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Testing for a generalized Pareto process

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Abstract: We investigate two models for the following setup: We consider a stochastic process $\boldsymbol{X} \in C[0,1]$ whose distribution belongs to a parametric family indexed by $\vartheta \in \Theta \subset \mathbb{R}$. In case $\vartheta = 0$, $\boldsymbol{X}$ is a generalized Pareto process. Based on n independent copies $\boldsymbol{X}^{(1)}, \dots, \boldsymbol{X}^{(n)}$ of $\boldsymbol{X}$, we establish local asymptotic normality (LAN) of the point process of exceedances among $\boldsymbol{X}^{(1)}, \dots, \boldsymbol{X}^{(n)}$ above an increasing threshold line in each model.

The corresponding central sequences provide asymptotically optimal sequences of tests for testing $H_0 : \vartheta = 0$ against a sequence of alternatives $H_n : \vartheta = \vartheta_n$ converging to zero as n increases. In one model, with an underlying exponential family, the central sequence is provided by the number of exceedances only, whereas in the other one the exceedances themselves contribute, too. However it turns out that, in both cases, the test statistics also depend on some additional and usually unknown model parameters.

We, therefore, consider an omnibus test statistic sequence as well and compute its asymptotic relative efficiency with respect to the optimal test sequence.

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1. Introduction

In the recent three decades, the focus of univariate extreme value theory has shifted from the investigation of maxima (minima) in a sample to the investigation of exceedances above a high threshold. This approach towards large observations eased accessing the field of extreme value theory and became a crucial tool for various applied disciplines, such as building dykes.

Since the publications of the articles by Balkema and de Haan (1974) and Pickands (1975) it is known that exceedances above a high threshold can reasonably be modeled only by (univariate) generalized Pareto distributions (GPD), resulting in the peaks-over-threshold approach (POT). Due to practical necessity, the focus of extreme value theory has moved in recent years to multivariate observations as well. Accordingly, the investigation of multivariate exceedances enforced the definition and investigation of multivariate GPD. This investigation is still lively continuing as even the definition of multivariate GPD is under debate; see, for instance, Tajvidi (1996, Paper B), Beirlant et al. (2004, Section 8.3), Rootzén and Tajvidi (2006) and Falk, Hüsler and Reiss (2010, Chapter 5).

As already mentioned by de Haan and Ferreira (2006, p. 293): *Infinite-dimensional extreme value theory is not just a theoretical extension of multivariate extreme value theory to a more abstract context. It serves to solve concrete problems as well.* Such concrete problems are, e.g., observing dykes and tides along their whole width and not only at a finite set of observation points. There is, consequently, the need for a POT approach for functional data and for generalized Pareto *processes* as well. Again, the data exceeding some kind of a high threshold are modeled by a functional counterpart of a GPD; see Aulbach, Falk and Hofmann (2012a) and Ferreira and de Haan (2012). The current paper deals with optimal tests that check for particular models whether those exceedances do, in fact, arise from such a kind of process.

1.1. Basic mathematical material

Following Buishand, de Haan and Zhou (2008) and Ferreira and de Haan (2012), a standard generalized Pareto process, i.e., a generalized Pareto process with ultimately uniform tails in the margins, is defined as follows. For convenience, we use bold font such as $\boldsymbol{V}$ for stochastic processes and default font such as f for non stochastic functions. All operations on functions such as $f \le 0$ are meant pointwise.

Definition 1.1. Let U be an on $(0,1)$ uniformly distributed random variable (rv) and let $\boldsymbol{Z} = (Z_t)_{t\in[0,1]} \in C[0,1]$ be a stochastic process on the interval $[0,1]$ having continuous sample paths. We require that U and $\boldsymbol{Z}$ are independent and choose an arbitrary constant $M < 0$. Then

$$\boldsymbol{V} := (V_t)_{t\in[0,1]} := \left(\max\left(-\frac{U}{Z_t}, M\right)\right)_{t\in[0,1]}. \tag{1}$$

defines a *standard generalized Pareto process* (GPP) if $0 \le Z_t \le m$, $E(Z_t) = 1$, $t \in [0,1]$, hold for some constant $m \ge 1$. A stochastic process $\boldsymbol{Z} \in C[0,1]$ with these two properties will be called a *generator*.

The role of the constant M is twofold: On the one hand it prevents division by zero in the definition (1) of a GPP, on the other hand it reflects the fact that the behavior of a GPP is prescribed only in its upper tail, i.e., above the constant M. Note that the finite-dimensional marginal distributions of $\boldsymbol{V}$ are multivariate GPD with ultimately uniform tails; see, e.g., Aulbach, Bayer and Falk (2012).

The process $\boldsymbol{V}$ in (1) is characterized by the fact that its *functional* distribution function (df) is given by

$$P(\boldsymbol{V} \le f) = 1 - E\left(\sup_{t\in[0,1]} (|f(t)|\, Z_t)\right), \quad f \in \bar{E}^-[0,1],\ \|f\|_\infty \le \min\left(|M|, \frac{1}{m}\right).$$

We set $\bar{E}^-[0,1] := \{f \in E[0,1] : f \le 0\}$ where $E[0,1]$ denotes the set of all bounded functions $f : [0,1] \to \mathbb{R}$ that have only a finite number of discontinuities. The space $C[0,1]$ of continuous functions is, obviously, a subset of $E[0,1]$. A suitable choice of $f \in \bar{E}^-[0,1]$ allows the immediate incorporation of the finite dimensional distributions of $\boldsymbol{V}$ in its functional df: Choose $0 \le t_1 < \dots < t_d \le 1$, $x_1, \dots, x_d \le 0$ and put $f(t) := \sum_{i=1}^d x_i 1_{\{t_i\}}(t)$. Then

$$P(\boldsymbol{V} \le f) = P(V_{t_i} \le x_i,\ i = 1, \dots, d).$$

It is, moreover, obvious that

$$\|f\|_D := E\left(\sup_{t\in[0,1]} (|f(t)|\, Z_t)\right), \quad f \in E[0,1],$$

defines a norm on $E[0,1]$, called *D-norm* with *generator* $\boldsymbol{Z}$; see Aulbach, Falk and Hofmann (2012b). This representation of the df of $\boldsymbol{V}$ in terms of a D-norm is in complete analogy with the multivariate case of a GPD. We refer again to Falk, Hüsler and Reiss (2010, Section 5.1).

For each standard GPP there is a corresponding *standard* max-stable process (MSP), i.e., a stochastic process $\boldsymbol{\eta} = (\eta_t)_{t\in[0,1]} \in C[0,1]$ such that

$$P(\boldsymbol{\eta} \le f) = \exp\left(-\|f\|_D\right), \qquad f \in \bar{E}^-[0,1]. \tag{2}$$

Note that this implies $P(\eta_t \le x) = \exp(x)$, $x \le 0$, $t \in [0,1]$. On the other hand, the df of each max-stable process $\boldsymbol{\eta}$ having standard negative exponential margins has a representation as in Equation (2); we refer to Aulbach, Falk and Hofmann (2012b) for details.

1.2. Overview of the current paper

We replace the rv U in equation (1) by a rv $W \ge 0$ which is independent of $\boldsymbol{Z}$, too. However, the distribution of W is different from the uniform one and, thus, the process

$$\boldsymbol{X} := (X_t)_{t\in[0,1]} := \left(\max\left(-\frac{W}{Z_t}, M\right)\right)_{t\in[0,1]} \tag{3}$$

is no longer a standard GPP.

This gives rise to the following problem: Based on the exceedances in a sample of n independent copies $\boldsymbol{X}^{(1)},\dots,\boldsymbol{X}^{(n)}$ of $\boldsymbol{X}$ above a high threshold line, how close can the df of W get to that of U with the difference still being detected?

The distance between the df of W and U will be measured in terms of their densities, i.e., we will assume parametric models for the distance of the density of W from the constant function one, see equations (4) and (5) below. As we consider exceedances above a high threshold, only the lower end of the density of W matters.

