Electronic Journal of Statistics
Vol. 11 (2017) 99–147
ISSN: 1935-7524
DOI: 10.1214/17-EJS1225

Bootstrap for the second-order analysis of Poisson-sampled almost periodic processes

Dominique Dehay

Institut de Recherche Mathématique de Rennes
UMR CNRS 6625, Université Rennes 2, Rennes, France
e-mail: dominique.dehay@univ-rennes2.fr

and

Anna E. Dudek*

Institut de Recherche Mathématique de Rennes, Université Rennes 2, Rennes, France
AGH University of Science and Technology, Department of Applied Mathematics,
al. Mickiewicza 30, 30-059 Krakow, Poland
e-mail: aedudek@agh.edu.pl

Abstract: In this paper we consider a continuous almost periodically correlated process $\{X(t), t \in \mathbb{R}\}$ that is observed at the jump moments of a stationary Poisson point process $\{N(t), t \geq 0\}$. The processes $\{X(t), t \in \mathbb{R}\}$ and $\{N(t), t \geq 0\}$ are assumed to be independent. We define the kernel estimators of the Fourier coefficients of the autocovariance function of $X(t)$ and investigate their asymptotic properties. Moreover, we propose a bootstrap method that provides consistent pointwise and simultaneous confidence intervals for the considered coefficients. Finally, to illustrate our results we provide a simulated data example.

MSC 2010 subject classifications: Primary 62M15; secondary 62G09, 62G05, 62G20.
Keywords and phrases: Block bootstrap, consistency, Fourier coefficients of autocovariance function, irregular sampling, nonstationary process.

Received July 2016.

1. Introduction

Periodicity and almost periodicity appear naturally in many real datasets. A wide variety of examples from different fields can be found in [1], [14], [19], [28] and [29]. Often this type of data has a structure of periodic or almost periodic correlation. To be more precise, a stochastic process that has finite second order moments is *almost periodically correlated* (APC) if its mean function and its shifted covariance function are almost periodic in time [16]. The notion of an almost periodic function was introduced in [3]. Recall that a function $f : \mathbb{R} \to \mathbb{R}$

*This project has received funding from the European Union's Horizon 2020 research and innovation programme under Marie Skłodowska-Curie grant agreement No 655394.

is called *almost periodic* if for every $\varepsilon > 0$, there exists a number l_ε such that for any interval of length greater than l_ε, there exists a number p_ε in this interval such that

$$\sup_{t\in\mathbb{R}} |f(t+p_\varepsilon) - f(t)| < \varepsilon.$$

Equivalently, the almost periodic functions can be defined as the uniform limits of trigonometric polynomials (see [3]). Periodic functions are the only class of almost periodic functions that have a period. In general, an almost periodic function has no period. There is a large amount of papers concerning periodically correlated (PC) processes (see e.g. [19] and references therein), but still relatively few on general APC processes. Thus, the analysis of APC processes is challenging. It is usually performed using Fourier expansions of the mean function and of the shifted autocovariance functions. The other important issue is the fact that the considered processes cannot be observed continuously and data are often sampled irregularly to avoid aliasing and folding phenomena (see e.g. [26]).

In this paper we deal with an APC continuous process $X = \{X(t), t \in \mathbb{R}\}$ that is observed at the jump moments T_k, $k \geq 1$, of a stationary Poisson point process $N = \{N(t), t \geq 0\}$ which is independent on X. This type of sampling was considered e.g. in [5] (see also [7], [22], [23] and [30]). The estimation problem of the Fourier coefficients of the mean function in this framework was considered in [7] where the standard estimators have been used. In the following we focus on the Fourier analysis of the shifted autocovariance function. In this case, the Fourier coefficients are computed from products of the values of the process where distance in time is equal to an arbitrary fixed lag $\tau \in \mathbb{R}$, i.e. $X(t)X(t+\tau)$, $t \in \mathbb{R}$ (see relation (1)). However, using Poisson sampling we do not have at our disposal the necessary information to perform some sampled versions of these products, thus we cannot use the standard estimators. To deal with this problem we are going to apply a kernel method. Similar ideas for stationary processes can be found in [27] (see also [22]), and for harmonizable APC processes in [9]. The second key issue is the construction of confidence intervals for the Fourier coefficients which we are estimating. The asymptotic covariance matrices are very complicated and depend on unknown parameters (see Proposition 6.1). Hence in practice it is very difficult or even impossible to estimate them. Thus to get confidence intervals, resampling methods are used. In the literature the number of bootstrap and subsampling consistency results for the Fourier coefficients of PC and APC processes is constantly growing. The subsampling validity for the Fourier coefficients is established for PC time series in [25], and for APC processes in [8]. Moreover, the consistency of the Moving Block Bootstrap (MBB) for the Fourier coefficients of the shifted autocovariance function for PC or APC time series was studied in [31] (see also [12]). Additionally, for PC time series the Generalized Seasonal Block Bootstrap (GSBB) was considered in [13]. The GSBB in contrary to the MBB requires knowledge of the period length and hence cannot be applied in the general case for APC processes. Finally, the modification of the MBB method for the Poisson sampled APC process was considered in [7] for the estimation problem of the cyclic mean.

In this paper we provide a consistent estimator for the Fourier coefficients of the shifted autocovariance function of a continuous time APC process from a Poisson sampled observation. Moreover, we obtain the validity of some bootstrap approach that is based on the MBB method. As a result construction of bootstrap confidence intervals is possible.

The paper is organized as follows. In Section 2 the necessary notation is introduced and the problem is formulated. Additionally, the kernel estimator of the Fourier coefficients is introduced. In Section 3 the assumptions are discussed and asymptotic properties of the considered estimator are derived. Moreover, the bootstrap method is described and its consistency is shown. Section 4 contains multidimensional results and the construction of the bootstrap simultaneous confidence intervals. In Section 5 a simulated data example is presented. The proofs of the results can be found in Section 6.

2. Problem formulation

Let $X = \{X(t), t \in \mathbb{R}\}$ be a real-valued zero-mean APC process that is observed at the jump moments T_k, $k \geq 1$, of a stationary Poisson point process $N = \{N(t), t \geq 0\}$ with intensity $\beta > 0$. The processes X and N are assumed to be independent. Moreover, let $K_X(t, t+\tau)$ be the shifted autocovariance function of the real-valued zero-mean process X, i.e. $K_X(t, t+\tau) := \mathrm{cov}\{X(t), X(t+\tau)\} = \mathrm{E}\{X(t)X(t+\tau)\}$. Since the process X is APC, the function $(t, \tau) \mapsto K(t, t+\tau)$ is uniformly continuous, and for each τ the function $t \mapsto K(t, t+\tau)$ is almost periodic (see e.g. [16] and [18]). Then the *cyclic covariance* of the process X is defined as Fourier-Bohr coefficients

$$\begin{aligned} a_X(\lambda, \tau) &:= \lim_{T\to\infty} \frac{1}{T} \int_0^T K_X(t, t+\tau) e^{-i\lambda t}\, dt \\ &= \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{E}\{X(t)X(t+\tau)\} e^{-i\lambda t}\, dt \end{aligned} \tag{1}$$

for any $\lambda \in \mathbb{R}$ and any $\tau \in \mathbb{R}$. The set of *cyclic frequencies* $\Lambda := \{\lambda \in \mathbb{R} : a_X(\lambda, \tau) \neq 0$ for some $\tau \in \mathbb{R}\}$ is known to be countable (see [18]).

The estimation of the cyclic covariance is based on the observations where the time distance is equal to τ, and $\tau \in \mathbb{R}$ is fixed. Let us recall that in the considered problem the process X is not observed continuously and the time distances between the observations in general are not even integer multiples of τ. To solve this problem we propose a kernel estimator of $a_X(\lambda, \tau)$.

Let $T > 0$. Assume that the observation of the process X is performed during the time interval $[0, T]$. This means that we observe the random sequence $X(T_k)$ at the sampling moments $0 \leq T_k \leq T$, $k = 1, \ldots, N(T)$.

Let us define $w_h(t) := w\big(t/h\big)/h$, where the window width $h = h_T > 0$ tends to 0 as $T \to \infty$ and the weight function $w : \mathbb{R} \to \mathbb{R}$ is a non-negative symmetric measurable function with support contained in $[-1, 1]$. Moreover assume that $\int_{-1}^{1} w(t)\, dt = 1$.

Recall that $\mathrm{E}\{X(t)\} = 0$. Then for all fixed λ and $|\tau| \le T - h_T$ we define the estimator of $a_X(\lambda,\tau)$ as follows

$$\widehat{a}_T = \widehat{a}_T(\lambda,\tau) := \frac{1}{\beta^2 T}\sum_{n_1,n_2=1}^{N(T)}\sum \mathbb{I}_{\{n_1\neq n_2\}} w_{h_T}(\tau - T_{n_2} + T_{n_1}) X(T_{n_1}) X(T_{n_2})\, e^{-i\lambda T_{n_1}}$$
$$= \frac{1}{\beta^2 T}\int_0^T\int_0^T w_{h_T}(\tau - t + s) X(s) X(t)\, e^{-i\lambda s}\, dN^{(2)}(s,t),$$

where $dN^{(2)}(s,t) := \mathbb{I}_{\{s\neq t\}}\, dN(s)dN(t)$. Hence

$$\widehat{a}_T = \frac{1}{\beta^2 T}\int_{-T}^{T}\int_{I(T,t)} w_{h_T}(\tau - t) X(s) X(s+t)\, e^{-i\lambda s}\, N^{(2)}(s+ds, s+t+dt),$$

where $I(T,t) := \big\{s : 0 \le s \le T, -t \le s \le T - t\big\} = [0,T]\cap[-t, T-t]$.

In this paper we assume that the intensity β of the Poisson process $\{N(t), t \ge 0\}$ is known. When β is unknown, in the definition of $\widehat{a}_T$ it may be replaced by $\widehat{\beta}_T = N(T)/T$. Then the consistency of this estimator of $a_X(\lambda,\tau)$ as well as the consistency of the bootstrap method can be easily established. However, this problem is beyond the scope of this paper.

3. Main results

In this section we present the asymptotic properties of the estimator $\widehat{a}_T$ and its bootstrap version $\widehat{a}_T^*$. At first we discuss the assumptions that will allow us to establish the results.

(AP$_2$) For each $t \in \mathbb{R}$, $\mathrm{E}\{X(t)\} = 0$ and $\mathrm{E}\{X(t)^2\} < \infty$; the function $(t,\tau) \mapsto \mathrm{E}\{X(t)X(t+\tau)\}\}$ is uniformly continuous on $\mathbb{R}\times\mathbb{R}$ and is almost periodic in t uniformly with respect to τ varying in $\mathbb{R}$.

(AP$_4$) For each $t\in\mathbb{R}$, $\mathrm{E}\{X(t)^4\} < \infty$; the function $(t,\tau_1,\tau_2,\tau_3) \mapsto \mathrm{E}\{X(t)X(t+\tau_1)X(t+\tau_2)X(t+\tau_3)\}$ is almost periodic in t uniformly with respect to τ_1,τ_2,τ_3 varying in $\mathbb{R}$.

(CS(λ,τ)) *Cycle separability:* For fixed $\lambda,\tau\in\mathbb{R}$

$$\sum_{\lambda'\neq\lambda}\frac{|a_X(\lambda',\tau)|}{|\lambda'-\lambda|} < \infty. \tag{2}$$

(Lip) *Lipschitz property of the shifted covariance:* There exists a constant $L > 0$ such that for all $s,\tau\in\mathbb{R}$ and $u\in[-1,1]$ we have

$$\big|\mathrm{E}\{X(s)X(s+\tau-u)\} - \mathrm{E}\{X(s)X(s+\tau)\}\big| \le L|u|. \tag{3}$$

(M) *Mixing property of the processs X:* $X(t)$ is α-mixing and

(i) either $\{X(t), t\in\mathbb{R}\}$ is bounded and $\alpha_X(\cdot)\in L^1([0,\infty))$,

(ii) or there exists $\delta > 0$ such that $\sup_t \mathrm{E}\big\{X(t)^{4+\delta}\big\} < \infty$ and $\alpha_X(\cdot)\in L^{\delta/(4+\delta)}([0,\infty))$.

Assumptions (AP_2) and (AP_4) denote almost periodicity of the second and fourth moments of the process X, respectively. Condition (CS(λ, τ)) is a separability condition on the cyclic frequencies. It is satisfied for all $\lambda, \tau \in \mathbb{R}$ whenever the set of cyclic frequencies Λ satisfies

$$\sum_{\lambda \in \Lambda \setminus \{0\}} |\lambda|^{-2} < \infty.$$

The last inequality is fulfilled when the cyclic frequencies are separated by at least a positive constant because

$$\left(\sum_{\lambda' \in \Lambda \setminus \{\lambda\}} \frac{|a_X(\lambda', \tau)|}{|\lambda' - \lambda|} \right)^2 \le \left(\sum_{\lambda' \in \Lambda \setminus \{\lambda\}} |a_X(\lambda', \tau)|^2 \right) \left(\sum_{\lambda' \in \Lambda \setminus \{\lambda\}} |\lambda' - \lambda|^{-2} \right)$$

and for any APC process we ave

$$\sum_{\lambda' \in \Lambda} |a_X(\lambda', \tau)|^2 < \infty.$$

Finally, the asymptotic results that we will present require some mixing assumption (M). To be precise the process X is assumed to be α-mixing (strong mixing) i.e. $\alpha_X(k) \to 0$ as $k \to \infty$, where

$$\alpha_X(u) = \sup_t \sup_{\substack{A \in \mathcal{F}_X(-\infty, t) \\ B \in \mathcal{F}_X(t+u, \infty)}} |P(A \cap B) - P(A)P(B)|,$$

and $\mathcal{F}_X(-\infty, t) = \sigma(\{X(s) : s \le t\})$ and $\mathcal{F}_X(t+u, \infty) = \sigma(\{X(s) : s \ge t + u\})$. For more details and examples of other dependence measures that we could considered we refer the reader to [10] and [6].

In the sequel we use the following symbols: $\mathcal{O}(\cdot)$, $o(\cdot)$ and $\ll$. Let $f(\cdot)$ and $g(\cdot)$ be real valued functions defined on $(0, \infty)$ or on $\mathbb{N}$. The notation $f(T) = \mathcal{O}(g(T))$ denotes that $|f(T)/g(T)|$ remains bounded as $T \to \infty$. The notation $f(T) = o(g(T))$ stands for $f(T)/g(T) \to 0$ as $T \to \infty$. We will also use the notation $f(T) \ll g(T)$ when $f(T)/g(T) \to 0$ as $T \to \infty$.

3.1. Properties of $\widehat{a}_T$

This section is dedicated to some asymptotic properties of $\widehat{a}_T$. From now on whenever we consider a complex number z we treat it as the bidimensional vector with coordinates equal to the real and the imaginary parts of z. At first we study the bias, the convergence in quadratic mean and the almost sure convergence of $\widehat{a}_T$, as well as the rate of convergence. The last result states the asymptotic normality of the considered estimator.

Proposition 3.1. *Let $\{X(t), t \ge 0\}$ be an APC process which satisfies condition (AP_2). Then for all $\lambda, \tau \in \mathbb{R}$*

$$\lim_{T \to \infty} E\left\{\widehat{a}_T(\lambda, \tau)\right\} = a_X(\lambda, \tau).$$

Furthermore, if in addition conditions (CS(λ,τ)) and (Lip) hold, and if $h_T \ll T^{-1/3}$, then

$$\lim_{T\to\infty} \sqrt{Th_T}\big(\mathrm{E}\left\{\widehat{a}_T(\lambda,\tau)\right\} - a_X(\lambda,\tau)\big) = 0.$$

Proposition 3.2. *Let $\{X(t), t \geq 0\}$ be such that the APC condition (AP_2) and the mixing condition (M) are fulfilled. Assume also that $T^{-1} \ll h_T \ll 1$. Then for all $\lambda, \tau \in \mathbb{R}$*

$$\lim_{T\to\infty} \mathrm{E}\left\{\left|\widehat{a}_T(\lambda,\tau) - a_X(\lambda,\tau)\right|^2\right\} = 0.$$

If in addition conditions ($CS_2(\lambda,\tau)$) and (Lip) are fulfilled and $h_T \ll T^{-1/3}$, then for all $\lambda, \tau \in \mathbb{R}$

$$\limsup_{T\to\infty} Th_T \mathrm{E}\left\{\left|\widehat{a}_T(\lambda,\tau) - a_X(\lambda,\tau)\right|^2\right\} < \infty$$

As a by-product of the rate of convergence in the quadratic mean, we state the almost sure convergence of the estimator $\widehat{a}_T(\lambda,\tau)$.

Proposition 3.3. *In addition to all the conditions of Proposition 3.2, assume that the window width h_T is non-increasing, $T^{-\kappa} \leq h_T \ll T^{-1/3}$ for some $1/3 < \kappa < 1$, there exists $x > (1-\kappa)^{-1}$ such that $h_{n^x} = h_{(n+1)^x}(1 + o(1))$ as $n \to \infty$, and the kernel function $w(\cdot)$ is non-increasing on $(0,\infty)$. Then $\widehat{a}_T(\lambda,\tau)$ converges almost surely to $a_X(\lambda,\tau)$ as $T \to \infty$.*

Furthermore under the hypotheses of the previous proposition, we can establish some rate of almost sure convergence. For instance when $h_T = T^{-\kappa}$, we have

$$\lim_{T\to\infty} T^{(1-\kappa)\epsilon}\big(\widehat{a}_T(\lambda,\tau) - a_X(\lambda,\tau)\big) = 0 \qquad \text{a.s.}$$

for any $0 < \epsilon < \min\left\{\big(2x(1-\kappa)\big)^{-1}, 1/2 - \big(2x(1-\kappa)\big)^{-1}\right\}$.

Next we state the asymptotic normality of the estimator.

Theorem 3.4. *Let $\{X(t), t \geq 0\}$ be an APC process and assume that conditions (CS(λ,τ)), (Lip), (AP_2) and (AP_4) as well as the mixing condition (M) are satisfied. Then $\sqrt{Th_T}\big(\widehat{a}_T(\lambda,\tau) - a_X(\lambda,\tau)\big)$ converges to a bidimensional Gaussian distribution provided $T^{-1} \ll h_T \ll T^{-1/3}$.*

The form of the asymptotic covariance matrix is very complicated (see Proposition 6.1) and hence very difficult to estimate. Thus, in practice to construct confidence intervals, resampling methods are used. In the sequel, we introduce a bootstrap approach and provide results stating its consistency.

3.2. Bootstrap method and consistency results

In this section we present a bootstrap technique that can be used to construct the consistence bootstrap confidence intervals for the cyclic covariances. The idea of this approach was introduced in [7] for the cyclic means. Our bootstrap method is the modification of the usual Moving Block Bootstrap. Before we describe the bootstrap algorithm let us introduce some additional notation.

Recall that we observe $(X(T_k), T_k)$ at the sampling moments T_k in the time interval $[0, T]$. In the sequel P^*, E^* and var^* denote respectively conditional probability, conditional expectation and conditional variance given the sample.

Let $0 < b_T < T$ be the block length, $h_T > 0$ be the window width. Without loss of generality we assume that the considered time interval $[0, T]$ can be split into l_T disjoint subintervals of the length b_T, i.e. $T = l_T b_T$, where $T, b_T, l_T \in \mathbb{N}$. To simplify the notation we write h, b and l for respectively h_T, b_T and l_T except when there is some risk of misunderstanding. Throughout the paper we assume that $h \to 0$, $b \to \infty$ and $l \to \infty$ as $T \to \infty$.

Bootstrap Algorithm:

1. Choose an integer number $0 < b < T$.
2. For $k = 0, \ldots T - b$ define the blocks of observations as following
$$B_k := B_{k,b} := \{(X(T_i), T_i) : k < T_i \leq k + b, i \in \mathbb{N}^*\},$$
where $\mathbb{N}^* = \{1, 2, \ldots\}$.
3. Select randomly with replacement l blocks from the set $\{B_0, \ldots, B_{T-b}\}$. This corresponds to the random selection with replacement of l numbers from the set $\{0, \ldots, T - b\}$. More precisely let $i_1^*, \ldots, i_l^*$ be i.i.d. random variables on the set $\{0, 1, \ldots, T - b\}$:
$$\mathrm{P}^*\left(i_j^* = k\right) = \frac{1}{T - b + 1}, \qquad k = 0, \ldots, T - b, \quad j = 1, \ldots, l.$$
For simplicity of notation we write $j^* := i_{j+1}^*$ for $j = 0, \ldots, l - 1$.
4. Join the l blocks $(B_0^*, B_1^*, \ldots, B_{l-1}^*)$ to obtain a bootstrap sample, where
$$B_j^* := B_{j^*}, \qquad j = 0, \ldots, l - 1.$$

Before we construct a bootstrap version of the estimator $\widehat{a}_T$ of $a_X(\lambda, \tau)$, we introduce an estimator $\widehat{a}_{k,b}$, which is the version of $\widehat{a}_T$ defined on the subinterval $(k, k + b]$, for $0 \leq k < k + b \leq T$. Let

$$\begin{aligned}
\widehat{a}_{k,b} &= \widehat{a}_{k,b}(\lambda, \tau) \\
&:= \frac{1}{\beta^2 b} \sum_{n_1, n_2 = N(k)+1}^{N(k+b)} \mathbb{I}_{\{n_1 \neq n_2\}} w_h\big(\tau - T_{n_2} + T_{n_1}\big) X(T_{n_1}) X(T_{n_2})\, e^{-i\lambda T_{n_1}} \\
&= \frac{1}{\beta^2 b} \int_k^{k+b}\!\!\int_k^{k+b} w_h(\tau - t + s)\, X(s) X(t)\, e^{-i\lambda s}\, dN^{(2)}(s, t) \\
&= \frac{1}{\beta^2 b} \int_0^b\!\!\int_0^b w_h(\tau - t + s)\, X(k+s) X(k+t)\, e^{-i\lambda(s+k)}\, N^{(2)}(k+s+ds, k+t+dt).
\end{aligned}$$

In [7] it is shown that in the case of cyclic mean estimation the original estimator can be equivalently expressed as the sum of the estimators defined on subintervals. Unfortunately, for the cyclic covariance estimation case such decomposition does not hold. In general

$$\widehat{a}_T \neq \frac{1}{l}\sum_{j=0}^{l-1}\widehat{a}_{jb,b}.$$

However, we consider the bootstrap version of the estimator $\widehat{a}_T$ defined as

$$\widehat{a}_T^* = \widehat{a}_T^*(\lambda,\tau) := \frac{1}{l}\sum_{j=0}^{l-1}\widehat{a}_{j^*,b}(\lambda,\tau),$$

where

$$\begin{aligned}
\widehat{a}_{j^*,b} &= \widehat{a}_{j^*,b}(\lambda,\tau) \\
&:= \frac{1}{\beta^2 b}\int_{j^*}^{j^*+b}\int_{j^*}^{j^*+b} w_h(\tau-t+s)X(s)X(t)\,e^{-i\lambda s}\,dN^{(2)}(s,t) \\
&= \frac{1}{\beta^2 b}\sum_{n_1=1}^{N(T)}\sum_{n_2=1}^{N(T)} \mathbb{I}_{\{n_1\neq n_2\}}\mathbb{I}_{\{j^*\leq T_{n_1},T_{n_2}\leq j^*+b\}} w_h(\tau-T_{n_1}+T_{n_2}) \\
&\quad \times X(T_{n_1})X(T_{n_2})e^{-i\lambda T_{n_2}}.
\end{aligned}$$

One can easily notice that $\widehat{a}_T^*$ (conditional on the observation) is a biased estimator of $\widehat{a}_T$.

$$\mathrm{E}^*\left\{\widehat{a}_T^*\right\} = \frac{1}{l}\sum_{j=0}^{l-1}\frac{1}{T-b+1}\sum_{k=0}^{T-b}\widehat{a}_{k,b} = \frac{1}{T-b+1}\sum_{k=0}^{T-b}\widehat{a}_{k,b} \neq \widehat{a}_T.$$

Moreover for each $j = 0,\ldots,l-1$, we have

$$\mathrm{E}^*\left\{\widehat{a}_T^*\right\} = \mathrm{E}^*\left\{\widehat{a}_{j^*,b}\right\} \quad \text{and} \quad \mathrm{var}^*\left\{\widehat{a}_T\right\} = \mathrm{var}^*\left\{\widehat{a}_{j^*,b}\right\}.$$

Next proposition states the asymptotic unbiasedness of $\widehat{a}_T^*$.

