considerably in the special case of absolute continuity; cf. Section II.4 in [3] for detailed derivations.

In the case of ties, however, the differing structure of both variation processes cause asymptotic (co)variance functions which differ from those in the absolutely continuous case. Therefore, the Greenwood-type variance estimators

$$\widehat{\sigma}_j^2(t) = n\int_0^t \frac{1-\Delta\widehat{A}_j}{Y}\mathrm{d}\widehat{A}_j$$

should be utilized which are uniformly consistent for the true asymptotic variance

$$\sigma_j^2(t) = \int_0^t \frac{1-\Delta A_j}{\bar{H}}\mathrm{d}A_j$$

on any compact interval $[0, K] \subset [0, \infty)$ such that $A_j(K) < \infty$ irrespective of the presence of ties. These variance estimators are obtained by replacing all unknown quantities in any of the variation processes of the martingale representation of $\widehat{A}_j$ by their canonical estimators. Note that ignoring the ties and using the estimator $n\int_0^t Y^{-1}\mathrm{d}\widehat{A}_j$ instead of $\widehat{\sigma}_j^2(t)$ would result in a consistent estimator for $\int_0^t \bar{H}^{-1}\mathrm{d}A_j \neq \sigma_j^2(t)$. The difference between both quantities obviously increases the more the survival times are discretized. See e.g. Example 3.9.19 in [24] for a textbook treatment of the Nelson-Aalen estimator in the presence of ties using empirical processes. The variances of Aalen-Johansen estimators show a similar behaviour when comparing the absolutely continuous case with the case allowing for ties.

Now, the typical wild bootstrap procedure, as applied in survival and event history analysis (see e.g. [18], [20], [5] or [6]) exhibits a similar shortcoming as the non-Greenwood-type estimators: Applied to Nelson-Aalen estimators, this widely used resampling procedure produces stochastic processes which, given all data, converge in distribution on Skorohod spaces to Gaussian processes with variance functions being the limits of $n\int_0^t Y^{-1}\mathrm{d}\widehat{A}_j \neq \widehat{\sigma}_j^2(t)$. This is due to the nature of how martingale residuals are replaced in the resampling step: the (independent) martingale increments $\mathrm{d}M_{ji}(u)$ in the martingale representation of $\sqrt{n}(\widehat{A}_j - A_j)$ are replaced by independently weighted counting process increments, i.e., by $\xi_i\mathrm{d}N_{ji}(u)$. Here, the wild bootstrap weights ξ_i are i.i.d. with zero mean and variance 1. Given $N_{ji}(u)$, however, the resampled residual has the conditional variance

$$var(\xi_i\mathrm{d}N_{ji}(u) \mid N_{ji}(u)) = var(\xi_i)\mathrm{d}N_{ji}(u) = \mathrm{d}N_{ji}(u) \neq (1-\Delta A_j(u))\mathrm{d}N_{ji}(u).$$

But only the latter would have succeeded in recovering the correct asymptotic variance structure. We see from these calculations that the usual wild bootstrap generally overestimates the variance of the Nelson-Aalen estimator in the presence of ties. A similar problem persists when considering asymptotic covariances between multiple components of the Nelson-Aalen estimator: In the case of ties, the Nelson-Aalen estimators are in general not asymptotically independent, whereas the commonly applied wild bootstrap only produces independent components; cf. [6]. However, a negative correlation between all Nelson-Aalen estimators arises quite naturally: if several type j events occur at a certain point of time, then this reduces the number of possible type $\ell \neq j$ events at the same time. All these properties obviously also have implications on Aalen-Johansen estimators and their resampling counterparts which will be treated later on.

3. Adjusted wild bootstrap: single Nelson-Aalen estimators

The non-trivial problem described in the previous section calls for a general solution. In the present article, we exemplify the subsequent solution in the right-censored competing risks set-up. Extensions and modifications to wild bootstrap versions of more general Nelson-Aalen estimators or of Aalen-Johansen estimators in general Markovian situations may be obtained in a similar manner, but the limit variances will be much more complicated. The crucial defect in the wild bootstrap resampling scheme described above is that the martingale increments $\mathrm{d}M_{ji}(u)$ should not be replaced by $\xi_i \mathrm{d}N_{ji}(u)$ but rather by something which – considered again as a martingale – reproduces the correct (co)variation structure. Therefore, we suggest to replace

$$\mathrm{d}M_{ji}(u) \quad \text{by} \quad \mathrm{d}\widehat{M}_{ji}(u) = \xi_i \sqrt{1 - \Delta \widehat{A}_j(u)} \mathrm{d}N_{ji}(u),$$

which has a promising conditional variance:

$$var(\mathrm{d}\widehat{M}_{ji}(u) \mid N_{ji}(u)) = var(\xi_i)(1 - \Delta \widehat{A}_j(u)) \mathrm{d}N_{ji}(u) = (1 - \Delta \widehat{A}_j(u)) \mathrm{d}N_{ji}(u);$$

remember that we desired variances approximately equal to $(1-\Delta A_j(u))\mathrm{d}N_{ji}(u)$. This results in the following wild bootstrap resampling version of the normalized Nelson-Aalen estimator:

$$\widehat{W}_j(t) = \sqrt{n} \sum_{i=1}^n \xi_i \int_0^t \sqrt{1 - \Delta \widehat{A}_j(u)} \frac{\mathrm{d}N_{ji}(u)}{Y(u)}.$$

In a way similar to [6], one can show that $(\widehat{W}_j(t))_{t \in [0,K]}$ is a martingale with respect to the filtration $(\mathcal{F}_t)_{t \in [0,K]}$ given by

$$\mathcal{F}_t = \sigma(\xi_i N_{ji}(u), N_{ji}(v), Y_i(v) : u \in [0,t], v \in [0,K], i = 1, \dots, n).$$

It is easy to see that its predictable and optional variation processes are given by

$$\begin{aligned} \widehat{\sigma}_j^2 : t \longmapsto \langle \widehat{W}_j \rangle(t) &= n \int_0^t (1 - \Delta \widehat{A}_j(u)) \frac{\mathrm{d}\widehat{A}_j(u)}{Y(u)} \\ \text{and} \quad \breve{\sigma}_j^2 : t \longmapsto [\widehat{W}_j](t) &= n \sum_{i=1}^n \xi_i^2 \int_0^t (1 - \Delta \widehat{A}_j(u)) \frac{\mathrm{d}N_{ji}(u)}{Y^2(u)}. \end{aligned}$$

As explained above, the Greenwood-type variance estimator $\widehat{\sigma}_j^2$ is uniformly consistent for σ_j^2. Assuming the existence of the fourth moments of ξ_i, a simple application of Chebyshev's inequality shows the (conditional) consistency of the second estimator; the uniformity in the conditional convergence in probability follows from a Pólya-type argument. The conditional weak convergence of the finite-dimensional marginal distributions of the wild bootstrapped Nelson-Aalen process follows easily by an application of Theorem A.1 in [5]. This also shows that the proposed wild bootstrap approach succeeds in maintaining the correct asymptotic covariance function which had been our aim in the first place.

Denote by $D = \{t \in [0, K] : \sum_{j=1}^k \Delta A_j(t) > 0\}$ the subset of time points for which ties among any event type are possible. Throughout the rest of the article, we assume the following technical condition in order to conclude the conditional tightness of the wild bootstrapped Nelson-Aalen process.

Condition 3.1. *The set of discontinuity time points D has finitely many elements.*

In practical applications, this assumption is naturally satisfied: A finite end-of-study time and measurements on a daily or weekly basis result in a finite lattice. A proof of conditional tightness finally yields the following conditional central limit theorem for the Nelson-Aalen process, where $\xrightarrow{d}$ denotes convergence in distribution:

Theorem 3.2. *Assume Condition 3.1. Given $\mathcal{F}_0$ and as $n \to \infty$, we have for each $j = 1, \dots, k$ the following conditional weak convergence*

$$\widehat{W}_j \xrightarrow{d} U_j \sim \mathit{Gauss}(0, \sigma_j^2) \qquad \text{in outer probability}$$

on the càdlàg function space $D[0, K]$ equipped with the supremum distance topology, where U_j is a Gaussian zero-mean martingale with variance function $t \mapsto \sigma_j^2(t)$.

The theorem entails that the modified wild bootstrap succeeds in reproducing the same limit process of the Nelson-Aalen process, in particular, if ties are present. Its proof is given in Appendix B. We conclude this section with an application of the present approach to the Kaplan-Meier estimator.

Remark 3.3. *Consider the case of only one risk, i.e. $k = 1$ and $W = W_1$. The Kaplan-Meier estimator for the survival function $S(t) = P(T > t)$ is $\widehat{S}(t) = \prod_{0 \leqslant u \leqslant t}(1 - \widehat{A}(\mathrm{d}u)),\ t \in [0, K]$, and it exhibits the martingale representation*

$$W_S(t) = \sqrt{n}(\widehat{S}(t) - S(t)) = S(t) \int_0^t \frac{W(\mathrm{d}u)}{1 - \Delta A(u)} + o_p(1).$$

Thus, the discontinuity-adjusted wild bootstrapped normalized Kaplan-Meier estimator is

$$\widehat{W}_S = \widehat{S} \int_0^{\cdot} \frac{\widehat{W}(\mathrm{d}u)}{1 - \Delta \widehat{A}(u)}$$

and Theorem 3.2 in combination with the continuous mapping theorem yields the correct limit process distribution, i.e. a zero-mean Gaussian process with covariance function given by

$$(s,t) \mapsto S(s)S(t) \int_0^{s\wedge t} \frac{\mathrm{d}A(u)}{(1-\Delta A(u))\bar{H}(u)},$$

where $s \wedge t = \min(s,t)$; cf. e.g. Example 3.9.31 in [24].

4. Resampling the multivariate Nelson-Aalen estimator

When resampling a functional of a multivariate Nelson-Aalen estimator such as the Aalen-Johansen estimator, it is mandatory to also take the covariance structure between all cause-specific Nelson-Aalen estimators into account. In order to reflect this in the resampling scheme, a further adjustment needs to be done as we will see below. In the absolutely continuous case, the asymptotic covariance function of two different cause-specific Nelson-Aalen estimators vanishes due to their asymptotic independence; cf. [3], Theorem IV.1.2. In the presence of ties, however, the situation is quite different: Here we have for the martingales M_{1i} and M_{2i} of Section 2 that both variation processes, $\langle M_{1i}, M_{2i}\rangle$ and $[M_{1i}, M_{2i}]$ do not vanish; cf. the derivations in Section II.4 in [3]. The following theorem describes the asymptotic dependence structure of a multivariate Nelson-Aalen estimator in detail.