Within these parametric models $\{H_\vartheta : \vartheta\in\Theta\}$ for the df H_ϑ of W we can derive optimal tests detecting the deviation of the distribution of the upper tail of $\boldsymbol{X}$ from that of $\boldsymbol{V}$, i.e., the deviation of ϑ from zero. This is the content of the present paper, which is organized as follows.

In Section 2 we require that the df H_ϑ of W has a density h_ϑ near zero, which satisfies for some $\delta>0$ the expansion

$$h_\vartheta(u) = 1 + \vartheta u^\delta + o(u^\delta) \qquad \text{as } u\downarrow 0 \tag{4}$$

with some parameter $\vartheta\in\Theta$, where zero is an inner point of $\Theta\subset\mathbb{R}$. The standard exponential df, for instance, satisfies this condition with $\delta=1$ and $\vartheta=-1$. The null-hypothesis $\vartheta=0$ is meant to be the uniform distribution on $(0,1)$.

In Section 3 we assume that the distribution of W belongs to an exponential family given by the probability density

$$h_\vartheta(u) = C(\vartheta)\exp(\vartheta T(u)), \qquad 0\le u\le 1,\ \vartheta\in\mathbb{R}. \tag{5}$$

In both models we establish local asymptotic normality (LAN) of the point process of exceedances among $\boldsymbol{X}^{(1)},\dots,\boldsymbol{X}^{(n)}$ above an increasing threshold line. The results, which are stated in Theorem 2.3 and Theorem 3.2, provide in each model the corresponding central sequence and, thus, optimal tests for testing $\vartheta=0$ against a sequence of alternatives ϑ_n converging to zero as the sample size increases. It turns out that the particular values of the exceedances contribute to the central sequence only in model (4), whereas in the exponential family (5) the number of exceedances alone yields the central sequence.

Different to the present paper, in which we focus on parametric models for the rv W and view $\boldsymbol{Z}$ as a functional nonparametric nuisance parameter, Aulbach and Falk (2012) considered a particular parametric model for the distribution of the generator $\boldsymbol{Z}$, indexed by $\beta>0$. This model goes back to de Haan and Pereira (2006). LAN of a point process of exceedances above a high constant threshold function was established within this setup. The central sequence turned out to be just the number of exceedances. As an application, obtained from LAN-theory, it was shown that within this parametric model the frequency estimator of the underlying β_0 is asymptotically efficient.

The fact that just the number of realizations in shrinking sets provides the central sequence was characterized for truncated processes in quite a general framework in Falk (1998) and Falk and Liese (1998).

It turns out that the central sequences and, thus, the asymptotically optimal tests within our setup depend on further parameters of the generator process $\boldsymbol{Z}$, which might be unknown in practice. We, therefore, consider an omnibus test for testing $\vartheta = 0$ as well. We compute its asymptotic relative efficiency (ARE) with respect to the optimal test in each model. While ARE is positive in model (4), it turns out to be zero in model (5).

To make the presentation more fluid, the proofs of our main results are postponed to Section 4.

2. Testing in δ-neighborhoods of a standard GPP

This section deals with optimal tests in the model introduced in (4). We assume that the df of the rv $W \geq 0$ in (3) belongs to a parametric family $\{H_\vartheta : \vartheta \in \Theta\}$ of distributions, where Θ is an open subset of $\mathbb{R}$ containing 0. By H_0 we denote the uniform distribution on the interval $(0,1)$. In addition to (4), it is required that there is some $u_0 \in (0,1)$ such that the density $h_\vartheta(u)$ of $H_\vartheta(u)$ exists for $u \in [0, u_0]$, $\vartheta \in \Theta$, and satisfies for some $\delta \in (0,1]$ the expansion

$$h_\vartheta(u) = 1 + \vartheta u^\delta + r_\vartheta(u), \qquad u \in [0, u_0], \tag{6}$$

where $r_\vartheta(0) = 0$, $\vartheta \in \Theta$, and

$$\sup_{0<|\vartheta|\leq\varepsilon_0} \left| \frac{r_\vartheta(u)}{\vartheta u^\delta} \right| = o(1) \tag{7}$$

as $u \downarrow 0$ for some $\varepsilon_0 > 0$. Obviously, (7) is equivalent with $r_\vartheta(u) = o(\vartheta u^\delta)$ as $u \downarrow 0$, uniformly for $|\vartheta| \leq \varepsilon_0$. Since we have $h_0 = 1$ and $r_0 = 0$, (6) and (7) imply in particular the representation

$$h_\vartheta(u) = h_0(u) \left(1 + O\left((H_0(u))^\delta\right)\right),$$

i.e., the lower tail of H_ϑ is in a δ-neighborhood of H_0; see Falk, Hüsler and Reiss (2010, Section 2.2).

Example 2.1. Take the exponential model

$$H_\vartheta(u) := \frac{\exp(\vartheta u) - 1}{\vartheta}, \qquad 0 \leq u \leq \log(1+\vartheta)/\vartheta,\ \vartheta \in [-1, \infty),$$

with the convention $H_0(u) = \lim_{\vartheta \to 0} H_\vartheta(u) = u$, $u \in [0,1]$. H_ϑ is the df of the rv $\log(1+\vartheta U)/\vartheta$, where U is uniformly on $(0,1)$ distributed. The density of H_ϑ satisfies

$$\begin{aligned} h_\vartheta(u) &= \exp(\vartheta u) \\ &= 1 + \vartheta u + (\exp(\vartheta u) - 1 - \vartheta u) \\ &=: 1 + \vartheta u + r_\vartheta(u), \qquad 0 \leq u \leq \log(1+\vartheta)/\vartheta, \end{aligned}$$

and, thus, condition (6) and (7) are satisfied with $\vartheta \in \Theta = (-1, 1)$, $u_0 = 1$ and $\delta = 1$.

Moreover, we assume

$$A := E\left(\inf_{t\in[0,1]} Z_t\right) > 0. \tag{8}$$

As $\inf_{t\in[0,1]} Z_t \geq 0$, this condition is equivalent with the assumption that $\inf_{t\in[0,1]} Z_t$ is not the constant function zero. Condition (8) is, for instance, satisfied if $\boldsymbol{Z} = 2\boldsymbol{U}$, where $\boldsymbol{U} = (U_t)_{t\in[0,1]} \in C[0,1]$ is a copula process such that $-\boldsymbol{U} \in \mathcal{D}(\boldsymbol{\eta})$, $\boldsymbol{\eta} \in C[0,1]$ being a standard MSP. This is implied by the fact that in this case $P\left(\inf_{t\in[0,1]} U_t > 0\right) = 1$, which can bee seen by elementary arguments.

Note that (8) and Hölder's inequality also give

$$B := E\left(\inf_{t\in[0,1]} Z_t^{1+\delta}\right) > 0. \tag{9}$$

2.1. Local asymptotic normality

In order to derive asymptotically optimal tests in this model, we first establish LAN of the point process of exceedances

$$N_{n,c}(B) := \sum_{i\leq n} \varepsilon_{\sup_{t\in[0,1]}\left(X_t^{(i)}/c\right)}(B \cap [0,1]), \qquad B \in \mathbb{B},$$

where $\boldsymbol{X}^{(i)}$, $i \leq n$, are independent copies of $\boldsymbol{X}$ in (3) and $c < 0$. $\mathbb{B}$ denotes the σ-field of Borel sets of $\mathbb{R}$ and ε_x is the point measure with mass one at x. Note that

$$\sup_{t\in[0,1]} \frac{X_t}{c} \leq u \iff \boldsymbol{X} \geq cu, \qquad u \in [0,1], \tag{10}$$

i.e., the random point measure $N_{n,c}$ actually represents those observations among $\boldsymbol{X}^{(1)}, \dots, \boldsymbol{X}^{(n)}$ which exceed the constant threshold function c.

