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Φ admissibility of stochastic regression coefficients in a general multivariate random effects model under a generalized balanced loss function

Ming-Xiang Cao*

Department of Mathematics
Hefei Normal University, Hefei 230601, P.R. China
and
School of Mathematics
Beijing Institute of Technology, Beijing 100081, P.R. China
e-mail: caomingx@163.com

Abstract: The definitions of Φ optimality and Φ admissibility of stochastic regression coefficients are given in a general multivariate random effects model under the generalized balanced loss function. Φ admissibility of linear estimators of stochastic regression coefficients is investigated. Sufficient and necessary conditions for linear estimators to be Φ admissible in classes of homogeneous and nonhomogeneous linear estimators are obtained, respectively.

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1. Introduction

Throughout this paper the general multivariate random effects model of the form

$$Y = XB + e \tag{1.1}$$

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will be considered. The Y is an $n \times q$ matrix of observations, X is an $n \times p$ known matrix of rank p, B is a $p \times q$ matrix of stochastic regression coefficients and e is an $n \times q$ error matrix, where $E\left(\begin{smallmatrix} B \\ e \end{smallmatrix}\right) = \left(\begin{smallmatrix} K\Theta \\ 0 \end{smallmatrix}\right)$ and $Cov\left(vec\left(\begin{smallmatrix} B \\ e \end{smallmatrix}\right)\right) = \sigma^2\Sigma \otimes V$. $\Sigma_{(n+p)\times(n+p)} \geq 0$, $V_{q\times q} \geq 0$ ($V \not\equiv 0$) and $K_{p\times r}$ are known matrices. In order to avoid identifiability problems, it assumes that the rank of K is r. Nevertheless $\Theta \in R^{r\times q}$ and $\sigma^2 > 0$ are unknown parameters. Furthermore, we rewrite Σ as a block matrix $\left(\begin{smallmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{smallmatrix}\right)$, where Σ_{11} and Σ_{22} are p and n nonnegative square matrices, respectively. Hence model (1.1) contains that B is non-stochastic or that some row vectors of B are stochastic and others are non-stochastic. Therefore, model (1.1) has extensive backgrounds and applications. It assumes here that Σ_{12} is zero matrix, which means that stochastic regression coefficient B is uncorrelated with random error e.

According to Zellner's idea of balanced loss ([29]) and the unified theory of least squares ([21]), the generalized balanced loss function ([2]) is proposed by

$$L(D,B) = w(Y - XD)'T^{-}(Y - XD) + (1 - w)(D - B)'S(D - B), \tag{1.2}$$

where w is a known scalar lying between 0 and 1. The choice of w reflects the relative weight which the experimenter wants to assign to goodness of fit of model and precision of estimation. The choice of w essentially depends on the experimenter and objective of the experiment. The experimenter may decide the value of w to be assigned on the basis of his experience with similar type of studies in the past, long association with the experiment or some prior information about the experiment. The extreme cases $w = 0$ and $w = 1$ refer solely to precision of an estimate and goodness of fit, respectively. $S > 0$ is a known matrix and $T = \Sigma_{22} + XUX'$ with $U \geq 0$ such that $rk(T) = rk(X : V)$ where T^{-} and $rk(T)$ denote $g-$inverse and rank of T, respectively. $D = D(Y)$ is an estimator of B. We often select $U = I$ when Σ_{22} is singular. For simplicity of exposition, we will consider this case in this paper. In the other cases of U, we can still get the similar results via the same method as one of the article.

Balanced loss function takes error of estimation and goodness of fit into account. Hence, compared to classical loss functions, it is a more comprehensive and reasonable standard measuring the estimates. It has been received much attention in the literature under different setups. For more details, the readers are referred to [1, 5, 6, 8, 9, 11, 12, 18, 19, 20], among many others. The risk function of estimator $D(Y)$ of stochastic regression coefficients B is defined by

$$R(D; \Theta, \sigma^2) := EL(D, B).$$

The optimality of an estimator, such as domination, admissibility and so forth, is evaluated by its risk in the range spaces of the risk function. When $q = 1$ (i.e. the univariate linear model), the risk function in model (1.1) is a scalar, so comparison of the risk size is carried out naturally according to the scalar size. In this case, when $\Sigma_{11} = 0$ and $\Sigma_{22} > 0$ (i.e. the nonsingular fixed effects linear model), [28] considered the admissibility of linear estimators of regression coefficients. [10] developed above results under $\Sigma_{11} = 0$ and $\Sigma_{22} \geq 0$ (i.e. the singular fixed effects linear model). [3] developed further the results with respect to $\Sigma_{11} \geq 0$ and $\Sigma_{22} \geq 0$ (i.e. singular random effects linear model). When $q \geq 2$, the risk function is a nonnegative definite matrix. There are many different comparison standards to the risk size. For $\Sigma_{11} = 0$ and $\Sigma_{22} = I$ (i.e. the nonsingular multivariate linear model with fixed effects), [4] discussed admissibility of linear estimators of regression coefficients under the matrix balanced loss function with respect to six different comparison standards.

For more comparison standards under other loss functions, the readers are referred to [14, 15, 16, 23, 24] and references cited therein. [25] gave a unified comparison criterion, namely Φ optimality (or general optimality). The definition of Φ optimality function is presented as follows.