Proposition 3.5. *Assume that conditions (AP$_2$) and (M) are satisfied. Then*

$$\lim_{T\to\infty}\mathrm{E}^*\left\{\widehat{a}_T^*(\lambda,\tau)\right\} = a_X(\lambda,\tau) \qquad \textit{in q.m.} \tag{4}$$

Assume in addition that conditions (CS(λ,τ)) and (Lip) are fulfilled, and that $T^{-1} \ll h \ll \min\{T^{-1/3}, b^2T^{-1}\}$, *then*

$$\limsup_{T\to\infty} Th\,\mathrm{E}\left\{\left|\mathrm{E}^*\left\{\widehat{a}_T^*(\lambda,\tau)\right\} - a_X(\lambda,\tau)\right|^2\right\} < \infty. \tag{5}$$

Moreover, assume that the window width $h = h_T$ is non-increasing, $T^{-\kappa} \leq h_T \ll T^{-1/3}$ where $1/3 < \kappa < 1$, the block length b_T is non-decreasing, $1 \ll b_T \ll T$ and there exists $x > 0$ such that $x(1-\kappa) > 1$, $h_{n^x} = h_{(n+1)^x}(1+o(1))$ and $b_{(n+1)^x} = b_{n^x}(1+\mathcal{O}(1/n))$ as $n\to\infty$. Assume also that the kernel function $w(\cdot)$ is non-increasing on $(0,\infty)$. Then

$$\lim_{T\to\infty}\mathrm{E}^*\left\{\widehat{a}_T^*(\lambda,\tau)\right\} = a_X(\lambda,\tau) \quad \textit{a.s.}$$

Furthermore, under the assumptions of the last proposition, we can obtain some rate of convergence. For instance when $h_T = T^{-\kappa}$

$$\lim_{T\to\infty} T^{(1-\kappa)\epsilon}\big(\mathrm{E}^*\left\{\widehat{a}^*_T(\lambda,\tau)\right\} - a_X(\lambda,\tau)\big) = 0 \quad \text{a.s.}$$

for any $0 < \epsilon < \min\left\{\big(2x(1-\kappa)\big)^{-1}, 1/2 - \big(2x(1-\kappa)\big)^{-1}\right\}$.

Then we state the convergence in P-probability of the bootstrap variance.

Proposition 3.6. *Let λ and τ be fixed. Assume that conditions (AP_2), (AP_4), ($CS(\lambda,\tau)$) and (Lip) are fulfilled. Assume that*

- *either the process $\{X(t), t\in\mathbb{R}\}$ is bounded and $\alpha_X(\cdot)\in L^1([0,\infty))$,*
- *or $\sup_t \mathrm{E}\left\{|X(t)|^{8+\delta}\right\} < \infty$ and $\alpha_X(\cdot)\in L^{\delta/(4+\delta)}([0,\infty))$.*

Let $T^{1/3}\ll b\ll T$ and $b^{-1}\ll h\ll \min\{T^{-1/3}, b^{-3/2}T^{1/2}\}$. Then $Th\,\mathrm{var}^*\left\{\widehat{a}^*_T\right\}$ converges in P-mean (i.e. in $L^1(\mathrm{P})$) so in P-probability to the variance matrix of the bidimension Gaussian limit distribution obtained in Theorem 3.4.

Finally, we present the consistency of the bootstrap method.

Theorem 3.7. *Let λ and τ be fixed. Assume that conditions (AP_2), (AP_4), ($CS(\lambda,\tau)$ and (Lip) are fulfilled. Assume also that*

- *either the process $\{X(t), t\geq 0\}$ is bounded and $\int_0^\infty t\,\alpha_X(t)\,dt < \infty$,*
- *or $\sup_t \mathrm{E}\left\{|X(t)|^{8+\delta}\right\} < \infty$ and $\int_0^\infty t\,\alpha_X(t)^{\delta/(4+\delta)}dt < \infty$.*

Let $0<\theta\leq 2/9$, $T^{1/3}\ll b\leq T^{\theta+1/3}$ and $\max\{b^{-1}, (bT^{-1})^{1/(2-3\theta)}\}\ll h\ll T^{-1/3}$. Then

$$\rho\left(\mathcal{L}\left\{\sqrt{Th}\left(\widehat{a}_T(\lambda,\tau) - a_X(\lambda,\tau)\right)\right\}, \mathcal{L}^*\left\{\sqrt{Th}\left(\widehat{a}^*_T(\lambda,\tau) - \widehat{a}_T(\lambda,\tau)\right)\right\}\right) \xrightarrow{p} 0,$$

as $T\to\infty$, where ρ is a metric metricizing weak convergence in $\mathbb{R}^2$.

In theorem above by $\mathcal{L}\left\{\sqrt{Th}\left(\widehat{a}_T(\lambda,\tau) - a_X(\lambda,\tau)\right)\right\}$ we denote a probability law of $\sqrt{Th}\left(\widehat{a}_T(\lambda,\tau) - a_X(\lambda,\tau)\right)$ and by $\mathcal{L}^*\left\{\sqrt{Th}\left(\widehat{a}^*_T(\lambda,\tau) - \widehat{a}_T(\lambda,\tau)\right)\right\}$ its bootstrap counterpart.

4. Multidimensional results

In this section we provide the multidimensional versions of Theorems 3.4 and 3.7. Moreover, we discuss construction of the bootstrap simultaneous confidence intervals. At first let us introduce some additional notation. Let $r\in\mathbb{N}$ be fixed and

$$\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_r)', \boldsymbol{\tau} = (\tau_1,\ldots,\tau_r)'$$

be vectors of frequencies and lags, respectively. Moreover, let

$$\Re a_X(\boldsymbol{\lambda},\boldsymbol{\tau}) = (\Re a_X(\lambda_1,\tau_1),\ldots,\Re a_X(\lambda_r,\tau_r))',$$

$$\Im a_X(\boldsymbol{\lambda},\boldsymbol{\tau}) = (\Im a_X(\lambda_1,\tau_1),\ldots,\Im a_X(\lambda_r,\tau_r))',$$

and

$$a_X(\boldsymbol{\lambda},\boldsymbol{\tau}) = (\Re a_X(\lambda_1,\tau_1),\Im a_X(\lambda_1,\tau_1),\ldots,\Re a_X(\lambda_r,\tau_r),\Im a_X(\lambda_r,\tau_r))'.$$

Additionally, by $\widehat{a}_T(\boldsymbol{\lambda},\boldsymbol{\tau})$ and $\widehat{a}_T^*(\boldsymbol{\lambda},\boldsymbol{\tau})$ we denote the estimator of $a_X(\boldsymbol{\lambda},\boldsymbol{\tau})$ and its bootstrap version.

Theorem 4.1. *Under the assumptions of Theorem 3.4,* $\sqrt{Th}\big(\widehat{a}_T(\boldsymbol{\lambda},\boldsymbol{\tau})-a_X(\boldsymbol{\lambda},\boldsymbol{\tau})\big)$ *converges to a 2r-dimensional Gaussian distribution with mean zero.*

The form of covariance matrix is presented in Proposition 6.1.

Theorem 4.2. *Under the assumptions of Theorem 3.7*

$$\rho\left(\mathcal{L}\left\{\sqrt{Th}\left(\widehat{a}_T(\boldsymbol{\lambda},\boldsymbol{\tau})-a_X(\boldsymbol{\lambda},\boldsymbol{\tau})\right)\right\},\mathcal{L}^*\left\{\sqrt{Th}\left(\widehat{a}_T^*(\boldsymbol{\lambda},\boldsymbol{\tau})-\widehat{a}_T(\boldsymbol{\lambda},\boldsymbol{\tau})\right)\right\}\right)\xrightarrow{p}0,$$

where ρ *is a metric metricizing weak convergence in* $\mathbb{R}^{2r}$*.*

Thanks to the continuous mapping theorem one can easily deduce the consistency of the bootstrap approach for smooth functions of $a_X(\boldsymbol{\lambda},\boldsymbol{\tau})$. This is a key result for obtaining the bootstrap consistent confidence intervals for the parameters of interest, which are very important in real data applications. Below we briefly recall the construction of the $(1-2\alpha)\%$ bootstrap equal-tailed percentile simultaneous confidence intervals. In the next section we use them to detect significant frequencies of some simulated signal. Let

$$K_{max}(x) := P^*\left(\sqrt{Th}\max_i \Re\left(\widehat{a}_T^*(\lambda_i,\tau)-\widehat{a}_T(\lambda_i,\tau)\right)\le x\right),$$
$$K_{min}(x) := P^*\left(\sqrt{Th}\min_i \Re\left(\widehat{a}_T^*(\lambda_i,\tau)-\widehat{a}_T(\lambda_i,\tau)\right)\le x\right)$$

for $x\in\mathbb{R}$, and we get the confidence region of the form

$$\left(\Re\widehat{a}_T(\lambda_i,\tau)-\frac{K_{max}^{-1}(1-\alpha)}{\sqrt{Th}},\Re\widehat{a}_T(\lambda_i,\tau)-\frac{K_{min}^{-1}(\alpha)}{\sqrt{Th}}\right)$$

for $i=1,\ldots,r$, $\lambda_1,\ldots,\lambda_r\in\mathbb{R}$ and $\tau\in\mathbb{R}$. The confidence intervals for imaginary case are defined correspondingly.

5. Simulated data example

In our study we consider the process X of the form

$$X(t) = 20\sin(2\pi t/5)\,\mathit{OU}(t),$$

where $\{\mathit{OU}(t), t\in\mathbb{R}\}$ is a zero-mean Ornstein-Uhlenbeck process generated with the following parameters: the time step is 0.01, the relaxation time 0.1, the

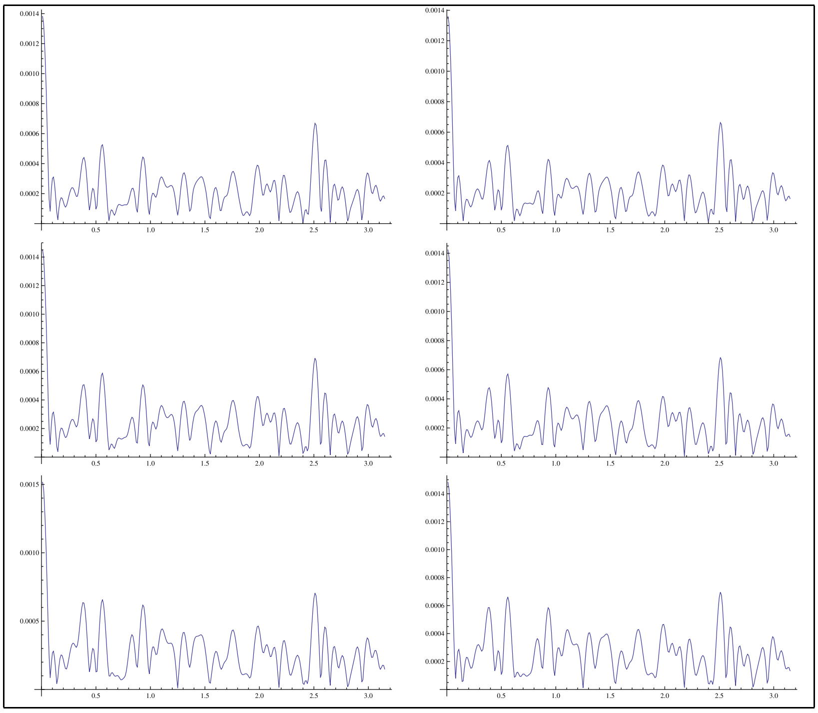

FIG 1. *Estimated values of* $|a_X(\lambda, 0)|$ *for* $\lambda \in \{0, 0.01, \ldots, 3.14\}$ *and sample size* $T = 100$, $b = 10$. *Results for* $h = 20T^{-0.35}$ *and* $h = 20T^{-0.37}$ *in the left and the right column, respectively. From top results for* $d = 0.5, 0.4, 0.2$, *respectively.*

diffusion constant 1 and the initial value $OU(0) = 0$ (for more details please see [15]). The intensity of the Poisson process $\{N(t), t \geq 0\}$ is set to $\beta = 5$.

Our aim is to identify significant frequencies for the shifted autocovariance function applying the methodology presented in the previous sections. In the considered example in the interval $[0, \pi]$ there are two such frequencies 0 and 0.8π. To detect them we construct 95% bootstrap pointwise and simultaneous equal-tailed confidence intervals for $a_X(\lambda, \tau)$, more precisely for its real part $\Re a_X(\lambda, \tau)$ and its imaginary part $\Im a_X(\lambda, \tau)$. We take two sample sizes $T \in \{100, 400\}$, the number of bootstrap resamples $B = 500$ and the block length $b = \lfloor \sqrt{T} \rfloor \in \{10, 20\}$. Thus $T^{1/3} \ll b \ll T^{5/9}$ (Theorem 3.7). The kernel function $w(\cdot)$ is of the form

$$w^d(t) = \begin{cases} t/d & \text{for } t \in [0, d] \\ 1 & \text{for } t \in [d, 1-d] \\ (1-t)/d & \text{for } t \in [1-d, 1]. \end{cases}$$

with $d = 0.5, 0.4, 0.2$. Depending on the value of the constant d, we get a trapezoidal ($d < 0.5$) or a triangular function ($d = 0.5$). Finally, the window width $h \in \{20T^{-0.35}, 20T^{-0.37}\}$. The considered frequencies λ belong to the set

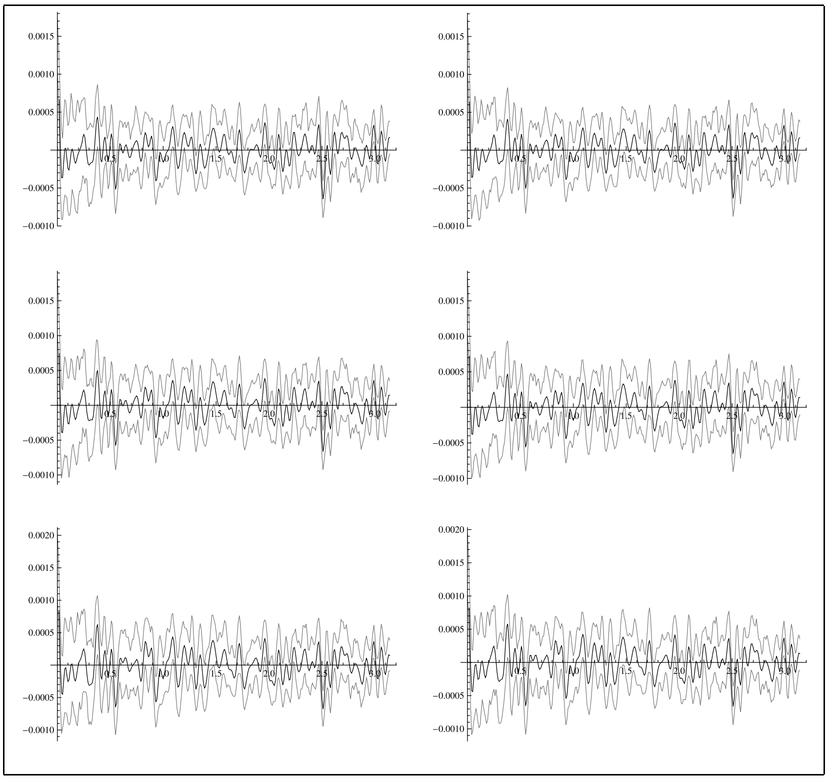

FIG 2. *Estimated values of* $\Re a_X(\lambda, 0)$ *for* $\lambda \in \{0, 0.01, \dots, 3.14\}$ *and sample size* $T = 100$, $b = 10$ *(black line) together with the 95% bootstrap pointwise equal-tailed confidence intervals (gray line) constructed with* $h = 20T^{-0.35}$ *(left column) and* $h = 20T^{-0.37}$ *(right column). From top results for* $d = 0.5, 0.4, 0.2$*, respectively.*

$\{0, 0.01, \dots, 3.14\}$. The lag is set $\tau = 0$. One should be aware that none of the values in the considered set of frequencies is precisely equal to the true frequency $0.8\pi \approx 2.51327$.

In Figure 1 we present the estimated values of $|a_X(\lambda, 0)|$ for different kernel functions and $T = 100$. The number of jumps of the generated Poisson process is 495. Independently on the values of d and h one may observe a high peak for $\lambda = 0$. There are also some other values that may be significant and they can be observed not only around the true frequency $\lambda = 0.8\pi$.

Next we present the figures only for the real part $\Re a_X(\lambda, 0)$ of $a_X(\lambda, 0)$, since they are very similar for the imaginary part $\Im a_X(\lambda, 0)$. Figure 2 illustrates the obtained 95% bootstrap pointwise equal-tailed confidence intervals. From the pointwise confidence intervals we detect many significant frequencies. By significant we mean λ for which the confidence interval constructed for $\Re a_X(\lambda, 0)$ or $\Im a_X(\lambda, 0)$ does not contain 0. For example for triangular kernel function ($d = 0.5$) and $h = 20T^{-0.37}$ the following frequencies have been detected for the

Table 1

Number of detected frequencies for sample size $T = 100$, $b = \lfloor\sqrt{T}\rfloor = 10$, $h_1 = 20T^{-0.35}$ and $h_2 = 20T^{-0.37}$. For each specified range of frequencies, each specified values of constant d and window width h, first and second row contain the number of frequencies detected from the 95% bootstrap pointwise equal-tailed confidence intervals for $\Re a_X(\lambda,0)$ and $\Im a_X(\lambda,0)$, respectively.

	$d=0.5$		$d=0.4$		$d=0.2$	
λ	h_1	h_2	h_1	h_2	h_1	h_2
$[0, 0.1]$	3	3	3	3	3	3
	4	4	4	3	3	3
$[0.11, 2.40]$	19	15	17	14	22	19
	18	16	17	17	15	16
$[2.41, 2.61]$	4	4	4	4	4	4
	6	7	7	7	7	7
$[2.62, 3.14]$	3	3	3	3	3	3
	1	2	2	1	1	3

Table 2

Number of detected frequencies for sample size $T = 400$, $b = \lfloor\sqrt{T}\rfloor = 20$, $h_1 = 20T^{-0.35}$ and $h_2 = 20T^{-0.37}$. For each specified range of frequencies, each specified values of constant d and window width h, first and second row contain the number of frequencies detected from the 95% bootstrap pointwise equal-tailed confidence intervals for $\Re a_X(\lambda,0)$ and $\Im a_X(\lambda,0)$, respectively.

	$d=0.5$		$d=0.4$		$d=0.2$	
λ	h_1	h_2	h_1	h_2	h_1	h_2
$[0, 0.1]$	1	1	1	1	1	1
	1	1	1	1	1	1
$[0.11, 2.40]$	14	11	14	14	25	19
	14	11	17	15	22	26
$[2.41, 2.61]$	1	1	1	1	1	2
	3	3	3	3	3	4
$[2.62, 3.14]$	1	1	2	1	4	1
	1	1	1	0	1	2

real and the imaginary part, respectively:

$$\{0, 0.01, 0.02, 0.38, 0.54, 0.55, 0.56, 0.93, 0.94, 1.09, 1.2, 1.3, 1.31,$$
$$1.76, 1.95, 1.96, 2.05, 2.23, 2.47, 2.5, 2.51, 2.52, 2.98, 2.99, 3.0\}$$
$$\{0.01, 0.02, 0.03, 0.04, 0.53, 0.54, 0.57, 0.9, 0.91, 0.92, 1.17, 1.28, 1.29, 1.73, 1.74,$$
$$1.97, 1.98, 1.99, 2.21, 2.22, 2.48, 2.49, 2.5, 2.53, 2.54, 2.59, 2.6, 2.96, 2.97\}\,.$$

Some of the listed frequencies are equal to or close to the true frequencies, but most of them are just incorrectly detected. In Table 1 we summarize the results obtained for different parameters using the pointwise confidence intervals. To make them more clear we aggregate detected frequencies into 4 intervals: $[0, 0.1], [0.11, 2.4], [2.41, 2.61], [2.62, 3.14]$. The first and the third interval contain frequencies equal or close to the true ones (0 and 0.8π). One can easily notice that independently of the choice of d and h the numbers of detected frequencies are similar. Most of the detected frequencies are in the wide interval $[0.11, 2.4]$. One may obtain similar conclusions looking at the corresponding results for $T = 400$ (see Table 2 and Figure 3). The number of jumps of the generated

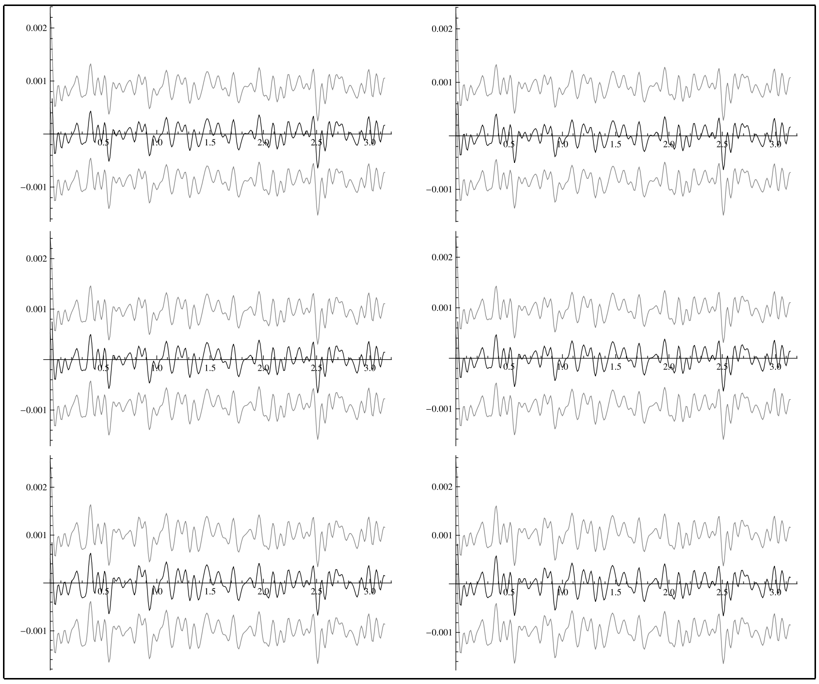

Fig 3. *Estimated values of* $\Re a_X(\lambda, 0)$ *for* $\lambda \in \{0, 0.01, \ldots, 3.14\}$ *and sample size* $T = 400$, $b = 20$ *(black line) together with the 95% bootstrap pointwise equal-tailed confidence intervals (gray line) constructed with* $h = 20T^{-0.35}$ *(left column) and* $h = 20T^{-0.37}$ *(right column). From top results for* $d = 0.5, 0.4, 0.2$, *respectively.*

Table 3

Frequencies detected from the 95% bootstrap simultaneous equal-tailed confidence intervals for $\Re a_X(\lambda, 0)$ *and* $\Im a_X(\lambda, 0)$ *for sample size* $T = 100$, $b = 10$, $h_1 = 20T^{-0.35}$, $h_2 = 20T^{-0.37}$ *and specified values of constant* d.