Denote those observations among $\sup_{t\in[0,1]}(X_t^{(i)}/c)$ with $\sup_{t\in[0,1]}(X_t^{(i)}/c) \leq 1$, $i \leq n$, by $Y_1, \dots, Y_{\tau(n)}$ in the order of their outcome. Then we have

$$N_{n,c}(B) = \sum_{k\leq\tau(n)} \varepsilon_{Y_k}(B), \qquad B \in \mathbb{B}.$$

By Theorem 1.4.1 in Reiss (1993) we may assume without loss of generality that $Y_1, Y_2, \dots$ are independent copies of a rv Y with df

$$P_\vartheta(Y \leq u) = \frac{P_\vartheta(\boldsymbol{X} \geq cu)}{P_\vartheta(\boldsymbol{X} \geq c)}, \qquad 0 \leq u \leq 1,$$

under parameter ϑ, and that they are independent of the total number $\tau(n)$, which is binomial $B\left(n, P_\vartheta\left(\boldsymbol{X} \geq c\right)\right)$-distributed.

In the next lemma we provide the density $f_{\vartheta,c}$ of $\sup_{t\in[0,1]}(X_t/c)$ and, thus, the density of Y under ϑ, which is $f_{\vartheta,c}/P_\vartheta(\boldsymbol{X}\geq c)$. By $P*Z$ we denote the distribution of a rv Z.

Lemma 2.2. *Suppose that the distribution of the rv W in (3) belongs to the family $\{H_\vartheta : \vartheta\in\Theta\}$. Then there is some $c_0<0$ such that the density $f_{\vartheta,c}(u)$, with respect to the Lebesgue measure, of the rv $\sup_{0\leq t\leq 1}(X_t/c)$ exists for $\vartheta\in\Theta$, $c\in[c_0,0)$, $u\in[0,1]$, and it is given by*

$$f_{\vartheta,c}(u)=|c|\int_0^m z\,h_\vartheta(|c|\,zu)\left(P*\inf_{t\in[0,1]}Z_t\right)(dz).$$

Furthermore there exists $\varepsilon_0>0$ such that

$$f_{\vartheta,c}(u)=|c|\,A+\vartheta\,|c|^{1+\delta}\,Bu^\delta+o\left(\vartheta\,|c|^{1+\delta}\right) \tag{11}$$

uniformly for $|\vartheta|\leq\varepsilon_0$ and $u\in[0,1]$ as $c\uparrow 0$; note that $f_{0,c}(u)=|c|\,A$.

Proof. Let m be given as in Definition 1.1 and u_0,ε_0,δ be given as in equations (6) and (7). Then we obtain for $c_0>\max\{M,-u_0/m\}$, $\vartheta\in\Theta$, $c\in[c_0,0)$ and $u\in[0,1]$ by conditioning on $\inf_{t\in[0,1]}Z_t=z$ and Fubini's theorem

$$\begin{aligned}P_\vartheta(\boldsymbol{X}\geq cu)&=P_\vartheta\left(\max\left(-\frac{W}{Z_t},M\right)\geq cu,\ t\in[0,1]\right)\\&=P_\vartheta\left(W\leq|c|\,u\inf_{t\in[0,1]}Z_t\right)\\&=\int_0^m H_\vartheta(|c|\,zu)\left(P*\inf_{t\in[0,1]}Z_t\right)(dz).\end{aligned}$$

This representation implies that $\sup_{t\in[0,1]}(X_t/c)$ has the density

$$\begin{aligned}f_{\vartheta,c}(u)&=|c|\int_0^m z\,h_\vartheta(|c|\,zu)\left(P*\inf_{t\in[0,1]}Z_t\right)(dz)\\&=|c|\,A+\vartheta\,|c|^{1+\delta}\,u^\delta B+o\left(\vartheta\,|c|^{1+\delta}\right)\end{aligned}$$

as $c\uparrow 0$, uniformly for $|\vartheta|\leq\varepsilon_0$ and $u\in[0,1]$. (Note that $r_\vartheta(|c|\,zu)=o(\vartheta\,|c|^\delta)$ uniformly for $|\vartheta|\leq\varepsilon_0$, $u\in[0,1]$ and $z\in[0,m]$ as $c\uparrow 0$.) As $H_0(u)=u$, we have $h_0(u)=1$, $u\in[0,1]$, which completes the proof. □

If $c_0\leq c<0$, then we have $P_0(Y\leq u)=u$, i.e., the distribution $\mathcal{L}_{\vartheta,c}(Y)$ of Y under ϑ is then dominated by $\mathcal{L}_{0,c}(Y)$ with pertaining density given by

$$\frac{d\mathcal{L}_{\vartheta,c}(Y)}{d\mathcal{L}_{0,c}(Y)}(u)=\frac{f_{\vartheta,c}(u)}{f_{0,c}(u)}\frac{P_0(\boldsymbol{X}\geq c)}{P_\vartheta(\boldsymbol{X}\geq c)},\qquad u\in[0,1].$$

The distribution $\mathcal{L}_\vartheta(N_{n,c})$ of $N_{n,c}$ under ϑ is, in this case, dominated by $\mathcal{L}_0(N_{n,c})$ – see, e.g., Theorem 3.1.2 in Reiss (1993) – and we obtain from Reiss (1993, Example 3.1.2) that the density of $\mathcal{L}_\vartheta(N_{n,c})$ with respect to $\mathcal{L}_0(N_{n,c})$ is given by

$$\frac{d\mathcal{L}_\vartheta(N_{n,c})}{d\mathcal{L}_0(N_{n,c})}(\mu) = \left(\prod_{i=1}^{\mu([0,1])} \frac{f_{\vartheta,c}(y_i)}{f_{0,c}(y_i)} \frac{P_0(\boldsymbol{X} \geq c)}{P_\vartheta(\boldsymbol{X} \geq c)}\right) \times \left(\frac{P_\vartheta(\boldsymbol{X} \geq c)}{P_0(\boldsymbol{X} \geq c)}\right)^{\mu([0,1])} \left(\frac{1 - P_\vartheta(\boldsymbol{X} \geq c)}{1 - P_0(\boldsymbol{X} \geq c)}\right)^{n-\mu([0,1])}$$

where $\mu = \sum_{i=1}^{\mu([0,1])} \varepsilon_{y_i}$, $0 \leq y_1, \dots, y_{\mu([0,1])} \leq 1$ and $\mu([0,1]) \leq n$. The loglikelihood ratio is, consequently,

$$\begin{aligned} L_{n,c}(\vartheta \mid 0) &:= \log\left\{\frac{d\mathcal{L}_\vartheta(N_{n,c})}{d\mathcal{L}_0(N_{n,c})}(N_{n,c})\right\} \\ &= \sum_{i\leq\tau(n)} \log\left(\frac{f_{\vartheta,c}(Y_i)}{f_{0,c}(Y_i)} \frac{P_0(\boldsymbol{X} \geq c)}{P_\vartheta(\boldsymbol{X} \geq c)}\right) + \tau(n)\log\left(\frac{P_\vartheta(\boldsymbol{X} \geq c)}{P_0(\boldsymbol{X} \geq c)}\right) \\ &\quad + (n - \tau(n))\log\left(\frac{1 - P_\vartheta(\boldsymbol{X} \geq c)}{1 - P_0(\boldsymbol{X} \geq c)}\right). \end{aligned} \tag{12}$$

We let in the sequel $c = c_n$ depend on the sample size n with $c_n \uparrow 0$ and, equally, $\vartheta = \vartheta_n$ with $\vartheta_n \to 0$ as $n \to \infty$. Precisely, we put with arbitrary $\xi \in \mathbb{R}$

$$\vartheta_n := \vartheta_n(\xi) := \frac{\xi}{\left(n\,|c_n|^{1+2\delta}\right)^{1/2}}. \tag{13}$$

The following result provides the desired LAN property of $N_{n,c}$; it is a crucial tool for deriving asymptotically optimal tests in the subsequent subsection. By $\to_{D_0}$ we denote ordinary weak convergence under $\vartheta = 0$; the constants A, B are given in equations (8), (9) and (11).