Definition 1.1. Let $\mathcal{W}$ be the set $\{W : W_{q\times q} \geq 0\}$, a function $\Phi : \mathcal{W} \to [0, +\infty)$, satisfies the following conditions:

(1) $\Phi(W) = 0$ if and only if $W = 0$;

(2) If $W_1 \leq W_2$, then $\Phi(W_1) \leq \Phi(W_2)$;

(3) $\Phi(cW) = c^{\varphi(W)}\Phi(W)$, where c is a nonnegative scalar and $\varphi(W)$ is a positive real function defined on $\mathcal{W}$;

(4) $\Phi(W)$ is continuous on $\frac{q(q+1)}{2}$ variables.

Remark 1.1. It is easy to verify that Φ function includes $\Phi_1(M) = tr(M)$ and $\Phi_\infty(M) = \lambda_1(M)$. The significance of subscripts can be seen in [13].

Then we give the following definition of Φ admissibility under the generalized balanced loss function (1.2).

Definition 1.2. Let $D_1(Y)$ and $D_2(Y)$ be two estimators of B, then $D_1(Y)$ is said to be Φ better than $D_2(Y)$ if

$$\Phi[R(D_1(Y); \Theta, \sigma^2)] \leq \Phi[R(D_2(Y); \Theta, \sigma^2)]$$

holds for every $(\Theta, \sigma^2) \in R^{r\times q} \times R^+$ with strict inequality holding for at least some point. If there does not exist estimators which are Φ better than $D_1(Y)$ in a class of some estimators $\mathcal{L}$, then $D_1(Y)$ will be said to be Φ admissible in $\mathcal{L}$, which is denoted by $D_1(Y) \overset{\mathcal{L}}{\underset{\Phi}{\sim}} B$.

Similar to the definition of optimum designs in [13], the following definition is given.

Definition 1.3. $D_1(Y)$ is said to be G admissible in $\mathcal{L}$, which is denoted by $D_1(Y) \overset{\mathcal{L}}{\underset{G}{\sim}} B$, if and only if there does not exist $D_2(Y) \in \mathcal{L}$ such that

$$R(D_2(Y); \Theta, \sigma^2) \leq R(D_1(Y); \Theta, \sigma^2)$$

for every $(\Theta, \sigma^2) \in R^{r\times q} \times R^+$ with $R(D_2(Y); \Theta, \sigma^2) - R(D_1(Y); \Theta, \sigma^2) \neq 0$ for at least some point.

Since the definition of Φ admissibility was given, there has been a lot of work about that. For more details, see [2, 25, 26, 27], among many others. In the present paper, a study of the problem of Φ admissibility for linear estimators on stochastic regression coefficients matrix B in model (1.1) under the generalized balanced loss function (1.2) is given in a general situation, and sufficient and necessary conditions that linear estimators are Φ admissible in $\mathcal{LH}$ and $\mathcal{LI}$ are obtained respectively, where $\mathcal{LH} = \{LY : L \in R^{p\times n}\}$ and $\mathcal{LI} = \{LY + \Gamma : L \in R^{p\times n}$ and $\Gamma \in R^{p\times q}\}$, which is an extension of [2, 3, 10, 28].

The remainder of the paper is organized as follows. In Section 2, we give our main results, some corollaries and remarks. Section 3 contains some lemmas playing important roles in the paper. Section 4 shows proofs of main results. Some examples are included in Section 5. Concluding remarks are given in Section 6.

2. Main results

Theorem 2.1. *Under model (1.1) and loss function (1.2), $LY \overset{\mathcal{LH}}{\underset{\Phi}{\sim}} B$ if and only if*
(a) $L\Sigma_{22} = LX(X'T^-X)^{-1}X'T^-\Sigma_{22}$,
(b) $\{L^*X-(1-w)P^{-1}S\Sigma_{11}[KK'+(X'T^-X)^{-1}+\Sigma_{11}-I]^+\}(I-N)[(X'T^-X)^{-1}+\Sigma_{11}-I] = 0$,
(c) $L^*XN[(X'T^-X)^{-1} + \Sigma_{11} - I]N'X'L^{*\prime} \leq (1-w)\{L^*XN[(X'T^-X)^{-1} + \Sigma_{11} - I]N'SP^{-1} + P^{-1}S\Sigma_{11}N'(LX-I)'\}$
and
(d) $\mu[(LX-I)K] = \mu[(LX-I)N(X'T^-X-I)]$
hold simultaneously, where $N = K\{K'[(X'T^-X)^{-1} + KK' + \Sigma_{11} - I]^+K\}^-K'[(X'T^-X)^{-1} + KK' + \Sigma_{11} - I]^+$, $P = wX'T^-X + (1-w)S$ *and* $L^* = L - wP^{-1}X'T^-$. $\mu[(LX-I)K]$ *denotes the range space of matrix* $(LX-I)K$.

Remark 2.1. From Theorem 2.1, we know that the result is independent of V of model (1.1). Hence, we can assume V is either known or unknown matrix.

Remark 2.2. It is easy to verify that $\widehat{B} = (X'T^-X)^{-1}X'T^-Y$ satisfies the conditions of Theorem 2.1. This shows that the best linear unbiased estimator (or predictor) $\widehat{B}$ of B is admissible in $\mathcal{LH}$.

Let (e) be $rk[(LX-I)K] = rk[(LX-I)N(X'T^-X-I)]$, then we can get the equivalent expression of Theorem 2.1, which is presented as follows.