$d = 0.5$		$d = 0.4$		$d = 0.2$	
h_1	h_2	h_1	h_2	h_1	h_2
0.00	0.00	0.00	0.00	0.00	0.00
0.01	0.01	0.01	0.01	0.01	0.01
0.02	0.02	0.02	0.02	0.02	0.02
0.03	0.03	0.03	0.03	0.03	0.03

Poisson process in this case is 2064. Figure 4, presenting the estimated values of $|a_X(\lambda, 0)|$, shows that for any set of parameters, two high peaks are observed. Moreover, the true frequencies belong to the regions with local maxima. As for $T = 100$, from bootstrap pointwise confidence intervals one detects too many frequencies. Most of them belong to $[0.11, 2.5]$ and the amount decreases when d is increasing.

In the second part of our study we constructed the 95% percentile simultaneous equal-tailed bootstrap confidence intervals for $\Re a_X(\lambda, 0)$ and $\Im a_X(\lambda, 0)$. The results for $T = 100$ and $T = 400$ can be found in Tables 3, 4 and Figures 5,

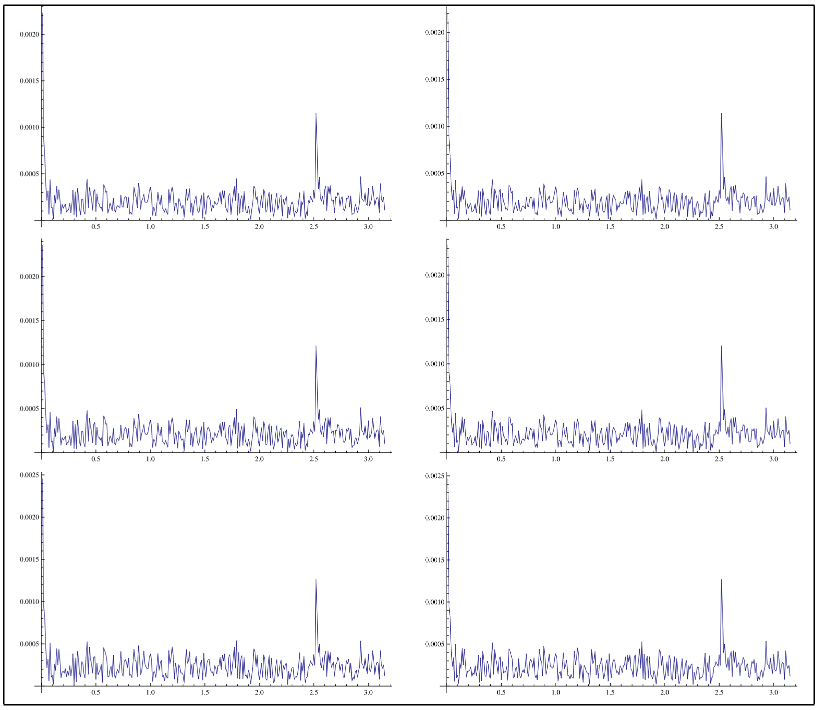

Fig 4. *Estimated values of* $|a_X(\lambda,0)|$ *for* $\lambda \in \{0, 0.01, \ldots, 3.14\}$ *and sample size* $T = 400$, $b = 20$. *Results for* $h = 20T^{-0.35}$ *(left column) and* $h = 20T^{-0.37}$ *(right column). From top results for* $d = 0.5, 0.4, 0.2$*, respectively.*

Table 4

Frequencies detected from the 95% bootstrap simultaneous equal-tailed confidence intervals for $\Re a_X(\lambda,0)$ *and* $\Im a_X(\lambda,0)$ *for sample size* $T = 400$, $b = 20$, $h_1 = 20T^{-0.35}$, $h_2 = 20T^{-0.37}$ *and specified values of constant* d.

$d = 0.5$		$d = 0.4$		$d = 0.2$	
h_1	h_2	h_1	h_2	h_1	h_2
0.00	0.00	0.00	0.00	0.00	0.00
				2.51	2.51

6, respectively. For $T = 100$, independently on the values of d and h we detect always frequency $\lambda = 0$ and some frequencies that are in its neighbourhood. For $T = 400$ in each considered case we detect $\lambda = 0$ and additionally for $d = 0.2$ frequency $\lambda = 2.51$. Hence, only in the last case we managed to detect correctly both true frequencies. To be more precise, we detect frequency $\lambda = 2.51$, which is the closest one from the considered set of frequencies to the true frequency equal to $0.8\pi \approx 2.51327$.

Additionally, we checked how the presented bootstrap approach is working for $\tau = 1$. The values of $|a_X(\lambda,1)|$ are definitely smaller than the corresponding ones of $|a_X(\lambda,0)|$, which makes detection more difficult. Below we discuss the

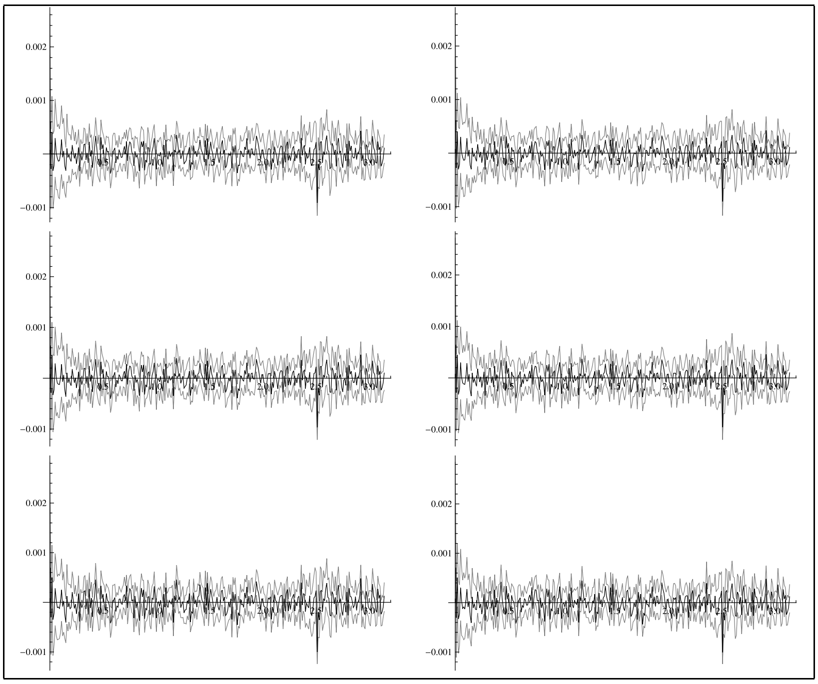

FIG 5. *Estimated values of* $\Re a_X(\lambda, 0)$ *for* $\lambda \in \{0, 0.01, \ldots, 3.14\}$ *and sample size* $T = 100$, $b = 10$ *(black line) together with the 95% bootstrap simultaneous equal-tailed confidence intervals (gray line) constructed with* $h = 20T^{-0.35}$ *(left column) and* $h = 20T^{-0.37}$ *(right column). From top results for* $d = 0.5, 0.4, 0.2$*, respectively.*

results for $d = 0.2$ and $h = 20T^{-0.37}$. The estimated values of $|a_X(\lambda, 1)|$ for $T = 100$ and $T = 400$ are presented in Figure 7. The results are similar to those obtained for $\tau = 0$. The main difference can be observed for $T = 400$. In this case from the simultaneous confidence intervals, additional frequencies were detected. Many of them belong to the neighbourhoods of both true frequencies (see Table 5) but some are incorrectly detected.

6. Appendix

6.1. Asymptotic results for $\widehat{a}_T$

Proofs of Propositions 3.1 and 3.2

Proof. Propositions 3.1 and 3.2 are particular cases of Lemmas 6.2 and 6.3, hence we refer the reader to the proofs of these lemmas which are below. □

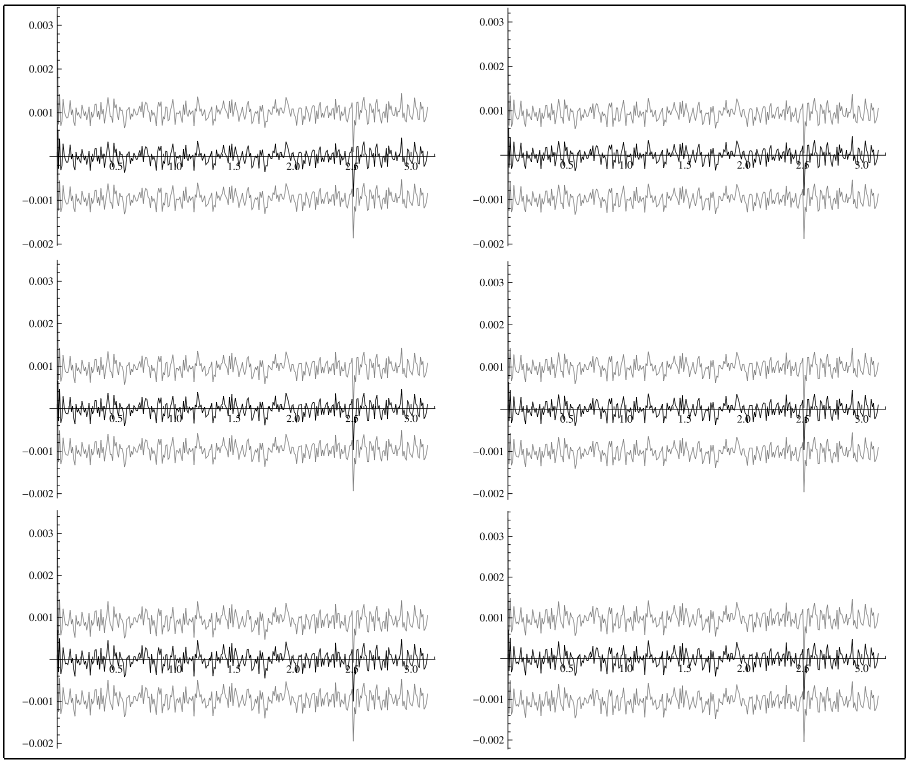

FIG 6. *Estimated values of* $\Re a_X(\lambda, 0)$ *for* $\lambda \in \{0, 0.01, \ldots, 3.14\}$ *and sample size* $T = 400$, $b = 20$ *(black line) together with the 95% bootstrap simultaneous equal-tailed confidence intervals (gray line) constructed with* $h = 20T^{-0.35}$ *(left column) and* $h = 20T^{-0.37}$ *(right column). From top results for* $d = 0.5, 0.4, 0.2$*, respectively.*

TABLE 5

Frequencies detected from 95% bootstrap simultaneous equal-tailed confidence intervals for $\Re a_X(\lambda, 1)$ *and* $\Im a_X(\lambda, 1)$ *taking* $b = \lfloor\sqrt{T}\rfloor$, $h = 20T^{-0.37}$ *and* $d = 0.2$.

$T = 100$	$T = 400$
0.00	0.00
0.01	0.01
0.02	0.02
0.03	0.03
	0.07
	0.33
	0.65
	0.72
	0.93
	2.51
	2.52
	2.92
	2.94

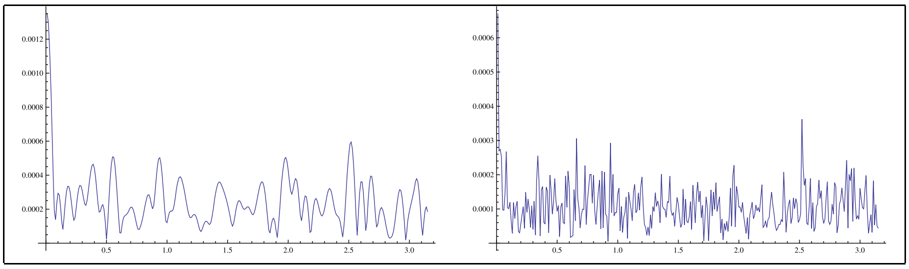

FIG 7. *Estimated values of* $|a_X(\lambda,1)|$ *for* $\lambda \in \{0, 0.01, \dots, 3.14\}$*, sample size* $T = 100$ *(left panel) and* $T = 400$ *(right panel),* $b = \lfloor T^{1/2} \rfloor$*,* $h = 20T^{-0.37}$ *and* $d = 0.2$*.*

Proof of Proposition 3.3

Proof. To establish the almost sure convergence of $\widehat{a}_T$ we apply the usual technique based on the Borel-Cantelli lemma and Markov inequality (Bienaymé-Chebychev inequality) to prove the almost sure convergence for the sequence $\{\widehat{a}_{n^x}\}$ for some $x > 1$ as $n \to \infty$. Then we establish the almost sure convergence to 0 of $\sup\{|\widehat{a}_T - \widehat{a}_{n^x}| : n^x \le T < (n+1)^x\}$.

Indeed, from the rate of convergence in quadratic mean obtained in Proposition 3.2 we easily obtain that the sequence $\{\widehat{a}_{n^x}\}$ converges almost surely to $a_X(\lambda, \tau)$ as $n \to \infty$ when $x(1-\kappa) > 1$. Next, for $n^x \le T \le (n+1)^x$, we have

$$
\begin{aligned}
&|\widehat{a}_{n^x} - \widehat{a}_T| \\
&\le \left| \frac{1}{\beta^2 n^x} - \frac{1}{\beta^2 (n+1)^x} \right| \int_0^{n^x} \int_0^{n^x} w_{h_{n^x}}(\tau - t + s) |X(s)X(t)| \, dN^{(2)}(s,t) \\
&+ \frac{1}{\beta^2 n^x} \left(\int_0^{(n+1)^x} \int_{n^x}^{(n+1)^x} + \int_{n^x}^{(n+1)^x} \int_0^{n^x} \right) w_{h_{n^x}}(\tau - t + s) |X(s)X(t)| \, dN^{(2)}(s,t) \\
&+ \frac{1}{\beta^2 n^x} \int_0^{(n+1)^x} \int_0^{(n+1)^x} |w_{h_{n^x}}(\tau - t + s) - w_{h_T}(\tau - t + s)| \, |X(s)X(t)| \, dN^{(2)}(s,t).
\end{aligned}
$$

Moreover

$$
\left| \frac{1}{n^x} - \frac{1}{(n+1)^x} \right| \le \frac{1}{n^x} \left(1 - \left(\frac{n}{n+1} \right)^x \right) = \frac{1}{n^x} \left(\frac{x}{n} + o\left(\frac{1}{n} \right) \right).
$$

Then since $\|X\|_4^4 := \sup_t \mathrm{E}\left\{X(t)^4\right\} < \infty$, the independence between the APC process X and the Poisson process N means that

$$
\begin{aligned}
&\mathrm{E}\left\{ \left(\left| \frac{1}{n^x} - \frac{1}{(n+1)^x} \right| \int_0^{n^x} \int_0^{n^x} w_{h_{n^x}}(\tau - t + s) |X(s)X(t)| \, dN^{(2)}(s,t) \right)^2 \right\} \\
&\quad \le \frac{c}{n^{2(1+x)}} \sum_{k_1=0}^{\lceil n^x \rceil} \sum_{k_2=0}^{\lceil n^x \rceil} \iint_{K_{k_1}} \iint_{K_{k_2}} w_{h_{n^x}}(\tau - t_1 + s_1) w_{h_{n^x}}(\tau - t_2 + s_2)
\end{aligned}
$$

$$\times \mathrm{E}\left\{dN^{(2)}(s_1,t_1)dN^{(2)}(s_2,t_2)\right\}$$

where $K_{k_i} = (k_i, k_i+1] \times \mathbb{R}$, $i = 1, 2$. By Lemma 6.11 the right-hand side of the last inequality is estimated by

$$\frac{c}{n^{2(1+x)}} \sum_{k_1=0}^{\lceil n^x \rceil} \sum_{k_2=0}^{\lceil n^x \rceil} \left(1 + \frac{1}{h_{n^x}} \mathbb{I}_{\{|k_1-k_2| \le h_{n_x} + |\tau| + 1\}}\right)$$
$$\le \frac{c}{n^2}\left(1 + \frac{2}{n^x}\right)\left(1 + \frac{2}{n^x} + \frac{2(2+|\tau|)}{n^x h_{n^x}}\right) = \mathcal{O}\left(\frac{1}{n^2}\right).$$

The last order of magnitude is due to the fact that $n^x h_{n^x} \to \infty$ as $n \to \infty$. Here and in the following, c denotes a positive constant whose value may differ from one expression to another. We deduce the almost sure convergence to 0 of

$$\left|\frac{1}{n^x} - \frac{1}{(n+1)^x}\right| \int_0^{n^x}\int_0^{n^x} w_{h_{n^x}}(\tau - t + s)|X(s)X(t)|\, dN^{(2)}(s,t).$$

Following a similar method we easily get the almost sure convergence to 0 of

$$\frac{1}{\beta^2 n^x}\left(\int_0^{(n+1)^x}\int_{n^x}^{(n+1)^x} + \int_{n^x}^{(n+1)^x}\int_0^{n^x}\right) w_{h_{n^x}}(\tau - t + s)|X(s)X(t)|\, dN^{(2)}(s,t).$$

Finally since the kernel function is symmetric on $\mathbb{R}$, non-increasing on $(0,\infty)$, and from the monotony properties of b_T and h_T, we have

$$\int_0^{(n+1)^x}\int_0^{(n+1)^x} |w_{h_{n^x}}(\tau - t + s) - w_{h_T}(\tau - t + s)|\, |X(s)X(t)|\, dN^{(2)}(s,t)$$
$$\le \int_0^{(n+1)^x}\int_0^{(n+1)^x}\left(\frac{1}{h_{(n+1)^x}} - \frac{1}{h_{n^x}}\right) w\left(\frac{\tau - t + s}{h_{n^x}}\right)|X(s)X(t)|\, dN^{(2)}(s,t)$$
$$+ \int_0^{(n+1)^x}\int_0^{(n+1)^x} \frac{1}{h_{n^x}}\left(w\left(\frac{\tau - t + s}{h_{n^x}}\right) - w\left(\frac{\tau - t + s}{h_{(n+1)^x}}\right)\right)|X(s)X(t)|\, dN^{(2)}(s,t).$$

As previously we readily obtain the almost sure convergence to 0 of the two terms of the right-hand side of the last inequality. This achieves the proof of Proposition 3.3.

In the proof we obtained the different rates of convergence and taking them under consideration one may state rate of almost sure convergence of $\widehat{a}_T$. □

Below we provide the limit covariances between the estimators $\widehat{a}_T$ calculated for different frequencies and lags. For simplicity of presentation we present the limit covariances for the estimators considered as complex random variables. The reader can readily deduce the covariance matrices for the estimators considered as bidimensional random vectors.

Proposition 6.1. *Assume conditions (AP_2), (AP_4) and (M) are fulfilled. If $\tau = \tau_1 = \pm\tau_2$, then*

$$\lim_{T\to\infty} Th\beta^4 \operatorname{cov}\{\widehat{a}_T(\lambda_1,\tau), \widehat{a}_T(\lambda_2,\pm\tau)\} = \Gamma^{\pm}\big(\lambda_1,\lambda_2;\tau\big).$$

If $|\tau_1| \neq |\tau_2|$, then

$$\lim_{T\to\infty} T\beta^4 \operatorname{cov}\{\widehat{a}_T(\lambda_1,\tau_1), \widehat{a}_T(\lambda_2,\tau_2)\} = \Gamma\big(\lambda_1,\lambda_2;\tau_1,\tau_2\big).$$

Here

$$\begin{aligned}
\Gamma^{+}\big(\lambda_1,\lambda_2;\tau\big) &:= a_4(\lambda_1-\lambda_2;\tau,0,\tau) + a^{(2)}(\lambda_1-\lambda_2;\tau,\tau),\\
\Gamma^{-}\big(\lambda_1,\lambda_2;\tau\big) &:= \Big(a_4(\lambda_1-\lambda_2;\tau,0,\tau) + a^{(2)}(\lambda_1-\lambda_2;\tau,\tau)\Big)\, e^{i\lambda_2\tau},\\
\Gamma\big(\lambda_1,\lambda_2;\tau_1,\tau_2\big) &:= C\big(\lambda_1,\lambda_2;\tau_1,\tau_2\big) + a_4(\lambda_1-\lambda_2;\tau_1,0,\tau_2) + a^{(2)}(\lambda_1-\lambda_2;\tau_1,\tau_2)\\
&\qquad + \big(a_4(\lambda_1-\lambda_2;\tau_1,0,-\tau_2) + a^{(2)}(\lambda_1-\lambda_2;\tau_1,-\tau_2)\big)e^{-i\lambda_2\tau_2}\\
&\qquad + \big(a_4(\lambda_1-\lambda_2;-\tau_1 0,\tau_2) + a^{(2)}(\lambda_1-\lambda_2;-\tau_1,\tau_2)\big)e^{i\lambda_1\tau_1}\\
&\qquad + \big(a_4(\lambda_1-\lambda_2;-\tau_1,0,-\tau_2)\\
&\qquad\quad + a^{(2)}(\lambda_1-\lambda_2;-\tau_1,-\tau_2)\big)e^{i(\lambda_1\tau_1-\lambda_2\tau_2)},\\
C\big(\lambda_1,\lambda_2;\tau_1,\tau_2\big) &:= 2\int_{\mathbb{R}} a_4\big(\lambda_1-\lambda_2;\tau_2,s,s+\tau_1\big)e^{-i\lambda_1 s}\,ds\\
&= 2\int_{\mathbb{R}} a_4\big(\lambda_1-\lambda_2;\tau_1,s,s+\tau_2\big)e^{i\lambda_2 s}\,ds,\\
a_4(\lambda;\tau_1,\tau_2,\tau_3) &:= \lim_{T\to\infty}\frac{1}{T}\int_0^T K_X(u,u+\tau_1;u+\tau_2,u+\tau_3)\,e^{-i\lambda u}\,du,\\
K_X(s_1,t_1;s_2,t_2) &:= \operatorname{cov}\{X(s_1)X(t_1), X(s_2)X(t_2)\}
\end{aligned}$$

and

$$a^{(2)}(\lambda;\tau_1,\tau_2) := \lim_{T\to\infty}\frac{1}{T}\int_0^T K_X(t,t+\tau_1)K_X(t,t+\tau_2)\,e^{-i\lambda t}dt.$$

Proof. See Lemma 6.4 □

6.2. Results for the estimators defined on the blocks of length b

In the next lemma we study the bias of the estimators constructed on the blocks of length b, i.e. $\widehat{a}_{k,b}$ (see Section 3.2).