Theorem 2.3. *Suppose that $c_n \uparrow 0$, $n\,|c_n|^{1+2\delta} \to \infty$ as $n \to \infty$. Then we obtain for ϑ_n as in (13) the expansion*

$$\begin{aligned} L_{n,c_n}(\vartheta_n \mid 0) &= \frac{\xi B}{(1+\delta)A^{1/2}}(Z_{n1} + Z_{n2}) - \frac{\xi^2 B^2}{2A(2\delta+1)} + o_{P_0}(1) \\ &\to_{D_0} N\left(-\frac{\xi^2 B^2}{2A(2\delta+1)}, \frac{\xi^2 B^2}{A(2\delta+1)}\right) \end{aligned}$$

with $Z_{n1} := \frac{\tau(n) - n|c_n|A}{(n|c_n|A)^{1/2}} \to_{D_0} N(0,1)$ and

$$Z_{n2} := \frac{1+\delta}{\tau(n)^{1/2}} \sum_{k\leq\tau(n)} \left(Y_k^\delta - \frac{1}{1+\delta}\right) \to_{D_0} N\left(0, \frac{\delta^2}{2\delta+1}\right)$$

being independent.

2.2. Testing $\vartheta = 0$ against $\vartheta = \vartheta_n$

Denote by $u_\alpha = \Phi^{-1}(1-\alpha)$ the $(1-\alpha)$-quantile of the standard normal df. By the Neyman-Pearson lemma and Theorem 2.3, the test statistic

$$\varphi_1\left(N_{n,c_n}\right) = 1_{(u_\alpha,\infty)}\left(\frac{(2\delta+1)^{1/2}}{1+\delta}(Z_{n1}+Z_{n2})\right)$$

defines an asymptotically optimal level-α test, based on N_{n,c_n}, for $H_0:\ \vartheta = 0$ against $\vartheta_n = \vartheta_n(\xi) = \xi/(n\,|c_n|^{1+2\delta})^{1/2}$ with $\xi > 0$. As $\varphi_1\left(N_{n,c_n}\right)$ does not depend on ξ, this test is asymptotically optimal, uniformly in $\xi > 0$, within the class of tests that depend on N_{n,c_n}.

The corresponding uniformly asymptotically optimal test for H_0 against $\vartheta_n(\xi)$ with $\xi < 0$ is

$$\varphi_2\left(N_{n,c_n}\right) = 1_{(-\infty,-u_\alpha)}\left(\frac{(2\delta+1)^{1/2}}{1+\delta}(Z_{n1}+Z_{n2})\right).$$

The asymptotic power functions of these tests are provided by Theorem 2.3 as well. By LeCam's third lemma we obtain that under $\vartheta_n = \vartheta_n(\xi)$

$$\begin{aligned} L_{n,c_n}(\vartheta_n \mid 0) &= \frac{\xi B}{(1+\delta)A^{1/2}}(Z_{n1}+Z_{n2}) - \frac{\xi^2 B^2}{2A(2\delta+1)} + o_{P_{\vartheta_n}}(1) \\ &\to_{D_{\vartheta_n}} N\left(\frac{\xi^2 B^2}{2A(2\delta+1)}, \frac{\xi^2 B^2}{A(2\delta+1)}\right) \end{aligned}$$

with

$$Z_{n1}+Z_{n2} \to_{D_{\vartheta_n}} N\left(\frac{\xi B(1+\delta)}{A^{1/2}(2\delta+1)}, \frac{(1+\delta)^2}{2\delta+1}\right).$$

The asymptotic power functions of φ_i are, consequently, given by

$$\lim_{n\to\infty} E_{\vartheta_n(\xi)}\left(\varphi_i\left(N_{n,c_n}\right)\right) = 1 - \Phi\left(u_\alpha - \frac{|\xi|\,B}{A^{1/2}(2\delta+1)^{1/2}}\right). \tag{14}$$

If $\sup_{t\in[0,1]} Z_t = m$ a.s., then $\sup_{t\in[0,1]} X_t = \max(-W/m, M)$. In this case we obtain a non censored observation Y by considering only those realizations of $\sup_{t\in[0,1]} X_t$ greater than c, i.e.,

$$\sup_{t\in[0,1]} X_t \ge c \iff W \le |c|\, m,$$

provided $c > M$. As M and m are assumed to be unknown, one has to let the threshold $c = c_n$ depend on n and shrink to zero as n increases. The Neyman-Pearson lemma for testing the density

$$h_\vartheta(u) = 1 + \vartheta u^\delta + o(u^\delta) \text{ against } h_{\vartheta_0}(u) = 1, \qquad 0 \le u \le 1,$$

based on the non censored observations Y_i, $i \le \tau(n)$, is then given by the second part Z_{n2} of the central sequence in Theorem 2.3. As the test $\varphi_i(N_{n,c_n})$ uses the complete process N_{n,c_n} and not only Y_i, $i \le \tau(n)$, also the *number* $\tau(n)$ of non censored observations contributes to the central sequence.

A disadvantage of the optimal test statistics $\varphi_i\left(N_{n,c_n}\right)$ is the fact that they require explicit knowledge of the constants A and δ. To overcome this disadvantage, we consider in the following an alternative test.

2.3. An omnibus test

Recall that the observations $Y_1, Y_2, \dots$ are independent and, under $\vartheta = 0$, uniformly on $(0,1)$ distributed rv if the threshold c is close to zero. Conditional on the assumption that there is at least one exceedance, i.e., conditionally on $\tau(n) > 0$, the test statistic

$$T_{n,c} := \frac{1}{\tau(n)^{1/2}} \sum_{k=1}^{\tau(n)} \Phi^{-1}(Y_k)$$

is under H_0 exactly $N(0,1)$-distributed. By Φ we denote the standard normal df. This test statistic is analogous to that in Falk and Michel (2009) for testing for a multivariate generalized Pareto distribution.

The next result provides the asymptotic distribution of T_{n,c_n} under the alternative $\vartheta_n = \vartheta_n(\xi)$ as $n \to \infty$.

Proposition 2.4. *Under the assumptions of Theorem 2.3 we have*

$$T_{n,c_n} \to_{D_{\vartheta_n}} N\left(\xi \frac{B}{A^{1/2}} \int_{-\infty}^{\infty} x(\Phi(x))^{\delta} \varphi(x)\, dx, 1\right).$$

From Proposition 2.4 we obtain that

$$\varphi_1^*\left(N_{n,c_n}\right) := 1_{(u_\alpha,\infty)}\left(T_{n,c_n}\right), \quad \varphi_2^*\left(N_{n,c_n}\right) := 1_{(-\infty,-u_\alpha)}\left(T_{n,c_n}\right)$$

are one-sided tests for testing $\vartheta > 0$ and $\vartheta < 0$, respectively, against 0. Their asymptotic power functions are given by

$$\begin{aligned} \beta(\xi) &:= \lim_{n\to\infty} E_{\vartheta_n(\xi)}\left(\varphi_i^*\left(N_{n,c_n}\right)\right) \\ &= 1 - \Phi\left(u_\alpha - |\xi| \frac{B}{A^{1/2}} \int_{-\infty}^{\infty} x(\Phi(x))^{\delta} \varphi(x)\, dx\right), \qquad \xi \in \mathbb{R}. \end{aligned} \tag{15}$$

The asymptotic relative efficiency of $\varphi_i^*\left(N_{n,c_n}\right)$ with respect to $\varphi_i\left(N_{n,c_n}\right)$ is, by (14) and (15), given by the ratio

$$\frac{\left(|\xi|\, B \int_{-\infty}^{\infty} x(\Phi(x))^{\delta} \varphi(x)\, dx / A^{1/2}\right)^2}{\left(|\xi|\, B / \left(A^{1/2}(2\delta+1)^{1/2}\right)\right)^2} = (2\delta+1)\left(\int_{-\infty}^{\infty} x(\Phi(x))^{\delta} \varphi(x)\, dx\right)^2,$$

which is independent of ξ.