Theorem 4.1. *As $n \to \infty$, we have on the product càdlàg function space $D^k[0,K]$, equipped with the* max-sup-*norm, that*

$$(W_1, W_2, \dots, W_k) \xrightarrow{d} (U_1, U_2, \dots, U_k),$$

where $U_1, U_2, \dots, U_k$ are zero-mean Gaussian martingales with variance functions

$$t \mapsto \sigma_j^2(t) = \int_0^t \frac{1-\Delta A_j(u)}{\bar{H}(u)} \mathrm{d}A_j(u), \quad j = 1, 2, \dots, k$$

and covariance functions (for $j \neq \ell$)

$$(s,t) \mapsto cov(U_j(s), U_\ell(t)) = -\int_0^{s\wedge t} \frac{\Delta A_\ell(u)}{\bar{H}(u)} \mathrm{d}A_j(u) =: \sigma_{j\ell}(s\wedge t).$$

We refer to Appendix C for a derivation of this asymptotic covariance function. In order to account for this dependence structure in a joint convergence consideration, the wild bootstrap of the previous section needs to be adjusted once more. The general aim is to find resampling versions of the martingales M_{ji} that will yield the asymptotic variances $\sigma_j^2(t)$ and also the negative correlation described by $\sigma_{j\ell}(s \wedge t)$. To this end, the wild bootstrap multipliers involved in one of the resampled Nelson-Aalen estimators need to be involved in the remaining resampled cause-specific Nelson-Aalen estimators as well. Furthermore, they are required to a appear with alternating signs. Therefore, let

$\xi_{j\ell i}, j, \ell = 1, \ldots, k,\ i = 1, \ldots, n$, be i.i.d. random variables with $E(\xi_{111}) = 0$ and $E(\xi_{111}^2) = 1$, which are also independent of the data. Denote by $N = \sum_{j=1}^k N_j$ the number of all kinds of events and by $\widehat{A} = \sum_{j=1}^k \widehat{A}_j$ the all-cause Nelson-Aalen estimator which estimates the all-cause cumulative hazard function $A = \sum_{j=1}^k A_j$. We propose the following wild bootstrap version of the multivariate Nelson-Aalen estimator $(W_1, \ldots, W_k)$ in terms of its single components:

$$\widehat{W}_j(t) = \sqrt{n} \sum_{i=1}^n \xi_{jji} \int_0^t \sqrt{1 - \Delta\widehat{A}(u)} \frac{\mathrm{d}N_{ji}(u)}{Y(u)} + \sqrt{\frac{n}{2}} \sum_{\ell=1}^k sign(\ell - j)$$
$$\times \sum_{i=1}^n \Big[\xi_{j\ell i} \int_0^t \sqrt{\Delta\widehat{A}_j(u)} \frac{\mathrm{d}N_{\ell i}(u)}{Y(u)} + \xi_{\ell ji} \int_0^t \sqrt{\Delta\widehat{A}_\ell(u)} \frac{\mathrm{d}N_{ji}(u)}{Y(u)}\Big],$$

where $sign(x) = 1\{x > 0\} - 1\{x < 0\}$ is the signum function. Ignoring, for the time being, the second part of the above representation of $\widehat{W}_j$, we see that the resulting asymptotic covariance function would be $(s,t) \mapsto \int_0^{s \wedge t} (1 - \Delta A(u)) \mathrm{d}A_j(u)$ which is too small and, additionally, the covariance structure of several cause-specific Nelson-Aalen estimates would not be reflected accordingly because of the mutual (conditional given $\mathcal{F}_0$) independence of the first parts of $\widehat{W}_1, \ldots, \widehat{W}_k$.

Now, the second parts of $\widehat{W}_j, j = 1, \ldots, k$, remedy both problems at the same time. First, further variability is introduced due to the addition of other conditionally independent terms. Second, the involved signum function ensures the required negative covariances between all resampled Nelson-Aalen estimators and the asymptotic covariances exactly equal the required ones.

Note that the same asymptotic effect would have been achieved by choosing the second part of $\widehat{W}_j(t)$ to be $\sqrt{n} \sum_{\ell=1}^k sign(\ell - j) \sum_{i=1}^n \xi_{j\ell i} \int_0^t \sqrt{\Delta\widehat{A}_j(u)} \frac{\mathrm{d}N_{\ell i}(u)}{Y(u)}$ (even more choices are possible) but we decided in favor of the variant in the previous display which is symmetric in $j, \ell = 1, \ldots, k$.

Theorem 4.2. *Assume Condition 3.1. Given $\mathcal{F}_0$ and as $n \to \infty$, we have the following conditional weak convergence on the product càdlàg function space $D^k[0, K]$, equipped with the* max-sup-*norm:*

$$(\widehat{W}_1, \widehat{W}_2, \ldots, \widehat{W}_k) \xrightarrow{d} (U_1, U_2, \ldots, U_k) \qquad \textit{in outer probability,}$$

where $(U_1, U_2, \ldots, U_k)$ is the same Gaussian martingale as in Theorem 4.1.

The proof is given in Appendix D. Note that, if we are interested in just a single univariate Nelson-Aalen estimator, the present approach yields the same limit distribution as the wild bootstrap technique proposed in Section 3. Hence, it does – asymptotically – not matter which of both techniques is applied to the univariate Nelson-Aalen estimator.

Variance and covariance estimators (also for the wild bootstrap versions) are again motivated by the predictable and optional covariation processes of the involved martingales. The resulting estimators turn out to be same as those obtained by the plug-in method:

$$\widehat{\sigma}_j^2(t) = n \int_0^t \frac{1 - \Delta \widehat{A}_j(u)}{Y(u)} \mathrm{d}\widehat{A}_j(u),$$
$$\widehat{\sigma}_{j\ell}(t) = -n \int_0^t \frac{\Delta \widehat{A}_j(u)}{Y(u)} \mathrm{d}\widehat{A}_\ell(u),$$

$1 \leqslant j \neq \ell \leqslant k$, are the usual Greenwood-type (co)variance estimators and

$$\begin{aligned}\breve{\sigma}_j^2(t) &= n \sum_{i=1}^n \xi_{jji}^2 \int_0^t (1 - \Delta \widehat{A}_j(u)) \frac{\mathrm{d}N_{ji}(u)}{Y^2(u)} \\ &\quad + \frac{n}{2} \sum_{\ell \neq j} \Big[\sum_{i=1}^n \xi_{j\ell i}^2 \int_0^t \Delta \widehat{A}_j(u) \frac{\mathrm{d}N_{\ell i}(u)}{Y^2(u)} + \sum_{i=1}^n \xi_{\ell j i}^2 \int_0^t \Delta \widehat{A}_\ell(u) \frac{\mathrm{d}N_{ji}(u)}{Y^2(u)} \Big], \\ \breve{\sigma}_{j\ell}(t) &= -\frac{n}{2} \Big[\sum_{i=1}^n \xi_{j\ell i}^2 \int_0^t \Delta \widehat{A}_j(u) \frac{\mathrm{d}N_{\ell i}(u)}{Y^2(u)} + \sum_{i=1}^n \xi_{\ell j i}^2 \int_0^t \Delta \widehat{A}_\ell(u) \frac{\mathrm{d}N_{ji}(u)}{Y^2(u)} \Big],\end{aligned}$$

$1 \leqslant j \neq \ell \leqslant k$, are the optional process-type (co)variance estimators motivated from the wild bootstrap martingale properties. Assume that all ξ_{111} have finite fourth moments. By applications of Glivenko-Cantelli theorems in combination with the continuous mapping theorem, the Greenwood-type (co)variance estimators $\widehat{\sigma}_j^2$ and $\widehat{\sigma}_{j\ell}$ are shown to be uniformly consistent for σ_j^2 and $\sigma_{j\ell}$, respectively. For the wild bootstrap-type (co)variance estimators, we can parallel the arguments in the proof of Theorem 3.2, after first assuming the existence of fourth moments $E(\xi_{111}^4) < \infty$: In points of continuity of all cumulative hazard functions, i.e. on $[0, K] \backslash D$, Rebolledo's martingale central limit theorem applies and it also implies the uniform consistency of the optional variation process increments. In points of discontinuity, which are finitely many by assumption, we approximate $\breve{\sigma}_j^2$ by $\widehat{\sigma}_j^2$ and apply the conditional Chebyshev inequality (given $\mathcal{F}_0$) in order to show the negligibility of the differences $\breve{\sigma}_j^2 - \widehat{\sigma}_j^2$ in probability. The last argument can be repeated for the covariance estimators. A final application of the continuous mapping theorem yields

$$\widehat{\sigma}_j^2, \ \breve{\sigma}_j^2 \xrightarrow{p} \sigma_j^2 \quad \text{and} \quad \widehat{\sigma}_{j\ell}, \ \breve{\sigma}_{j\ell} \xrightarrow{p} \sigma_{j\ell}$$

uniformly on $[0, K]$ in (conditional outer) probability for all $j \neq \ell$ as $n \to \infty$, where $\xrightarrow{p}$ denotes convergence in probability.

5. Resampling the deduced Aalen-Johansen estimator

We wish to extend the results of the previous section to functionals of the multivariate Nelson-Aalen estimator. In particular, we focus on resampling the Aalen-Johansen estimator for a cumulative incidence function in the presence of ties. Denote these cumulative incidence functions by $F_j(t) = \int_0^t S(u-)\mathrm{d}A_j(u), j = 1, \dots, k$, which specify the probabilities to have a type j event during the time interval $[0, t]$. For ease of presentation, we consider the situation of $k = 2$ competing risks which is achieved by aggregating all but the first risk to be the second competing risk. Utilizing the functional delta-method in combination with the weak convergence results for the Nelson-Aalen estimator, we as well obtain a weak convergence theorem for the Aalen-Johansen estimator $\widehat{F}_1(t) = \int_0^t \widehat{S}(u-)\mathrm{d}\widehat{A}_1(u)$ for $F_1(t)$:

Theorem 5.1. *As $n \to \infty$, we have on the càdlàg space $D[0, K]$*

$$\begin{aligned} W_{F_1} &= \sqrt{n}(\widehat{F}_1 - F_1) \xrightarrow{d} U_{F_1} \\ &= \int_0^{\cdot} \frac{1 - F_2(u-) - F_1(\cdot)}{1 - \Delta A(u)} \mathrm{d}U_1(u) + \int_0^{\cdot} \frac{F_1(u-) - F_1(\cdot)}{1 - \Delta A(u)} \mathrm{d}U_2(u), \end{aligned}$$

where U_{F_1} is a zero-mean Gaussian process with covariance function

$$\begin{aligned} \sigma^2_{F_1} : (s,t) \mapsto &\int_0^{s\wedge t} \frac{(1 - F_2(u-) - F_1(s))(1 - F_2(u-) - F_1(t))}{\bar{H}(u)} \frac{\mathrm{d}A_1(u)}{1 - \Delta A(u)} \\ &+ \int_0^{s\wedge t} \frac{(F_1(u-) - F_1(s))(F_1(u-) - F_1(t))}{\bar{H}(u)} \frac{\mathrm{d}A_2(u)}{1 - \Delta A(u)} \\ &+ \sum_{\substack{u \in D \\ u \leqslant s,t}} \frac{S^2(u-)}{\bar{H}(u)} \frac{\Delta A_1(u)\Delta A_2(u)}{(1 - \Delta A(u))^2}. \end{aligned}$$

Remark 5.2. *The general result for more than two competing risks is obtained by replacing U_2, A_2, and F_2 in the above representation by $U - U_1$, $A - A_1$, and $1 - S - F_1$, respectively.*

For the application of the functional delta-method, note that the Aalen-Johansen estimator in the present competing risks framework is a combination of the Wilcoxon and the product integral functional applied to the multivariate Nelson-Aalen estimator. Both of these functionals are Hadamard-differentiable as shown for example in Section 3.9 of [24]. A derivation of the above asymptotic covariance function is presented in Appendix E.

Now, an appropriate wild bootstrap version of $\sqrt{n}(\widehat{F}_1 - F_1)$ is given by plugging in the canonical estimators for all unknown quantities and the wild bootstrapped martingales for all unknown counting process martingales, i.e.

$$\widehat{W}_{F_1}(t) = \int_0^{\cdot} \frac{1 - \widehat{F}_2(u-) - \widehat{F}_1(\cdot)}{1 - \Delta\widehat{A}(u)} \mathrm{d}\widehat{W}_1(u) + \int_0^{\cdot} \frac{\widehat{F}_1(u-) - \widehat{F}_1(\cdot)}{1 - \Delta\widehat{A}(u)} \mathrm{d}\widehat{W}_2(u),$$

where $\widehat{W}_1$ and $\widehat{W}_2$ are again the wild bootstrap versions of the Nelson-Aalen estimators as presented in Section 4. Using similar martingale arguments as in Appendix B, we obtain the following conditional central limit theorem for the wild bootstrap version of the Aalen-Johansen estimator:

Theorem 5.3. *Assume Condition 3.1. Given $\mathcal{F}_0$ and as $n \to \infty$, we have the following weak convergence on the càdlàg function space $D[0, K]$, equipped with* *the* sup-*norm:*

$$\widehat{W}_{F_1} \xrightarrow{d} U_{F_1} \qquad \textit{in outer probability,}$$

where U_{F_1} *is the same Gaussian process as in Theorem 5.1.*

If based on the same wild bootstrapped multivariate Nelson-Aalen estimator $(\widehat{W}_1, \ldots, \widehat{W}_k)$, one can similarly argue that the joint convergence of all resampled Aalen-Johansen estimators, say $(\widehat{W}_{F_1}, \ldots, \widehat{W}_{F_k})$, towards the same limit distribution of $(\sqrt{n}(\widehat{F}_j - F_j))_{j=1}^k$ holds.