Lemma 6.2. *Let $\{X(t), t \geq 0\}$ be an APC process and suppose that condition (AP_2) is fulfilled, and $h \to 0$ as $b \to \infty$. Then for all $\lambda, \tau \in \mathbb{R}$*

$$\lim_{b\to\infty} E\big\{\widehat{a}_{k,b}\big\} = a_X(\lambda,\tau)$$

uniformly with respect to $k \in \mathbb{N}$. Furthermore, if in addition conditions (CS(λ,τ)) and (Lip) hold, and $h \ll b^{-1/3}$ as $b \to \infty$, then

$$\lim_{T\to\infty} \sup_k \sqrt{bh}\left|\mathrm{E}\left\{\widehat{a}_{k,b}(\tau)\right\} - a_X(\lambda,\tau)\right| = 0.$$

Proof. Let $k \geq 0$ be fixed. Notice that

$$\mathrm{E}\left\{N^{(2)}(s+ds, s+t+dt)\right\} = \mathrm{E}\left\{\mathbb{I}_{\{t\neq 0\}} N(s+ds)N(s+t+dt)\right\} = \beta^2 ds dt.$$

Then

$$\begin{aligned}
&\mathrm{E}\left\{\widehat{a}_{k,b}\right\}\\
&= \frac{1}{\beta^2 b}\int_k^{k+b}\int_k^{k+b} w_h(\tau - t + s)\mathrm{E}\left\{X(s)X(t)\right\} e^{-i\lambda s}\,\mathrm{E}\left\{dN^{(2)}(s,t)\right\}\\
&= \frac{1}{\beta^2 b}\int_{-b}^{b}\int_{I(b,t)} w_h(\tau - t)\mathrm{E}\left\{X(k+s)X(k+s+t)\right\} e^{-i\lambda(k+s)}\\
&\quad \times\ \mathrm{E}\left\{N^{(2)}(k+s+ds, k+s+t+dt)\right\}\\
&= \int_{-b}^{b} w_h(\tau - t)\left(\frac{1}{b}\int_{I(b,t)} \mathrm{E}\left\{X(k+s)X(k+s+t)\right\} e^{-i\lambda(k+s)}\,ds\right) dt\\
&= \int_{\tau-b}^{\tau+b} w_h(u)\left(\frac{1}{b}\int_{I(b,\tau-u)} \mathrm{E}\left\{X(k+s)X(k+s+\tau-u)\right\} e^{-i\lambda(k+s)}\,ds\right) du.
\end{aligned}$$

Since the support of the weight function $w(\cdot)$ is contained in $[-1,1]$ we get

$$\begin{aligned}
&\mathrm{E}\left\{\widehat{a}_{k,b}\right\}\\
&= \int_{-1}^{1} w(u)\left(\frac{1}{b}\int_{I(b,\tau-uh)} \mathrm{E}\{X(k+s)X(k+s+\tau-uh)\}e^{-i\lambda(k+s)}ds\right) du\\
&= \int_{-1}^{1} w(u)\left(\frac{1}{b}\int_{I(b,\tau-uh)}\Big(\mathrm{E}\left\{X(k+s)X(k+s+\tau-uh)\right\}\right.\\
&\quad \left. - \mathrm{E}\left\{X(k+s)X(k+s+\tau)\right\}\Big)e^{-i\lambda(k+s)}ds\right)du + \int_{-1}^{1} w(u)\left(1 - \frac{|\tau - uh|}{b}\right)\\
&\quad \times\left(\frac{1}{b - |\tau - uh|}\int_{I(b,\tau-uh)} \mathrm{E}\left\{X(k+s)X(k+s+\tau)\right\} e^{-i\lambda(k+s)}\,ds\right)du.
\end{aligned}$$

Assumption (AP_2) with the Lebesgue dominated convergence theorem implies that

$$\lim_{b\to\infty} \mathrm{E}\left\{\widehat{a}_{k,b}\right\} = \left(\int_{-1}^{1} w(u)\,du\right) a_x(\lambda,\tau) = a_X(\lambda,\tau)$$

uniformly with respect to $k \in \mathbb{R}$. Moreover, under (CS(λ,τ)) and (Lip) we obtain the rate of convergence uniform with respect to k. Indeed

$$\mathrm{E}\left\{\widehat{a}_{k,b}\right\} - a_X(\lambda,\tau)$$

$$= \int_{-1}^{1} w(u) \left(\frac{1}{b} \int_{I(b,\tau-uh)} \mathrm{E}\{X(k+s)X(k+s+\tau-uh)\} e^{-i\lambda(k+s)} ds - a_X(\lambda,\tau) \right) du$$
$$= \int_{-1}^{1} w(u) \left(\frac{1}{b} \int_{I(b,\tau-uh)} \Big(\mathrm{E}\{X(k+s)X(k+s+\tau-uh)\} \right.$$
$$\left. - \mathrm{E}\{X(k+s)X(k+s+\tau)\} \Big) e^{-i\lambda(k+s)} ds \right) du + \int_{-1}^{1} w(u) \left(1 - \frac{|\tau-uh|}{b}\right)$$
$$\times \left(\frac{1}{b-|\tau-uh|} \int_{I(b,\tau-uh)} \mathrm{E}\{X(k+s)X(k+s+\tau)\} e^{-i\lambda(k+s)} ds - a_X(\lambda,\tau) \right) du$$
$$- \left(\int_{-1}^{1} w(u) \frac{|\tau-uh|}{b} du \right) a_X(\lambda,\tau).$$

From the almost periodicity of the shifted covariance function, we have

$$\left| \frac{1}{b-|\tau-uh|} \int_{I(b,\tau-uh)} \mathrm{E}\{X(k+s)X(k+s+\tau)\} e^{-i\lambda(k+s)} ds - a_X(\lambda,\tau) \right|$$
$$\leq \frac{1}{b-|\tau-uh|} \sum_{\lambda' \neq \lambda} \frac{|a_X(\lambda',\tau)|}{|\lambda'-\lambda|}.$$

Thus,

$$\left| \mathrm{E}\{\widehat{a}_{k,b}\} - a_X(\lambda,\tau) \right|$$
$$\leq \sup_s \int_{-1}^{1} w(u) \left| \mathrm{E}\{X(s)X(s+\tau-uh)\} - \mathrm{E}\{X(s)X(s+\tau)\} \right| du$$
$$+ \frac{1}{b} \sum_{\lambda' \neq \lambda} \frac{|a_X(\lambda',\tau)|}{|\lambda'-\lambda|} + \frac{1}{b} \left(\int_{-1}^{1} w(u)|\tau-uh| \, du \right) \left| a_X(\lambda,\tau) \right| \leq c\,(h + b^{-1}).$$

Hence, if $h \ll b^{-1/3}$ then $\sqrt{bh} \sup_k |\mathrm{E}\{\widehat{a}_{k,b}\} - a_X(\lambda,\tau)| = o(1)$. □

Below we present a few additional results concerning the consistency and the rate of convergence of the estimators defined on the blocks.

Lemma 6.3. *Let $\{X(t), t \geq 0\}$ be an APC process which satisfies conditions (AP_2) and (M). Assume also that $b^{-1} \ll h \ll 1$ as $b \to \infty$. Then for all $\lambda, \tau \in \mathbb{R}$*

$$\lim_{b\to\infty} \sup_k \mathrm{E}\left\{ \left| \widehat{a}_{k,b}(\lambda,\tau) - a_X(\lambda,\tau) \right|^2 \right\} = 0.$$

If, additionally, conditions (CS(λ,τ)) and (Lip) are fulfilled, and if $h \ll b^{-1/3}$, then for all $\lambda, \tau \in \mathbb{R}$

$$\limsup_{b\to\infty} \sup_k bh\, \mathrm{E}\left\{ \left| \widehat{a}_{k,b}(\lambda,\tau) - a_X(\lambda,\tau) \right|^2 \right\} < \infty.$$

Proof. The covariance mixing inequalities established in Lemma 6.10, means that

$$\left|\operatorname{var}\left\{\widehat{a}_{k,b}(\tau)\right\}\right| \leq \frac{c}{b^2}\left(1+\frac{1}{h}\right)\left(h+|\tau|\right)^2 + \frac{c(2|\tau|+3)}{bh} + \frac{c}{b}\sum_{s=0}^{b-1}\alpha_X(s) \tag{6}$$

when the process X is bounded. In the case when the process X is not bounded, but $\sup_t \mathrm{E}\left\{|X(t)|^{4+\delta}\right\} < \infty$, in the relation above $\alpha_X(s)$ will be replaced by $\alpha_X(s)^{\delta/(4+\delta)}$. Finally, to finish the proof Lemma 6.2 needs to be used. □

Similarly to $\widehat{a}_T$ case (Proposition 3.3) the almost sure convergence of $\widehat{a}_{k,b}$ can be obtained. The technical details are left to the reader.

Next, we compute the limit covariances between the estimators $\widehat{a}_{k,b}(\lambda,\tau)$ calculated for different frequencies λ and lags τ.

Lemma 6.4. *Assume that conditions (AP$_2$), (AP$_4$) and (M) are fulfilled, and $h \to 0$ as $b \to \infty$.*
If $\tau = \tau_1 = \pm\tau_2$ then

$$\lim_{b\to\infty}\sup_k \left|bh\beta^4 \operatorname{cov}\left\{\widehat{a}_{k,b}(\lambda_1,\tau), \widehat{a}_{k,b}(\lambda_2,\pm\tau)\right\} - \Gamma^{\pm}\left(\lambda_1,\lambda_2;\tau\right)\right| = 0.$$

If $|\tau_1| \neq |\tau_2|$ then

$$\lim_{b\to\infty}\sup_{k\geq 0} \left|b\beta^4 \operatorname{cov}\left\{\widehat{a}_{k,b}(\lambda_1,\tau_1), \widehat{a}_{k,b}(\lambda_2,\tau_2)\right\} - \Gamma\left(\lambda_1,\lambda_2;\tau_1,\tau_2\right)\right| = 0.$$

Proof. From the independence of the process $\{X(t) : t \in \mathbb{R}\}$ and the Poisson process $\{N(t) : t > 0\}$ we have that

$$\begin{aligned} &b\beta^4 \operatorname{cov}\left\{\widehat{a}_{k,b}(\lambda_1,\tau_1), \widehat{a}_{k,b}(\lambda_2,\tau_2)\right\} \\ &\quad= \frac{1}{b}\int_0^b\!\!\int_0^b\!\!\int_0^b\!\!\int_0^b w_h(\tau_1 - t_1 + s_1)w_h(\tau_2 - t_2 + s_2)\, e^{-i\lambda_1(k+s_1)+i\lambda_2(k+s_2)} \\ &\qquad\times \operatorname{cov}\left\{Z(k+s_1+ds_1, k+t_1+dt_1), Z(k+s_2+ds_2, k+t_2+dt_2)\right\}, \end{aligned} \tag{7}$$

where

$$dZ(s,t) = Z(s+ds, t+dt) := X(s)X(t)\, dN^{(2)}(s,t).$$

Note that

$$\begin{aligned} &\operatorname{cov}\left\{Z(k+s_1+ds_1, k+t_1+dt_1), Z(k+s_2+ds_2, k+t_2+dt_2)\right\} \\ &= \beta^4 K_X(k+s_1, k+t_1; k+s_2, k+t_2)\, ds_1dt_1ds_2dt_2 \\ &\quad+ \mathrm{E}\left\{X(k+s_1)X(k+t_1)X(k+s_2)X(k+t_2)\right\} \times \Big\{\beta^3\delta_{\{s_1=s_2\}}(ds_1)dt_1ds_2dt_2 \\ &\quad+ \beta^3\delta_{\{s_1=t_2\}}(ds_1)dt_1ds_2dt_2 + \beta^3\delta_{\{t_1=s_2\}}(dt_1)ds_1ds_2dt_2 \\ &\quad+ \beta^3\delta_{\{t_1=t_2\}}(dt_1)ds_1ds_2dt_2 + \beta^2\delta_{\{s_1=s_2\}}(ds_1)ds_2\delta_{\{t_1=t_2\}}(dt_1)dt_2 \\ &\quad+ \beta^2\delta_{\{s_1=t_2\}}(ds_1)dt_2\delta_{\{t_1=s_2\}}(dt_1)ds_2\Big\}. \end{aligned}$$

Since the reasoning for all summands on the right hand side of equality (7) is quite similar and requires the same techniques, we study in detail only two of them, which illustrate how we obtain the two parts of the proposition: the one containing expression $\delta_{\{s_1=s_2\}}(ds_1)dt_1ds_2dt_2$ and the one containing $\delta_{\{s_1=s_2\}}(ds_1)\delta_{\{t_1=t_2\}}(dt_1)ds_2dt_2$. In particular, analyzing the last term provides the information why the factor h appears when $\tau_1=\tau_2$. Recall that $h\to 0$ as $b\to\infty$.

(i) The summand containing $\delta_{\{s_1=s_2\}}(ds_1)dt_1ds_2dt_2$ is equal to

$$\begin{aligned}
&\frac{1}{b}\int_0^b\!\!\int_0^b\!\!\int_0^b\!\!\int_0^b w_h(\tau_1-t_1+s_1)w_h(\tau_2-t_2+s_2)\,e^{-i\lambda_1(k+s_1)+i\lambda_2(k+s_2)}\\
&\qquad\times \mathrm{E}\left\{X(k+s_1)X(k+t_1)X(k+s_2)X(k+t_2)\right\}\delta_{\{s_1=s_2\}}(ds_1)dt_1ds_2dt_2\\
&\quad=\frac{1}{b}\int_0^b\!\!\int_0^b\!\!\int_0^b w_h(\tau_1-t_1+s)w_h(\tau_2-t_2+s)\,e^{-i(\lambda_1-\lambda_2)(k+s)}\\
&\qquad\times K_X(k+s,k+t_1;k+s,k+t_2)\,dsdt_1dt_2\\
&\qquad+\frac{1}{b}\int_0^b\!\!\int_0^b\!\!\int_0^b w_h(\tau_1-t_1+s)w_h(\tau_2-t_2+s)\,e^{-i(\lambda_1-\lambda_2)(k+s)}\\
&\qquad\times \mathrm{E}\left\{X(k+s)X(k+t_1)\right\}\mathrm{E}\left\{X(k+s)X(k+t_2)\right\}\,dsdt_1dt_2.
\end{aligned}$$

Changing the variables to $u_1:=t_1-s_1$ and $u_2:=t_2-s_2$, the first term of the right-hand side is equal to

$$\begin{aligned}
&\int_{-b}^b\!\!\int_{-b}^b w_h(\tau_1-u_1)w_h(\tau_2-u_2)\\
&\times\left(\frac{1}{b}\int_{J(b,u_1,u_2)} K_X(k+s,k+s+u_1;k+s,k+s+u_2)\,e^{-i(\lambda_1-\lambda_2)(k+s)}\,ds\right)du_1du_2
\end{aligned}$$

and converges to $a_4(\lambda_1-\lambda_2;\tau_1,0,\tau_2)$ as $T\to\infty$. Here $J(b,u_1,u_2):=\{s:\max\{0,-u_1,-u_2\}\le s\le\min\{b,b-u_1,b-u_2\}\}$. Additionally, the second term coincides with

$$\begin{aligned}
&\int_{-b}^b\!\!\int_{-b}^b w_h(\tau_1-u_1)w_h(\tau_2-u_2)\left(\frac{1}{b}\int_{J(b,u_1,u_2)}\mathrm{E}\left\{X(k+s)X(k+s+u_1)\right\}\right.\\
&\qquad\left.\times\,\mathrm{E}\left\{X(k+s)X(k+s+u_2)\right\}e^{-i(\lambda_1-\lambda_2)(k+s)}\,ds\right)du_1du_2,
\end{aligned}$$

which converges to

$$\sum_\lambda a_X(\lambda,\tau_1)\,\overline{a_X(\lambda-\lambda_1+\lambda_2,\tau_2)}=\sum_\lambda a_X(\lambda,\tau_1)\,a_X(\lambda_1-\lambda_2-\lambda,\tau_2).$$

(ii) The summand of equality (7) containing $\delta_{\{s_1=s_2\}}(ds_1)\delta_{\{t_1=t_2\}}(dt_1)ds_2dt_2$ is equal to

$$\frac{1}{b}\int_0^b\!\!\int_0^b w_h(\tau_1-t+s)w_h(\tau_2-t+s)\mathrm{E}\big\{X(k+s)X(k+t)^2\big\}\,e^{-i(\lambda_1-\lambda_2)(k+s)}dsdt.$$

Taking $u := t - s$, we get

$$\int_{-b}^{b} w_h(\tau_1 - u) w_h(\tau_2 - u) \times \left(\frac{1}{b} \int_{I(b,u)} \mathrm{E}\left\{ X(k+s)^2 X(k+s+u)^2 \right\} e^{-i(\lambda_1-\lambda_2)(k+s)}\, ds \right) du,$$

which converges to 0 when $\tau_1 \neq \tau_2$. Indeed it is null for $2h \leq |\tau_1 - \tau_2|$ since the support of $w(\cdot)$ is contained in $[-1,1]$, and $h \to 0$ as $b \to \infty$. When $\tau = \tau_1 = \tau_2$, with the change of variable $t := (\tau - u)/h$, the summand coincides with

$$\frac{1}{h} \int_{(\tau-b)/h}^{(\tau+b)/h} w(t)^2 \times \left(\frac{1}{b} \int_{I(b,\tau-th)} \mathrm{E}\left\{ X(k+s)^2 X(k+s+\tau-th)^2 \right\} e^{-i(\lambda_1-\lambda_2)(k+s)}\, ds \right) dt.$$

From conditions (AP_2) and (AP_4) this term multiplied by the factor h converges to

$$\int_{-1}^{1} w(t)^2 dt \left(a_4(\lambda_1 - \lambda_2; \tau, 0, \tau) + \sum_{\lambda} a_X(\lambda,\tau)\, \overline{a_X(\lambda - \lambda_1 + \lambda_2, \tau)} \right)$$
$$= \int_{-1}^{1} w(t)^2 dt \left(a_4(\lambda_1 - \lambda_2; \tau, 0, \tau) +, \sum_{\lambda} a_X(\lambda,\tau_1)\, a_X(\lambda_1 - \lambda_2 - \lambda, \tau) \right).$$

This achieves the proof of Lemma 6.4. □

6.3. Proof of Theorems 3.4 and 4.1

The proof of Theorem 4.1 is a direct consequence of the Cramér-Wold device and the reasoning presented in the proof of the one-dimensional case in Theorem 3.4. Thus, we skip the technical details and we only present the proof for the one-dimensional case. Moreover to get the desired convergence to the bidimensional normal distribution in Theorem 3.4, it is enough to prove the corresponding result for the estimator $\widehat{a}_{k,b}$.

Lemma 6.5. *Under assumptions of Theorem 3.4,* $\sqrt{bh}\big(\widehat{a}_{k,b}(\lambda,\tau) - a_X(\lambda,\tau)\big)$ *converges to a bidimensional Gaussian distribution as* $b \to \infty$ *provided that* $b^{-1} \ll h \ll b^{-1/3}$.

The limit covariance matrix may be deduced from Lemma 6.4.

Proof. Note that

$$\sqrt{bh}\big(\widehat{a}_{k,b} - a_X(\lambda,\tau)\big) = \sqrt{bh}\big(\widehat{a}_{k,b}(\tau) - \mathrm{E}\{\widehat{a}_{k,b}\}\big) + \sqrt{bh}\big(\mathrm{E}\{\widehat{a}_{k,b}\} - a_X(\lambda,\tau)\big).$$

Lemma 6.2 means that the second term of the right-hand side converges to 0. The convergence in law of the first term is a consequence of Lemma 6.4 and the following central limit result. □

Proposition 6.6. *Let $\{X(t), t \geq 0\}$ be a real-valued APC process and $\{N(t), t \geq 0\}$ be a Poisson point process with intensity β, which is independent on $\{X(t) : t \geq 0\}$ and $g : \mathbb{R} \to \mathbb{R}$ be an almost periodic function. Let $\tau \in \mathbb{R}$ be fixed. Denote*

$$S_T := \int_0^T \int_0^T w_h(\tau - t + s) g(s) X(s) X(t) \, dN^{(2)}(s,t)$$

and $\sigma_T^2 := \operatorname{var}\{S_T\}$. Assume that that the mixing condition (M) is satisfied. Then $\limsup_T T^{-1} h \, \sigma_T^2 < \infty$ as $h \to 0$ and $T \to \infty$.

If additionally $\liminf_T T^{-1} h \, \sigma_T^2 > 0$, then

$$\sigma_T^{-1}\big(S_T - \mathrm{E}\{S_T\}\big) \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)$$

as $h \to 0$ and $T \to \infty$ with $Th \to \infty$.

Proof. The idea of the proof is based on Stein's Lemma (see relation (8)) applied in a similar way to the central limit theorem in [4] (see also Theorem 3.2 in [17]). To simplify the presentation we consider the following random variable

$$\mathcal{S}_T := \int_0^T \int_{\mathbb{R}} w_h(\tau - t + s) g(s) X(s) X(t) \, dN^{(2)}(s,t)$$

and we establish its asymptotic equivalence with S_T. Note that

$$\mathcal{S}_T - S_T = \int_0^{0 \vee (-\tau + h)} \int_{\tau + s - h}^{0} w_h(\tau - t + s) g(s) X(s) X(t) \, dN^{(2)}(s,t) \\ + \int_{T \wedge (T - \tau - h)}^{T} \int_T^{\tau + s + h} w_h(\tau - t + s) g(s) X(s) X(t) \, dN^{(2)}(s,t).$$

From Lemma 6.11 we get

$$\mathrm{E}\left(\int_0^{-\tau + h} \int_{\tau + s - h}^{0} w_h(\tau - t + s) g(s) X(s) X(t) \, dN^{(2)}(s,t) \right)^2 \\ \leq c \, \|X\|_4^4 (|\tau| + h)^2 (1 + h^{-1})$$

for $h > \tau$ and

$$\mathrm{E}\left(\int_{T - \tau - h}^{T} \int_T^{\tau + s + h} w_h(\tau - t + s) g(s) X(s) X(t) \, dN^{(2)}(s,t) \right)^2 \\ \leq c \, \|X\|_4^4 (|\tau| + h)^2 (1 + h^{-1})$$

for $T - \tau - h < T$, i.e. $h > -\tau$. Here and thereafter $\|X\|_4^4 := \sup_t \mathrm{E}\{X(t)^4\}$. Hence

$$\lim_{T \to \infty} T^{-1} h \, \mathrm{E}\left\{ (S_T - \mathcal{S}_T)^2 \right\} = 0$$

and as a consequence

$$\lim_{T \to \infty} T^{-1} h (\mathrm{E}\{S_T\} - \mathrm{E}\{\mathcal{S}_T\})^2 \quad \text{and} \quad \lim_{T \to \infty} T^{-1} h (\sigma_T^2 - \varsigma_T^2) = 0,$$

where $\varsigma_T^2 := \text{var}\{\mathcal{S}_T\}$. Thus, to prove the convergence in law of $\sigma_T^{-1}(S_T - \mathrm{E}\{S_T\})$ we can replace S_T by $\mathcal{S}_T$, thus we are going to study $\varsigma_T^{-1}(\mathcal{S}_T - \mathrm{E}\{\mathcal{S}_T\})$.

For simplicity of notation, let $T = n+1$, $0 < h = h_n < 1$ $K_k :=]k, k+1] \times \mathbb{R}$, $k \in D_n := \{0, \dots, n\} \cap [-\tau+1, n-\tau-1]$. The random variable $\mathcal{Y}_k$ is defined as

$$\mathcal{Y}_k := \iint_{K_k} w_h(\tau - t + s) g(s) X(s) X(t) \, dN^{(2)}(s,t).$$

To simplify the presentation put $\alpha_\tau(k) := \alpha_X(k - 2|\tau| - 3)$ when $k \geq 2|\tau| + 3$, and $\alpha_\tau(k) = c$ when $k \leq 2|\tau| + 3$, for some $c \geq 1$.