Denote by $k_n := \min\left\{k \in \mathbb{N} : E_{\vartheta_n(\xi)}\left(\varphi_i^*\left(N_{k,c_k}\right)\right) \geq E_{\vartheta_n(\xi)}\left(\varphi_i\left(N_{n,c_n}\right)\right)\right\}$ the least sample size, for which $\varphi_i^*(N_{k_n,c_{k_n}})$ performs, at $\vartheta_n(\xi)$, at least as good as $\varphi_i\left(N_{n,c_n}\right)$, $i = 1, 2$. The *relative efficiency* of $\varphi_i^*(N_{k_n,c_{k_n}})$ with respect to $\varphi_i\left(N_{n,c_n}\right)$ is then defined as n/k_n. From (14) and (15) we obtain that

$$\lim_{n\to\infty} \frac{n\left|c_n\right|^{1+2\delta}}{k_n\left|c_{k_n}\right|^{1+2\delta}} = (2\delta+1)\left(\int_{-\infty}^{\infty} x(\Phi(x))^{\delta}\varphi(x)\,dx\right)^2 =: \mathrm{ARE}(\delta), \qquad (16)$$

see Section 10.2 in Pfanzagl (1994) for the underlying reasoning. If we put, for example, $c_n := -n^{-\varepsilon/(1+2\delta)}$, $n \in \mathbb{N}$, for some $\varepsilon \in (0, 1)$, then equation (16) yields

$$\lim_{n\to\infty} \frac{n^{1-\varepsilon}}{k_n^{1-\varepsilon}} = \mathrm{ARE}(\delta),$$

or

$$k_n \sim \frac{n}{\mathrm{ARE}(\delta)^{1/(1-\varepsilon)}}$$

as $n \to \infty$. By $\sim$ we denote asymptotic equivalence, i.e., the ratio of left and right hand side converges to one.

Figure 1 displays the function $\mathrm{ARE}(\delta)$ for $\delta \in [0, 1]$. While it is nearly linear for $\delta \in [0.25, 1]$ with approximate maximum value 0.24 and minimum value

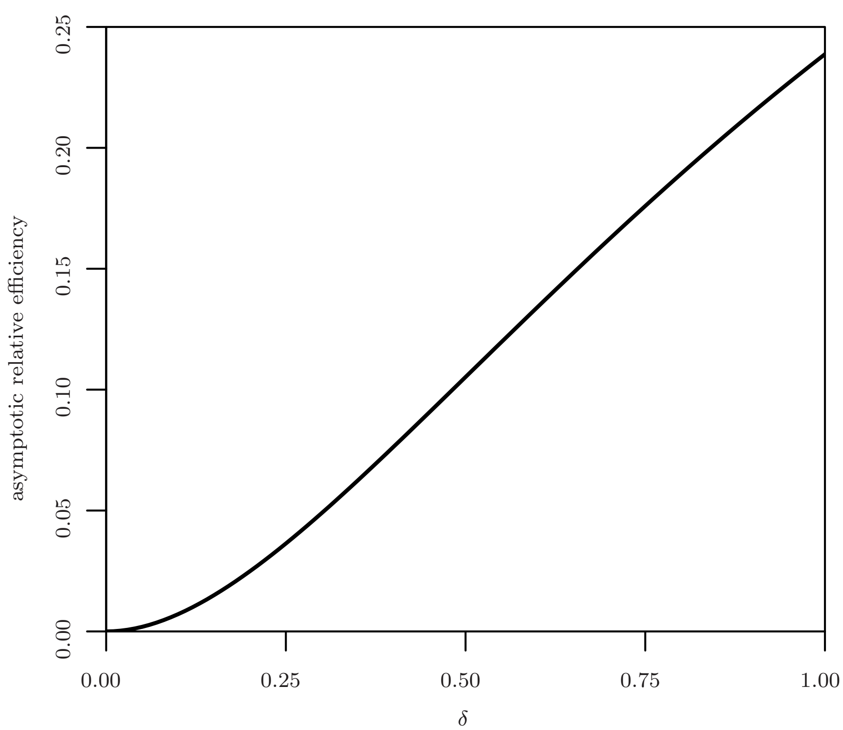

Fig 1. *Asymptotic relative efficiency* $\mathrm{ARE}(\delta)$ *as defined in* (16).

0.035, roughly, $\mathrm{ARE}(\delta)$ quickly converges to zero for δ less than 0.25. With c_n chosen as above, this means that the minimum sample size k_n, for which $\varphi_i^*(N_{k_n,c_n})$ performs, at $\vartheta_n(\xi)$, as good as the optimal test $\varphi_i(N_{n,c_n})$ with sample size n, is asymptotically equivalent to $n/\mathrm{ARE}(\delta)^{1/(1-\varepsilon)}$ and, thus, an increasing multiple of n as δ shrinks to zero. This explains the significance of the asymptotic relative efficiency defined above.

3. Testing in an exponential family model

In this section we assume that the distribution of W belongs to an exponential family given by the probability densities on the interval $[0,1]$

$$h_\vartheta(u) = C(\vartheta)\exp(\vartheta T(u)), \qquad 0 \le u \le 1,\ \vartheta \in \mathbb{R},$$

where $T : [0,1] \to \mathbb{R}$ is a bounded Borel-measurable function satisfying

$$\lim_{u \downarrow 0} T(u) =: C \in \mathbb{R},$$

and $C(\vartheta)$ is defined by

$$C(\vartheta) := \frac{1}{\int_0^1 \exp(\vartheta T(u))\,du}, \qquad \vartheta \in \mathbb{R}.$$

An obvious example is the family of truncated exponential distributions having the densities

$$h_\vartheta(u) = C(\vartheta)\exp(\vartheta u), \qquad 0 \le u \le 1,\ \vartheta \in \mathbb{R},$$

where $C(\vartheta) = \vartheta/(\exp(\vartheta) - 1)$ with the convention $C(0) = \lim_{\vartheta \to 0} \vartheta/(\exp(\vartheta) - 1) = 1$.

Remark 3.1. From the arguments in the proof of Lemma 2.2 we obtain that the rv $\sup_{0 \le t \le 1}(X_t/c)$ has for $c < 0$ close to zero and each $\vartheta \in \mathbb{R}$ on $[0,1]$ the Lebesgue-density

$$f_{\vartheta,c}(u) = |c| \int_0^m z h_\vartheta(|c|\,zu) \left(P * \inf_{0 \le t \le 1} Z_t\right)(dz), \qquad 0 \le u \le 1.$$

In what follows we put with arbitrary $\xi \in \mathbb{R}$

$$\vartheta_n := \vartheta_n(\xi) := \frac{\xi}{(n\,|c_n|)^{1/2} A^{1/2}\left(C - \int_0^1 T(u)\,du\right)},$$

where we require that $C \neq \int_0^1 T(u)\,du$.

Theorem 3.2. *Suppose that $|c_n| \to 0$, $n\,|c_n| \to \infty$ as $n \to \infty$. Then we obtain for the loglikelihood ratio in (12) the expansion*

$$L_{n,c_n}(\vartheta_n \mid 0) = \xi Z_{n1} - \frac{\xi^2}{2} + o_{P_0}(1).$$

The test statistic

$$\phi_1(N_{n,c_n}) := 1_{(u_\alpha,\infty)}(Z_{n1})$$

defines by the Neyman-Pearson lemma an asymptotically optimal level-α test, based on N_{n,c_n}, for the null-hypothesis $\vartheta = 0$ against the sequence of alternatives $\vartheta_n = \vartheta_n(\xi) = \xi/((n\,|c_n|)^{1/2}A^{1/2}(C-\int_0^1 T(u)\,du))$ with $\xi > 0$. As $\phi_1(N_{n,c_n})$ does not depend on ξ, this test is asymptotically optimal uniformly in $\xi > 0$, within the class of tests that depend on N_{n,c_n}. The corresponding uniformly optimal test for $\vartheta = 0$ against $\vartheta_n(\xi)$ with $\vartheta < 0$ is

$$\phi_2(N_{n,c_n}) := 1_{(-\infty,-u_\alpha)}(Z_{n1}).$$

From LeCam's third lemma we obtain that under $\vartheta_n = \vartheta_n(\xi)$

$$L_{n,c_n}(\vartheta_n \mid 0) = \xi Z_{n1} - \frac{\xi^2}{2} + o_{P_n}(1) \to_{D_{\vartheta_n}} N\left(\frac{\xi^2}{2}, \xi^2\right),$$

with

$$Z_{n1} \to_{D_{\vartheta_n}} N(\xi, 1).$$

The asymptotic power functions of ϕ_i, $i = 1, 2$, are, consequently, given by

$$\lim_{n\to\infty} E_{P_{\vartheta_n}}\left(\phi_i(N_{n,c_n})\right) = 1 - \Phi(u_\alpha - |\xi|), \qquad i = 1, 2.$$

Next we compute the performance of the statistic $T_{n,c} = \tau(n)^{-1/2} \times \sum_{k=1}^{\tau(n)} \Phi^{-1}(Y_k)$ for the testing problem $\vartheta = 0$ against $\vartheta_n(\xi)$.