Remark 5.4 (The weird bootstrap). *Note that the very same proofs may be applied to verify that the above conditional central limit theorems hold for the weird bootstrap as well. This resampling scheme corresponds to choosing* $\xi_{j\ell i} + 1 \sim Bin(Y(X_i), \max(1, Y(X_i))^{-1})$, *where* X_i *is the censoring or event time of individual* i, *whichever comes first. This is a particular choice of the data-dependent multiplier bootstrap of [12]. In their article, heuristic arguments for the second order correctness under absolute continuity of the data have shown that centered unit Poisson variates and weird bootstrap multipliers perform favorably in comparison to standard normal wild bootstrap weights. In order to also check the preference of either of the first two resampling procedures in the present set-up, where ties are allowed, we included the weird bootstrap in the subsequent simulation study, yielding competing inference methods.*

An estimator for $\sigma^2_{F_1}$ and its wild bootstrap variant are obtained similarly as such estimators for the Nelson-Aalen (co)variances, i.e. via plug-in:

$$\begin{aligned}
\widehat{\sigma}^2_{F_1} : (s,t) \mapsto & \int_0^{s\wedge t} \frac{(1 - \widehat{F}_2(u-) - \widehat{F}_1(s))(1 - \widehat{F}_2(u-) - \widehat{F}_1(t))}{(1 - \Delta\widehat{A}(u))^2} \mathrm{d}\widehat{\sigma}^2_1(u) \\
& + \int_0^{s\wedge t} \frac{(\widehat{F}_1(u-) - \widehat{F}_1(s))(\widehat{F}_1(u-) - \widehat{F}_1(t))}{(1 - \Delta\widehat{A}(u))^2} \mathrm{d}\widehat{\sigma}^2_2(u) \\
& + \int_0^{s\wedge t} \frac{(1 - \widehat{F}_2(u-) - \widehat{F}_1(s))(\widehat{F}_1(u-) - \widehat{F}_1(t))}{(1 - \Delta\widehat{A}(u))^2} \mathrm{d}\widehat{\sigma}_{12}(u) \\
& + \int_0^{s\wedge t} \frac{(1 - \widehat{F}_2(u-) - \widehat{F}_1(t))(\widehat{F}_1(u-) - \widehat{F}_1(s))}{(1 - \Delta\widehat{A}(u))^2} \mathrm{d}\widehat{\sigma}_{21}(u).
\end{aligned}$$

Similarly, $\breve{\sigma}^2_{F_1}$ is obtained by replacing all estimators $\widehat{\sigma}^2_j$ and $\widehat{\sigma}_{j\ell}$, $j \neq \ell$, by their wild bootstrap counterparts $\breve{\sigma}^2_j$ and $\breve{\sigma}_{j\ell}$, respectively. Their uniform (conditional) consistencies for $\sigma^2_{F_1}$ on $[0, K]$ follow immediately by the uniform consistency of the Nelson-Aalen (co)variance estimators and the continuous mapping theorem.

Remark 5.5 (Deduced confidence bands). *Following the lines of [5], time-simultaneous confidence bands for* F_1 *can be deduced. In particular, let* $\phi(s) = \log(-\log(1-s))$ *be a transformation applied to* F_1 *in order to ensure band boundaries between 0 and 1 and let* $g_1(s) = \log(1 - \widehat{F}_1(s))/\widehat{\rho}(s)$ *and* $g_2(s) = \log(1 -$

$\widehat{F}_1(s))/(1+\widehat{\rho}^2(s))$ be weight functions leading to the usual equal precision and Hall-Wellner bands, respectively; see [3]. Here $\widehat{\rho}^2(s) = \widehat{\sigma}^2_{F_1}(s)/(1-\widehat{F}_1(s))^2$. Let $\widehat{g}_1$ and $\widehat{g}_2$ be their wild bootstrap counterparts, i.e. the variance estimates $\widehat{\sigma}^2_{F_1}$ are replaced by $\breve{\sigma}^2_{F_1}$. The confidence bands for F_1 are then derived from the asymptotics of the supremum distance $Z_{1\ell} = \sup_{u\in[t_1,t_2]} |\sqrt{n} g_\ell(u)(\phi(\widehat{F}_1(u)) - \phi(F_1(u)))|$ and its wild bootstrap counterpart $\widehat{Z}_{1\ell} = \sup_{u\in[t_1,t_2]} |\widehat{g}_\ell(u)\phi'(\widehat{F}_1(u))\widehat{W}_{F_1}(u)|$, where $[t_1,t_2] \subset [0,K]$ and $\ell \in \{1,2\}$. Let $q_{0.95,\ell}$ be the conditional 95% quantile of $\widehat{Z}_{1\ell}$ given the data. The resulting asymptotic 95% confidence bands are $1-(1-\widehat{F}_1(s))^{\exp(\pm n^{-1/2} q_{0.95,\ell}/g_\ell(s))}$, $s \in [t_1,t_2]$, $\ell \in \{1,2\}$.

6. Small sample behaviour

We empirically assess the difference between the common wild bootstrap approach and the adjusted wild bootstrap proposed in this article via simulation studies. We simulated the wild bootstrap procedures based on standard normal and centered unit Poisson multipliers as well as the weird bootstrap of Remark 5.4. These methods are compared in terms of the simulated coverage probabilities of the confidence bands described in Remark 5.5. We consider a simulation set-up motivated by [10], i.e. we chose the cause-specific hazard rates $\alpha_1(t) = \exp(-t)$ and $\alpha_2(t) = 1-\exp(-t)$ which yield the cumulative function of the first risk $F_1(t) = 0.5(1-\exp(-2t))$. In order to allow for tied data, we pre-specify different discretization lattices and round different proportions of the population to the nearest discretization point. In particular, we choose the discretization lattices to be $\{0, \frac{1}{k}, \frac{2}{k}, \dots\}$, where $k \in \{5, 10, 20\}$, and the discretization probabilities to be $p \in \{0, 0.25, 0.5, 0.75, 1\}$. The resulting theoretic cumulative incidence functions

$$F_1^{p,k}(t) = pF_1\Big(\frac{[kt-0.5]}{k} + \frac{0.5}{k}\Big) + (1-p)F_1(t)$$

for $p > 0$ are presented in Figure 1. Here $[s]$ denotes the integer closest to $s \in \mathbb{R}$. For simulating data, which have the desired cumulative incidence function $F_1^{p,k}$, it is mandatory to first round the event times T_i, and then generate the event types ε_i in a second step, according to the formula

$$P\Big(\varepsilon_i = 1 \mid \frac{[T_i k]}{k} = u\Big) = \frac{F_1(u+\frac{1}{2k}) - F_1(\max(u-\frac{1}{2k},0))}{S(\max(u-\frac{1}{2k},0)) - S(u+\frac{1}{2k})},$$

where $S: t \mapsto \exp(-t)$ denotes the survival function of the continuous random variables T_i.

Censoring is introduced by i.i.d. standard exponentially distributed random variables. If the ith survival time is discretized, then we discretize the ith censoring time as well. Finally, we take the minimum out of each such pair and mark an individual as censored whenever the (discretized) censoring time precedes the (discretized) survival time. The sample size increases from $n = 50$ to $n = 250$ in steps of 25. We choose the time interval, along which asymptotic

95% confidence bands shall be constructed, to be $[0.25, 0.75]$. The simulations have been conducted using `R` version 3.2.3 [22] using 10,000 outer Monte Carlo iterations and 999 wild bootstrap replicates.

Tables 1 to 3 contain the simulated coverage probabilities of equal precision and Hall-Wellner bands for simulation set-ups with a discrete component in the cumulative incidence function, i.e. $p > 0$. The columns of simulation results corresponding to the common wild / weird bootstrap procedures are entitled *old*, whereas the columns showing the results of the respective adjusted wild / weird bootstrap are entitled *new*.

At first, we start with a discussion on the choice of multipliers. For equal precision bands and in almost all set-ups, there is a pronounced superiority of the wild bootstrap with centered unit Poisson multipliers and the weird bootstrap over the respective coverage probabilities of the bands based on standard normal weights. This is true for the common resampling procedures as well as for the proposed adjusted bootstraps. For Hall-Wellner bands, this superiority is not as much pronounced and sometimes even the confidence bands based on standard normal multipliers yield the most accurate coverage probabilities. But in cases, where this is so, the deviance is only very small. The phenomenon, that standard normal multipliers yield a worse performance than those with skewness equal to one, is in line with the findings of [12] where also heuristic theoretic arguments for a second-order correctness of both superior resampling procedures are provided. As there is, all in all, not much of a difference between the simulated coverage probabilities of the centered unit Poisson wild bootstrap and the weird bootstrap, we only focus on the results of the Poisson choice. In general, the equal precision bands appear to be more accurate than the Hall-Wellner bands, except for some very small sample set-ups or sometimes for $p = 1$.

In almost any scenario, the discretization-adjusted wild bootstrap yielded coverage probabilities closer to the nominal level in comparison to the unadjusted wild bootstrap. The deviances between these coverage probabilities of each of those two resampling procedures appear to be larger the higher the discretization probability p and the coarser the discretization lattice is. For instance, this difference even amounts to 4.1 percentage points in case of the Hall-Wellner bands, $n = 50$, $k = 5$, and $p = 1$ and to 3.8 percentage points in case of equal precision bands and the same n, k and p.

In case of $k \in \{5, 10\}$, the coverage probabilities of the common wild bootstrap do not appear to converge at all towards 95% as the sample size increases. Instead, the simulated probabilities fluctuate around 93% or even 92%. On the other hand, the discretization-adjusted equal precision wild bootstrap bands yield much better coverage probabilities which are greater than 94% or at least in the high 93%-region for larger sample sizes. In contrast to the unadjusted procedure, we observe for small samples and for the adjusted confidence bands coverage probabilities closer to the nominal level for higher discretization probabilities p. This is only reasonable as $p = 100\%$ corresponds to a multivariate, but not an infinite-dimensional statistical problem. We do not see this tendency for the unadjusted procedure in case of $k \in \{5, 10\}$, which again stresses that it is not suitable for these kinds of tied data regimes.

The better coverage probabilities of the adjusted wild bootstrap may be explained by means of the asymptotic covariance function $\sigma^2_{F_1}(s,t)$ given in Theorem 5.1: This covariance is generally increased in comparison to the case of no ties which would lead to the covariance

$$\int_0^{s\wedge t} (1 - F_2(u) - F_1(s))(1 - F_2(u) - F_1(t)) \frac{\mathrm{d}A_1(u)}{\bar{H}(u)} \\ + \int_0^{s\wedge t} (F_1(u) - F_1(s))(F_1(u) - F_1(t)) \frac{\mathrm{d}A_2(u)}{\bar{H}(u)}.$$

The unadjusted wild bootstrap, however, only results in processes with limit distributions having the latter covariance. That is, the unadjusted wild bootstrap does not account for the increased variability due to tied data and this is reflected in its bad performance in the simulation study. Similarly, when focus is on inference methods for cumulative hazard functions, the common wild bootstrap generally overestimates the variability of the Nelson-Aalen estimator.