1) *Boundedness of the variance* ς_T^2. The covariance mixing inequalities in Lemma 6.10 and the fact that the function $g(\cdot)$ is bounded means that

$$T^{-1} h \, \varsigma_T^2 \leq \frac{c}{T}(h+1)(h+|\tau|)^2 + c(2|\tau|+3) + ch \sum_{s=0}^{T-1} \alpha_X(s)$$

when the process X is bounded. When it is not bounded, we replace $\alpha_X(s)$ by $\alpha_X(s)^{\delta/(4+\delta)}$. Then the mixing hypothesis (M) implies that

$$\limsup_{T \to \infty} T^{-1} h \, \varsigma_T^2 < \infty.$$

2) *Central limit theorem in the case of a bounded process.* Here we assume that the process $\{X(t), t \geq 0\}$ is bounded: $\|X\|_\infty := \sup_t \operatorname{ess\,sup} |X(t)| < \infty$.

(i) Let $(r_n)_n \subset \mathbb{N}$ be such that $1 \lll r_n^2 \lll n \lll \alpha_X(r_n)^{-2}$ as $n \to \infty$. Such sequence $(r_n)_n$ exists. Indeed, since the function $\alpha_X(\cdot)$ is non-increasing and integrable in $[0, \infty)$, we have that $\alpha_X(t) \lll t^{-1}$ as $t \to \infty$. Hence, we can define

$$r_n := \min\{r \in \mathbb{N} : r \leq n \leq r \alpha_X(r)^{-1}\},$$

which has the required behaviour. Moreover we can assume that $r_n \geq 2|\tau| + 3$.

(ii) Let us define

$$S_{k,n} := \sum_{l \in D_n, |l-k| \leq r_n} \mathcal{Y}_l, \qquad \gamma_n := \sum_{k \in D_n} \text{cov}\left\{\mathcal{Y}_k, S_{k,n}\right\},$$

$$\bar{S}_n := \gamma_{h,n}^{-1/2} S_n, \qquad \bar{S}_{k,n} := \gamma_n^{-1/2} S_{k,n}.$$

Thus,

$$\begin{aligned} \varsigma_n^2 := \text{var}\{S_n\} &= \sum_{k \in D_n} \text{cov}\left\{\mathcal{Y}_k, S_n\right\} \\ &= \gamma_n + \sum_{k \in D_n} \text{cov}\left\{\mathcal{Y}_k, S_n - S_{k,n}\right\}. \end{aligned}$$

Since $0 < h < 1$ and $r_n > 2|\tau| + 3 \geq 2h + 2|\tau| + 1$, from the mixing covariance inequality (20) we get that

$$\left| \sum_{k \in D_n} \text{cov}\{\mathcal{Y}_k, S_n - S_{k,n}\} \right| \leq \sum \sum_{k_1, k_2 \in D_n, |k_1 - k_2| > r_n} \left| \text{cov}\{\mathcal{Y}_{k_1}, \mathcal{Y}_{k_2}\} \right|$$

$$\leq c \, \|X\|_\infty^4 \, n \sum_{k > r_n} \alpha_\tau(k).$$

We used the fact that the function $g(\cdot)$ is almost periodic, so bounded. Since $r_n \to \infty$ and $\sum_k \alpha_\tau(k) < \infty$, this expression is $o(n)$ as $n \to \infty$, and we have $\varsigma_n^2 = \gamma_n + o(n)$.

When $0 < \liminf_n n^{-1} h \, \varsigma_n^2 \leq \limsup_n n^{-1} h \, \varsigma_n^2 < \infty$ the previous computations imply that $0 < \liminf_n n^{-1} h \gamma_n \leq \limsup_n n^{-1} h \gamma_n < \infty$ and $\varsigma_n^2 = \gamma_n (1 + o(1))$.

(iii) *Asymptotic normality of* $\bar{S}_n$. As $0 < \liminf_n n^{-1} h \gamma_n \leq \limsup_n n^{-1} h \gamma_n < \infty$, we have that $\sup_n \mathrm{var}\{\bar{S}_n\} = \sup_n \mathrm{E}\Big\{\big(\widetilde{\bar{S}}_n\big)^2\Big\} < \infty$ and the asymptotic normality will follow from the Stein Lemma. We show that

$$\lim_{n \longrightarrow \infty} \mathrm{E}\left\{ (i\lambda - \widetilde{\bar{S}}_n) e^{i\lambda \widetilde{\bar{S}}_n} \right\} = 0. \tag{8}$$

For that purpose we use the following decomposition

$$(i\lambda - \widetilde{\bar{S}}_n) e^{i\lambda \widetilde{\bar{S}}_n} = A_1 - A_2 - A_3,$$

where

$$\begin{aligned}
A_1 &:= i\lambda \, e^{i\lambda \widetilde{\bar{S}}_n} \left(1 - \gamma_n^{-1} \sum_{k \in D_n} \widetilde{\mathcal{Y}}_k \widetilde{S}_{k,n} \right), \\
A_2 &:= \gamma_n^{-\frac{1}{2}} e^{i\lambda \widetilde{\bar{S}}_n} \sum_{k \in D_n} \widetilde{\mathcal{Y}}_k \left(1 - i\lambda \widetilde{\bar{S}}_{k,n} - e^{-i\lambda \widetilde{\bar{S}}_{k,n}} \right), \\
A_3 &:= \gamma_n^{-\frac{1}{2}} \sum_{k \in D_n} \widetilde{\mathcal{Y}}_k e^{i\lambda (\widetilde{\bar{S}}_n - \widetilde{\bar{S}}_{k,n})}.
\end{aligned}$$

Below we present the reasoning only for A_1. The two other cases, A_2 and A_3, are similar thus we skip the technical details. We have that

$$\mathrm{E}\left\{ |A_1|^2 \right\} \leq \lambda^2 \gamma_{h,n}^{-2} \sum_{k_1, k_2, l_1, l_2 \in D_n, \, |k_1 - l_1|, |k_2 - l_2| \leq r_n} \left| \mathrm{cov}\{ \widetilde{\mathcal{Y}}_{k_1} \widetilde{\mathcal{Y}}_{l_1}, \widetilde{\mathcal{Y}}_{k_2} \widetilde{\mathcal{Y}}_{l_2} \} \right|.$$

Using the covariance inequality for bounded variables we deduce that:

- if $r := |k_1 - k_2| - 2h - 2|\tau| - 1 > 2r_n$, $|k_1 - l_1| \leq r_n$ and $|k_2 - l_2| \leq r_n$ then by inequality (21) in Lemma 6.10,

$$\left| \mathrm{cov}\left\{ \widetilde{\mathcal{Y}}_{k_1} \widetilde{\mathcal{Y}}_{l_1}, \widetilde{\mathcal{Y}}_{k_2} \widetilde{\mathcal{Y}}_{l_2} \right\} \right| \leq c \, \|X\|_\infty^8 \, \alpha_\tau(r) \big(1 + h^{-1} + h^{-2} \big);$$

- if $r := \min\{l_1, k_2, l_2\} - k_1 - 2h - 2|\tau| - 1 > 0$ then by inequalities (18), (20) and (22) in Lemma 6.10

$$\begin{aligned}
&\left| \mathrm{cov}\{ \widetilde{\mathcal{Y}}_{k_1} \widetilde{\mathcal{Y}}_{l_1}, \widetilde{\mathcal{Y}}_{k_2} \widetilde{\mathcal{Y}}_{l_2} \} \right| \\
&\quad \leq \left| \mathrm{E}\{ \widetilde{\mathcal{Y}}_{k_1} \widetilde{\mathcal{Y}}_{l_1} \} \times \mathrm{E}\{ \widetilde{\mathcal{Y}}_{k_2} \widetilde{\mathcal{Y}}_{l_2} \} \right| + \left| \mathrm{cov}\{ \widetilde{\mathcal{Y}}_{k_1}, \widetilde{\mathcal{Y}}_{l_1} \widetilde{\mathcal{Y}}_{k_2} \widetilde{\mathcal{Y}}_{l_2} \} \right| \\
&\quad \leq c \, \|X\|_\infty^8 \, \alpha_\tau(r) \times \left(1 + h^{-1} \right) + c \, \|X\|_\infty^8 \, \alpha_\tau(r) \left(1 + h^{-1} + h^{-2} \right);
\end{aligned}$$

– in any case, by Lemma 6.11

$$\begin{aligned}&\left|\operatorname{cov}\{\widetilde{\mathcal{Y}}_{k_1}\widetilde{\mathcal{Y}}_{l_1},\widetilde{\mathcal{Y}}_{k_2}\widetilde{\mathcal{Y}}_{l_2}\}\right|\\&\quad\le c\,\|X\|_\infty^8\left(1+\left(h^{-1}+h^{-2}+h^{-3}\right)\mathbb{I}_{\{\max_{i,i'}\{|k_i-k_{i'}|,|k_i-l_{i'}|\}\le h+|\tau|+1\}}\right).\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\mathrm{E}\left\{|A_1|^2\right\}&\le c\,\lambda^2\gamma_n^{-2}nr_n^2\|X\|_\infty^8\left(\sum_{r=2r_n+1}^{\infty}\alpha_\tau(r-2r_n)+\sum_{r=1}^{2r_n}\alpha_\tau(r)+1\right)\\&\qquad\times\left(1+h^{-1}+h^{-2}\right)+c\lambda^2\gamma_n^{-2}h^{-3}\|X\|_\infty^8(h+|\tau|+1)^4\\&\le c\,\lambda^2\,\gamma_n^{-2}nr_n^2\|X\|_\infty^8\sum_{r=0}^{\infty}\alpha_\tau(r)\left(1+h^{-1}+h^{-2}\right)\\&\qquad+c\,\lambda^2\gamma_n^{-2}h^{-3}\|X\|_\infty^8(h+|\tau|+1)^4\\&\le c\,\lambda^2\,\gamma_n^{-2}nr_n^2h^{-2}+c\,\lambda^2\,\gamma_n^{-2}h^{-3}.\end{aligned}$$

Since $\gamma_n^{-1}=\mathcal{O}(n^{-1}h)$ and $r_n=o(n^{1/2})$, we have $\gamma_n^{-2}nr_n^2h^{-2}=\mathcal{O}(n^{-1}r_n^2)=o(1)$ and $\gamma_n^{-2}h^{-3}=\mathcal{O}(n^{-1}h^{-1})=o(1)$. Moreover we know that $\sum_r\alpha_\tau(r)<\infty$, so we deduce that $|\mathrm{E}\{A_1\}|^2\le\mathrm{E}\left\{A_1^2\right\}=o(1)$ as $n\to\infty$.

The reasoning behind A_2 and A_3 is similar. In the case of A_2 the following inequality is used: $|1-ix-e^{ix}|\le x^2$ for any $x\in\mathbb{R}$. Finally, we get

$$\begin{aligned}\mathrm{E}\{|A_2|\}&\le c\gamma_n^{-3/2}\lambda^2\|X\|_\infty^6 n\left(r_n+r_nh^{-1}+h^{-2}\right),\\|\mathrm{E}\{A_3\}|&\le c\gamma_{h,n}^{-1/2}\|X\|_\infty n\,\alpha_\tau(r_n).\end{aligned}$$

Then we easily state the convergence to 0 of $\mathrm{E}\{|A_2|\}$ and $\mathrm{E}\{A_3\}$ as $n\to\infty$. Hence convergence (8) is proved. This achieves the proof of the theorem in the case of a bounded APC process X.

3) In the last step of the proof we assume that the process $\{X(t),t\ge0\}$ is not necessarily bounded and we show that the problem can be reduced to the bounded case following the well-known truncation method (see e.g. [21]).

(i) *Truncation.* Let $C>0$ be fixed. Define

$$f^{(C)}(x):=x\quad\text{if }|x|\le C,\qquad f^{(C)}(x):=0\quad\text{otherwise.}$$

Denote $\mathcal{V}^{(C)}(s,t):=f^{(C)}(X(s))f^{(C)}(X(t))$ and $\mathcal{V}^{(Cc)}(s,t):=\mathcal{V}(s,t)-\mathcal{V}^{(C)}(s,t)$ and

$$\begin{aligned}\mathcal{Y}_k^{(C)}&:=\iint_{K_k}w_h(\tau-t+s)g(s)\mathcal{V}^{(C)}(s,t)\,dN^{(2)}(s,t),\\\mathcal{Y}_k^{(Cc)}&:=\iint_{K_k}w_h(\tau-t+s)g(s)\mathcal{V}^{Cc}(s,t)\,dN^{(2)}(s,t).\end{aligned}$$

Notice that the process $X^{(C)}:=\{X^{(C)}(t):t\ge0\}$ is bounded by C. From inequality (18) we deduce that

$$\left|\operatorname{cov}\{\mathcal{Y}_{k_1}^{(C)},\mathcal{Y}_{k_2}^{(C)}\}\right|\le c\,\|X^{(C)}\|_\infty^4\times\left(1+h^{-1}\mathbb{I}_{\{|k_1-k_2|\le h+|\tau|+1\}}\right)$$

$$\leq c\,C^4 \times \left(1 + h^{-1}\mathbb{I}_{\{|k_1-k_2|\leq h+|\tau|+1\}}\right).$$

Moreover, note that

$$\begin{aligned}\left|\mathcal{V}^{(Cc)}(s,t)\right| &= \left|X(s)X(t) - X^{(C)}(s)X^{(C)}(t)\right| \\ &= |X(s)X(t)|\left(1 - \mathbb{I}_{\{|X(s)|\leq C\}}\mathbb{I}_{\{|X(t)\leq C\}}\right) \\ &\leq |X(s)X(t)| \times \frac{|X(s)X(t)|^{\delta/4}}{C^{\delta/2}} \leq C^{-\delta/2}|X(s)X(t)|^{1+\delta/4}.\end{aligned}$$

Hence

$$\left\|\mathcal{V}^{(Cc)}(s,t)\right\|_2^2 \leq C^{-\delta}\left\|X(s)X(t)\right\|_{2+\delta/2}^{2+\delta/2} \leq C^{-\delta}\|X\|_{4+\delta}^{4+\delta}$$

and finally

$$\left|\mathrm{cov}\{\mathcal{Y}_{k_1}^{(Cc)}, \mathcal{Y}_{k_2}^{(Cc)}\}\right| \leq c\,C^{-\delta}\|X\|_{4+\delta}^{4+\delta} \times \left(1 + h^{-1}\mathbb{I}_{\{|k_1-k_2|\leq h+|\tau|+1\}}\right).$$

Here and thereafter $\|X\|_{4+\delta}^{4+\delta} := \sup_t \mathrm{E}\left\{|X(t)|^{4+\delta}\right\}$. Moreover, from inequality (20) we have that

$$\left|\mathrm{cov}\{\mathcal{Y}_{k_1}^{(C)}, \mathcal{Y}_{k_2}^{(C)}\}\right| \leq c\alpha_\tau(|k_1-k_2|)\|X^{(C)}\|_\infty^4 \leq c\alpha_\tau(|k_1-k_2|)C^4.$$

Additionally, from inequality (19),

$$\left|\mathrm{cov}\{\mathcal{Y}_{k_1}^{(Cc)}, \mathcal{Y}_{k_2}^{(Cc)}\}\right| \leq c\alpha_\tau(|k_1-k_2|)^{\frac{\delta}{4+\delta}}\|X\|_{4+\delta}^4,$$

for $|k_1 - k_2| \geq 2h + 2|\tau| + 1$.

(ii) Since $\mathcal{Y}_k = \mathcal{Y}_k^{(C)} + \mathcal{Y}_k^{(Cc)}$, we can decompose S_n as follows

$$S_n = S_n^{(C)} + S_n^{(Cc)} := \sum_{k\in D_n}\mathcal{Y}_k^{(C)} + \sum_{j\in D_n}\mathcal{Y}_k^{(Cc)}.$$

Following the technique of truncation (see proof of Theorem 17.2.2 in [21]) we get

$$\begin{aligned}\mathrm{var}\left\{S_n^{(Cc)}\right\} &= \sum_{k_1,k_2\in D_n}\mathrm{cov}\left\{\mathcal{Y}_{k_1}^{(Cc)}, \mathcal{Y}_{k_2}^{(Cc)}\right\} = \sum_{j\in\mathbb{Z}}\sum_{k\in D_n(j)}\mathrm{cov}\left\{\mathcal{Y}_k^{(Cc)}, \mathcal{Y}_{j+k}^{(Cc)}\right\} \\ &= \sum_{|j|<A}\sum_{k\in D_n(j)}\mathrm{cov}\left\{\mathcal{Y}_k^{(Cc)}, \mathcal{Y}_{j+k}^{(Cc)}\right\} + \sum_{|j|\geq A}\sum_{k\in D_n(j)}\mathrm{cov}\left\{\mathcal{Y}_k^{(Cc)}, \mathcal{Y}_{j+k}^{(Cc)}\right\},\end{aligned}$$

where $D_n(j) := \{k \in D_n : k + j \in D_n\}$ and A is any integer such that $A \geq 2h + 2|\tau| + 1$. Note that $\mathrm{card}D_n(j) \leq n$. Then, for any integer $A \geq 2h + 2|\tau| + 1$

$$\begin{aligned}\mathrm{var}\left\{S_n^{(Cc)}\right\} &\leq c\sum_{|j|<A} n\|X\|_4^4\left(1 + h^{-1}\mathbb{I}_{\{|j|\leq h+|\tau|+1\}}\right) + c\sum_{|j|\geq A} n\alpha_\tau(|j|)^{\delta/(4+\delta)}\|X\|_{4+\delta}^4 \\ &\leq cn\left(\frac{hA + (h+|\tau|+1)}{h}C^{-\delta}\|X\|_{4+\delta}^{4+\delta} + \sum_{j=A}^{\infty}\alpha_\tau(j)^{\delta/(4+\delta)}\|X\|_{4+\delta}^4\right).\end{aligned}$$

Taking $A = \lfloor C^\delta \rfloor$, and since $h = h_n \ll 1$ we obtain

$$\lim_{C\to\infty} \sup_n n^{-1} h \operatorname{var}\{S_n^{(Cc)}\} = 0.$$

On the other hand, we have that

$$\left|\operatorname{var}\left\{S_n^{(C)} + S_n^{(Cc)}\right\}^{1/2} - \operatorname{var}\left\{S_n^{(C)}\right\}^{1/2}\right| \le \operatorname{var}\left\{S_n^{(Cc)}\right\}^{1/2}.$$

Since $\sup_n n^{-1} h \operatorname{var}\left\{S_n^{(Cc)}\right\} \to 0$ as $C \to \infty$ we deduce that

$$\lim_{C\to\infty} \sup_n n^{-1} h \left|\operatorname{var}\left\{S_n^{(C)} + S_n^{(Cc)}\right\} - \operatorname{var}\left\{S_n^{(C)}\right\}\right| = 0.$$

Thus, if in addition

$$s^2 := \liminf_n n^{-1} h \varsigma_T^2 = \liminf_n n^{-1} h \operatorname{var}\left\{S_n^{(C)} + S_n^{(Cc)}\right\} > 0$$

then there exists $C_1 > 0$ such that for any $C > C_1$, we have

$$\liminf_n n^{-1} h \operatorname{var}\left\{S_n^{(C)}\right\} \ge \frac{s^2}{2} > 0.$$

(iii) *Convergence of the sequence of characteristic functions.* Let

$$\varsigma_n^2 = \operatorname{var}\{S_n\}, \qquad (\varsigma_n^{(C)})^2 = \operatorname{var}\left\{S_n^{(C)}\right\}, \qquad (\varsigma_n^{(Cc)})^2 = \operatorname{var}\left\{S_n^{(Cc)}\right\}.$$

Then

$$\begin{aligned}
&\left|\mathrm{E}\left\{\exp\left\{\frac{it}{\varsigma_n}\widetilde{S}_n\right\}\right\} - \exp\left\{-\frac{t^2}{2}\right\}\right| \\
&\quad \le \mathrm{E}\left\{\left|\exp\left\{\frac{it}{\varsigma_n}\widetilde{S}_n\right\} - \exp\left\{\frac{it}{\varsigma_n}\widetilde{S}_n^{(C)}\right\}\right|\right\} \\
&\qquad + \mathrm{E}\left\{\left|\exp\left\{\frac{it}{\varsigma_n}S_n^{(Cc)}\right\} - \exp\left\{\frac{it}{\varsigma_n^{(C)}}S_n^{(Cc)}\right\}\right|\right\} \\
&\qquad + \left\{\left|\mathrm{E}\left\{\exp\left\{\frac{it}{\varsigma_n^{(C)}}S_n^{(Cc)}\right\}\right\} - \exp\left\{\frac{-t^2}{2}\right\}\right|\right\} \\
&\quad \le \mathrm{E}\left|\exp\left\{\frac{it}{\varsigma_n}S_n^{(Cc)}\right\} - 1\right| + \mathrm{E}\left|\exp\left\{\frac{it}{\varsigma_n^{(C)}}\Big(\frac{\varsigma_n^{(C)}}{\varsigma_n} - 1\Big)\widetilde{S}_n^{(C)}\right\} - 1\right| + g_{C,n}(t) \\
&\quad \le |t|\mathrm{E}\left\{\frac{1}{\varsigma_n}\left|S_n^{(Cc)}\right|\right\} + |t|\left|\frac{\varsigma_n^{(C)}}{\varsigma_n} - 1\right|\mathrm{E}\left\{\frac{1}{\varsigma_n}\left|S_n^{(Cc)}\right|\right\} + g_{C,n}(t) \\
&\quad \le |t|\frac{\varsigma_n^{(Cc)}}{\varsigma_n} + |t|\left|\frac{\varsigma_n^{(C)}}{\varsigma_n} - 1\right|\frac{\varsigma_n^{(C)}}{\varsigma_n} + g_{C,n}(t)
\end{aligned}$$

for any $C > 0$. The last two inequalities are due to the Cauchy-Schwarz inequality and to the fact that $|e^{it} - 1| \leq |t|$. Hence it remains to prove that for each t

$$g_{C,n}(t) := \left| \mathrm{E}\left\{ \exp\left\{ \frac{it}{\varsigma_n^{(C)}} \widetilde{S}_n^{(C)} \right\} \right\} - \exp\left\{ \frac{-t^2}{2} \right\} \right|$$

converges to 0 as $n \to \infty$, i.e. $\left(\varsigma_n^{(C)}\right)^{-1} \widetilde{S}_n^{(C)}$ converges in law to $\mathcal{N}(0,1)$, but this was already shown in the second part of the proof. Thus the proposition is proved. □

6.4. Bootstrap consistency

For the sake of clarity of the proof of Theorem 3.7 at first we present a few lemmas and we introduce some additional notation. Let $\widehat{\widehat{a}}_T$ be the average of the estimators defined on the blocks:

$$\widehat{\widehat{a}}_T := \frac{1}{T-b+1} \sum_{k=0}^{T-b} \widehat{a}_{k,b}. \tag{9}$$

Notice that $\widehat{\widehat{a}}_T = \mathrm{E}^* \left\{ \widehat{a}_T^* \right\}$ (see Section 3.2).

The next lemma investigates the relationship between the estimator $\widehat{a}_T$ and the estimator based on the blocks of length b.