Theorem 2.2. *Under model (1.1) and loss function (1.2), $LY \overset{\mathcal{LH}}{\underset{\Phi}{\sim}} B$ if and only if (a), (b), (c) and (e) hold simultaneously.*

If $K = I$, the simple result of Theorem 2.1 can be obtained.

Corollary 2.1. *Under model (1.1) with $K = I$ and loss function (1.2), $LY \overset{\mathcal{LH}}{\underset{\Phi}{\sim}} B$ if and only if (a), (d′)* $rk(LX-I) = rk[(LX-I)(X'T^-X-I)]$ *and*
(c′) $L^*X[(X'T^-X)^{-1} + \Sigma_{11} - I]X'L^{*\prime} \leq (1-w)\{L^*X[(X'T^-X)^{-1} + \Sigma_{11} - I]SP^{-1} + P^{-1}S\Sigma_{11}(LX-I)'\}$ *hold simultaneously.*

When $\Sigma_{11} = 0$, (1.1) is a general multivariate linear model with fixed effects. For $K = I$, the following result can be gotten, which is the main result of [2].

Corollary 2.2. *Under model (1.1) with above assumption and loss function (1.2), $LY \overset{\mathcal{LH}}{\underset{\Phi}{\sim}} B$ if and only if (a), (d′) and*
(c″) $L^*X[(X'T^-X)^{-1} - I]X'L^{*\prime} \le (1-w)L^*X[(X'T^-X)^{-1} - I]SP^{-1}$ hold simultaneously.

When $\Sigma_{22} > 0$ and $\Sigma_{11} \not\equiv 0$, (1.1) is called nonsingular multivariate random effects model. This model is more produced in practical problems than singular multivariate random effects model (1.1) appearing in the theoretical studies. For the nonsingular model, the generalized balanced loss function (1.2) turns into the following

$$L_0(D,B) = w(Y - XD)'\Sigma_{22}^{-1}(Y - XD) + (1-w)(D-B)'S(D-B). \quad (2.1)$$

The corresponding result can be obtained by matrix calculations according to Theorem 2.1, which is given as follows.

Corollary 2.3. *Under the nonsingular model and loss function (2.1), $LY \overset{\mathcal{LH}}{\underset{\Phi}{\sim}} B$ if and only if*
(a‴) $L\Sigma_{22} = LX(X'\Sigma_{22}^{-1}X)^{-1}X'$,
(b‴) $\{L^*X - (1-w)P^{-1}S\Sigma_{11}[KK' + (X'\Sigma_{22}^{-1}X)^{-1} + \Sigma_{11} - I]^+\}(I-N)[\Sigma_{11} - I + (X'\Sigma_{22}^{-1}X)^{-1}] = 0$,
(c‴) $L^*XN[(X'\Sigma_{22}^{-1}X)^{-1} + \Sigma_{11} - I]N'X'L^{*\prime} \le (1-w)\{L^*XN[(X'\Sigma_{22}^{-1}X)^{-1} + \Sigma_{11} - I]N'SP^{-1} + P^{-1}S\Sigma_{11}N'(LX - I)'\}$
and
(d‴) $\mu[(LX-I)K] = \mu[(LX-I)N(X'\Sigma_{22}^{-1}X - I)]$,
hold simultaneously, where $N = K\{K'[KK' + (X'\Sigma_{22}^{-1}X)^{-1} + \Sigma_{11} - I]^+K\}^-K'[KK' + (X'\Sigma_{22}^{-1}X)^{-1} + \Sigma_{11} - I]^+$.

From Lemma 3.5, it gets the sufficient and necessary conditions for a linear estimator of B being G admissible.

Corollary 2.4. *Under model (1.1) and loss function (1.2), $LY \overset{\mathcal{LH}}{\underset{G}{\sim}} B$ if and only if (a), (b), (c) and (d) in Theorem 2.1 hold simultaneously.*

The following corollary depicts the relationships between Φ admissibility and G admissibility.

Corollary 2.5. *$LY \overset{\mathcal{LH}}{\underset{\Phi}{\sim}} B$ is equivalent to $LY \overset{\mathcal{LH}}{\underset{G}{\sim}} B$ from Theorem 2.1 and Corollary 2.4.*

When $q = 1$, model (1.1) is a general Gauss-Markov random effects model, where the risk is a scalar. Hence, Φ admissibility is a classical admissibility when Φ mapping is an identity mapping. The further result is given as follows according to Theorem 2.1, which is the main result of [3].

Corollary 2.6. *Under model (1.1) and loss function (1.2) where $q = 1$, $LY \overset{\mathcal{LH}}{\underset{\Phi}{\sim}} B$ if and only if (a), (b), (c) and (d) in Theorem 2.1 hold simultaneously.*

Corollary 2.7. *Under the conditions of Corollary 2.6, if it further assumes that* $K = I, \Sigma_{11} = 0, \Sigma_{22} \geq 0$ *and* $K = I, \Sigma_{11} = 0, \Sigma_{22} > 0$ *respectively, it can get the main results of [10] and [28] from Theorem 2.1, which we omit here.*

Theorem 2.3. *Under model (1.1) and loss function (1.2),* $LY + \Gamma \underset{\Phi}{\overset{\mathcal{LI}}{\sim}} B$ *if and only if* $LY \underset{\Phi}{\overset{\mathcal{LH}}{\sim}} B$ *and* $\Gamma \in \mu[(LX - I)K]$.