Lemma 3.3. *We have*

$$E_{\vartheta_n,c_n}\left(\Phi^{-1}(Y)\right) = o(\vartheta_n), \quad Var_{\vartheta_n,c_n}\left(\Phi^{-1}(Y)\right) = 1 + o(\vartheta_n^2).$$

Lemma 3.3 together with the central limit theorem implies that $T_{n,c_n} \to_{D_{\vartheta_n,c_n}} N(0, 1)$ independent of ξ and, thus, different to the optimal test based on Z_{n1}, the test statistic T_{n,c_n} is not capable to detect the alternative $\vartheta_n = \vartheta_n(\xi)$.

4. Proofs

The proofs of the main results of this paper are given in this section.

Proof of Theorem 2.3. First we compile several facts that will be used in the proof. From Lemma 2.2 we obtain

$$P_0(\boldsymbol{X} \geq c_n) = P(\boldsymbol{V} \geq c_n) = |c_n|\,A \qquad \text{(Fact 1)}$$

and, thus, a suitable version of the central limit theorem implies

$$\frac{\tau(n) - n\,|c_n|\,A}{(n\,|c_n|\,A)^{1/2}} \to_{D_0} N(0, 1). \qquad \text{(Fact 2)}$$

Moreover, we conclude from Lemma 2.2 for $|\vartheta| \le \varepsilon_0$

$$P_\vartheta(\boldsymbol{X} \ge c_n) - P_0(\boldsymbol{X} \ge c_n) = |c_n|^{1+\delta} \frac{\vartheta}{1+\delta} B + o\left(\vartheta \, |c_n|^{1+\delta}\right) \qquad \text{(Fact 3)}$$

and, thus,

$$\frac{P_{\vartheta_n}(\boldsymbol{X} \ge c_n) - P_0(\boldsymbol{X} \ge c_n)}{P_0(\boldsymbol{X} \ge c_n)} = \frac{1}{(n\,|c_n|)^{1/2}} \frac{\xi B}{(1+\delta)A} + o\left(\frac{1}{(n\,|c_n|)^{1/2}}\right). \qquad \text{(Fact 4)}$$

Hence, Taylor expansion $\log(1+\varepsilon) = \varepsilon - \varepsilon^2/2 + o(\varepsilon^2)$ as $\varepsilon \to 0$ implies

$$\begin{aligned}
&\tau(n) \log\left(\frac{P_{\vartheta_n}(\boldsymbol{X} \ge c_n)}{P_0(\boldsymbol{X} \ge c_n)}\right) + (n - \tau(n)) \log\left(\frac{1 - P_{\vartheta_n}(\boldsymbol{X} \ge c_n)}{1 - P_0(\boldsymbol{X} \ge c_n)}\right) \\
&= \tau(n) \Bigg\{ \frac{P_{\vartheta_n}(\boldsymbol{X} \ge c_n) - P_0(\boldsymbol{X} \ge c_n)}{P_0(\boldsymbol{X} \ge c_n)} - \frac{1}{2}\left(\frac{P_{\vartheta_n}(\boldsymbol{X} \ge c_n) - P_0(\boldsymbol{X} \ge c_n)}{P_0(\boldsymbol{X} \ge c_n)}\right)^2 \\
&\qquad + O\left(\left|\frac{P_{\vartheta_n}(\boldsymbol{X} \ge c_n) - P_0(\boldsymbol{X} \ge c_n)}{P_0(\boldsymbol{X} \ge c_n)}\right|^3\right)\Bigg\} \\
&\quad + (n - \tau(n)) \Bigg\{ \frac{P_0(\boldsymbol{X} \ge c_n) - P_{\vartheta_n}(\boldsymbol{X} \ge c_n)}{1 - P_0(\boldsymbol{X} \ge c_n)} \\
&\qquad + O\left(\left|P_{\vartheta_n}(\boldsymbol{X} \ge c_n) - P_0(\boldsymbol{X} \ge c_n)\right|^2\right)\Bigg\} \\
&= \tau(n) \frac{P_{\vartheta_n}(\boldsymbol{X} \ge c_n) - P_0(\boldsymbol{X} \ge c_n)}{P_0(\boldsymbol{X} \ge c_n)} + (n - \tau(n)) \frac{P_0(\boldsymbol{X} \ge c_n) - P_{\vartheta_n}(\boldsymbol{X} \ge c_n)}{1 - P_0(\boldsymbol{X} \ge c_n)} \\
&\quad - \frac{\tau(n)}{2}\left(\frac{P_{\vartheta_n}(\boldsymbol{X} \ge c_n) - P_0(\boldsymbol{X} \ge c_n)}{P_0(\boldsymbol{X} \ge c_n)}\right)^2 + o_{P_0}(1)
\end{aligned}$$

as

$$\tau(n) \left|\frac{P_{\vartheta_n}(\boldsymbol{X} \ge c_n) - P_0(\boldsymbol{X} \ge c_n)}{P_0(\boldsymbol{X} \ge c_n)}\right|^3 \sim n P_0(\boldsymbol{X} \ge c_n)\, O\left(\frac{1}{(n\,|c_n|)^{3/2}}\right) \to_{n\to\infty} 0,$$

where $\sim$ denotes asymptotic equivalence. The preceding convergence to zero follows from the condition $n\,|c_n|^{1+2\delta} \to_{n\to\infty} \infty$ and the equivalence

$$(n - \tau(n))\, |P_{\vartheta_n}(\boldsymbol{X} \ge c_n) - P_0(\boldsymbol{X} \ge c_n)|^2 \sim O\left(|c_n|\right) \to_{n\to\infty} 0.$$

From the law of large numbers and Fact 1 we obtain

$$\begin{aligned}
&\tau(n)\left(\frac{P_{\vartheta_n}(\boldsymbol{X} \ge c_n) - P_0(\boldsymbol{X} \ge c_n)}{P_0(\boldsymbol{X} \ge c_n)}\right)^2 \\
&\sim n\,|c_n|\, A\left(\frac{1}{(n\,|c_n|)^{1/2}} \frac{\xi B}{(1+\delta)A} + o\left(\frac{1}{(n\,|c_n|)^{1/2}}\right)\right)^2 \to_{n\to\infty} \frac{\xi^2 B^2}{(1+\delta)^2 A}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\tau(n)\frac{P_{\vartheta_n}(\boldsymbol{X}\geq c_n)-P_0(\boldsymbol{X}\geq c_n)}{P_0(\boldsymbol{X}\geq c_n)}+(n-\tau(n))\frac{P_0(\boldsymbol{X}\geq c_n)-P_{\vartheta_n}(\boldsymbol{X}\geq c_n)}{1-P_0(\boldsymbol{X}\geq c_n)}\\
&=\frac{P_{\vartheta_n}(\boldsymbol{X}\geq c_n)-P_0(\boldsymbol{X}\geq c_n)}{P_0(\boldsymbol{X}\geq c_n)(1-P_0(\boldsymbol{X}\geq c_n))}(\tau(n)-nP_0(\boldsymbol{X}>c_n))\\
&=(n\,|c_n|\,A)^{1/2}\left(\frac{1}{(n\,|c_n|)^{1/2}}\frac{\xi B}{(1+\delta)A(1+o(1))}+o\left(\frac{1}{(n\,|c_n|)^{1/2}}\right)\right)\\
&\quad\times\frac{\tau(n)-nP_0(\boldsymbol{X}\geq c_n)}{(n\,|c_n|\,A)^{1/2}}\\
&=\frac{\xi B}{(1+\delta)A^{1/2}}Z_{n1}+o_{P_0}(1).
\end{aligned}$$