Finally, Table 4 shows the corresponding results for the scenario in which the continuous F_1 is the true cumulative incidence function and the usual wild bootstrap technique yields asymptotically exact inference procedures. Here, it surprises to see that the adjusted wild bootstrap again yields more accurate confidence bands than the unadjusted procedure. Therefore, there is apparently no loss at all in utilizing the discretization adjustment, even if the data contain no ties.

All in all, we conclude that the proposed discontinuity adjustment should always be applied in order to greatly improve the coverage probabilities of confidence bands for $F_1^{p,k}$. The present simulation results show this improvement, which amounts to up to two or three percentage points for smaller samples and in many conducted simulation scenarios. As the standard normal variate-based wild bootstrap disappoints in general, our final advice is to combine the present discontinuity adjustment with the wild bootstrap based on the Poisson-distributed random variables or the weird bootstrap. Additionally, equal precision bands should be preferred to Hall-Wellner confidence bands due to the slight but frequent superiority of the first in terms of coverage probabilities.

Note that, even with the best combination of the kind of confidence band and the choice of multipliers, the adjusted wild bootstrap proposed in this article still produces slightly too narrow confidence bands. For instance, this phenomenon has been observed similarly by [4], [12] and [11]. In the latter it is seen that the Kolmogorov-Smirnov-type statistic applied to Aalen-Johansen estimators in right-censored competing risks set-ups and in combination with the wild bootstrap yields a quite liberal test procedure when testing equality of cumulative incidence functions. In their simulation studies, moderately large sample sizes are required in order to obtain empirical type I error rates close to the nominal level. This behaviour may be due to the infinite dimensionality of the problem of constructing nonparametric confidence bands for cumulative incidence functions.

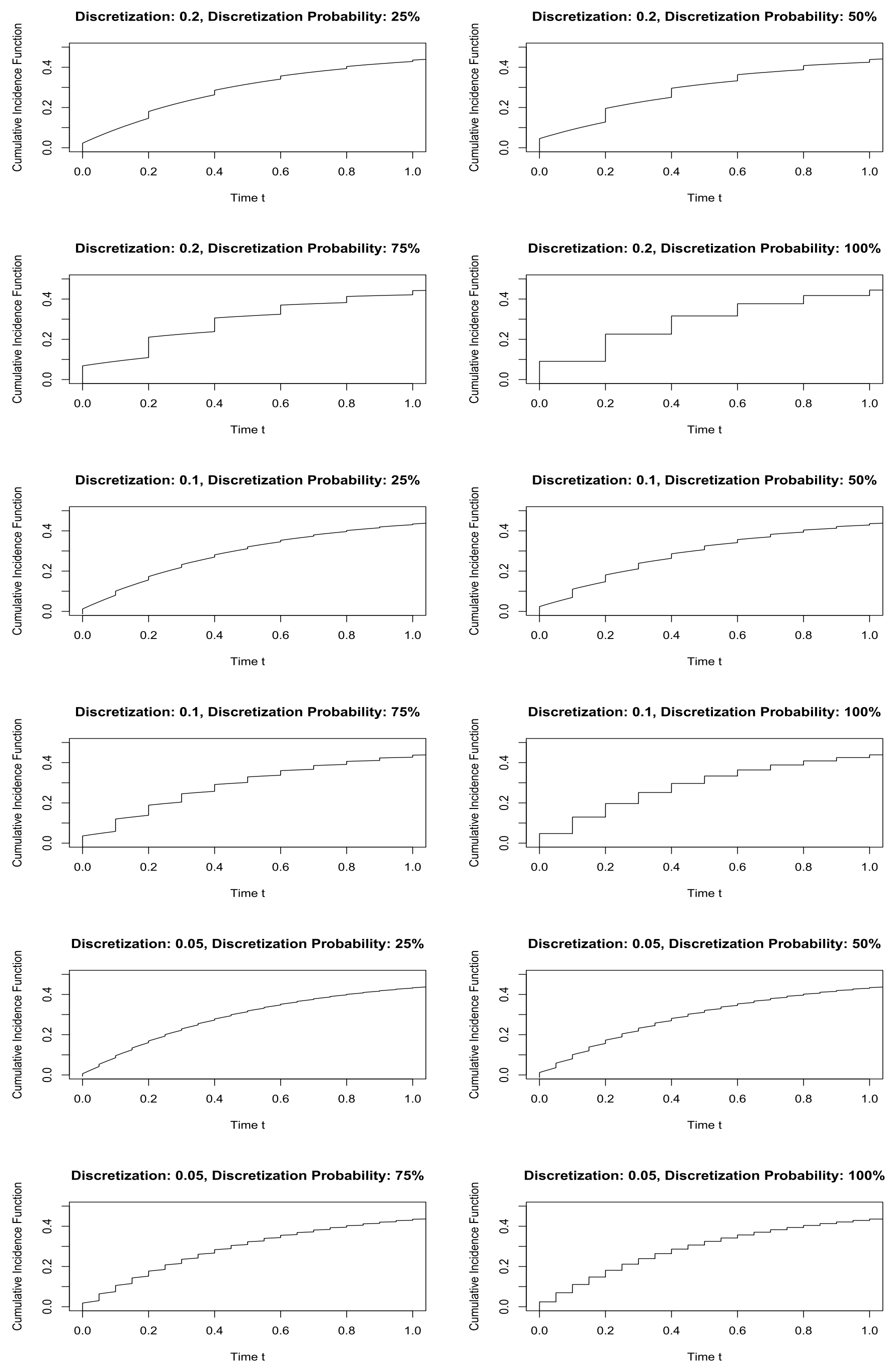

FIG 1. *Cumulative incidence functions $F_1^{p,k}$ underlying the present simulations.*

		equal precision						Hall-Wellner					
set-ups		$N(0,1)$		$Poi(1)-1$		weird		$N(0,1)$		$Poi(1)-1$		weird	
p	n	old	new	old	new	old	new	old	new	old	new	old	new
25%	50	87.3	88.5	89.4	90.6	88.9	90.4	88.7	89.9	89.3	**90.8**	88.9	90.2
	75	89.5	90.4	91.6	**92.2**	91.1	92.0	89.8	90.6	90.6	91.5	90.4	91.3
	100	90.6	91.3	92.3	**93.0**	92.2	92.6	90.5	91.1	90.9	91.8	91.1	91.9
	125	91.2	92.0	92.4	93.1	93.4	**93.8**	90.9	91.7	91.1	91.8	91.1	91.9
	150	91.1	91.7	92.5	93.0	93.4	**93.8**	91.4	91.7	91.5	92.2	92.2	92.7
	175	91.7	92.1	93.1	93.2	93.2	**93.6**	91.8	92.2	92.8	93.2	92.7	93.2
	200	92.0	92.5	93.3	93.8	93.6	**94.1**	92.2	92.7	92.2	92.6	92.3	92.7
	225	92.5	93.0	93.3	93.7	93.6	**94.0**	92.2	92.7	92.8	93.2	92.6	93.1
	250	92.5	92.9	93.7	**93.9**	93.4	93.8	92.4	93.0	92.5	93.1	92.4	92.7
50%	50	87.4	89.3	89.1	**91.3**	89.8	**91.3**	87.7	90.4	88.5	90.9	88.8	91.1
	75	89.4	91.0	91.3	**92.7**	91.0	92.6	89.2	91.1	89.5	91.4	89.9	91.9
	100	90.0	91.5	92.4	**93.6**	91.9	93.3	89.9	91.6	90.3	91.9	90.6	92.2
	125	90.6	91.9	93.0	**93.9**	92.3	93.4	90.9	92.3	91.3	92.7	91.0	92.4
	150	90.6	91.9	92.4	93.4	93.0	**94.1**	91.0	92.3	91.3	92.8	91.2	92.6
	175	91.2	92.2	92.6	93.5	92.7	**93.7**	91.4	92.7	91.8	92.8	91.1	92.4
	200	91.2	92.3	92.3	93.3	93.2	**94.2**	91.8	93.0	92.2	93.6	91.8	92.9
	225	91.4	92.6	92.7	93.7	93.3	**94.2**	91.7	92.9	91.7	93.0	91.9	93.1
	250	91.7	92.8	93.1	**94.3**	93.1	94.1	91.4	92.8	92.0	93.2	91.7	93.0
75%	50	87.2	90.6	89.1	91.6	89.5	**92.1**	87.4	90.8	88.7	**92.1**	88.3	91.8
	75	90.1	92.6	91.0	93.2	91.3	**93.8**	90.0	92.4	90.6	93.4	90.1	93.2
	100	90.6	92.9	92.1	**94.2**	91.6	93.8	89.9	92.6	90.3	93.3	90.2	93.0
	125	90.8	93.0	92.1	**94.3**	91.8	94.1	90.5	93.0	91.0	93.4	91.0	93.6
	150	90.8	93.1	92.0	93.9	92.3	**94.4**	90.6	93.0	91.2	93.8	91.2	93.7
	175	90.8	93.1	91.7	93.9	92.3	**94.0**	91.0	93.4	91.5	**94.0**	90.9	93.4
	200	90.9	93.1	92.5	**94.5**	92.5	**94.5**	91.3	93.5	91.0	93.5	91.3	93.4
	225	91.2	93.4	92.5	**94.5**	92.2	94.1	91.6	93.8	91.7	93.5	91.1	93.6
	250	91.8	93.8	92.4	94.4	92.7	**94.5**	91.4	93.6	91.6	93.8	92.0	94.2
100%	50	87.7	92.0	89.1	92.9	89.6	93.2	89.0	92.9	89.6	**93.7**	89.7	93.4
	75	88.8	92.7	90.3	93.7	90.2	93.6	90.3	93.7	90.7	**94.3**	90.2	94.0
	100	89.9	93.3	90.9	94.1	90.5	93.8	90.6	94.0	91.2	**94.4**	90.6	94.1
	125	89.8	93.1	90.9	93.9	90.4	93.6	91.3	94.6	91.8	**94.8**	91.6	94.6
	150	89.6	93.2	90.8	94.0	91.0	93.9	91.6	**94.6**	91.4	94.5	91.2	94.6
	175	90.1	93.2	90.5	93.5	91.3	94.0	91.5	94.6	91.95	**94.8**	91.0	94.2
	200	90.3	93.6	91.1	94.0	91.1	94.1	91.8	**94.7**	91.8	**94.7**	91.8	94.7
	225	90.0	93.1	91.3	94.2	91.1	94.3	91.9	**94.9**	91.9	94.8	91.6	94.6
	250	89.7	93.2	90.9	93.9	90.8	93.9	92.4	**94.9**	91.9	94.8	91.4	94.5

TABLE 1

Simulated coverage probabilities of equal precision (left) and Hall-Wellner bands (right) in per cent where $k = 5$. Those closest to the nominal level of 95% are printed in bold-type.