Remark 2.3. It is easy to get all results of this paper are independent of the selection of T^-. Hence, we often select Moore-Penrose inverse T^+.

Remark 2.4. In the loss function (1.2), we can select appropriate S according to practical problems or theoretical studies. On the other hand, in order to avoid uncertainty of results because of arbitrary S, we can select a fixed S, such as $S = X'T^-X$ or $S = I$.

3. Some important preliminaries

In order to prove our main results, some lemmas are given firstly.

Lemma 3.1. *([19]) Consider the multivariate random effects linear model*

$$Z = H\Xi + \epsilon, \quad E\begin{pmatrix} \Xi \\ \epsilon \end{pmatrix} = \begin{pmatrix} Q\theta \\ 0 \end{pmatrix}, \quad Cov\left(vec\begin{pmatrix} \Xi \\ \epsilon \end{pmatrix}\right) = \sigma^2 \Lambda \otimes \Delta,$$

where $H \in R^{n\times p}$, $Q \in R^{n\times m}$, $\Lambda = \left(\begin{smallmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{smallmatrix}\right) \geq 0$ *and* $\Delta_{q\times q} \geq 0$ $(\Delta \not\equiv 0)$ *are known matrices, while* $\theta \in R^{m\times q}$ *and* $\sigma^2 > 0$ *are unknown parameters. Write* $\mathcal{L}^* = \{AZ : A \in R^{k\times n}\}$. *If* $J\Xi$ *is estimable, then* $AZ \underset{G}{\overset{\mathcal{L}^*}{\sim}} J\Xi$ *under the quadratic loss function* $L(AZ, J\Xi) = (AZ - J\Xi)'(AZ - J\Xi)$ *if and only if*
(i) $A\Omega = AHQ(Q'H'G^+HQ)^-Q'H'G^+\Omega + J\Lambda_{11}H'G^+[I - HQ(Q'H'G^+HQ)^- Q'H'G^+]\Omega$,
(ii) $AHQCQ'H'A' \leq (AH-J)Q(Q'H'G^+HQ)^-Q'H'G^+H\Lambda_{11}J' + JQCQ'H'A'$,
(iii) $\mu[(AH - J)Q] = \mu(W)$,
hold simultaneously, where $C = (Q'H'G^+HQ)^-Q'H'G^+\Omega G^+HQ(Q'H'G^+HQ)^-$, $\Omega = H\Lambda_{11}H' + \Lambda_{22}$, $G = \Omega + HQQ'H'$ *and* $W = (AH-J)Q[(Q'H'G^+HQ)^-Q'H' G^+H\Lambda_{11}J' - CQ'H'A'] + (AH - J)QCQ'(AH - J)'$.

Lemma 3.2. *Let* $\mathcal{L}_0 = \{DX'T^-Y : D \in R^{p\times p}\}$, *then* $\mathcal{L}_0$ *is a complete class of* $\mathcal{LH}$.

Proof. It easily gets $LP_XY \in \mathcal{L}_0$ from $LY \in \mathcal{LH}$, where $P_X = X(X'T^-X)^{-1}X'T^-$.
Note that

$$\begin{aligned} Cov\left(vec\begin{pmatrix} B \\ Y \end{pmatrix}\right) &= Cov\left(\begin{pmatrix} I & 0 \\ X & I \end{pmatrix}\begin{pmatrix} B \\ e \end{pmatrix}\right) \\ &= \sigma^2 \begin{pmatrix} \Sigma_{11} & \Sigma_{11}X' \\ X\Sigma_{11} & \Sigma_{22} + X\Sigma_{11}X' \end{pmatrix} \otimes V. \end{aligned}$$

Thus, there has

$$R(LY;\Theta,\sigma^2)=\sigma^2\{tr[(LX-I)\Sigma_{11}(LX-I)'P+L\Sigma_{22}L'P-2w\cdot L\Sigma_{22}T^-X] \\ +w\cdot tr(\Sigma_{22}T^-)\}V+\Theta'K'(LX-I)'P(LX-I)K\Theta. \quad (3.1)$$

Therefore, we get

$$\begin{aligned} R(LY;\Theta,\sigma^2)-R(LP_XY;\Theta,\sigma^2) &= \sigma^2\cdot tr[(L\Sigma_{22}L'-LP_X\Sigma_{22}P_X'L')P]V \\ &= \sigma^2\cdot tr[L(I-P_X)\Sigma_{22}(I-P_X)'L'P]V \\ &\geq 0 \end{aligned}$$

for all $(\Theta,\sigma^2)\in R^{r\times q}\times R^+$, from which and (2) of Definition 1.1, it easily gets $\Phi[R(LP_XY;\ \Theta,\sigma^2)]\leq\Phi[R(LY;\Theta,\sigma^2)]$ for all $(\Theta,\sigma^2)\in R^{r\times q}\times R^+$. By (3) of Definition 1.1 again, we easily get the equality holds if and only if $L\Sigma_{22}=LP_X\Sigma_{22}$. This completes the proof of Lemma 3.2. □

The following lemmas play crucial roles in proving main results.