Altogether we have shown so far that

$$\begin{aligned}
&\tau(n)\log\left(\frac{P_{\vartheta_n}(\boldsymbol{X}\geq c_n)}{P_0(\boldsymbol{X}\geq c_n)}\right)+(n-\tau(n))\log\left(\frac{1-P_{\vartheta_n}(\boldsymbol{X}\geq c_n)}{1-P_0(\boldsymbol{X}\geq c_n)}\right)\\
&=\frac{\xi B}{(1+\delta)A^{1/2}}Z_{n1}-\frac{\xi^2B^2}{2(1+\delta)^2A}+o_{P_0}(1).
\end{aligned}$$

Next we show

$$\begin{aligned}
&\sum_{k\leq\tau(n)}\log\left(\frac{f_{\vartheta_n,c_n}(Y_k)}{f_{0,c_n}(Y_k)}\frac{P_0(\boldsymbol{X}\geq c_n)}{P_{\vartheta_n}(\boldsymbol{X}\geq c_n)}\right)\\
&=\frac{\xi B}{A^{1/2}(1+\delta)}Z_{n2}-\frac{\xi^2B^2\delta^2}{2A(2\delta+1)(1+\delta)^2}+o_{P_0}(1).
\end{aligned}$$

We have by Lemma 2.2

$$\frac{f_{\vartheta_n,c_n}(Y_k)}{f_{0,c_n}(Y_k)}=1+\frac{\xi}{(n\,|c_n|)^{1/2}}\frac{B}{A}Y_k^\delta+r_0(Y_k,\vartheta_n,c_n),$$

where $r_0(Y_k,\vartheta_n,c_n)=o((n\,|c_n|)^{-1/2})$ uniformly for k and n with

$$\begin{aligned}
E_0\left(r_0(Y_1,\vartheta_n,c_n)\right)&=\int_0^1 r_0(t,\vartheta_n,c_n)\frac{f_{0,c_n}(t)}{P_0(\boldsymbol{X}\geq c_n)}\,dt\\
&=\int_0^1\left(\frac{f_{\vartheta_n,c_n}(t)}{f_{0,c_n}(t)}-1-\frac{\xi}{(n\,|c_n|)^{1/2}}\frac{B}{A}t^\delta\right)\frac{f_{0,c_n}(t)}{P_0(\boldsymbol{X}\geq c_n)}\,dt\\
&=\frac{P_{\vartheta_n}(\boldsymbol{X}\geq c_n)-P_0(\boldsymbol{X}\geq c_n)}{P_0(\boldsymbol{X}\geq c_n)}-\frac{\xi}{(n\,|c_n|)^{1/2}}\frac{B}{(1+\delta)A}
\end{aligned}$$

and

$$Var_0\left(r_0(Y_1,\vartheta_n,c_n)\right)\leq E_0\left(r_0^2(Y_1,\vartheta_n,c_n)\right)=o\left(1/(n\,|c_n|)\right).$$

Using again the Taylor expansion $\log(1+\varepsilon) = \varepsilon - \varepsilon^2/2 + O(\varepsilon^3)$ as $\varepsilon \to 0$, we deduce

$$
\begin{aligned}
&\sum_{k\le\tau(n)} \log\left(\frac{f_{\vartheta_n,c_n}(Y_k)}{f_{0,c_n}(Y_k)} \frac{P_0(\boldsymbol{X}\ge c_n)}{P_{\vartheta_n}(\boldsymbol{X}\ge c_n)}\right)\\
&= \sum_{k\le\tau(n)} \log\left(1+\frac{\xi}{(n\,|c_n|)^{1/2}}\frac{B}{A}Y_k^\delta + r_0(Y_k,\vartheta_n,c_n)\right)\\
&\quad - \tau(n)\log\left(1+\frac{P_{\vartheta_n}(\boldsymbol{X}\ge c_n)-P_0(\boldsymbol{X}\ge c_n)}{P_0(\boldsymbol{X}\ge c_n)}\right)\\
&= \sum_{k\le\tau(n)} \left(\frac{\xi}{(n\,|c_n|)^{1/2}}\frac{B}{A}Y_k^\delta + r_0(Y_k,\vartheta_n,c_n) - \frac{\xi^2}{2n\,|c_n|}\frac{B^2}{A^2}Y_k^{2\delta}\right)\\
&\quad - \tau(n)\left(\frac{P_{\vartheta_n}(\boldsymbol{X}\ge c_n)-P_0(\boldsymbol{X}\ge c_n)}{P_0(\boldsymbol{X}\ge c_n)} - \frac{1}{2}\left(\frac{P_{\vartheta_n}(\boldsymbol{X}\ge c_n)-P_0(\boldsymbol{X}\ge c_n)}{P_0(\boldsymbol{X}\ge c_n)}\right)^2\right)\\
&\quad + o_{P_0}(1)\\
&= \sum_{k\le\tau(n)} \left(\frac{\xi}{(n\,|c_n|)^{1/2}}\frac{B}{A}\left(Y_k^\delta - \frac{1}{1+\delta}\right) + r_0(Y_k,\vartheta_n,c_n) - E_0(r_0(Y_1,\vartheta_n,c_n))\right)\\
&\quad - \frac{\xi^2}{2n\,|c_n|}\frac{B^2}{A^2}\sum_{k\le\tau(n)} Y_k^{2\delta} + \frac{\tau(n)}{n\,|c_n|}\frac{\xi^2 B^2}{2(1+\delta)^2 A^2} + o_{P_0}(1)\\
&= \frac{\tau(n)^{1/2}}{(n\,|c_n|)^{1/2}}\frac{1}{\tau(n)^{1/2}}\frac{\xi B}{A}\sum_{k\le\tau(n)}\left(Y_k^\delta - \frac{1}{1+\delta}\right)\\
&\quad - \frac{\tau(n)}{2n\,|c_n|}\frac{\xi^2 B^2}{A^2}\frac{1}{\tau(n)}\sum_{k\le\tau(n)} Y_k^{2\delta} + \frac{\tau(n)}{n\,|c_n|}\frac{\xi^2 B^2}{2(1+\delta)^2 A^2} + o_{P_0}(1)\\
&= \frac{\xi B}{A^{1/2}}\frac{1}{\tau(n)^{1/2}}\sum_{k\le\tau(n)}\left(Y_k^\delta - \frac{1}{1+\delta}\right) - \frac{\xi^2 B^2 \delta^2}{2A(2\delta+1)(1+\delta)^2} + o_{P_0}(1)\\
&\to_{D_0} N\left(-\frac{\xi^2 B^2\delta^2}{2A(2\delta+1)(1+\delta)^2}, \frac{\xi^2 B^2\delta^2}{A(2\delta+1)(1+\delta)^2}\right)
\end{aligned}
$$

by the law of large numbers and the central limit theorem. This completes the proof of Theorem 2.3. □

Proof of Proposition 2.4. First we compute the asymptotic mean and variance of $\Phi^{-1}(Y)$ under ϑ_n and c_n for $n \to \infty$. From Lemma 2.2 we obtain that the density of Y under ϑ_n is for $0 \le u \le 1$ and $c_n \ge c_0$ given by

$$
\begin{aligned}
p_{\vartheta_n,c_n}(u) &= \frac{f_{\vartheta_n,c_n}(u)}{P_{\vartheta_n}(\boldsymbol{X}\ge c_n)}\\
&= \frac{|c_n|}{P_{\vartheta_n}(\boldsymbol{X}\ge c_n)}\int_0^m z h_{\vartheta_n}(|c_n|\,uz)\left(P * \inf_{0\le t\le 1} Z_t\right)(dz).
\end{aligned}
$$

From Fubini's theorem and the substitution $u \mapsto \Phi(x)$ we, therefore, obtain

$$\begin{aligned}
&E_{\vartheta_n,c_n}\left(\Phi^{-1}(Y)\right)\\
&= \int_0^1 \Phi^{-1}(u) p_{\vartheta_n,c_n}(u)\,du\\
&= \frac{|c_n|}{P_{\vartheta_n}(\boldsymbol{X} \ge c_n)} \int_0^m z \int_0^1 \Phi^{-1}(u) h_{\vartheta_n}(|c_n|\,uz)\,du \left(P * \inf_{0\le t\le 1} Z_t\right)(dz)\\
&= \frac{|c_n|}{P_{\vartheta_n}(\boldsymbol{X} \ge c_n)} \int_0^m z \int_{-\infty}^{\infty} x h_{\vartheta_n}(|c_n|\,\Phi(x) z)\varphi(x)\,dx \left(P * \inf_{0\le t\le 1} Z_t\right)(dz)
\end{aligned}$$

where $\varphi(x) = \Phi'(x) = (2\pi)^{-1/2}\exp(-x^2/2)$, $x \in \mathbb{R}$, is the density of the standard normal df Φ.