		equal precision						Hall-Wellner					
set-ups		$N(0,1)$		$Poi(1)-1$		weird		$N(0,1)$		$Poi(1)-1$		weird	
p	n	old	new	old	new	old	new	old	new	old	new	old	new
	50	87.7	88.6	89.4	90.7	89.9	**90.8**	87.2	88.7	89.3	90.5	89.0	90.4
	75	89.2	90.2	91.9	**92.6**	91.8	92.4	89.2	90.1	90.9	91.6	90.7	91.5
	100	90.7	91.2	92.4	93.1	93.1	**93.3**	90.5	90.9	91.9	92.6	91.4	91.9
	125	91.1	91.6	93.1	93.5	93.4	**93.8**	91.5	92.0	92.5	92.9	91.6	92.3
25%	150	91.6	92.2	93.5	**93.8**	93.5	**93.8**	92.5	92.8	92.3	92.9	92.2	92.7
	175	91.8	92.1	93.9	**94.2**	93.8	93.9	92.6	92.9	92.9	93.1	92.7	93.2
	200	91.9	92.4	93.8	**93.9**	93.7	**93.9**	92.3	92.8	92.7	92.9	92.8	93.2
	225	92.4	92.8	93.5	93.9	93.7	**94.0**	92.9	93.4	93.0	93.3	93.0	93.4
	250	92.2	92.6	94.0	94.3	94.2	**94.6**	92.4	92.8	92.9	93.3	93.2	93.5
	50	87.2	88.6	89.2	90.7	89.2	90.5	87.3	89.1	88.9	90.5	89.2	**90.8**
	75	89.8	90.8	91.5	**92.5**	91.2	92.2	89.9	91.2	91.1	92.3	90.7	91.8
	100	90.2	91.3	92.6	93.3	92.5	**93.5**	91.2	92.4	91.1	92.3	91.5	92.6
	125	91.2	92.1	93.0	93.7	93.1	**93.8**	91.1	92.2	91.6	92.5	91.9	92.8
50%	150	91.6	92.5	93.3	**94.0**	93.2	**94.0**	92.0	92.8	91.9	92.8	92.0	93.0
	175	92.7	93.3	92.7	93.3	93.1	**93.7**	92.4	93.1	92.5	93.2	92.7	93.4
	200	92.2	92.9	93.4	94.0	93.5	**94.2**	92.2	92.8	92.4	93.4	92.6	93.3
	225	92.2	93.0	94.1	**94.8**	93.1	94.0	92.3	93.2	92.8	93.4	92.9	93.6
	250	93.0	93.5	93.7	**94.2**	93.4	93.9	92.1	92.7	92.7	93.4	92.1	92.7
	50	87.6	89.8	89.2	91.4	89.0	90.9	88.2	90.6	88.3	91.0	89.5	**91.7**
	75	89.5	91.5	92.0	**93.4**	91.4	93.0	90.4	92.2	91.3	93.0	90.4	92.3
	100	90.3	91.9	92.5	93.8	92.5	**94.0**	91.3	92.7	91.0	92.7	91.2	93.0
	125	91.4	92.9	92.9	94.0	92.8	**94.1**	91.8	93.2	92.0	93.3	91.6	93.1
75%	150	91.5	92.8	93.2	**94.5**	93.2	94.2	91.7	92.9	91.7	93.0	91.9	93.3
	175	91.5	92.8	93.0	94.2	93.7	**94.6**	92.3	93.6	92.5	93.8	92.1	93.3
	200	92.0	93.1	93.9	**94.9**	93.8	**94.9**	92.3	93.5	92.1	93.5	92.1	93.5
	225	92.2	93.1	93.6	**94.6**	93.2	94.1	92.4	93.6	92.4	93.9	92.9	94.1
	250	92.5	93.8	93.5	94.6	93.7	**94.8**	92.7	93.8	92.4	93.7	92.6	93.7
	50	88.3	91.1	90.2	**92.8**	90.3	92.7	89.0	92.0	89.5	92.7	89.6	**92.8**
	75	89.8	92.2	91.7	**93.9**	91.8	93.6	91.1	93.6	91.1	93.6	90.9	93.8
	100	90.5	92.7	92.1	94.0	92.1	94.0	90.8	93.1	92.1	**94.3**	91.7	93.8
	125	91.1	93.2	92.6	94.4	92.7	**94.5**	92.1	93.8	92.1	94.2	92.3	94.3
100%	150	91.4	93.4	92.8	94.6	92.8	**94.8**	92.0	93.9	92.6	94.7	92.6	94.6
	175	91.6	93.6	92.8	94.6	92.8	**94.8**	92.4	94.2	92.9	94.7	92.3	94.4
	200	91.5	93.5	93.0	94.8	93.3	**95.0**	92.3	94.2	93.1	94.9	93.0	94.9
	225	92.3	94.0	92.9	94.7	93.4	**95.0**	92.7	94.3	93.3	94.9	93.1	94.9
	250	91.8	93.8	92.8	94.6	93.0	**94.7**	93.3	94.9	92.7	94.4	92.9	94.6

TABLE 2

Simulated coverage probabilities of equal precision (left) and Hall-Wellner bands (right) in per cent where $k = 10$. Those closest to the nominal level of 95% are printed in bold-type.

		equal precision						Hall-Wellner					
set-ups		$N(0,1)$		$Poi(1)-1$		weird		$N(0,1)$		$Poi(1)-1$		weird	
p	n	old	new	old	new	old	new	old	new	old	new	old	new
	50	87.2	88.0	89.9	90.9	90.2	90.7	88.4	89.5	89.9	**91.2**	89.1	90.1
	75	89.8	90.3	92.2	**92.5**	91.8	92.4	90.4	91.0	90.7	91.4	90.7	91.5
	100	91.0	91.6	93.1	**93.5**	92.7	93.1	91.4	91.9	91.8	92.3	91.9	92.3
	125	92.1	92.3	94.0	**94.2**	93.3	93.7	91.7	92.2	92.5	93.0	92.5	92.9
25%	150	91.9	92.4	93.6	93.8	93.7	**93.9**	92.1	92.5	92.6	92.8	92.9	93.3
	175	92.4	92.7	94.0	**94.4**	94.2	**94.4**	92.2	92.6	92.8	93.0	92.8	93.1
	200	92.8	93.0	94.2	**94.4**	93.8	94.1	93.0	93.3	92.6	93.0	93.0	93.2
	225	93.2	93.4	94.5	**94.8**	94.0	94.2	93.1	93.3	93.0	93.3	93.2	93.4
	250	92.8	93.1	93.9	94.1	94.2	**94.4**	92.9	93.0	93.6	93.7	93.1	93.3
	50	88.0	89.2	89.6	**90.9**	89.6	90.8	88.8	90.0	89.3	90.7	89.5	**90.9**
	75	89.9	90.7	92.4	**93.3**	92.3	93.0	90.4	91.4	91.3	92.3	90.9	91.9
	100	91.0	91.7	93.2	**93.7**	93.0	**93.7**	91.7	92.5	92.0	92.7	91.9	92.6
	125	92.0	92.5	93.4	93.8	93.9	**94.2**	92.1	92.8	92.5	93.1	92.1	92.6
50%	150	91.9	92.5	93.6	94.0	94.0	**94.3**	92.1	92.6	93.1	93.6	92.6	93.2
	175	93.0	93.4	94.0	94.4	94.1	**94.5**	92.4	93.0	93.0	93.3	93.0	93.5
	200	92.8	93.1	94.1	**94.4**	93.7	94.0	92.9	93.2	93.1	93.5	92.7	93.3
	225	92.8	93.3	94.2	94.5	94.3	**94.7**	92.7	93.0	93.2	93.9	93.5	93.9
	250	93.0	93.3	93.9	**94.3**	93.8	94.0	93.4	93.6	93.0	93.5	92.7	93.3
	50	88.4	89.9	90.8	**92.1**	90.5	92.0	89.2	90.8	89.7	91.5	89.7	91.4
	75	90.3	91.5	92.7	**93.9**	92.9	93.7	91.0	92.2	91.3	92.4	91.2	92.3
	100	91.4	92.5	93.3	**94.2**	93.0	93.9	91.2	92.1	92.4	93.2	92.3	93.4
	125	91.4	92.3	93.8	**94.6**	93.6	94.1	92.2	93.2	92.6	93.4	92.5	93.5
75%	150	92.7	93.3	94.0	**94.8**	93.5	94.3	92.6	93.4	93.1	94.0	92.7	93.5
	175	92.4	93.1	94.4	**95.0**	94.1	94.8	92.5	93.2	93.3	93.9	92.7	93.4
	200	93.0	93.8	94.1	**94.7**	94.0	94.6	92.7	93.4	93.3	94.2	93.1	93.8
	225	93.3	93.9	94.4	**95.0**	94.1	94.7	92.9	93.5	93.2	94.0	93.6	94.1
	250	93.0	93.6	94.3	**94.9**	94.3	94.8	93.3	94.0	93.8	94.4	92.9	93.7
	50	88.9	90.7	91.0	92.8	91.2	92.7	90.1	92.3	90.7	92.8	90.5	**92.9**
	75	91.0	92.6	93.2	**94.3**	93.0	94.2	91.2	92.7	92.6	94.1	91.9	93.4
	100	92.0	93.3	93.7	**95.0**	93.7	94.7	92.0	93.6	93.1	94.4	92.5	93.8
	125	92.9	93.9	93.7	**94.9**	94.1	**95.1**	92.4	93.5	93.3	94.6	93.6	94.6
100%	150	92.8	93.9	94.2	**95.1**	94.6	95.5	92.8	94.1	93.5	94.6	93.3	94.4
	175	93.2	94.3	**94.9**	95.6	**94.9**	95.7	93.2	94.4	93.7	94.8	94.0	**94.9**
	200	93.5	94.5	94.4	95.3	94.6	95.3	93.9	94.7	93.8	**94.8**	93.9	**94.8**
	225	94.1	94.8	94.6	95.6	94.7	95.4	93.7	94.6	94.3	**95.1**	93.7	94.7
	250	93.1	94.3	**95.0**	95.8	94.8	95.6	93.6	94.6	93.9	**95.0**	94.2	95.1

TABLE 3

Simulated coverage probabilities of equal precision (left) and Hall-Wellner bands (right) in per cent where $k = 20$. Those closest to the nominal level of 95% are printed in bold-type.

	equal precision						Hall-Wellner					
set-ups	$N(0,1)$		$Poi(1)-1$		weird		$N(0,1)$		$Poi(1)-1$		weird	
n	old	new	old	new	old	new	old	new	old	new	old	new
50	87.1	88.1	89.6	90.4	89.4	**90.5**	87.6	88.6	88.8	90.0	88.7	89.8
75	89.4	90.2	91.8	**92.3**	91.6	92.1	90.2	90.8	90.7	91.5	90.7	91.5
100	90.7	91.2	93.2	**93.4**	92.4	92.7	90.8	91.3	91.1	91.6	91.8	92.2
125	91.4	91.8	93.0	93.4	93.2	**93.5**	91.7	92.1	92.0	92.3	91.6	92.2
150	91.3	91.5	93.6	**93.9**	93.5	93.8	92.3	92.5	92.6	92.9	92.9	93.2
175	92.5	92.6	93.9	94.1	94.1	**94.4**	92.3	92.5	92.5	92.8	92.9	93.2
200	92.9	93.0	93.7	**93.8**	93.5	**93.8**	92.7	92.9	92.9	93.0	93.2	93.3
225	92.7	92.9	93.9	94.1	**94.2**	**94.2**	93.0	93.1	92.6	92.7	93.3	93.7
250	93.2	93.4	93.7	93.8	**94.3**	**94.3**	92.8	93.0	92.9	93.1	93.1	93.3

TABLE 4

Simulated coverage probabilities of confidence bands for F_1 in per cent where $p = 0$. Those closest to the nominal level of 95% are printed in bold-type.

7. Real data example

We applied the present discretization adjustment to the `sir.adm` data-set of the `R` package *mvna*. It consists of competing risks data of patients who are in an intensive care unit (ICU), where the event of primary interest, "alive discharge out of ICU", competes against the secondary event "death in ICU". For seeing the difference between the common and the new approach more clearly, we analyzed the subset of all male patients suffering from pneumonia. Out of these $n = 63$ individuals, five have been right-censored and 41 out of all 44 type 1 events fell into the time interval $[5, 55]$, which we chose for confidence band construction. Due to the worse performance of the wild bootstrap based on standard normal multipliers as seen in Section 6, we only derived centered unit Poisson variate-based bands. In order to minimize the computational error in the quantile-finding process, 99,999 wild bootstrap iterations have been conducted. The confidence bands resulting from the weird bootstrap almost coincide with those just described. Therefore, they are not shown.