Lemma 3.3. *For the multivariate random effects linear model*

$$Z=(X'T^-X)B+\xi,$$

where $Cov\left(vec\begin{pmatrix}B\\ \xi\end{pmatrix}\right)=\sigma^2\begin{pmatrix}\Sigma_{11} & 0\\ 0 & X'T^-\Sigma_{22}T^-X\end{pmatrix}\otimes V$ *and* $E\begin{pmatrix}B\\ \xi\end{pmatrix}=\begin{pmatrix}K\Theta\\ 0\end{pmatrix}$*. Let* $C_1=(1-w)P^{-1}S$, $\mathcal{L}_1=\{DZ: D\in R^{p\times p}\}$, *the loss function* $L_P(d,C_1B)=(d-C_1B)'P(d-C_1B)$ *and* $D^*=D-wP^{-1}$*. In the above model,* C_1B *is estimable. Then under the loss function* $L_P(d,C_1B)$, $D^*Z\overset{\mathcal{L}_1}{\underset{G}{\sim}}C_1B$ *holds if and only if* $DX'T^-Y\overset{\mathcal{L}_0}{\underset{G}{\sim}}B$ *holds under model (1.1) and the generalized balanced loss function (1.2).*

Proof. From (3.1), it is easy to get

$$R(DX'T^-Y;\Theta,\sigma^2)=\sigma^2\{tr[(DX'T^-X-I)\Sigma_{11}(DX'T^-X-I)'P+DX'T^- \\ \cdot\Sigma_{22}T^-XD'P-2w\cdot DX'T^-\Sigma_{22}T^-X]+w\cdot tr(\Sigma_{22}T^-)\} \\ \cdot V+\Theta'K'(DX'T^-X-I)'P(DX'T^-X-I)K\Theta. \quad (3.2)$$

Furthermore, it gets

$$EL_P(D^*Z,C_1B)=\sigma^2\{tr[(DX'T^-X-I)\Sigma_{11}(DX'T^-X-I)'P+DX'T^-\Sigma_{22} \\ \cdot T^-XD'P-2w\cdot DX'T^-\Sigma_{22}T^-X]+w^2\cdot tr(X'T^-\Sigma_{22}T^- \\ \cdot XP^{-1})\}V+\Theta'K'(DX'T^-X-I)'P(DX'T^-X-I)K\Theta. \quad (3.3)$$

If $DX'T^-Y\overset{\mathcal{L}_0}{\underset{G}{\sim}}B$, then for arbitrary $MX'T^-Y\in\mathcal{L}_0$, there has

$$R(DX'T^-Y;\Theta,\sigma^2)\leq R(MX'T^-Y;\Theta,\sigma^2) \quad (3.4)$$

for every $(\Theta, \sigma^2) \in R^{r\times q} \times R^+$, and there exists one point (Θ_0, σ_0^2) such that

$$R(MX'T^-Y; \Theta_0, \sigma_0^2) - R(DX'T^-Y; \Theta_0, \sigma_0^2) \neq 0. \tag{3.5}$$

By (3.2) and (3.3), (3.4) and (3.5) hold if and only if

$$EL_P(D^*Z, C_1B) \leq EL_P(M^*Z, C_1B) \tag{3.6}$$

holds for every $(\Theta, \sigma^2) \in R^{r\times q} \times R^+$ with

$$EL_P(M^*Z, C_1B) - EL_P(D^*Z, C_1B) \neq 0 \tag{3.7}$$

at (Θ_0, σ_0^2), where $M^* = M - wP^{-1}$.

Thus (3.6) and (3.7) are equivalent to $D^*Z \underset{G}{\overset{\mathcal{L}_1}{\sim}} C_1B$. This completes the proof of Lemma 3.3. □

It is easy to get the following lemma from [7].

Lemma 3.4. *Under assumption of Lemma 3.3, $D^*Z \underset{G}{\overset{\mathcal{L}_1}{\sim}} C_1B$ holds with respect to loss function $L_P(D^*Z, C_1B)$ if and only if $D^*Z \underset{G}{\overset{\mathcal{L}_1}{\sim}} C_1B$ with respect to loss function $L_I(D^*Z, C_1B)$.*

Lemma 3.5. *$DX'T^-Y$ is Φ better than $MX'T^-Y$ if and only if $DX'T^-Y$ is G better than $MX'T^-Y$.*

Proof. First, it is easy to prove that $DX'T^-Y$ is G better than $MX'T^-Y$ if and only if

$$\begin{aligned} &tr[(DX'T^-X - I)\Sigma_{11}(DX'T^-X - I)'P + DX'T^-\Sigma_{22}T^-XD'P \\ &\quad - 2w \cdot DX'T^-\Sigma_{22}T^-X] \\ &\leq tr[(MX'T^-X - I)\Sigma_{11}(MX'T^-X - I)'P + MX'T^-\Sigma_{22}T^-XM'P \\ &\quad - 2w \cdot MX'T^-\Sigma_{22}T^-X], \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &K'(DX'T^-X - I)'P(DX'T^-X - I)K \\ &\leq K'(MX'T^-X - I)'P(MX'T^-X - I)K, \end{aligned} \tag{3.9}$$

moreover, when the equality holds in (3.8), there has

$$\begin{aligned} &K'(MX'T^-X - I)'P(MX'T^-X - I)K \\ &\quad \neq K'(DX'T^-X - I)'P(DX'T^-X - I)K. \end{aligned} \tag{3.10}$$

Therefore, we only need to prove that $DX'T^-Y$ is Φ better than $MX'T^-Y$ if and only if (3.8), (3.9) and (3.10) with the equality holding in (3.8) hold simultaneously.