Lemma 6.7. *Let* $1 \ll b \ll T$ *and* $h \ll \min\{1, b^2 T^{-1}\}$. *If* $\sup_t \mathrm{E}\{X(t)^2\} < \infty$, *then*

$$\lim_{T\to\infty} \sqrt{Th}\, \mathrm{E}\left\{ \left| \widehat{a}_T(\lambda,\tau) - \frac{1}{l} \sum_{j=0}^{l-1} \widehat{a}_{jb,b}(\lambda,\tau) \right| \right\} = 0$$

and

$$\lim_{T\to\infty} \sup_{r=0,\ldots,b-1} \sqrt{Th}\, \mathrm{E}\left\{ \left| \widehat{a}_T(\lambda,\tau) - \frac{1}{l-1} \sum_{j=0}^{l-2} \widehat{a}_{jb+r,b}(\lambda,\tau) \right| \right\} = 0.$$

When $\sup_t \mathrm{E}\{X(t)^4\} < \infty$, *then*

$$\lim_{T\to\infty} Th\, \mathrm{E}\left\{ \left| \widehat{a}_T(\lambda,\tau) - \frac{1}{l} \sum_{j=0}^{l-1} \widehat{a}_{jb,b}(\lambda,\tau) \right|^2 \right\} = 0$$

and

$$\lim_{T\to\infty} \sup_{r=0,\ldots,b-1} Th\, \mathrm{E}\left\{ \left| \widehat{a}_T(\lambda,\tau) - \frac{1}{l-1} \sum_{j=0}^{l-2} \widehat{a}_{jb+r,b}(\lambda,\tau) \right|^2 \right\} = 0.$$

Proof. Since $T = lb$ with $l \in \mathbb{N}$, we have

$$\widehat{a}_T = \frac{b}{\beta^2 T} \sum_{j=0}^{l-1} \widehat{a}_{jb,b}(\lambda, \tau) + \frac{1}{\beta^2 T} \sum_{j_1,\, j_2=0}^{l-1} \mathbb{I}_{\{j_1 \neq j_2\}} \int_{j_1 b}^{j_1 b+b} \int_{j_2 b}^{j_2 b+b} w_h(\tau - t + s)\, e^{-i\lambda s}\, dZ(s,t).$$

Moreover, one can note that

$$\int_{j_1 b}^{j_1 b+b} \int_{j_2 b}^{j_2 b+b} w_h(\tau - t + s)\, e^{-i\lambda s}\, dZ(s,t) = \int_{j_1 b}^{j_1 b+b} \int_{I(j_2 b, b, s+\tau, h)} w_h(\tau - t + s)\, e^{-i\lambda s}\, dZ(s,t),$$

where $I(k, b, s+\tau, h) := \{t : k \leq t \leq k+b \text{ and } s+\tau-h \leq t \leq s+\tau+h\}$. Remark that if $-b < \tau - h < \tau + h < b$, $|j_2 - j_1| > 1$ and $j_1 b \leq s \leq j_1 b + b$ then the domain of integration $I(j_2 b, b, s+\tau, h)$ is empty, so the integral is null.

To simplify the computations from now on we assume that T is large enough so that $0 < h < 1$, $l \geq 2$ and $-b < \tau - 1 < \tau + 1 < b$. Then we get

$$\widehat{a}_T = \frac{1}{l} \sum_{j=0}^{l-1} \widehat{a}_{jb,b}(\tau) + \frac{1}{\beta^2 T} \sum_{j=0}^{l-2} \int_{jb}^{jb+b} \int_{jb+b}^{jb+2b} w_h(\tau - t + s)\, e^{-i\lambda s}\, dZ(s,t) + \frac{1}{\beta^2 T} \sum_{j=0}^{l-2} \int_{jb+b}^{jb+2b} \int_{jb}^{jb+b} w_h(\tau - t + s)\, e^{-i\lambda s}\, dZ(s,t). \tag{10}$$

Moreover, for $r = 0, \ldots, b-1$ we can also decompose $\widehat{a}_T$ as follows

$$\begin{aligned}
\widehat{a}_T = {} & \frac{r}{T}\widehat{a}_{0,r} + \frac{1}{l} \sum_{j=0}^{l-2} \widehat{a}_{jb+r,b}(\lambda, \tau) + \frac{b-r}{T} \widehat{a}_{(l-1)b+r, b-r} \qquad (11) \\
& + \frac{1}{\beta^2 T} \int_0^r \int_r^{r+b} w_h(\tau - t + s)\, e^{-i\lambda s}\, dZ(s,t) \\
& + \frac{1}{\beta^2 T} \int_r^{r+b} \int_0^r w_h(\tau - t + s)\, e^{-i\lambda s}\, dZ(s,t) \\
& + \frac{1}{\beta^2 T} \sum_{j=0}^{l-3} \int_{jb+r}^{jb+b+r} \int_{jb+r+b}^{jb+r+2b} w_h(\tau - t + s)\, e^{-i\lambda s}\, dZ(s,t) \\
& + \frac{1}{\beta^2 T} \sum_{j=0}^{l-3} \int_{jb+r+b}^{jb+r+2b} \int_{jb+r}^{jb+r+b} w_h(\tau - t + s)\, e^{-i\lambda s}\, dZ(s,t) \\
& + \frac{1}{\beta^2 T} \int_{(l-2)b+r}^{(l-1)b+r} \int_{(l-1)b+r}^{lb} w_h(\tau - t + s)\, e^{-i\lambda s}\, dZ(s,t)
\end{aligned}$$

$$+\frac{1}{\beta^2 T}\int_{(l-1)b+r}^{lb}\int_{(l-2)b+r}^{(l-1)b+r} w_h(\tau-t+s)\,e^{-i\lambda s}\,dZ(s,t).$$

In the decomposition (10) of the estimator $\widehat{a}_T$, the 2nd and the 3rd term are negligible. In the second decomposition all the terms except the 2nd one are negligible. For simplicity of presentation, we only study the first cross term in decomposition (10) of $\widehat{a}_T$ taking $r=0$.

Let $j=0,\dots,l-2$ and

$$A_j := \int_{jb}^{jb+b}\int_{jb+b}^{jb+2b} w_h(\tau-t+s)\,e^{-i\lambda s}\,dZ(s,t).$$

Then

$$|A_j| \le \int_{jb}^{jb+b}\int_{jb+b}^{jb+2b} w_h(\tau-t+s)\,\big|X(s)X(t)\big|\,dN^{(2)}(s,t).$$

From the independence property of the increments of the Poisson process, and since the intervals $(jb, jb+b]$ and $(jb+b, jb+2b]$ are disjoint, we deduce that

$$\begin{aligned}
\mathrm{E}\{|A_j|\} &\le \int_{jb}^{jb+b}\int_{jb+b}^{jb+2b} w_h(\tau-t+s)\,\mathrm{E}\big\{\big|X(s)X(t)\big|\big\}\,\beta^2\,dsdt \\
&\le \beta^2 \sup_t \mathrm{E}\big\{X(t)^2\big\}\int_{jb}^{jb+b}\int_{jb+b}^{jb+2b} w_h(\tau-t+s)\,dsdt \\
&= \beta^2 \sup_t \mathrm{E}\big\{X(t)^2\big\}\int_{-b}^{0}\left(\int_{I(0,b,s+\tau,h)} w_h(\tau-t+s)\,dt\right)ds,
\end{aligned}$$

where $I(0,b,s+\tau,h)=\{t: 0\le t\le b, s+\tau-h\le t\le s+\tau+h\}$.

We now investigate the last integral with respect to (s,t) for $-b\le s\le 0\le t\le b$ and $\tau-h<t-s<\tau+h$. For that purpose we consider three cases: (i) $-b<\tau-h<\tau+h<0$, (ii) $-b<\tau-h<0\le\tau+h<0$ and (iii) $0\le\tau-h<\tau+h<0$.

(i) In this case the integration domain in empty, so the integral is null.

(ii) Note that $-h<\tau<h$. The integration domain is $\{(s,t): -\tau-h\le s\le 0$ and $0\le t\le s+\tau+h\}$ and

$$\int_{-\tau-h}^{0}\left(\int_{0}^{s+\tau+h} w_h(\tau-t+s)dt\right)ds = \int_{-\tau-h}^{0}\left(\int_{-\frac{s+\tau}{h}}^{1} w(u)du\right)ds \le |\tau|+h.$$

(iii) Note that $h\le\tau<b-h$. The integration domain can be split into two parts: $\{(s,t): -\tau-h\le s\le -\tau+h$ and $0\le t\le s+\tau+h\}$ and $\{(s,t): -\tau+h\le s\le 0$ and $s+\tau-h\le t\le s+\tau+h\}$. For the first part we have

$$\int_{-\tau-h}^{-\tau+h}\left(\int_{0}^{s+\tau+h} w_h(\tau-t+s)\,dt\right)ds = \int_{-\tau-h}^{-\tau+h}\left(\int_{-\frac{s+\tau}{h}}^{1} w(u)\,du\right)ds \le 2h$$

and for the second part

$$\int_{-\tau-h}^{0}\left(\int_{s+\tau-h}^{s+\tau+h} w_h(\tau-t+s)\,dt\right)ds=\int_{-\tau-h}^{0}\left(\int_{-1}^{1} w(u)\,du\right)ds\le|\tau|+h.$$

Then we obtain that $\mathrm{E}\{|A_j|\}\le\beta^2\sup_t\mathrm{E}\{X(t)^2\}\,(|\tau|+3h)$. The same technique applied to the other summands of the decomposition (11) of $\widehat{a}_T$ gives

$$\mathrm{E}\left\{\left|\widehat{a}_T-\frac{1}{l}\sum_{j=0}^{l-1}\widehat{a}_{jb,b}\right|\right\}\le\frac{2l}{T}(|\tau|+3h)\sup_t\mathrm{E}\{X(t)^2\}\le\frac{2}{b}(|\tau|+3h)\sup_t\mathrm{E}\{X(t)^2\}$$

and

$$\sqrt{Th}\mathrm{E}\left\{\left|\widehat{a}_T-\frac{1}{l}\sum_{j=0}^{l-1}\widehat{a}_{jb+r,b}\right|\right\}\le c\frac{\sqrt{Th}}{b}(|\tau|+h).$$

This entails the first part of the lemma. The proof of second part follows the same way. □

The lemma below is a direct consequence of Lemma 6.7.

Lemma 6.8. *Let* $1\ll b\ll T$ *and* $h\ll\min\{1,b^2T^{-1}\}$. *If* $\sup_t\mathrm{E}\{X(t)^2\}<\infty$, *then*

$$\lim_{T\to\infty}\sqrt{Th}\,\mathrm{E}\left\{\left|\widehat{a}_T-\widehat{\widehat{a}}_T\right|\right\}=0.$$

If $\sup_t\mathrm{E}\{X(t)^4\}<\infty$, *then*

$$\lim_{T\to\infty}Th\,\mathrm{E}\left\{\left|\widehat{a}_T(\tau)-\widehat{\widehat{a}}_T(\tau)\right|^2\right\}=0. \tag{12}$$

Proof of Proposition 3.5

Recall that $\widehat{\widehat{a}}_T=\mathrm{E}^*\{\widehat{a}_T^*\}$. Then $\left|\mathrm{E}^*\{\widehat{a}_T^*\}-a_X(\lambda,\tau)\right|\le\left|\widehat{\widehat{a}}_T-\widehat{a}_T\right|+\left|\widehat{a}_T-a_X(\lambda,\tau)\right|$. The asymptotic unbiasedness (4) in the quadratic mean of $\widehat{a}_T^*$ and the rate of convergence (5) are consequences of Propositions 3.1, 3.2 and Lemma 6.8. Moreover, from Proposition 3.3 the estimator $\widehat{a}_T$ converges almost surely to $a_X(\lambda,\tau)$. To state the almost sure convergence of $\widehat{\widehat{a}}_T$ we decompose $\widehat{\widehat{a}}_T-\widehat{a}_T$ as follows

$$\widehat{\widehat{a}}_T-\widehat{a}_T=\left(\widehat{\widehat{a}}_T-\widehat{\widehat{a}}_{n^x}\right)+\left(\widehat{\widehat{a}}_{n^x}-\widehat{a}_{n^x}\right)+\left(\widehat{a}_{n^x}-\widehat{a}_T\right)$$

for $n^x\le T\le(n+1)^x$. From the proof of Proposition 3.3 we know that $\sup\{|\widehat{a}_{n^x}-\widehat{a}_T|:n^x\le T\le n^{x+1}\}$ converges almost surely to 0 as $n\to\infty$. Applying again the Bienaymé-Chebychev inequality as well as the Borel-Cantelli lemma with relation (12), we obtain that $\widehat{\widehat{a}}_{n^x}-\widehat{a}_{n^x}$ converges almost surely to 0. Finally, the almost sure convergence to 0 of $\sup\{|\widehat{a}_T-\widehat{a}_{n^x}|:n^x\le T\le n^{x+1}\}$ as $n\to\infty$ is deduced from the decomposition

$$\left|\widehat{\widehat{a}}_{n^x}-\widehat{\widehat{a}}_T\right|\le\left|\frac{1}{n^x-b_{n^x}+1}-\frac{1}{T-b_T+1}\right|\left|\sum_{k=0}^{n^x-b_{n^x}}\widehat{a}_{k,b_{n^x}}\right|$$

$$+ \frac{1}{T-b_T+1} \sum_{k=0}^{n^x - b_{n^x}} \left| \widehat{a}_{k,b_{n^x}} - \widehat{a}_{k,b_T} \right|$$
$$+ \frac{1}{T-b_T+1} \left| \sum_{k=n^x-b_{n^x}+1}^{T-b_T} \widehat{a}_{k,b_T} \right| .$$

Then similar arguments to the last part of the proof of Proposition 3.3 achieve the proof of the lemma. The details are left to the reader. Furthermore, taking under consideration the different rates of convergence along with the previous computations, it is easy to deduce some rate of almost sure convergence. □

Proof of Proposition 3.6

Since the random variables $\widehat{a}_{j^*,b}$, $j = 1, \ldots, l$ are P*-independent and $\widehat{\widehat{a}}_T = \mathrm{E}^* \left\{ \widehat{a}_{j^*,b} \right\}$, we have

$$l \operatorname{var}^* \left\{ \widehat{a}^*_T \right\} = \frac{1}{l} \sum_{j=0}^{l-1} \operatorname{var}^* \left\{ \widehat{a}_{j^*,b} \right\} = \frac{1}{T-b+1} \sum_{k=0}^{T-b} \left(\widehat{a}_{k,b} - \widehat{\widehat{a}}_T \right) \left(\widehat{a}_{k,b} - \widehat{\widehat{a}}_T \right)'$$

Recall that we consider the complex numbers as bidimensional real vectors and the notation $(v)'$ indicates the transpose of the column vector v. Thus $\operatorname{var}^* \left\{ \widehat{a}^*_T \right\}$ is 2×2 square matrix. Define the random matrix

$$V_{k,b} := \left(\widehat{a}_{k,b} - \mathrm{E} \left\{ \widehat{a}_{k,b} \right\} \right) \left(\widehat{a}_{k,b} - \mathrm{E} \left\{ \widehat{a}_{k,b} \right\} \right)'.$$

Then we have

$$\mathrm{E} \left\{ \left| \operatorname{var}^* \left\{ \widehat{a}_{j^*,b} \right\} - \frac{1}{T-b+1} \sum_{k=0}^{T-b} V_{k,b} \right| \right\}$$
$$\leq \frac{2}{T-b+1} \sum_{k=0}^{T-b} \mathrm{E} \left\{ \left| \mathrm{E} \left\{ \widehat{a}_{k,b} \right\} - \widehat{\widehat{a}}_T \right|^2 \right\}^{1/2} \times \mathrm{E} \left\{ \left| \widehat{a}_{k,b} - \mathrm{E} \left\{ \widehat{a}_{k,b} \right\} \right|^2 \right\}^{1/2}$$
$$+ \frac{1}{T-b+1} \sum_{k=0}^{T-b} \mathrm{E} \left\{ \left| \mathrm{E} \left\{ \widehat{a}_{k,b} \right\} - \widehat{\widehat{a}}_T \right|^2 \right\} .$$

Since $T^{1/3} \ll b \ll T$ and $b^{-1} \ll h \ll T^{-1/3} \ll b^2 T^{-1}$, by Proposition 3.1 and Lemmas 6.2, 6.3 and 6.8, we deduce that

$$\lim_{T \to \infty} \mathrm{E} \left\{ \left| Th \operatorname{var}^* \left\{ \widehat{a}^*_T \right\} - \frac{bh}{T-b+1} \sum_{k=0}^{T-b} V_{k,b} \right| \right\}$$
$$= \lim_{T \to \infty} bh \mathrm{E} \left\{ \left| \operatorname{var}^* \left\{ \widehat{a}_{j^*,b} \right\} - \frac{1}{T-b+1} \sum_{k=0}^{T-b} V_{k,b} \right| \right\} = 0.$$

Now we study the behaviour of $V_{k,b}$. First by Lemma 6.3

$$\lim_{T\to\infty} \mathrm{E}\left\{\frac{bh}{T-b+1}\sum_{k=0}^{T-b} V_{k,b}\right\} = \beta^{-4}\Gamma^{+}(\lambda,\lambda;\tau) := \Sigma.$$

Next, to state the convergence in P-probability of $\frac{bh}{T-b+1}\sum_{k=0}^{T-b} V_{k,b}$, we are going to show that its covariance is converging to 0. From Lemma 6.10 we can state that:

– $|\mathrm{cov}\{V_{k,b}, V_{k+r,b}\}| \le c\,\|X\|_8^8$, for $|r| \le b-2|\tau|-2$;
– $|\mathrm{cov}\{V_{k,b}, V_{k+r,b}\}| \le c\,\alpha_X\big(r-b-2|\tau|-2\big)\|X\|_\infty^8$, for $|r| > b-2|\tau|-2$ when the process X is bounded ($\|X\|_\infty < \infty$). Hence, in this case

$$\left|\mathrm{var}\left\{\frac{bh}{T-b+1}\sum_{k=0}^{T-b} V_{k,b}\right\}\right| = \frac{(bh)^2}{(T-b+1)^2}\left|\sum_{k_1=0}^{T-b}\sum_{k_2=0}^{T-b}\mathrm{cov}\{V_{k_1,b}, V_{k_2,b}\}\right|$$
$$\le \frac{(bh)^2}{(T-b+1)^2}\sum_{r=-T+b}^{T-b}\sum_{k\in I(T-b,r)}|\mathrm{cov}\{V_{k,b}, V_{k+r,b}\}|$$
$$\le \frac{c\,b^3h^2\|X\|_8^8}{(T-b+1)} + \frac{c\,(bh)^2\|X\|_\infty^8}{(T-b+1)}\sum_{r=0}^{T-b}\alpha_X(r),$$

which converges to 0 since $h \ll T^{1/2}b^{-3/2}$. This achieves the proof of the Lemma when the process X is bounded. Otherwise, when $\|X\|_{8+\delta}^{8+\delta} := \sup_t \mathrm{E}\big\{|X(t)|^{8+\delta}\big\} < \infty$, we replace $\|X\|_\infty$ by $\|X\|_{8+\delta}$, and $\alpha_X(r)$ by $\alpha_X(r)^{\delta/(4+\delta)}$. □

Finally, we state a Rosenthal-type inequality which will be useful for the proof of the consistency of the bootstrap method (Theorems 3.7 and 4.2).

Lemma 6.9. *Assume that*

– *either the process* $\{X(t), t\ge 0\}$ *is bounded and* $\int_0^\infty t\,\alpha_X(t)\,dt < \infty$,
– *or* $\sup_t \mathrm{E}\left\{|X(t)|^{8+\delta}\right\} < \infty$ *and* $\int_0^\infty t\,\alpha_X(t)^{\delta/(4+\delta)}dt < \infty$ *for some* $\delta > 0$.

Then there exists $K > 0$ *such that for all* $b > 1$ *and* $0 < h < b^{-1/3}$,

$$\sup_k (bh)^2\mathrm{E}\left\{\left|\widehat{a}_{k,b} - \mathrm{E}\big\{\widehat{\widehat{a}}_T\big\}\right|^4\right\} \le Kh^{-1}.$$

Proof. By the inequality $|x+y|^4 \le 8|x|^4 + 8|y|^4$ fo all x and y, we have

$$(bh)^2\mathrm{E}\left\{\left|\widehat{a}_{k,b} - \mathrm{E}\big\{\widehat{\widehat{a}}_T\big\})\right|^4\right\}$$
$$\le 8(bh)^2\mathrm{E}\left\{\left|\widehat{a}_{k,b} - \mathrm{E}\{\widehat{a}_{k,b}\}\right|^4\right\} + 8(bh)^2\left|\mathrm{E}\{\widehat{a}_{k,b}\} - \mathrm{E}\big\{\widehat{\widehat{a}}_T\big\}\right|^4.$$

From the proof of Lemma 6.2 we know that

$$\left|\mathrm{E}\{\widehat{a}_{k,b}\}-\mathrm{E}\{\widehat{\widehat{a}}_T\}\right| \leq \left|\mathrm{E}\left\{\widehat{a}_{k,b}\right\}-a_X(\lambda,\tau)\right| + \left|\mathrm{E}\{\widehat{\widehat{a}}_T\}-a_X(\lambda,\tau)\right| \leq c\,(h+b^{-1}),$$

for some $c>0$ which does not depend on k,h,b and T. Since $0<h<b^{-3}<1$, we deduce that

$$\sqrt{bh}\left|\mathrm{E}\{\widehat{a}_{k,b}\}-\mathrm{E}\{\widehat{\widehat{a}}_T\}\right| \leq c\,(bh^3)^{1/2}+c\,b^{-1/2}h^{1/2}\leq 2c.$$

Thus, it remains to study

$$\sup_k (bh)^2 \mathrm{E}\left\{\left|\widehat{a}_{k,b}-\mathrm{E}\left\{\widehat{a}_{k,b}\right\}\right|^4\right\}.$$

For this purpose we adapt to our framework the method developed in [11] (see also [10] and [24]). By Lemma 6.11 we have

$$\begin{aligned}
&\mathrm{E}\left\{\left|\mathcal{Y}_{j_1}\mathcal{Y}_{j_2}\mathcal{Y}_{j_3}\mathcal{Y}_{j_4}\right|^4\right\}\\
&\leq \sup_t \mathrm{E}\left\{|X(t)|^8\right\}\mathrm{E}\left\{\prod_{i=1}^4\int_{j_i}^{j_i+1}\int_{\mathbb{R}} w_h(\tau-t_i+s_i)dN^{(2)}(s_i,t_j)\right\}\\
&\leq c\,\sup_t \mathrm{E}\left\{|X(t)|^8\right\}\left(1+\left(h^{-1}+h^{-2}+h^{-3}\right)\mathbb{I}_{\{\max_{i,i'}|j_i-j_{i'}|\leq h+|\tau|+1\}}\right),
\end{aligned}$$

for some $c>0$ which does not depend on k,h,b and T. Following the arguments presented in [11] and applying mixing inequalities (21), (22) and (23) we obtain that

$$\begin{aligned}
b^4\beta^8\mathrm{E}\left\{\left|\widehat{a}_{k,b}-\mathrm{E}\left\{\widehat{a}_{k,b}\right\}\right|^4\right\} &= \mathrm{E}\left\{\left|\sum_{j=k}^{k+b-1}\left(\mathcal{Y}_k-\mathrm{E}\left\{\mathcal{Y}_k\right\}\right)\right|^4\right\}\\
&\leq c\,b^2\sup_t \mathrm{E}\left\{|X(t)|^8\right\}\left(1+h^{-3}\right)\left(2h+2|\tau|+1+\sum_{j=1}^b j\alpha_X(j)\right),
\end{aligned}$$

when the process $\{X(t),t\geq 0\}$ is bounded. Otherwise, when $\{X(t),t\geq 0\}$ is not bounded but $\sup_t \mathrm{E}\{|X(t)|^{8+\delta}\}<\infty$, we get that

$$\begin{aligned}
&b^4\beta^8\mathrm{E}\left\{\left|\widehat{a}_{k,b}-\mathrm{E}\left\{\widehat{a}_{k,b}\right\}\right|^4\right\} = \mathrm{E}\left\{\left|\sum_{j=k}^{k+b-1}\left(\mathcal{Y}_k-\mathrm{E}\left\{\mathcal{Y}_k\right\}\right)\right|^4\right\}\\
&\leq c\,b^2\sup_t \mathrm{E}\left\{|X(t)|^{8+\delta}\right\}^{\frac{4}{4+\delta}}\left(1+h^{-3}\right)\left(2h+2|\tau|+1+\sum_{j=1}^b j\alpha_X(j)^{\frac{\delta}{4+\delta}}\right).
\end{aligned}$$

Then we can readily complete the proof of the lemma. □

Proof of Theorems 3.7 and 4.2

Since the proof of Theorem 4.2 is a direct consequence of the Cramér-Wold device and the reasoning used in one-dimensional case, we skip the technical details and we concentrate only on the proof of Theorem 3.7.