The resulting confidence bands are presented in Figure 2. For both kinds of bands, (equal precision bands in the upper panel, Hall-Wellner bands in the lower panel), we see that the discretization adjustment leads to a widening in comparison to the unadjusted bands. This is in line with the discussion and the simulation results of Section 6, where the unadjusted bands appeared to be the most liberal, i.e. the narrowest. In particular, the adjusted equal precision bands cover an additional area of 2.1 percentage points at the terminal time point $t = 55$, whereas this deviance even amounts to 3.3 percentage points for the Hall-Wellner bands. This might not appear to be much at a first glance at the plots in Figure 2. But in fact, it may be the cause for a formidable improvement of the bands' coverage probability: The simulation results of Section 6 for $k = 20$, discretization probability $p = 100\%$, and sample sizes $n \in \{50, 75\}$ suggest that the adjusted wild bootstrap procedure might improve the coverage probabilities of both kinds of bands by approximately two percentage points. With a view towards the liberal behaviour of the unadjusted bands, these enhancements of the coverage probabilities are highly worthwhile.

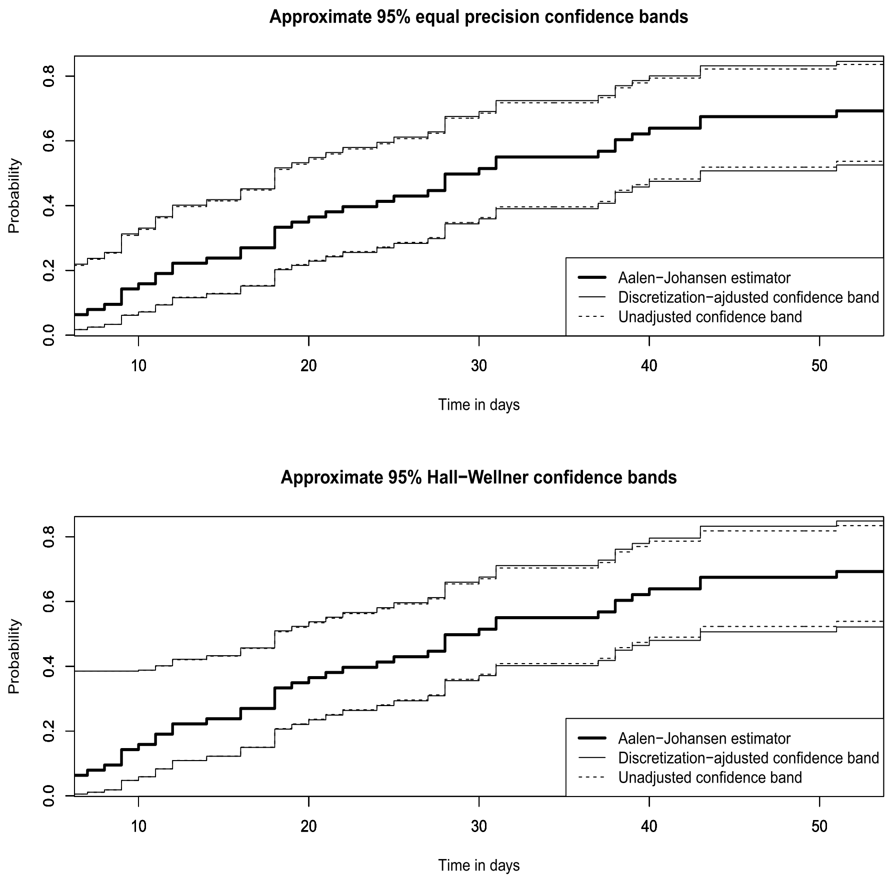

FIG 2. *Asymptotic* 95% *equal precision (upper) and Hall-Wellner bands (lower panel) for the cumulative incidence function of the competing risk "alive discharge out of ICU" for male patients suffering from pneumonia.*

8. Discussion and future research

In this article, we analyzed a discontinuity adjustment of the common wild bootstrap applied to right-censored competing risks data. This adjustment is absolutely necessary, as ties in the data introduce an asymptotic dependence between multiple cause-specific Nelson-Aalen estimators and the asymptotic variances of univariate Nelson-Aalen estimators are decreased in general. The common wild bootstrap fails in reproducing these effects since it establishes independence for all sample sizes. The problem is even more involved for Aalen-Johansen estimators of cumulative incidence functions, which are non-linear functionals of all cause-specific hazards. Simulation results reported the striking liberality of the unadjusted bands which also fail to keep the nominal level asymptotically. Instead, the discretization-adjusted wild bootstrap greatly improves the coverage probability. This effect is more pronounced the more discrete the event times are. But even in the absolutely continuous case, the suggested procedure appears to perform preferably. Therefore, we advise to always use the adjustment when right-censored competing risks data shall be analyzed. The real data example reveals that the discontinuity adjustment does actually only lead to slight widening of the common wild bootstrap-based bands which is already enough to improve the coverage accuracy greatly.

The presented wild bootstrap approach may be extended to more general Markovian multi-state models since the martingale arguments of Appendix B still apply. A still open question is, whether the common wild bootstrap also fails in case of tied survival data which are assumed to follow the Cox proportional hazards model [8]. See [19] for the wild bootstrap applied to absolutely continuous survival data following the Cox model. And, if it fails, whether the method proposed in this article requires further modification.

Appendix A: Detailed derivation of the asymptotic variance of the Nelson-Aalen estimator

Define $H^{uc}(t) = \sum_{j=1}^k EN_{j1}(t)$ as the probability that an uncensored event due to any cause occurs until time t. According to [24], p. 383f, we have $\sqrt{n}(\widehat{A} - A) \xrightarrow{d} \int_0^\cdot M^{uc}(\mathrm{d}u)/\bar{H}(u)$, where $M^{uc}(t) = G^{uc}(t) - \int_0^t \bar{G}(u)\mathrm{d}A(u)$ is a zero-mean Gaussian martingale. Its variance function is determined by

$$EG^{uc}(s)G^{uc}(t) = H^{uc}(s \wedge t) - H^{uc}(s)H^{uc}(t),$$

$$E\bar{G}(s)\bar{G}(t) = \bar{H}(s \vee t) - \bar{H}(s)\bar{H}(t),$$

$$EG^{uc}(s)\bar{G}(t) = (H^{uc}(s) - H^{uc}(t-))1\{t \leqslant s\} - H^{uc}(s)\bar{H}(t).$$

Note that $A(t) = \int_0^t H^{uc}(\mathrm{d}u)/\bar{H}(u)$. Thus, for $s \leqslant t$, the covariance function of M^{uc} at (s,t) is

$$\begin{aligned}
&E(M^{uc}(s)M^{uc}(t)) \\
=&H^{uc}(s) - H^{uc}(s)H^{uc}(t) + \int_0^s \int_0^t \frac{\bar{H}(u \vee v) - \bar{H}(u)\bar{H}(v)}{\bar{H}(u)\bar{H}(v)} \mathrm{d}H^{uc}(v)\mathrm{d}H^{uc}(u) \\
&- \int_0^t [(H^{uc}(s) - H^{uc}(u-))1\{u \leqslant s\} - H^{uc}(s)\bar{H}(u)] \frac{\mathrm{d}H^{uc}(u)}{\bar{H}(u)} \\
&- \int_0^s [(H^{uc}(t) - H^{uc}(v-))1\{v \leqslant t\} - H^{uc}(t)\bar{H}(v)] \frac{\mathrm{d}H^{uc}(v)}{\bar{H}(v)} \\
=&H^{uc}(s) - H^{uc}(s)H^{uc}(t) + \int_0^s \int_u^t \Big[\frac{1}{\bar{H}(u)} - 1\Big] \mathrm{d}H^{uc}(v)\mathrm{d}H^{uc}(u) \\
&+ \int_0^s \int_0^u \Big[\frac{1}{\bar{H}(v)} - 1\Big] \mathrm{d}H^{uc}(v)\mathrm{d}H^{uc}(u) \\
&- (H^{uc}(s) + H^{uc}(t))A(s) + 2\int_0^s H^{uc}(u-)\mathrm{d}A(u) + 2H^{uc}(s)H^{uc}(t)
\end{aligned}$$

$$
\begin{aligned}
=&H^{uc}(s) + H^{uc}(s)H^{uc}(t) + \int_0^s (H^{uc}(t) - H^{uc}(u))\Big[\frac{1}{\bar{H}(u)} - 1\Big]\mathrm{d}H^{uc}(u) \\
&+ \int_0^s (A(u) - H^{uc}(u))\mathrm{d}H^{uc}(u) - (H^{uc}(s) + H^{uc}(t))A(s) \\
&+ 2\int_0^s H^{uc}(u-)\mathrm{d}A(u) \\
=&H^{uc}(s) + H^{uc}(s)H^{uc}(t) + H^{uc}(t)A(s) - H^{uc}(t)H^{uc}(s) - \int_0^s H^{uc}(u)\mathrm{d}A(u) \\
&+ \int_0^s H^{uc}(u)\mathrm{d}H^{uc}(u) + \int_0^s (A(u) - H^{uc}(u))\mathrm{d}H^{uc}(u) \\
&- (H^{uc}(s) + H^{uc}(t))A(s) + 2\int_0^s H^{uc}(u-)\mathrm{d}A(u) \\
=&H^{uc}(s) - H^{uc}(s)A(s) - \int_0^s H^{uc}(u)\mathrm{d}A(u) + \int_0^s A(u)\mathrm{d}H^{uc}(u) \\
&+ 2\int_0^s H^{uc}(u-)\mathrm{d}A(u) \\
=&H^{uc}(s) - \int_0^s H^{uc}(u)\mathrm{d}A(u) + \int_0^s H^{uc}(u-)\mathrm{d}A(u) \\
=&\int_0^s \bar{H}(u)\mathrm{d}A(u) - \int_0^s \Delta H^{uc}(u)\mathrm{d}A(u) = \int_0^s \bar{H}(u)(1 - \Delta A(u))\mathrm{d}A(u)
\end{aligned}
$$

We conclude, as in [24], that

$$
\sqrt{n}(\widehat{A} - A) \xrightarrow{d} \int_0^{\cdot} \frac{1}{\bar{H}(u)}\mathrm{d}M^{uc}(u) \sim \mathit{Gauss}\Big(0, \int_0^{\cdot} \frac{1 - \Delta A(u)}{\bar{H}(u)}\mathrm{d}A(u)\Big)
$$

as $n \to \infty$, where *Gauss* again indicates that the limit process is a Gaussian martingale.

The very same calculations hold true if each H^{uc} is replaced with H_j^{uc} for the subdistribution function of an uncensored type j event, $j = 1, \dots, k$. Therefore, we have for Nelson-Aalen estimators for cause-specific cumulative hazard functions that, as $n \to \infty$,

$$
\sqrt{n}(\widehat{A}_j - A_j) \xrightarrow{d} \int_0^{\cdot} \frac{1}{\bar{H}(u)}\mathrm{d}M_j^{uc}(u) \sim \mathit{Gauss}\Big(0, \sigma_j^2(\cdot)\Big),
$$

where $M_j^{uc}(t) = G_j^{uc}(t) - \int_0^t \bar{G}(u)\mathrm{d}A_j(u)$ involves similar quantities as M^{uc}, but which are now cause-specific.

Appendix B: Consistency of the wild bootstrap for the univariate Nelson-Aalen estimator

Proof. Without loss of generality, assume that $0, K \in D$ for simplifying notation. Write $D = \{d_0, d_1, \dots, d_J\}$ with the natural ordering $d_j < d_{j+1}$ for all $j =$

$0, 1, \ldots, J-1$. Then $[0, K] \backslash D = \bigcup_{j=1}^{J} (d_{j-1}, d_j)$. For simplifying the notation, we subsequently only consider the Nelson-Aalen estimator corresponding to the first competing risk. As argued in [6], it is now straightforward to show that each process $(\widehat{W}_1(t) - \widehat{W}_1(d_{j-1}))_t$ on each interval $[d_{j-1}, d_j)$, $j = 1, \ldots, J$, defines a square-integrable martingale. Since such martingales can be extended to the right boundary of the time interval, we may *define* the boundary values $\widehat{W}_1(d_j) - \widehat{W}_1(d_{j-1}) := \lim_{t \uparrow d_j} \widehat{W}_1(t) - \widehat{W}_1(d_{j-1})$, and this procedure introduces square-integrable martingales on the whole $[d_{j-1}, d_j]$.