On the other hand, $DX'T^-Y$ is Φ better than $MX'T^-Y$ if and only if

$$\Phi[R(DX'T^-Y;\Theta,\sigma^2)] \le \Phi[R(MX'T^-Y;\Theta,\sigma^2)] \tag{3.11}$$

holds for every $(\Theta,\sigma^2) \in R^{r\times q}\times R^+$ with inequality strictly holding for at least some point, hence the following work is to prove that (3.11) holds for every (Θ,σ^2) with strict inequality holding for at least some point if and only if (3.8), (3.9) and (3.10) with the equality holding in (3.8) hold simultaneously.

Sufficiency. From (3.8) and (3.9), it easily gets

$$R(DX'T^-Y;\Theta,\sigma^2) \le R(MX'T^-Y;\Theta,\sigma^2)$$

holds for every $(\Theta,\sigma^2) \in R^{r\times q}\times R^+$. And by (2) of Definition 1.1, (3.11) evidently holds for every $(\Theta,\sigma^2) \in R^{r\times q}\times R^+$.

If the inequality strictly holds in (3.8), taking $(\Theta,\sigma^2)=(0,1)$, thus by (1) and (3) of Definition 1.1, there has

$$\begin{aligned}
&\Phi[R(DX'T^-Y;0,1)]\\
&= \{tr[(DX'T^-X-I)\Sigma_{11}(DX'T^-X-I)'P + DX'T^-\Sigma_{22}T^-XD'P\\
&\quad - 2w\cdot DX'T^-\Sigma_{22}T^-X] + w\cdot tr(\Sigma_{22}T^-)\}^{\varphi(V)}\Phi(V)\\
&< \{tr[(MX'T^-X-I)\Sigma_{11}(MX'T^-X-I)'P + MX'T^-\Sigma_{22}T^-XM'P\\
&\quad - 2w\cdot MX'T^-\Sigma_{22}T^-X] + w\cdot tr(\Sigma_{22}T^-)\}^{\varphi(V)}\Phi(V)\\
&= \Phi[R(MX'T^-Y;0,1)],
\end{aligned} \tag{3.12}$$

which means that the strict inequality in (3.11) holds at $(0,1)$.

If the equality holds in (3.8), then from (3.9) and (3.10), there exists $\alpha_0 \in R^r$ such that

$$\begin{aligned}
\zeta_0 &= \alpha_0'K'(DX'T^-X-I_p)'P(DX'T^-X-I_p)K\alpha_0\\
&< \alpha_0'K'(MX'T^-X-I_p)'P(MX'T^-X-I_p)K\alpha_0 = \eta_0.
\end{aligned}$$

Thus, taking $\Theta_0=(\alpha_0,0,\ldots,0)$, it gets by (1) and (3) of Definition 1.1

$$\begin{aligned}
&\Phi[\Theta_0'K'(DX'T^-X-I_p)'P(DX'T^-X-I_p)K\Theta_0]\\
&= (\zeta_0)^{\varphi(E_{11})}\Phi(E_{11}) < (\eta_0)^{\varphi(E_{11})}\Phi(E_{11})\\
&= \Phi[\Theta_0'K'(MX'T^-X-I_p)'P(MX'T^-X-I_p)K\Theta_0],
\end{aligned} \tag{3.13}$$

where E_{11} denotes a $q\times q$ square matrix whose $(1,1)$th is one and others are all zeros. Selecting satisfyingly small σ_0^2, from (3.13) and (4) of Definition 1.1, the strict inequality in (3.11) holds at (Θ_0,σ_0^2). Hence, the sufficiency is proved.

Necessity. If (3.11) holds for every $(\Theta,\sigma^2) \in R^{r\times q}\times R^+$, we take again $(\Theta,\sigma^2)=(0,1)$ in (3.11) and get (3.8) evidently holds from (1) of Definition 1.1 and (3.12). Let $\Theta=(\alpha,0,\ldots,0)$ again in (3.11), where $\alpha\in R^r$, by (3) and (4) of Definition 1.1, there is

$$\begin{aligned}
&\lim_{\sigma^2\to 0^+}\Phi[R(DX'T^-Y;(\alpha,0,\dots,0),\sigma^2)]\\
&=\Phi[\alpha'K'(DX'T^-X-I_p)'P(DX'T^-X-I_p)K\alpha E_{11}]\\
&=[\alpha'K'(DX'T^-X-I_p)'P(DX'T^-X-I_p)K\alpha]^{\varphi(E_{11})}\Phi(E_{11})\\
&\le \lim_{\sigma^2\to 0^+}\Phi[R(MX'T^-Y;(\alpha,0,\dots,0),\sigma^2)]\\
&=\Phi[\alpha'K'(MX'T^-X-I_p)'P(MX'T^-X-I_p)K\alpha E_{11}]\\
&=[\alpha'K'(MX'T^-X-I_p)'P(MX'T^-X-I_p)K\alpha]^{\varphi(E_{11})}\Phi(E_{11}).
\end{aligned}$$

Using (1) and (3) of Definition 1.1 and arbitrariness of α, (3.9) is obtained. Furthermore, if the equality in (3.8) holds and (3.10) does not hold, by (3.2) and (3.9), we get $R(DX'T^-Y;\Theta,\sigma^2)\equiv R(MX'T^-Y;\Theta,\sigma^2)$, which evidently results in

$$\Phi[R(DX'T^-Y;\Theta,\sigma^2)]\equiv\Phi[R(MX'T^-Y;\Theta,\sigma^2)]$$

for every (Θ,σ^2), contradictory to strict inequality in (3.11) holding for at least some point. Therefore, (3.10) holds when the equality holds in (3.8). The necessity is proved.