In the following we consider distribution functions defined on $\mathbb{R}^2$. We use the following notation: for $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{y} = (y_1, y_2)$ in $\mathbb{R}^2$ we write $\boldsymbol{x} \preceq \boldsymbol{y}$, when $x_1 \le y_1$ and $x_2 \le y_2$.

Moreover, it is worth keeping in mind that the condition $T^{1/3} \ll b \le T^{\theta+1/3}$ for some $0 < \theta \le 2/9$, implies that $T^{1/3} \ll b \le T^{5/9}$, $T^{-1/3} \le b^{-3/2}T^{1/2}$ and $T^{-1/3} \ll b^2T^{-1}$.

Furthermore from Theorem 3.4 we know that $\sqrt{Th}\,(\widehat{a}_T - a_X(\lambda, \tau))$ converges in law to a bidimensional Gaussian distribution which is denoted by F. To simplify the presentation of the proof, we assume that this bidimensional Gaussian distribution F is non-degenerate, i.e. the determinant of its covariance matrix is positive, thus its distribution function $F(\cdot)$ is uniformly continuous on $\mathbb{R}^2$.

1) At first we state that the bootstrap distribution $\mathrm{P}^*\left\{\sqrt{Th}\,(\widehat{a}_T^* - \widehat{a}_T) \preceq \boldsymbol{x}\right\}$ can be equivalently replaced by $\mathrm{P}^*\left\{\sqrt{Th}\left(\widehat{a}_T^* - \widehat{\widehat{a}}_T\right) \preceq \boldsymbol{x}\right\}$, for $\boldsymbol{x} \in \mathbb{R}^2$. Indeed let $\boldsymbol{\epsilon} = (\varepsilon, \varepsilon)/\sqrt{2}$, $\varepsilon > 0$ be fixed. Then we have

$$\mathrm{P}^*\left\{\sqrt{Th}\,(\widehat{a}_T^* - \widehat{a}_T) \preceq \boldsymbol{x}\right\} \le \mathrm{P}^*\left\{\sqrt{Th}(\widehat{a}_T^* - \widehat{\widehat{a}}_T) \preceq \boldsymbol{x} + \boldsymbol{\epsilon}\right\} + \mathbb{I}_{\{\sqrt{Th}|\widehat{\widehat{a}}_T - \widehat{a}_T| > \varepsilon\}}$$

and

$$\mathrm{P}^*\left\{\sqrt{Th}\,(\widehat{a}_T^* - \widehat{a}_T) \preceq \boldsymbol{x}\right\} \ge \mathrm{P}^*\left\{\sqrt{Th}(\widehat{a}_T^* - \widehat{\widehat{a}}_T) \preceq \boldsymbol{x} - \boldsymbol{\epsilon}\right\} \mathbb{I}_{\{\sqrt{Th}|\widehat{\widehat{a}}_T - \widehat{a}_T| \le \varepsilon\}}.$$

Hence

$$\begin{aligned}
&\left|\mathrm{P}^*\left\{\sqrt{Th}\,(\widehat{a}_T^* - \widehat{a}_T) \preceq \boldsymbol{x}\right\} - F(\boldsymbol{x})\right| \\
&\quad \le \left|\mathrm{P}^*\left\{\sqrt{Th}(\widehat{a}_T^* - \widehat{\widehat{a}}_T) \preceq \boldsymbol{x} - \boldsymbol{\epsilon}\right\} - F(\boldsymbol{x} - \boldsymbol{\epsilon})\right| + \left|F(\boldsymbol{x} - \boldsymbol{\epsilon}) - F(\boldsymbol{x})\right| \\
&\qquad + \left|\mathrm{P}^*\left\{\sqrt{Th}(\widehat{a}_T^* - \widehat{\widehat{a}}_T) \preceq \boldsymbol{x} + \boldsymbol{\epsilon}\right\} - F(\boldsymbol{x} + \boldsymbol{\epsilon})\right| + \left|F(\boldsymbol{x} + \boldsymbol{\epsilon}) - F(\boldsymbol{x})\right| \\
&\qquad + \mathbb{I}_{\{\sqrt{Th}|\widehat{\widehat{a}}_T - \widehat{a}_T| > \varepsilon\}}.
\end{aligned}$$

Thus, for each $\eta > 0$

$$\begin{aligned}
&\mathrm{P}\left\{\sup_{\boldsymbol{x}}\left|\mathrm{P}^*\left\{\sqrt{Th}\,(\widehat{a}_T^* - \widehat{a}_T) \preceq \boldsymbol{x}\right\} - F(\boldsymbol{x})\right| > \eta\right\} \\
&\quad \le \mathrm{P}\left\{2\sup_{\boldsymbol{x}}\left|\mathrm{P}^*\left\{\sqrt{Th}(\widehat{a}_T^* - \widehat{\widehat{a}}_T) \preceq \boldsymbol{x}\right\} - F(\boldsymbol{x})\right| > \frac{\eta}{4}\right\} \\
&\qquad + \mathrm{P}\left\{\mathbb{I}_{\{\sqrt{Th}|\widehat{\widehat{a}}_T - \widehat{a}_T| \le \varepsilon\}} > \frac{\eta}{4}\right\} \\
&\qquad + \mathbb{I}_{\{\sup_{\boldsymbol{x}}|F(\boldsymbol{x}-\boldsymbol{\epsilon}) - F(\boldsymbol{x})| > \frac{\eta}{4}\}} + \mathbb{I}_{\{\sup_{\boldsymbol{x}}|F(\boldsymbol{x}+\boldsymbol{\epsilon}) - F(\boldsymbol{x})| > \frac{\eta}{4}\}}.
\end{aligned}$$

Since $\sqrt{Th}\big(\widehat{a}_T - a_X(\lambda,\tau)\big)$ converges in law to the same bidimensional Gaussian distribution F, by Lemma 6.8 and by the uniform continuity of the distribution function $F(\cdot)$, to prove Theorem 3.7 it remains to show the convergence to 0 of the first summand on the right-hand side of the last inequality.

2) To prove the convergence in P-probability of $\mathcal{L}^*\left\{\sqrt{Th}\left(\widehat{a}^*_T - \widehat{\widehat{a}}_T\right)\right\}$ to the bidimensional Gaussian distribution F, we are going to apply Corollary 2.4.8 from [2]. Let us define

$$Z_{k,T} := \frac{\sqrt{Th}}{l}\left(\widehat{a}_{k,b} - \mathrm{E}\big\{\widehat{\widehat{a}}_T\big\}\right) = \frac{\sqrt{bh}}{\sqrt{l}}\left(\widehat{a}_{k,b} - \mathrm{E}\big\{\widehat{\widehat{a}}_T\big\}\right)$$

for $k = 0, 1, \ldots, T-b$. Then

$$Z_{j^*,T} = \frac{\sqrt{Th}}{l}\sum_{k=0}^{T-b}\left(\widehat{a}_{k,b} - \mathrm{E}\big\{\widehat{\widehat{a}}_T\big\}\right)\mathbb{I}_{\{j^*=k\}} = \frac{\sqrt{Th}}{l}\left(\widehat{a}_{j^*,b} - \mathrm{E}\big\{\widehat{\widehat{a}}_T\big\}\right).$$

Note that the random variables $Z_{j^*,T}$ are conditionally independent. Moreover,

$$\mathrm{E}^*\left\{Z_{j^*,T}\right\} = \frac{\sqrt{Th}}{l(T-b+1)}\sum_{k=0}^{T-b}\left(\widehat{a}_{k,b} - \mathrm{E}\big\{\widehat{\widehat{a}}_T\big\}\right) = \frac{\sqrt{Th}}{l}\left(\widehat{\widehat{a}}_T - \mathrm{E}\big\{\widehat{\widehat{a}}_T\big\}\right)$$

and hence $\mathrm{E}\left\{Z_{j^*,T}\right\} = \mathrm{E}\left\{\mathrm{E}^*\left\{Z_{j^*,T}\right\}\right\} = 0$. Furthermore

$$\sum_{j=0}^{l-1} Z_{j^*,T} = \frac{\sqrt{Th}}{l}\sum_{j=0}^{l-1}\left(\widehat{a}_{j^*,b} - \mathrm{E}\big\{\widehat{\widehat{a}}_T\big\}\right) = \sqrt{Th}\left(\widehat{a}^*_T - \mathrm{E}\big\{\widehat{\widehat{a}}_T\big\}\right).$$

Corollary 2.4.8 in [2] requires the following three conditions, to ensure the desired limit:

(i) for every $\epsilon > 0$

$$\lim_{T\to\infty}\sum_{j=0}^{l-1}\mathrm{P}^*\left\{|Z_{j^*,T}| > \epsilon\right\} = 0 \quad \text{in P} - \text{probability};$$

(ii) for some $\eta > 0$

$$\lim_{T\to\infty}\sum_{j=0}^{l-1}\mathrm{E}^*\left\{Z_{j^*,T}\,\mathbb{I}_{\{|Z_{j^*,T}|\le\eta\}}\right\} - \sqrt{Th}\left(\widehat{\widehat{a}}_T - \mathrm{E}\left\{\widehat{\widehat{a}}_T\right\}\right) = 0 \quad \text{in P} - \text{probability};$$

(iii) for some $\eta > 0$

$$\lim_{T\to\infty}\sum_{j=0}^{l-1}\mathrm{var}^*\left\{Z_{j^*,T}\,\mathbb{I}_{\{|Z_{j^*,T}|\le\eta\}}\right\} = \beta^{-4}\Gamma^+(\lambda,\lambda;\tau) := \Sigma \quad \text{in P} - \text{probability}.$$

Convergence (i). Let $\epsilon > 0$ be fixed and notice that

$$\sum_{j=0}^{l-1} \mathrm{P}^* \left\{ |Z_{j^*,T}| > \epsilon \right\} = \frac{l}{T-b+1} \sum_{k=0}^{T-b} \mathbb{I}_{\{|Z_{k,T}|>\epsilon\}}.$$

Let $U_{k,T} := l\, \mathbb{I}_{\{|Z_{k,T}|>\epsilon\}}$ and $A_T > 0$. Then

$$U_{k,T} = \widetilde{\widetilde{U}}_{k,T} + \mathrm{E}\left\{ U_{k,T} \mathbb{I}_{\{U_{k,T}<A_T\}} \right\} + U_{k,T} \mathbb{I}_{\{U_{k,T} \geq A_T\}}, \tag{13}$$

where $\widetilde{\widetilde{U}}_{k,T} := U_{k,T} \mathbb{I}_{\{U_{k,T}<A_T\}} - \mathrm{E}\left\{ U_{k,T} \mathbb{I}_{\{U_{k,T}<A_T\}} \right\}$. For every $\gamma > 0$, by the Bienaymé-Chebychev inequality

$$\begin{aligned} \mathrm{P}\left\{ \left| \frac{1}{T-b+1} \sum_{k=0}^{T-b} \widetilde{\widetilde{U}}_{k,T} \right| > \gamma \right\} &\leq \frac{1}{(T-b+1)^2\gamma^2} \sum_{k_1=0}^{T-b} \sum_{k_2=0}^{T-b} \left| \mathrm{cov}\left\{ \widetilde{\widetilde{U}}_{k_1,T}, \widetilde{\widetilde{U}}_{k_2,T} \right\} \right| \\ &\leq \frac{1}{(T-b+1)^2\gamma^2} \sum_{k_1=0}^{T-b} \sum_{k_2=0}^{T-b} \left| \mathrm{cov}\left\{ U_{k_1,T} \mathbb{I}_{\{|U_{k_1,T}|<A_T\}}, U_{k_2,T} \mathbb{I}_{\{U_{k_2,T}|<A_T\}} \right\} \right|. \end{aligned}$$

Then using the mixing covariance inequality (Lemma 6.12) for bounded random variables we deduce that

$$\begin{aligned} \mathrm{P}\left\{ \left| \frac{1}{T-b+1} \sum_{k=0}^{T-b} \widetilde{\widetilde{U}}_{k,T} \right| > \gamma \right\} &\leq \frac{c}{(T-b+1)^2\gamma^2} \sum_{k_1=0}^{T-b} \sum_{k_2=0}^{T-b} \alpha_X(|k_1-k_2|-b-2|\tau|-3) A_T^2 \\ &\leq \frac{c\,|A_T|^2}{(T-b+1)\gamma^2} \left(b + 2|\tau| + 3 + \sum_{k=0}^{T-b} \alpha_X(k) \right). \end{aligned}$$

The right-hand side tends to 0 if $A_T \ll l^{1/2}$. Moreover, we have

$$\mathrm{E}\left\{ U_{k,T} \mathbb{I}_{\{U_{k,T}<A_T\}} \right\} = l\, \mathrm{P}\left\{ |Z_{k,T}| > \epsilon \text{ and } l < A_T \right\}.$$

The right-hand side of the last equality is equal to 0 for T large enough when $A_T \ll l^{1/2}$. As for the third term of decomposition (13), the Markov inequality means that

$$\mathrm{P}\left\{ \left| \frac{1}{T-b+1} \sum_{k=0}^{T-b} U_{k,T} \mathbb{I}_{\{U_{k,T} \geq A_T\}} \right| > \gamma \right\} \leq \frac{1}{(T-b+1)\gamma} \sum_{k_1=0}^{T-b} \mathrm{E}\left\{ U_{k,T} \mathbb{I}_{\{U_{k,T} \geq A_T\}} \right\}.$$

Remark that $0 \leq U_{k,T} \leq l|Z_{k,T}|/\epsilon$ so $0 \leq \mathbb{I}_{\{U_{k,T}>A_T\}} \leq U_{k,T}/A_T$. Then for all $\mu, \nu > 0$ we get

$$\begin{aligned} \mathrm{P}\left\{ \left| \frac{1}{T-b+1} \sum_{k=0}^{T-b} U_{k,T} \mathbb{I}_{\{U_{k,T} \geq A_T\}} \right| > \gamma \right\} &\leq \frac{1}{(T-b+1)\gamma A_T^\mu} \sum_{k=0}^{T-b} \mathrm{E}\left\{ U_{k,T}^{1+\mu} \right\} \\ \leq \frac{l^{1+\mu}}{(T-b+1)\gamma A_T^\mu} \sum_{k=0}^{T-b} \mathrm{E}\left\{ \mathbb{I}_{\{|Z_{k,T}|>\epsilon\}} \right\} &\leq \frac{l^{\mu-\nu/2}}{(T-b+1)\gamma \epsilon^\nu A_T^\mu} \sum_{k=0}^{T-b} \mathrm{E}\left\{ l^{1+\frac{\nu}{2}} |Z_{k,T}|^{2+\nu} \right\} \end{aligned}$$

$$\leq \frac{l^{\mu-\nu/2}}{(T-b+1)\gamma\epsilon^{\nu}A_T^{\mu}} \sum_{k=0}^{T-b} \mathrm{E}\left\{\left|\sqrt{bh}\left(\widehat{a}_{k,b} - \mathrm{E}\{\widehat{\widehat{a}}_T\}\right)\right|^{2+\nu}\right\}. \tag{14}$$

Using Lemma 6.9, for $0 < \nu \leq 2$ we know that

$$\mathrm{E}\left|\sqrt{bh}\left(\widehat{a}_{k,b} - \mathrm{E}\{\widehat{\widehat{a}}_T\}\right)\right|^{2+\nu} \leq \mathrm{E}\left\{\left|\sqrt{bh}\left(\widehat{a}_{k,b} - \mathrm{E}\{\widehat{\widehat{a}}_T\}\right)\right|^4\right\}^{\frac{2+\nu}{4}} \leq K^{\frac{2+\nu}{4}} h^{-\frac{2+\nu}{4}}.$$

Hence expression (14) tends to 0 when $l^{\mu-\nu/2}h^{-(2+\nu)/4} \ll A_T^{\mu}$.

Consequently to state (i), choose $A_T \to \infty$ such that $l^{\mu-\nu/2}h^{-(2+\nu)/4} \ll A_T^{\mu} \ll l^{\mu/2}$. This is possible only when $l^{2(\mu-\nu)}h^{-(2+\nu)} \ll 1$. Hence $l^{2(\mu-\nu)/(2+\nu)} \ll h$. Thus, it suffices to take $0 < \mu < \nu < 2$ such that $(2+\nu)/2(\nu-\mu) = 2 - 3\theta$. This ends the proof of (i).

Convergence (ii). Remark that for each $\eta > 0$

$$\sum_{j=0}^{l-1} \mathrm{E}^*\left\{Z_{j^*,T}\,\mathbb{I}_{\{|Z_{j^*,T}|\leq\eta\}}\right\} - \sqrt{bh}\left(\widehat{\widehat{a}}_T - \mathrm{E}\left\{\widehat{\widehat{a}}_T\right\}\right)$$
$$= \sum_{j=0}^{l-1} \mathrm{E}^*\left\{Z_{j^*,T}\,\mathbb{I}_{\{|Z_{j^*,T}|>\eta\}}\right\} = \frac{l}{T-b+1}\sum_{k=0}^{T-b} Z_{k,T}\,\mathbb{I}_{\{|Z_{k,T}|>\eta\}}.$$

The expectation of the norm of the last expression can be bounded by

$$\frac{l}{(T-b+1)}\sum_{k=0}^{T-b}\mathrm{E}\left\{|Z_{k,T}|\;\mathbb{I}_{\{|Z_{k,T}|>\eta\}}\right\} \leq \frac{l}{(T-b+1)\eta^{1+\rho}}\sum_{k=0}^{T-b}\mathrm{E}\left\{|Z_{k,T}|^{2+\rho}\right\}$$
$$\leq \frac{l^{-\rho/2}}{(T-b+1)\eta^{1+\rho}}\sum_{k=0}^{T-b}\mathrm{E}\left\{\left|\sqrt{bh}\left(\widehat{a}_{k,b} - \mathrm{E}\{\widehat{\widehat{a}}_T\}\right)\right|^{2+\rho}\right\}.$$

From Lemma 6.9, for $0 < \rho \leq 2$ we have

$$\mathrm{E}\left\{\left|\sqrt{bh}\left(\widehat{a}_{k,b} - \mathrm{E}\{\widehat{\widehat{a}}_T\}\right)\right|^{2+\rho}\right\} \leq K^{\frac{2+\rho}{4}} h^{-\frac{2+\rho}{4}}.$$

When $l^{-\rho/2}h^{-(2+\rho)/4} \ll 1$ as $T \to \infty$, we obtain (ii). Taking $\rho = 2/3(1-2\theta)$ gives convergence (ii).

Convergence (iii). The definition of the random variables $Z_{j^*,T}$, $j = 0,\ldots,l-1$, and the fact that they are conditionally independent imply that

$$\sum_{j=0}^{l-1}\mathrm{var}^*\left\{Z_{j^*,T}\right\} = \mathrm{var}^*\left\{\sum_{j=0}^{l-1} Z_{j^*,T}\right\} = Th\,\mathrm{var}^*\left\{\widehat{a}_T^*\right\},$$

which converges in P-mean to Σ by Proposition 3.6. Now let $\eta > 0$. Then

$$\left|\mathrm{var}^*\left\{Z_{j^*,T}\mathbb{I}_{\{|Z_{j^*,T}|\leq\eta\}}\right\} - \mathrm{var}^*\left\{Z_{j^*,T}\right\}\right|$$

$$\leq \left|\mathrm{var}^*\left\{Z_{j^*,T}\mathbb{I}_{\{|Z_{j^*,T}|>\eta\}}\right\}\right| + 2\left|\mathrm{cov}^*\left\{Z_{j^*,T}, Z_{j^*,T}\mathbb{I}_{\{|Z_{j^*,T}|>\eta\}}\right\}\right|.$$

From the Bienaymé-Chebychev inequality

$$\left|\mathrm{cov}^*\left\{Z_{j^*,T}, Z_{j^*,T}\mathbb{I}_{\{|Z_{j^*,T}|>\eta\}}\right\}\right|^2 \leq c\left|\mathrm{var}^*\left\{Z_{j^*,T}\right\}\right| \times \left|\mathrm{var}^*\left\{Z_{j^*,T}\mathbb{I}_{\{|Z_{j^*,T}|>\eta\}}\right\}\right|.$$

On the one hand we know that $l^2\mathrm{var}^*\left\{Z_{j^*,T}\right\} = Th\,\mathrm{var}^*\left\{\widehat{a}_T^*\right\}$ converges in P-mean. On the other hand

$$\begin{aligned}
\sum_{j=0}^{l-1}\left|\mathrm{var}^*\left\{Z_{j^*,T}\mathbb{I}_{\{|Z_{j^*,T}|>\eta\}}\right\}\right| &\leq c\sum_{j=0}^{l-1}\mathrm{E}^*\left\{\left|Z_{j^*,T}\right|^2\mathbb{I}_{\{|Z_{j^*,T}|>\eta\}}\right\} \\
&\leq \frac{c\,l}{T-b+1}\sum_{k=0}^{T-b}\mathrm{E}\left\{\left|Z_{k,T}\right|^2\mathbb{I}_{\{|Z_{k,T}|>\eta\}}\right\} \leq \frac{c\,l}{(T-b+1)\eta^\rho}\sum_{k=0}^{T-b}\mathrm{E}\left\{\left|Z_{k,T}\right|^{2+\rho}\right\} \\
&= \frac{c\,l^{-\rho/2}}{(T-b)\eta^\rho}\sum_{k=0}^{T-b}\mathrm{E}\left\{\left|\sqrt{bh}\left(\widehat{a}_{k,b} - \mathrm{E}\{\widehat{\widehat{a}}_T\}\right)\right|^{2+\rho}\right\}. \qquad (15)
\end{aligned}$$

Lemma 6.9 implies for $0 < \rho \leq 2$ that

$$\mathrm{E}\left\{\left|\sqrt{bh}\left(\widehat{a}_{k,b} - \mathrm{E}\{\widehat{\widehat{a}}_T\}\right)\right|^{2+\rho}\right\} \leq K^{\frac{2+\rho}{4}}h^{-\frac{2+\rho}{4}}.$$

When $l^{-\rho/2}h^{-(2+\rho)/4} \ll 1$, the right-hand side of inequality (15) converges to 0 as $T \to \infty$. Consequently we choose $0 < \rho < 2$ such that $l^{-2\rho/(2+\rho)} \ll h \ll T^{-1/3}$. It suffices to take $\rho = 2/3(1-2\theta)$.