First, we notice that the conditional weak convergence in probability of the processes $(\widehat{W}_1(t) - \widehat{W}_1(d_{j-1}))_t$ on each interval $[d_{j-1}, d_j]$, $j = 1, \ldots, J$, is already implied by exactly the same Rebolledo's martingale central limit theorem arguments as in [6]. Denote the limit Gaussian martingale processes as $(\tilde{U}_{1j}(t))_{t \in [d_{j-1}, d_j]}$, $j = 1, \ldots, J$. Due to the martingale extension above, Rebolledo's limit theorem implies the almost sure continuity of $\tilde{U}_{1j}$ on each time interval. Furthermore, these are zero-mean processes with variance function $t \mapsto \sigma_1^2(t) - \sigma_1^2(d_{j-1})$.

Due to the continuity of the limit processes $\tilde{U}_{1j}$ on the intervals $[d_{j-1}, d_j]$, we are able to switch from the Skorohod topology to the more convenient sup-norm metrization; see the discussion in Section II.8 in [3]. At each $t = d_j$, the weak conditional convergence in distribution of $\Delta \widehat{W}_1(d_j)$ holds in probability by the already argued convergence of all finite-dimensional conditional distributions. Therefore, the independence of the (bootstrapped) Nelson-Aalen increments imply that, as $n \to \infty$, the conditional distribution of

$$\Big(\Delta \widehat{W}_1(d_0), \widehat{W}_1(t_0) - \widehat{W}_1(d_0), \Delta \widehat{W}_1(d_1), \\ \ldots, \widehat{W}_1(t_{J-1}) - \widehat{W}_1(d_{J-1}), \Delta \widehat{W}_1(d_J)\Big)_{t_0 \in [d_0, d_1], \ldots, t_{J-1} \in [d_{J-1}, d_J]}$$

given $\mathcal{F}_0$ converges weakly in probability to the distribution of

$$(\tilde{U}_1(d_0), \tilde{U}_{11}(t_1), \tilde{U}_1(d_1), \ldots, \tilde{U}_{1J}(t_J), \tilde{U}_1(d_J))_{t_1 \in [d_0, d_1], \ldots, t_J \in [d_{J-1}, d_J]}$$

on the product Space $\mathbb{R} \times D[d_0, d_1] \times \mathbb{R} \times \cdots \times D[d_{J-1}, d_J] \times \mathbb{R}$ equipped with the sup-max-norm. Here, all components are independent, and the normally distributed random variables $\tilde{U}_1(d_j)$ have mean zero and variance $\Delta \sigma_1^2(d_j)$, $j = 1, \ldots, J$.

Applying the following functional, which is continuous with respect to the max-sup-norm,

$$\begin{aligned} \psi : &\mathbb{R} \times D[d_0, d_1] \times \mathbb{R} \times \cdots \times D[d_{J-1}, d_J] \times \mathbb{R} \\ &\qquad \longrightarrow D[d_0, d_1] \times \cdots \times D[d_{J-1}, d_J] \times \mathbb{R}, \\ &(x_0, y_1(t_1), x_1, \ldots, y_J(t_J), x_J)_{t_1, \ldots, t_J} \end{aligned}$$

$$\longmapsto \Big(x_0 + y_1(t_1),\ x_0 + y_1(d_1) + x_1 + y_2(t_2), \dots,$$
$$x_0 + \sum_{j=1}^{J-1}(y_j(d_j) + x_j) + y_J(t_J),\ x_0 + \sum_{j=1}^{J}(y_j(d_j) + x_j)\Big)_{t_1,\dots,t_J}$$

to the previous limit theorem, the continuous mapping theorem implies that, given $\mathcal{F}_0$ and as $n \to \infty$, the conditional distribution of

$$\Big(\widehat{W}_1(t_1), \dots, \widehat{W}_1(t_J), \widehat{W}_1(d_J)\Big)_{t_1 \in [d_0,d_1],\dots,t_J \in [d_{J-1},d_J]}$$

converges weakly (on the product-function space $D[d_0, d_1] \times \cdots \times D[d_{J-1}, d_J] \times \mathbb{R}$ equipped with the max-sup-norm) in probability to the distribution of

$$\Big(U_1(t_1), \dots, U_1(t_J), U_1(d_J)\Big)_{t_1 \in [d_0,d_1],\dots,t_J \in [d_{J-1},d_J]}.$$

Here the right boundary values are again considered as the left-hand limits given by the martingale extension theorem. The process $(U_1(t))_{t\in[0,K]}$ is a zero-mean Gaussian martingale with variance function $t \mapsto \sigma_1^2(t)$ with, in general, discontinuous sample paths.

Finally, we apply the continuous functional

$$\phi:\ D[d_0, d_1] \times \cdots \times D[d_{J-1}, d_J] \times \mathbb{R} \longrightarrow D[0, K],$$
$$(y_1(t_1), \dots, y_J(t_J), x_J)_{t_1,\dots,t_J} \longmapsto \Big(\sum_{j=1}^{J} y_j(t) \cdot 1_{[d_{j-1},d_j)}(t) + x_J \cdot 1_{\{K\}}(t)\Big)_{t\in[0,K]}$$

in order to obtain the desired conditional weak convergence for $(\widehat{W}_1(t))_{t\in[0,K]}$ in probability. □

Appendix C: Derivation of the asymptotic covariance of multiple Nelson-Aalen estimators for cumulative cause-specific hazards

Consider the situation of two competing risks. In order to derive the asymptotic covariance of two cause-specific Nelson-Aalen estimators, we first note that, as $n \to \infty$,

$$\sqrt{n}(\widehat{A} - A) = \sum_{j=1}^{2} \sqrt{n}(\widehat{A}_j - A_j) \xrightarrow{d} U_1 + U_2 = U$$

by the continuous mapping theorem. The covariance function of U is given by

$$cov(U(s), U(t)) = \sigma^2(s \wedge t) = \int_0^{s\wedge t} \frac{1 - \Delta A(u)}{\bar{H}(u)} \mathrm{d}A(u)$$

but on the other hand

$$cov(U(s), U(t)) = \sum_{j=1}^{2} cov(U_j(s), U_j(t)) + cov(U_1(s), U_2(t)) + cov(U_1(t), U_2(s)).$$

Solving for the unknown covariances on the right-hand side of the previous display, we obtain

$$\begin{aligned}
&cov(U_1(s), U_2(t)) + cov(U_1(t), U_2(s)) \\
&= \int_0^{s\wedge t} \frac{1-\Delta A(u)}{\bar H(u)} \mathrm{d}A(u) - \sum_{j=1}^{2} \int_0^{s\wedge t} \frac{1-\Delta A_j(u)}{\bar H(u)} \mathrm{d}A_j(u) \\
&= -\int_0^{s\wedge t} \frac{\Delta A_1(u)}{\bar H(u)} \mathrm{d}A_2(u) - \int_0^{s\wedge t} \frac{\Delta A_2(u)}{\bar H(u)} \mathrm{d}A_1(u) \\
&= -2 \sum_{u \leqslant s\wedge t} \frac{\Delta A_1(u)\Delta A_2(u)}{\bar H(u)}.
\end{aligned}$$

Due to symmetry and inductively, it follows that

$$\sigma_{j\ell}(s\wedge t) = cov(U_j(s), U_\ell(t)) = -\int_0^{s\wedge t} \frac{\Delta A_j(u)}{\bar H(u)} \mathrm{d}A_\ell(u)$$

for $j \neq \ell,\ j, \ell = 1, \dots, k$, even in the situation of $k \in \mathbb{N}$ competing risks.

Appendix D: Consistency of the discretization-adjusted wild bootstrap for the multivariate Nelson-Aalen estimator

The proof of tightness follows along the same lines as that of Theorem 3.2 because the wild bootstrapped Nelson-Aalen estimators retain their martingale properties. It only remains to calculate the finite-dimensional marginal limit distributions. These are calculated with the help of Theorem A.1 in [5]: Therefore, we abbreviate

$$\widehat W_j(t) = \sum_{i=1}^n \xi_{jji} Z_{n,jji} + \sum_{\ell=1}^k sign(\ell - j) \sum_{i=1}^n \xi_{j\ell i} \tilde Z_{n,j\ell i} + \sum_{\ell=1}^k sign(\ell - j) \sum_{i=1}^n \xi_{\ell ji} \tilde Z_{n,\ell ji}.$$

Further, consider arbitrary points of time $0 \leqslant t_1 \leqslant \dots \leqslant t_m \leqslant K$ and the vector

$$(\widehat W_1(t_1), \dots, \widehat W_1(t_m), \widehat W_2(t_1), \dots, \widehat W_2(t_m) \dots, \widehat W_k(t_1), \dots, \widehat W_k(t_m)).$$

Due to analogy, it is enough to calculate the entries corresponding to both pairs $(\widehat W_1(t_1), \widehat W_1(t_2))$ and $(\widehat W_1(t_1), \widehat W_2(t_2))$ of the limit of the matrix

$$\begin{aligned}
\widehat\Gamma := \Big(& 1\{j = \tilde j\} \sum_{i=1}^n \Big[Z_{n,jji}(t_a) Z_{n,jji}(t_b) \\
& + \sum_{\ell \neq j} \tilde Z_{n,j\ell i}(t_a) \tilde Z_{n,j\ell i}(t_b) + \sum_{\ell \neq j} \tilde Z_{n,\ell ji}(t_a) \tilde Z_{n,\ell ji}(t_b) \Big] \\
- & 1\{j \neq \tilde j\} \sum_{i=1}^n \Big[\tilde Z_{n,j\tilde j i}(t_a) \tilde Z_{n,j\tilde j i}(t_b) + \tilde Z_{n,\tilde j ji}(t_a) \tilde Z_{n,\tilde j ji}(t_b) \Big] \Big)_{a,b=1,\dots,m;\ j,\tilde j = 1,\dots,k}.
\end{aligned}$$

We start by calculating the entry for $a = 1, b = 2, j = \tilde{j} = 1$, that is,

$$\begin{aligned}
&\sum_{i=1}^{n} \Big[Z_{n,11i}(t_1) Z_{n,11i}(t_2) + \sum_{\ell=2}^{k} \tilde{Z}_{n,1\ell i}(t_1) \tilde{Z}_{n,1\ell i}(t_2) + \sum_{\ell=2}^{k} \tilde{Z}_{n,\ell 1i}(t_1) \tilde{Z}_{n,\ell 1i}(t_2) \Big] \\
&= n \sum_{i=1}^{n} \int_0^{t_1} \frac{Y(u) - \Delta N(u)}{Y(u)} \frac{\mathrm{d}N_{1i}(u)}{Y^2(u)} \\
&\quad + \frac{1}{2} \sum_{\ell=2}^{k} \Big[n \sum_{i=1}^{n} \int_0^{t_1} \frac{\Delta N_1(u)}{Y(u)} \frac{\mathrm{d}N_{\ell i}(u)}{Y^2(u)} + n \sum_{i=1}^{n} \int_0^{t_1} \frac{\Delta N_\ell(u)}{Y(u)} \frac{\mathrm{d}N_{1i}(u)}{Y^2(u)} \Big] \\
&= n \int_0^{t_1} \frac{Y(u) - \Delta N(u)}{Y^2(u)} \frac{\mathrm{d}N_1(u)}{Y(u)} \\
&\quad + \frac{1}{2} \sum_{\ell=2}^{k} \Big[n \int_0^{t_1} \frac{\Delta N_1(u)}{Y^2(u)} \frac{\mathrm{d}N_\ell(u)}{Y(u)} + n \int_0^{t_1} \frac{\Delta N_\ell(u)}{Y^2(u)} \frac{\mathrm{d}N_1(u)}{Y(u)} \Big] \\
&= n \int_0^{t_1} \frac{Y(u) - \Delta N_1(u)}{Y^2(u)} \frac{\mathrm{d}N_1(u)}{Y(u)}.
\end{aligned}$$

Here, the last equality follows from $\Delta N_1 \mathrm{d}N_\ell = \Delta N_\ell \mathrm{d}N_1$ for all ℓ. By the Glivenko-Cantelli theorem in combination with the continuous mapping theorem, it follows that the quantity in the previous display converges to $\sigma_1^2(t_1)$ in probability as $n \to \infty$.