This completes the proof of Lemma 3.5. □

4. Proofs of main results

Proof of Theorem 2.1. First, from Lemma 3.3 and Lemma 3.5, under the loss function (1.2) $DX'T^-Y \underset{\Phi}{\overset{\mathcal{L}_0}{\sim}} B$ holds if and only if $D^*Z \underset{G}{\overset{\mathcal{L}_1}{\sim}} C_1B$ holds under the loss function $L_P(d;C_1B)=(d-C_1B)'P(d-C_1B)$.

Then from Lemma 3.1, 3.2, 3.4 and matrix calculations, it gets under the loss function (1.2), $LY \underset{\Phi}{\overset{\mathcal{LH}}{\sim}} B$ if and only if $(a),(b),(c)$ and (d) hold simultaneously. Therefore, this completes the proof of Theorem 2.1. □

Proof of Theorem 2.3. We only need to prove that $LY+\Gamma \underset{\Phi}{\overset{\mathcal{LI}}{\sim}} B$ holds if and only if $LY \underset{\Phi}{\overset{\mathcal{LH}}{\sim}} B$ and $\Gamma\in\mu[(LX-I)K]$ hold simultaneously.

Necessity. First, we prove $\Gamma\in\mu[(LX-I)K]$ if $LY+\Gamma \underset{\Phi}{\overset{\mathcal{LI}}{\sim}} B$.

Let T_0 be an orthogonal projection matrix onto $\mu[P^{\frac12}(LX-I)K]$ and $\Upsilon = P^{-\frac12}T_0P^{\frac12}\Gamma$, then evidently $\Upsilon\in\mu[(LX-I)K]$.

Notice that

$$\begin{aligned}
R(LY+\Gamma;\Theta,\sigma^2)=\sigma^2\{tr[(LX-I)\Sigma_{11}(LX-I)'P+L\Sigma_{22}L'P-2w\cdot L\Sigma_{22}T^-X]\\
+w\cdot tr(\Sigma_{22}T^-)\}V+[(LX-I)K\Theta+\Gamma]'P[(LX-I)K\Theta+\Gamma].
\end{aligned}$$

Thus, it gets

$$\begin{aligned} R(LY+\Gamma;\Theta,\sigma^2) - R(LY+\Upsilon;\Theta,\sigma^2) &= \Gamma' P\Gamma - \Upsilon' P\Upsilon \\ &= \Gamma' P^{\frac{1}{2}}(I-T_0)P^{\frac{1}{2}}\Gamma \geq 0, \end{aligned}$$

from which and (2) of Definition 1.1, it is easy to get

$$\Phi[R(LY+\Upsilon;\Theta,\sigma^2)] \leq \Phi[R(LY+\Gamma;\Theta,\sigma^2)] \tag{4.1}$$

for all (Θ,σ^2).

By $\Upsilon \in \mu[(LX-I)K]$, there exists a $p\times q$ matrix Υ_0 such that $\Upsilon = (LX-I)K\Upsilon_0$.

If $\Gamma \notin \mu[(LX-I)K]$, selecting $\Theta = -\Upsilon_0$ and satisfyingly small σ^2, we get the strict inequality in (4.1) holds, from which and (4.1), $LY+\Upsilon$ is Φ better than $LY+\Gamma$, contradictory to $LY+\Gamma \underset{\Phi}{\overset{\mathcal{LI}}{\sim}} \Theta$. Hence, there has $\Gamma \in \mu[(LX-I)K]$.

Next, we prove $LY \underset{\Phi}{\overset{\mathcal{LH}}{\sim}} \Theta$ if $LY+\Gamma \underset{\Phi}{\overset{\mathcal{LI}}{\sim}} \Theta$.

If $LY \underset{\Phi}{\overset{\mathcal{LH}}{\not\sim}} \Theta$, there exists an estimator L_1Y such that

$$\Phi[R(L_1Y;\Theta,\sigma^2)] \leq \Phi[R(LY;\Theta,\sigma^2)] \tag{4.2}$$

for all (Θ,σ^2) with strict inequality holding for at least one point (Θ_0,σ_0^2). By $\Gamma \in \mu[(LX-I)K]$, there exists a $p\times q$ matrix Γ_0 such that $\Gamma = (LX-I)K\Gamma_0$.

So by (4.2), there has

$$\begin{aligned} \Phi[R(LY+\Gamma;\Theta,\sigma^2)] &\equiv \Phi[R(LY;\Theta+\Gamma_0,\sigma^2)] \\ &\geq \Phi[R(L_1Y;\Theta+\Gamma_0,\sigma^2)] \\ &\equiv \Phi[R(L_1Y+(L_1X-I)\Gamma_0;\Theta,\sigma^2)] \end{aligned}$$

for all (Θ,σ^2) with strict inequality holding at $(\Theta_0-\Gamma_0,\sigma_0^2)$. This means that $L_1Y+(L_1X-I)\Gamma_0$ is Φ better than $LY+\Gamma$, contradictory to $LY+\Gamma \underset{\Phi}{\overset{\mathcal{LI}}{\sim}} \Theta$. Hence, the necessity is proved.