This ends the proof of (iii) and simultaneously the proof of the theorem. □

6.5. Covariance mixing inequalities

In this subsection we provide some covariance inequalities for the observation, which are the consequence of the mixing property of the process X. For the sake of simplicity and clarity we introduce some additional notation. Let

$$\begin{aligned}
dZ(s,t) &:= X(s)X(t)\,dN^{(2)}(s,t), \\
\mathcal{V}(s,t) &:= X(s)X(t), \\
\mathcal{Y}_k &:= \iint_{K_k} w_h(\tau - t + s)g(s)\,dZ(s,t), \qquad (16)
\end{aligned}$$

where $K_k :=]k, k+1] \times \mathbb{R},\ k \in \mathbb{N}$ and $g : \mathbb{R} \to \mathbb{R}$ is any bounded measurable function. Recall that the kernel function $w(\cdot)$ is nonnegative, its support is contained in $[-1,1]$ and $\int_{-1}^{1} w(t)\,dt = 1$.

Lemma 6.10. *Assume that $0 < h < 1$. Then the following inequalities are valid.*

1) For all k_1 and k_2

$$\left|\mathrm{E}\{\mathcal{Y}_{k_1}\mathcal{Y}_{k_2}\}\right| \leq c\|X\|_4^4 \times \left(1 + h^{-1}\mathbb{I}_{\{|k_1-k_2|\leq h+|\tau|+1\}}\right) \tag{17}$$

and

$$\left|\mathrm{cov}\{\mathcal{Y}_{k_1}, \mathcal{Y}_{k_2}\}\right| \leq c\|X\|_4^4 \times \left(1 + h^{-1}\mathbb{I}_{\{|k_1-k_2|\leq h+|\tau|+1\}}\right), \tag{18}$$

where c depends only on $w(\cdot)$ and β.

2) Assume that there exists $\delta > 0$ such that

$$\|X\|_{4+\delta} := \sup_t \mathrm{E}\{|X(t)|^{4+\delta}\}^{1/(4+\delta)} < \infty.$$

Then for $|k_1 - k_2| \geq 2h + 2|\tau| + 1$, we have

$$\left|\mathrm{cov}\{\mathcal{Y}_{k_1}, \mathcal{Y}_{k_2}\}\right| \leq c\,\|X\|_{4+\delta}^4\, \alpha_X(|k_2 - k_1| - 2h - 2|\tau| - 1)^{\delta/(4+\delta)}. \tag{19}$$

3) Assume that the process $\{X(t), t \geq 0\}$ is bounded:

$$\|X\|_\infty := \sup_t \mathit{ess}\sup|X(t) < \infty.$$

Then

(i) for $|k_1 - k_2| \geq 2h + 2|\tau| + 1$ we have

$$\left|\mathrm{cov}\{\mathcal{Y}_{k_1}, \mathcal{Y}_{k_2}\}\right| \leq c\,\|X\|_\infty^4\, \alpha_X\left(|k_2 - k_1| - 2h - 2|\tau| - 1\right); \tag{20}$$

(ii) furthermore for $\min\{k_3, k_4\} - \max\{k_1, k_2\} \geq 2h + 2|\tau| + 1$

$$\begin{aligned}&\left|\mathrm{cov}\left\{\mathcal{Y}_{k_1}\mathcal{Y}_{k_2}, \mathcal{Y}_{k_3}\mathcal{Y}_{k_4}\right\}\right| \\ &\quad\leq c\,\|X\|_\infty^8\, \alpha_X\left(\min\{k_3, k_4\} - \max\{k_1, k_2\} - 2h - 2|\tau| - 1\right) \\ &\qquad\times \left(1 + h^{-1}\mathbb{I}_{\{|k_1-k_2|\leq h+|\tau|+1\}}\right)\left(1 + h^{-1}\mathbb{I}_{\{|k_3-k_4|\leq h+|\tau|+1\}}\right);\end{aligned} \tag{21}$$

(iii) for $\min\{k_2, k_3, k_4\} - k_1 \geq 2h + 2|\tau| + 1$

$$\begin{aligned}&\left|\mathrm{cov}\left\{\mathcal{Y}_{k_1}, \mathcal{Y}_{k_2}\mathcal{Y}_{k_3}\mathcal{Y}_{k_4}\right\}\right| \\ &\quad\leq c\,\|X\|_\infty^8\, \alpha_X\left(\min\{k_2, k_3, k_4\} - k_1 - 2h - 2|\tau| - 1\right)\left(1 + h^{-1} + h^{-2}\right);\end{aligned} \tag{22}$$

(iv) for $k_1 - \max\{k_2, k_3, k_4\} \geq 2h + 2|\tau| + 1$

$$\begin{aligned}&\left|\mathrm{cov}\left\{\mathcal{Y}_{k_1}, \mathcal{Y}_{k_2}\mathcal{Y}_{k_3}\mathcal{Y}_{k_4}\right\}\right| \\ &\quad\leq c\,\|X\|_\infty^8\, \alpha_X\left(k_1 - max\{k_2, k_3, k_4\} - 2h - 2|\tau| - 1\right)\left(1 + h^{-1} + h^{-2}\right).\end{aligned} \tag{23}$$

Proof. To establish these inequalities, we readily develop an expression for the different covariances using the independence between the APC process X and the Poisson process N (see the proof of Lemma 6.4). Then we apply the classical mixing covariance inequalities to the process X (see e.g. [10], [20], [32]), and Lemma 6.11 below. The details of the proof are left to the reader. □

Lemma 6.11. *For $j \in \{1, 2, 3\}$*

$$\begin{aligned} &\mathrm{E}\left\{\prod_{i=1}^{j+1} \iint_{K_i} w_h(\tau - t_i + s_i) dN^{(2)}(s_i, t_i)\right\} \\ &\leq c + c\left(h^{-1} + \cdots + h^{-j}\right)\mathbb{I}_{\{\max_{i_1,i_2} |k_{i_1} - k_{i_2}| \leq h + |\tau| + 1\}}. \end{aligned}$$

Proof. The idea of the proof is to consider all possible relations between $s_1, \ldots, s_4$ and $t_1, \ldots, t_4$. Since the reasoning in all cases is similar we present the details for only one of them. Let $s := s_1 = s_2 = s_3 = s_4$ and $t := t_1 = t_2 = t_3 = t_4$. Then we obtain

$$\begin{aligned} &\iint_{K_1}\iint_{K_2}\iint_{K_3}\iint_{K_4} w_h(\tau - t_1 + s_1) w_h(\tau - t_2 + s_2) w_h(\tau - t_3 + s_3) w_h(\tau - t_4 + s_4) \\ &\quad \times\ d\delta_{\{s_4\}}(s_1) d\delta_{\{t_4\}}(t_1)\, d\delta_{\{s_4\}}(s_2) d\delta_{\{t_4\}}(t_2)\, d\delta_{\{s_4\}}(s_3) d\delta_{\{t_4\}}(t_3)\, ds_4 dt_4 \\ &\quad = \frac{1}{h^4}\int_{k_1}^{k_1+1}\int_{\mathbb{R}} w\left(\frac{\tau - t + s}{h}\right)^4 ds dt\, \mathbb{I}_{\{k_1 = k_2 = k_3 = k_4\}} \\ &\quad = \frac{1}{h^3}\left(\int_{-1}^{1} w(u)^4\, du\right)\mathbb{I}_{\{k_1 = k_2 = k_3 = k_4\}}. \end{aligned}$$

Thus, one may understand how the terms h^{-3} appears in the final inequality. □

The following lemma provides another covariance mixing inequality that is useful in the proof of the central limit proposition 6.6 as well in the proof of the consistency of the bootstrap method (Theorem 3.7).

Lemma 6.12. *Let j, l, b be nonzero fixed integers, and let ξ and ζ be two bounded real valued random variables, ξ being $\mathcal{F}^j(\mathcal{Y})$-measurable and ζ being $\mathcal{F}_{j+l}^{j+l+b}(\mathcal{Y})$-measurable where $\mathcal{Y} = \{\mathcal{Y}_k\}$ has been defined by (16). Then for $l \geq 2h + 2|\tau| + 1$ we have*

$$|\mathrm{cov}\,\{\xi, \zeta\}| \leq c\alpha_X(l - 2h - 2|\tau| - 1)\|\xi\|_\infty \|\zeta\|_\infty. \tag{24}$$

Remark. Notice that the sequence $\{\mathcal{Y}_k\}_{k\in\mathbb{N}}$ is not necessarily bounded even if the process $\{X(t), t \geq 0\}$ is bounded. Thus, inequalities (20) and (21) are not direct consequences of inequality (24).

Proof. It is well known that since ξ is a real-valued random variable which is measurable with respect to the real-valued random variables $\mathcal{Y}_0, \ldots, \mathcal{Y}_j$, there exists a measurable function $f : \mathbb{R}^{j+1} \to \mathbb{R}$ such that $\xi = f(\mathcal{Y}_0, \ldots, \mathcal{Y}_j)$. Then for $k \geq 0$ we have

$$\mathcal{Y}_k = \int_k^{k+1}\left(\int_{s+\tau-h}^{s+\tau+h} w_h(\tau - t + s) g(s) X(s) X(t) \mathbb{I}_{\{s \neq t\}}\, N(t + dt)\right) N(s + ds).$$

Since $\{(s,t) : k \leq s \leq k+1, s+\tau-h \leq t \leq s+\tau+h\} \subset [k - |\tau| - 1, k + |\tau| + 2]^2$, (recalling that $0 < h \leq 1$) we deduce that there exist a constant value ψ_0 and some measurable functions $\psi_k : \mathbb{R}^{2k} \to \mathbb{R}$, $k \in \mathbb{N}^*$, such that

$$\xi = \psi_0 \mathbb{I}_{\{N(j+|\tau|+2)=0\}} + \sum_{k\in\mathbb{N}^*} \psi_k\Big(X(T_1), \dots, X(T_k); T_1, \dots, T_k\Big)\mathbb{I}_{N(j+|\tau|+2)=k\}}. \tag{25}$$

Similarly there exist a constant value ϕ_0 and some measurable functions ϕ_κ : $\mathbb{R}^{2\kappa} \to \mathbb{R}$, $\kappa \in \mathbb{N}^*$, such that

$$\begin{aligned}\zeta = \phi_0 \mathbb{I}_{\{N(j+r+b+|\tau|+2)-N(j+r-|\tau|-1)=0\}} + \sum_{\kappa\in\mathbb{N}^*} \phi_\kappa\Big(X(T_{N(j+r-|\tau|-1)+1}),\\ \dots, X(T_{N(j+r-|\tau|-1)+\kappa}); T_{N(j+r-|\tau|-1)+1}, \dots, T_{N(j+r-|\tau|-1)+\kappa}\Big)\\ \times \mathbb{I}_{\{N(j+r+b+|\tau|+2)-N(j+r-|\tau|-1)=\kappa\}}.\end{aligned} \tag{26}$$

For the sake of simplicity we denote

$$\Psi_k\big(t_1, \dots, t_k\big) := \psi_k\Big(X(t_1), \dots, X(t_k); t_1, \dots, t_k\Big)$$

and

$$\Phi_\kappa\big(u_1, \dots, u_\kappa\big) := \phi_\kappa\Big(X(u_1), \dots, X(u_\kappa); u_1, \dots, u_\kappa\Big)\Big),$$

where $0 \le t_1 \le t_2 \le \dots \le t_k$, and $0 \le u_1 \le u_2 \le \dots \le u_\kappa$.

As a consequence, since $|\tau| + 2 \le r + b$, we deduce that

$$\begin{aligned}\mathrm{E}\left[\xi \,\middle|\, \mathcal{F}(N)\right] = \mathrm{E}\Big[\xi \,\Big|\, \mathcal{F}_0^{j+r+b}(N)\Big] = \psi_0 \mathbb{I}_{\{N(j+|\tau|+2)=0\}}\\ + \sum_{k\in\mathbb{N}^*} \mathrm{E}\Big[\Psi_k\big(T_1, \dots, T_k\big) \,\Big|\, \mathbb{I}_{\{N(j+|\tau|+2)=k\}}, T_1, \dots, T_k\Big]\, \mathbb{I}_{\{N(j+|\tau|+2)=k\}}\end{aligned}$$

and

$$\begin{aligned}\mathrm{E}\left[\zeta \,\middle|\, \mathcal{F}(N)\right] = \mathrm{E}\Big[\zeta \,\Big|\, \mathcal{F}_0^{j+r+b+|\tau|+2}(N)\Big] = \mathrm{E}\Big[\zeta \,\Big|\, \mathcal{F}_{j+r-|\tau|-1}^{j+r+b+|\tau|+2}(N)\Big] =\\ = \phi_0 \mathbb{I}_{A_0} + \sum_{\kappa\in\mathbb{N}^*} \mathrm{E}\Big[\Phi_\kappa\big(T_{N(j+r-|\tau|-1)+1}, \dots, T_{N(j+r-|\tau|-1)+\kappa}\big) \,\Big|\\ \mathbb{I}_{A_\kappa} T_{N(j+r-|\tau|-1)+1}, \dots, T_{N(j+r-|\tau|-1)+\kappa}\Big]\, \mathbb{I}_{A_\kappa},\end{aligned}$$

where

$$A_\kappa := \{N(j + r + b + |\tau| + 2) - N(j + r - |\tau| - 1) = \kappa\}.$$

Moreover,

$$\begin{aligned}\mathcal{F}(N) &:= \mathcal{F}\{N(t) : t > 0\} = \mathcal{F}\{T_1, T_2, \dots\},\\ \mathcal{F}_0^j(N) &:= \mathcal{F}\{N(t) : 0 < t \le j\}\\ &= \mathcal{F}\{\mathbb{I}_{\{N(j)=0\}}\} \vee \mathcal{F}\big\{\mathbb{I}_{\{N(j)>0\}}, N(j), T_p : i = 1, \dots, N(j)\big\}\end{aligned}$$

and

$$
\begin{aligned}
&\mathcal{F}_{j+r}^{j+r+b}(N)\\
&\quad := \mathcal{F}\{N(t) : j+r < t \leq j+r+b\}\\
&\quad = \mathcal{F}\{\mathbb{I}_{A_0}\} \vee \mathcal{F}\Big\{ \prod_{\kappa \in \mathbb{N}^*} \mathbb{I}_{A_\kappa}, N(j+r), T_q : q = N(j+r)+1, \ldots, N(j+r+b)\Big\}.
\end{aligned}
$$

Additionally,

$$
\operatorname{cov}\{\xi, \zeta\} = \operatorname{cov}\left\{\mathrm{E}\left[\xi \,\middle|\, \mathcal{F}(N)\right], \mathrm{E}\left[\zeta \,\middle|\, \mathcal{F}(N)\right]\right\} + \mathrm{E}\left\{\operatorname{cov}\left[\xi, \zeta \,\middle|\, \mathcal{F}(N)\right]\right\}.
$$

Using properties of Poisson process and the independence between $X(t)$ and $N(t)$ one can show that the random variables $\mathrm{E}\left[\xi \,\middle|\, \mathcal{F}(N)\right]$ and $\mathrm{E}\left[\zeta \,\middle|\, \mathcal{F}(N)\right]$ are not correlated, i.e.

$$
\operatorname{cov}\left\{\mathrm{E}\left[\xi \,\middle|\, \mathcal{F}(N)\right], \mathrm{E}\left[\zeta \,\middle|\, \mathcal{F}(N)\right]\right\} = 0. \tag{27}
$$

Finally, the covariance inequality for bounded random variables (see [20]), the mixing hypothesis on the process $\{X(t), t \geq 0\}$ and the independence between $\{X(t), t \geq 0\}$ and the Poisson point process $\{N(t), t \geq 0\}$ mean that

$$
\left|\mathrm{E}\left\{\operatorname{cov}\left[\xi, \zeta \,\middle|\, \mathcal{F}(N)\right]\right\}\right| \leq 4\alpha_X(l - 2h - 2|\tau| - 1)\|\xi\|_\infty \|\zeta\|_\infty. \tag{28}
$$

Indeed

$$
\begin{aligned}
&\mathrm{E}\left\{\operatorname{cov}\left[\xi, \zeta \,\middle|\, \mathcal{F}(N)\right]\right\}\\
&\quad = \sum_{k \in \mathbb{N}} \sum_{\kappa_0 \in \mathbb{N}} \sum_{\kappa \in \mathbb{N}} \int \cdots \int_{\mathbb{R}_+^k \times \mathbb{R}_+^\kappa} \operatorname{cov}\{\Psi_k(t_.), \Phi_\kappa(j+\tau+u_.)\}\\
&\qquad \times d\mathrm{P}_{(T_1,\ldots,T_k,U_1,\ldots U_\kappa)} \Big|_{B(k,\kappa_0,\kappa)} [t_1, \ldots, t_k, u_1, \ldots, u_\kappa]\\
&\qquad \times \mathrm{P}\{N(j+2|\tau|+3) = k, N(r-2|\tau|-3) = \kappa_0, N(b+2|\tau|+3) = \kappa\}.
\end{aligned} \tag{29}
$$

Since the random variables ξ and ζ are bounded, the functions ψ_k and ϕ_κ can be chosen such that $|\Psi_k(t_.)| = |\psi_k(\xi, t_.)| \leq \|\xi\|_\infty$ and $|\Phi_\kappa(j+r+u_.)| \leq \|\zeta\|_\infty$. Then from the definition of the functions ψ_k and ϕ_κ, the mixing property of the process $\{X(t) : t \geq 0\}$ we obtain that

$$
\begin{aligned}
&|\operatorname{cov}\{\Psi_k(t_.), \Phi_\kappa(j+\tau+u_.)\}| \leq 4\alpha_X(l-2h-2|\tau|-1)\|\Psi_k(t_.)\|_\infty \|\Phi_\kappa(j+r+u_.)\|_\infty\\
&\quad \leq 4\alpha_X(l-2|\tau|-3)\|\xi\|_\infty\|\zeta\|_\infty.
\end{aligned}
$$

Thus, using relation (29) one obtains inequality (28). Additionally, taking into account equality (27), we deduce inequality (24). □

References

[1] Antoni, J. (2009) Cyclostationarity by examples. *Mech. Syst. Sig. Process.*, **23**(4), 987–1036.

[2] Araujo, A. and Giné, E. (1980) *The Central Limit Theorem for Real and Banach Valued Random Variables.* Wiley, New York . MR0576407

[3] Besicovitch, A.S. (1932) *Almost Periodic Functions.* University Press, Cambridge.

[4] Bolthausen, E. (1982) On the central limit theorem for stationary random fields. *Ann. Probab.*, **10(4)**, 1047–1050. MR0672305

[5] Brillinger, D.R. (1972) The spectral analysis of stationary interval functions. *Proc. Sixth Berkeley Symp. Probability and Statistics*, **1**, 483–513. MR0407972

[6] Dedecker, J., Doukhan, P., Lang, G., Leon, J.R., Louhichi, S. and Prieur, C. (2007) *Weak Dependence: With Examples and Applications.* Lecture Notes in Statistics 190, Springer, New York. MR2338725

[7] Dehay, D. and Dudek, A.E. (2015) Bootstrap method for Poisson sampling almost periodic process. *J. Time Ser. Anal.*, **36**(3), 327–351. MR3343005

[8] Dehay, D., Dudek, A. and Leśkow, J. (2014) Subsampling for continuous-time nonstationary stochastic processes. *J. Stat. Plan. Inf.*, **150**, 142–158.

[9] Dehay, D. and Monsan, V. (1996) Random sampling estimation for almost periodically correlated processes. *J. Time Ser. Anal.*, **17**(5), 425–445.

[10] Doukhan, P. (1994) *Mixing: Properties and Examples.* Lecture Notes in Statistics 85, Springer, New York.

[11] Doukhan, P., Portal, F. (1983) Moments de variables aléatoires mélangeantes. *C. R. Acad. Sc. Paris*, Série I, **297**, 129–132.

[12] Dudek, A.E. (2015) Circular block bootstrap for coefficients of autocovariance function of almost periodically correlated time series. *Metrika*, **78(3)**, 313–335.

[13] Dudek, A.E., Maiz, S. and El Badaoui, M. (2014b) Generalized Seasonal Block Bootstrap in frequency analysis of cyclostationary signals. *Signal Process.*, **104C**, 358–368.

[14] Gardner, W.A., Napolitano, A. and Paura, L. (2006) Cyclostationarity: Half a century of research. *Signal Process.*, **86**(4), 639–697.

[15] Gillespie D.T. (1996) Exact numerical simulation of the Ornstein-Uhlenbeck process and its integral. *Physical Review E*, **54**(2), 2084–2091.

[16] Gladyshev, E. (1963) Periodically and almost periodically correlated random processes with continuous time parameter. *Th. Probability Appl.* **8**, 173–177.

[17] Guyon, X. (1995) *Random Fields on a Network.* New York: Springer-Verlag. MR1344683

[18] Hurd, H.L. (1991) Correlation theory for the almost periodically correlated processes with continuous time parameter. *J. Multivariate Analysis*, **37**(1), 24–45. MR1097303

[19] Hurd, H.L. and Miamee, A.G. (2007) *Periodically Correlated Random Sequences: Spectral. Theory and Practice*, John Wiley.

[20] Ibragimov, I.A. (1962) Some limit theorems for stationary processes. *Theor. Probability Appl.*, **7**, 349–381.

[21] Ibragimov, I.A. and Linnik, Y.V. (1971) *Independent and Stationary Sequences of Random Variables*. Wolters-Noordhoff, Groninger.

[22] Karr, A.E. (1986) Inference for stationary random fileds given Poisson samples. *Adv. Appl. Probab.*, **18**, 406–422.

[23] Karr, A.E. (1991) *Point Processes and their Statistical Inference, 2nd ed.*, Marcel Dekker, New York.

[24] Kim, T. (1994) Moment bounds for non-stationary dependent sequences. *J. Appl. Probab.*, **31**(3), 731–742. MR1285511

[25] Lenart, Ł., Leśkow, J. and Synowiecki, R. (2008) Subsampling in testing autocovariance for periodically correlated time series. *J. Time Ser. Anal.*, **29**, 995–1018.

[26] Masry, E. (1978) Alias-free sampling: an alternative conceptualization and its applications. ieee *Trans. Inform. Theory.* **IT-24**, 317–324.

[27] Masry, E. (1983) Non-parametric covariance estimation for irregularly-spaced data. *Adv. Appl. Prob.*, **15**, 113–132.

[28] Napolitano, A. (2012) *Generalizations of Cyclostationary Signal Processing: Spectral Analysis and Applications*. Wiley-IEEE Press.

[29] Napolitano, A. (2016) Cyclostationarity: New trends and applications. *Signal Process.*. **120**, 385–408.

[30] Parzen, B. (1983) *Design of Crystal and Other Harmonic Oscillators*, Wiley, New York

[31] Synowiecki, R. (2007) Consistency and application of moving block bootstrap for nonstationary time series with periodic and almost periodic structure. *Bernoulli,* **13**(4), 1151–1178. MR2364230

[32] Volkonski, V.A. and Rozanov, Y.A. (1959). Some limit theorems for random functions. *I. Teor. Veroyatn. Primen.*, **4**, 186–207.