Now, consider the entry of $\widehat{\Gamma}$ for $a = 1, b = 2, j = \tilde{j} = 2$:

$$\begin{aligned}
&- \sum_{i=1}^{n} \Big[\tilde{Z}_{n,12i}(t_1) \tilde{Z}_{n,12i}(t_2) + \tilde{Z}_{n,21i}(t_1) \tilde{Z}_{n,21i}(t_2) \Big] \\
&= -\frac{1}{2} \Big[n \int_0^{t_1} \frac{\Delta N_1(u)}{Y^2(u)} \frac{\mathrm{d}N_2(u)}{Y(u)} + n \int_0^{t_1} \frac{\Delta N_2(u)}{Y^2(u)} \frac{\mathrm{d}N_1(u)}{Y(u)} \Big].
\end{aligned}$$

By the same arguments as before, this is a consistent estimator for $\sigma_{12}(t_1)$ as $n \to \infty$.

Appendix E: Derivation of the asymptotic covariance function of an Aalen-Johansen estimator in the competing risks set-up

In order to derive the the covariance function of

$$\int_0^{\cdot} \frac{1 - F_2(u-) - F_1(\cdot)}{1 - \Delta A(u)} \mathrm{d}U_1(u) + \int_0^{\cdot} \frac{F_1(u-) - F_1(\cdot)}{1 - \Delta A(u)} \mathrm{d}U_2(u),$$

at any $(s,t) \in [0,K]^2$, we exemplarily calculate the covariance function of the first integral and the covariance function between both integrals. Hence, as the covariance function $(s,t) \mapsto \sigma_1^2(s \wedge t)$ of U_1 only increases along the diagonal,

$$cov\Big(\int_0^s \frac{1 - F_2(u-) - F_1(s)}{1 - \Delta A(u)} \mathrm{d}U_1(u), \int_0^t \frac{1 - F_2(u-) - F_1(t)}{1 - \Delta A(u)} \mathrm{d}U_1(u) \Big)$$

$$= \int_0^{s\wedge t} \frac{(1-F_2(u-)-F_1(s))(1-F_2(u-)-F_1(t))}{(1-\Delta A(u))^2} \mathrm{d}\sigma_1^2(u)$$
$$= \int_0^{s\wedge t} \frac{(1-F_2(u-)-F_1(s))(1-F_2(u-)-F_1(t))}{\bar H(u)} \frac{1-\Delta A_1(u)}{(1-\Delta A(u))^2} \mathrm{d}A_1(u).$$

Furthermore, we similarly have for the covariance between both integrals that

$$cov\Big(\int_0^s \frac{1-F_2(u-)-F_1(s)}{1-\Delta A(u)} \mathrm{d}U_1(u), \int_0^t \frac{F_1(u-)-F_1(t)}{1-\Delta A(u)} \mathrm{d}U_2(u)\Big)$$
$$= \int_0^{s\wedge t} \frac{(1-F_2(u-)-F_1(s))(F_1(u-)-F_1(t))}{(1-\Delta A(u))^2} \mathrm{d}\sigma_{12}(u)$$
$$= -\int_0^{s\wedge t} \frac{(1-F_2(u-)-F_1(s))(F_1(u-)-F_1(t))}{\bar H(u)} \frac{\Delta A_1(u)}{(1-\Delta A(u))^2} \mathrm{d}A_2(u).$$

Finally, including also the remaining two analogous terms, we obtain the following asymptotic covariance function of the Aalen-Johansen estimator for the first cumulative incidence function as the sum of all four covariance functions:

$$\begin{aligned}(s,t) \mapsto & \int_0^{s\wedge t} \frac{(1-F_2(u-)-F_1(s))(1-F_2(u-)-F_1(t))}{\bar H(u)} \frac{1-\Delta A_1(u)}{(1-\Delta A(u))^2} \mathrm{d}A_1(u) \\ & + \int_0^{s\wedge t} \frac{(F_1(u-)-F_1(s))(F_1(u-)-F_1(t))}{\bar H(u)} \frac{1-\Delta A_2(u)}{(1-\Delta A(u))^2} \mathrm{d}A_2(u) \\ & - \int_0^{s\wedge t} \frac{(1-F_2(u-)-F_1(s))(F_1(u-)-F_1(t))}{\bar H(u)} \frac{\Delta A_1(u)}{(1-\Delta A(u))^2} \mathrm{d}A_2(u) \\ & - \int_0^{s\wedge t} \frac{(1-F_2(u-)-F_1(t))(F_1(u-)-F_1(s))}{\bar H(u)} \frac{\Delta A_2(u)}{(1-\Delta A(u))^2} \mathrm{d}A_1(u).\end{aligned}$$

Expanding the terms $1-\Delta A_1 = 1-\Delta A+\Delta A_2$ (similarly for $j=2$) and rearranging all integrals, the covariance function simplifies to

$$\begin{aligned}& \int_0^{s\wedge t} \frac{(1-F_2(u-)-F_1(s))(1-F_2(u-)-F_1(t))}{\bar H(u)} \frac{\mathrm{d}A_1(u)}{1-\Delta A(u)} \\ & + \int_0^{s\wedge t} \frac{(F_1(u-)-F_1(s))(F_1(u-)-F_1(t))}{\bar H(u)} \frac{\mathrm{d}A_2(u)}{1-\Delta A(u)} \\ & + \int_0^{s\wedge t} [(1-F_2(u-)-F_1(s))-(F_1(u-)-F_1(s))] \\ & \quad \times [(1-F_2(u-)-F_1(t))-(F_1(u-)-F_1(t))] \frac{\Delta A_1(u)}{\bar H(u)(1-\Delta A(u))^2} \mathrm{d}A_2(u) \\ & = \int_0^{s\wedge t} \frac{(1-F_2(u-)-F_1(s))(1-F_2(u-)-F_1(t))}{\bar H(u)} \frac{\mathrm{d}A_1(u)}{1-\Delta A(u)} \\ & + \int_0^{s\wedge t} \frac{(F_1(u-)-F_1(s))(F_1(u-)-F_1(t))}{\bar H(u)} \frac{\mathrm{d}A_2(u)}{1-\Delta A(u)} \\ & + \sum_{u\leqslant s\wedge t} \frac{S^2(u-)\Delta A_1(u)\Delta A_2(u)}{\bar H(u)(1-\Delta A(u))^2}.\end{aligned}$$

The latter representation of the covariance function shows that this covariance function in general increases when ties in the data are present.

Acknowledgements

The author likes to thank Markus Pauly for helpful discussions. He also appreciates the support by the German Research Foundation (DFG).

References

[1] M. G. Akritas. Bootstrapping the Kaplan-Meier Estimator. *Journal of the American Statistical Association*, 81(396):1032–1038, 1986. MR0867628

[2] A. Allignol, M. Schumacher, and J. Beyersmann. A note on variance estimation of the Aalen-Johansen estimator of the cumulative incidence function in competing risks, with a view towards left-truncated data. *Biom. J.*, 52 (1):126–137, 2010. MR2756598

[3] P. K. Andersen, Ø. Borgan, R. D. Gill, and N. Keiding. *Statistical Models Based on Counting Processes*. Springer, New York, 1993. MR1198884

[4] R. Bajorunaite and J. P. Klein. Two-sample tests of the equality of two cumulative incidence functions. *Computational Statistics & Data Analysis*, 51:4269–4281, 2007. MR2364444

[5] J. Beyersmann, M. Pauly, and S. Di Termini. Weak Convergence of the Wild Bootstrap for the Aalen-Johansen Estimator of the Cumulative Incidence Function of a Competing Risk. *Scandinavian Journal of Statistics*, 40(3), 2013. MR3091688

[6] T. Bluhmki, D. Dobler, J. Beyersmann, and M. Pauly. The Wild Bootstrap for Multivariate Nelson-Aalen Estimators. *arXiv preprint:1602.02071v2*, 2016.

[7] T. Bluhmki, C. Schmoor, D. Dobler, M. Pauly, J. Finke, M. Schumacher, and J. Beyersmann. A Wild Bootstrap Approach for the Aalen-Johansen Estimator. *preprint*, 2017.

[8] D. R. Cox. Regression Models and Life-Tables. *Journal of the Royal Statistical Society, Series B (Methodological)*, 34(2):187–220, 1972. MR0341758

[9] D. R. Cox and D. Oakes. *Analysis of Survival Data*. Chapman & Hall/CRC, Boca Raton, Florida, USA, 1984. MR0751780

[10] D. Dobler and M. Pauly. Bootstrapping Aalen-Johansen processes for competing risks: Handicaps, solutions, and limitations. *Electronic Journal of Statistics*, 8(2):2779–2803, 2014. MR3299122

[11] D. Dobler and M Pauly. Approximate tests for the equality of two cumulative incidence functions of a competing risk. *Statistics*, `http://dx.doi.org/10.1080/02331888.2017.1336171`, 2017.

[12] D. Dobler, J. Beyersmann, and M Pauly. Non-strange weird resampling for complex survival data. *Biometrika*, `https://doi.org/10.1093/biomet/asx026`, 2017. MR3694591

[13] B. Efron. Censored Data and the Bootstrap. *Journal of the American Statistical Association*, 76(374):312–319, 1981. MR0624333

[14] B. Efron. Logistic Regression, Survival Analysis, and the Kaplan-Meier Curve. *Journal of the American Statistical Association*, 83(402):414–425, 1988. MR0971367

[15] R. D. Gill and S. Johansen. A Survey of Product-Integration with a View Toward Application in Survival Analysis. *The Annals of Statistics*, 18(4): 1501–1555, 1990. MR1074422

[16] J. D. Kalbfleisch and R. L. Prentice. *The Statistical Analysis of Failure Time Data*. John Wiley & Sons, Hoboken, New Jersey, USA, 2nd edition, 2002. MR1924807

[17] E. L. Kaplan and P. Meier. Nonparametric Estimation from Incomplete Observations. *Journal of the American statistical association*, 53(282):457–481, 1958. MR0093867

[18] D. Y. Lin. Non-parametric inference for cumulative incidence functions in competing risks studies. *Statistics and Medicine*, 16:901–910, 1997.

[19] D. Y. Lin, L. J. Wei, and Z. Ying. Checking the Cox model with cumulative sums of martingale-based residuals. *Biometrika*, 80(3):557–572, 1993. MR1248021

[20] T. Martinussen and T. H. Scheike. *Dynamic regression models for survival data*. Statistics for Biology and Health. Springer, New York, 2006. MR2214443

[21] D. F. Moore. *Applied Survival Analysis Using R*. Springer, Switzerland, 2016.

[22] R Development Core Team. *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria, 2016. URL `http://www.R-project.org`.

[23] G. Tutz and M. Schmid. *Modeling Discrete Time-to-Event Data*. Springer, Cham, 2016. MR3497009

[24] A. W. van der Vaart and J. A. Wellner. *Weak Convergence and Empirical Processes*. Springer, New York, 1996. MR1385671