Sufficiency. If $LY+\Gamma \underset{\Phi}{\overset{\mathcal{LI}}{\not\sim}} \Theta$, there exists an estimator $L_2Y+\Pi$ such that

$$\Phi[L_2Y+\Pi;\Theta,\sigma^2)] \leq \Phi[R(LY+\Gamma;\Theta,\sigma^2)] \tag{4.3}$$

for all (Θ,σ^2) with strict inequality holding at (Θ_1,σ_1^2).

Notice that

$$\Phi[R(LY+\Gamma;\Theta,\sigma^2)] \equiv \Phi[R(LY;\Theta+\Gamma_0,\sigma^2)]. \tag{4.4}$$

Hence, from (4.3) and (4.4), we get

$$\Phi[L_2Y+\Pi;\Theta,\sigma^2)] \leq \Phi[R(LY;\Theta+\Gamma_0,\sigma^2)] \tag{4.5}$$

for all (Θ,σ^2) with strict inequality holding at (Θ_1,σ_1^2).

Selecting $\Theta = -\Gamma_0$ in (4.5), by (2) and (4) of Definition 1.1, there has

$$\begin{aligned}\Phi\{[\Pi - (L_2X - I)K\Gamma_0]'P[\Pi - (L_2X - I)K\Gamma_0]\} &= \lim_{\sigma^2\to 0^+}\Phi[L_2Y + \Pi; -\Gamma_0, \sigma^2)]\\ &\le \lim_{\sigma^2\to 0^+}\Phi[R(LY; 0, \sigma^2)] = 0.\end{aligned} \tag{4.6}$$

Thus, by (4.6), (1) of Definition 1.1 and $\Phi : \mathcal{W} \to [0, +\infty)$, there is

$$\Pi = (L_2X - I)K\Gamma_0. \tag{4.7}$$

From (4.5) and (4.7), we get

$$\begin{aligned}\Phi[L_2Y + \Pi; \Theta, \sigma^2)] &\equiv \Phi[L_2Y + (L_2X - I)K\Gamma_0; \Theta, \sigma^2)]\\ &\equiv \Phi[R(L_2Y; \widetilde{\Theta}, \sigma^2)] \le \Phi[R(LY; \widetilde{\Theta}, \sigma^2)]\end{aligned}$$

for all $(\widetilde{\Theta}, \sigma^2)$ with strict inequality holding at $(\Theta_1 + \Gamma_0, \sigma_1^2)$, where $\widetilde{\Theta} = \Theta + \Gamma_0$. This means that L_2Y is Φ better than LY, which is contradictory to $LY \underset{\Phi}{\overset{\mathcal{LH}}{\sim}} \Theta$. Therefore, the sufficiency is proved.

This completes the proof of Theorem 2.3. $\square$

5. Examples

In this section, we give some examples to illustrate applications to our results.

Example 5.1. From Remark 2.2, we know that the best linear unbiased predictor $\widehat{B}$ is a Φ admissible estimator of stochastic regression coefficients B in a class of homogeneous linear estimators.

Example 5.2. Let $X = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $\Sigma_{22} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $L = \begin{pmatrix} -1 & 1 \\ 0.875 & 0 \end{pmatrix}$, $K = \Sigma_{11} = I$, $S = X'T^-X$, $w = 0.5$ and arbitrary nonnegative definite matrix V in model (1.1), then it is easy to verify that conditions (a) and (b) in Theorem 2.1 evidently hold, and furthermore to get $N = I$ and $P = X'T^-X$. Then the left of condition (c) is $\begin{pmatrix} 0.25 & 0 \\ 0 & 0.28125 \end{pmatrix}$ and the right is $\begin{pmatrix} 0.25 & 0 \\ 0 & 0.3125 \end{pmatrix}$, hence condition (c) holds. Finally, it is easy to get condition (d) holds. Therefore, LY is a Φ admissible estimator of B in a class of homogeneous linear estimators.

On the other hand, if $\Gamma \in \mu(LX - I)$ where $\mu(LX - I) = \begin{pmatrix} 0 & 0 \\ 0 & -0.125 \end{pmatrix}$, then $LY + \Gamma$ is Φ admissible in a class of nonhomogeneous linear estimators.

Remark 5.1. Under the assumption of model (1.1) in the second example, if $L = \begin{pmatrix} -1 & 1 \\ c & 0 \end{pmatrix}$ where $0.75 < c < 1$, we can still verify that LY is Φ admissible in a class of homogeneous linear estimators. Moreover, if $\Gamma \in \mu(LX - I)$ where $\mu(LX - I) = \begin{pmatrix} 0 & 0 \\ 0 & c-1 \end{pmatrix}$, then $LY + \Gamma$ is Φ admissible in a class of nonhomogeneous linear estimators.

6. Concluding remarks

In this article, we investigate Φ admissible estimators of stochastic regression coefficients in a multivariate random effects model with respect to generalized balanced loss function. The sufficient and necessary conditions are obtained. Throughout this article, $\Sigma_{12} = 0$ is assumed in model (1.1). So, it is interesting to study Φ admissibility when $\Sigma_{12} \not\equiv 0$. We hope that this limitation will be removed as a result of further work in this area.

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