From condition (6) we obtain the expansion

$$\begin{aligned}
&\int_0^m z \int_{-\infty}^{\infty} x h_{\vartheta_n}(|c_n|\,\Phi(x) z)\varphi(x)\,dx \,\left(P * \inf_{0\le t\le 1} Z_t\right)(dz)\\
&= \int_0^m z \int_{-\infty}^{\infty} x \left(1 + \vartheta_n \left(|c_n|\,\Phi(x) z\right)^{\delta} + r_{\vartheta_n}\left(|c_n|\,\Phi(x) z\right)\right)\varphi(x)\,dx\\
&\qquad\qquad\qquad\qquad \left(P * \inf_{0\le t\le 1} Z_t\right)(dz)\\
&= \vartheta_n\,|c_n|\,B \int_{-\infty}^{\infty} x\left(\Phi(x)\right)^{\delta}\varphi(x)\,dx + o\left(\vartheta_n\,|c_n|^{\delta}\right).
\end{aligned}$$

From Fact 1 and Fact 3 we obtain

$$\begin{aligned}
P_{\vartheta_n}(\boldsymbol{X} \ge c_n) &= P_0(\boldsymbol{X} \ge c_n) + |c_n|^{1+\delta}\,\frac{\vartheta_n}{1+\delta} + o\left(\vartheta_n\,|c_n|^{1+\delta}\right)\\
&= |c_n|\,A + |c_n|^{1+\delta}\,\frac{\vartheta_n}{1+\delta} B + o\left(\vartheta_n\,|c_n|^{1+\delta}\right)
\end{aligned}$$

and, thus,

$$\begin{aligned}
E_{\vartheta_n,c_n}\left(\Phi^{-1}(Y)\right) &= \frac{\vartheta_n\,|c_n|^{\delta}\,B \int_{-\infty}^{\infty} x\left(\Phi(x)\right)^{\delta}\varphi(x)\,dx + o\big(\vartheta_n\,|c_n|^{\delta}\big)}{A + |c_n|^{\delta}\,\frac{\vartheta_n}{1+\delta} B + o\big(\vartheta_n\,|c_n|^{\delta}\big)}\\
&= \frac{\frac{\xi}{(n|c_n|)^{1/2}} B \int_{-\infty}^{\infty} x\left(\Phi(x)\right)^{\delta}\varphi(x)\,dx + o\big(\vartheta_n\,|c_n|^{\delta}\big)}{A + |c_n|^{\delta}\,\frac{\vartheta_n}{1+\delta} B + o\big(\vartheta_n\,|c_n|^{\delta}\big)}\\
&= \frac{\xi}{(n\,|c_n|)^{1/2}}\,\frac{B}{A} \int_{-\infty}^{\infty} x\left(\Phi(x)\right)^{\delta}\varphi(x)\,dx + o\left(\frac{1}{(n\,|c_n|)^{1/2}}\right)
\end{aligned}$$

Equally, we obtain

$$\begin{aligned}
&E_{\vartheta_n,c_n}\left(\left(\Phi^{-1}(Y)\right)^2\right)\\
&= \frac{|c_n|}{P_{\vartheta_n}(\boldsymbol{X} \ge c_n)} \int_0^m z \int_{-\infty}^{\infty} x^2 h_{\vartheta_n}(|c_n|\,\Phi(x) z)\varphi(x)\,dx \left(P * \inf_{0\le t\le 1} Z_t\right)(dz)
\end{aligned}$$

$$= \frac{|c_n|}{P_{\vartheta_n}(\boldsymbol{X} \geq c_n)} \int_0^m z \int_{-\infty}^{\infty} x^2 \left(1 + \vartheta_n \left(|c_n| \, \Phi(x) z\right)^{\delta} + r_{\vartheta_n}\left(|c_n| \, \Phi(x) z\right)\right) \times \varphi(x) \, dx \left(P * \inf_{0 \leq t \leq 1} Z_t\right)(dz)$$

$$\sim 1$$

and, thus, the asymptotic variance of $\Phi^{-1}(Y)$ is under ϑ_n and c_n for $n \to \infty$ equivalent to 1.

Finally we have the expansion

$$\begin{aligned} E_{\vartheta_n, c_n}(\tau(n)) &= n P_{\vartheta_n}(\boldsymbol{X} \geq c_n) \\ &= n \, |c_n| \, A + (n \, |c_n|)^{1/2} \frac{B}{1+\delta} + o\left(n \, |c_n|^{1+\delta}\right) \\ &= n \, |c_n| \, A(1 + o(1)). \end{aligned}$$

Now we can compute the asymptotic distribution of T_{n,c_n} under ϑ_n. We have

$$\begin{aligned} T_{n,c_n} &= \frac{1}{\tau(n)^{1/2}} \sum_{k=1}^{\tau(n)} \Phi^{-1}(Y_k) \\ &= \frac{1}{\tau(n)^{1/2}} \sum_{k=1}^{\tau(n)} \left(\Phi^{-1}(Y_k) - E_{\vartheta_n, c_n}\left(\Phi^{-1}(Y)\right)\right) + \tau(n)^{1/2} E_{\vartheta_n, c_n}\left(\Phi^{-1}(Y)\right), \end{aligned}$$

where the first term is by a suitable version of the central limit theorem asymptotically standard normal distributed, and

$$\begin{aligned} \tau(n)^{1/2} E_{\vartheta_n, c_n}\left(\Phi^{-1}(Y)\right) &\sim E_{\vartheta_n, c_n}\left(\tau(n)^{1/2}\right) E_{\vartheta_n, c_n}\left(\Phi^{-1}(Y)\right) \\ &\sim (n \, |c_n| \, A)^{1/2} \frac{\xi}{(n \, |c_n|)^{1/2}} \frac{B}{A} \int_{-\infty}^{\infty} x \left(\Phi(x)\right)^{\delta} \varphi(x) \, dx \\ &\sim \xi \frac{B}{A^{1/2}} \int_{-\infty}^{\infty} x \left(\Phi(x)\right)^{\delta} \varphi(x) \, dx, \end{aligned}$$

which completes the proof of Proposition 2.4. □

Proof of Theorem 3.2. Again we compile several facts first.

$$C(\vartheta_n) = 1 - \vartheta_n \int_0^1 T(u) \, du + o(\vartheta_n), \qquad n \in \mathbb{N}. \qquad \text{(Fact 5)}$$

This follows from the expansion $\exp(x) = 1 + x + o(x)$ as $x \to 0$:

$$C(\vartheta_n) = \frac{1}{\int_0^1 \exp(\vartheta T(u)) \, du} = \frac{1}{1 + \vartheta_n \int_0^1 T(u) \, du + o(\vartheta_n)}.$$

Moreover, we have

$$\begin{aligned} & P_{\vartheta_n}(\boldsymbol{X} \geq c_n) - P_0(\boldsymbol{X} \geq c_n) \\ & = \vartheta_n \, |c_n| \, A \left(C - \int_0^1 T(u) \, du\right) + o(\vartheta_n \, |c_n|) \end{aligned}$$

$$= \left(\frac{|c_n|}{n}\right)^{1/2} A^{1/2}\xi + o\left(\left(\frac{|c_n|}{n}\right)^{1/2}\right). \qquad \text{(Fact 6)}$$

This can be seen as follows. From Remark 3.1 and Fact 5 we obtain

$$\begin{aligned}
&P_{\vartheta_n}(\boldsymbol{X} \geq c_n) - P_0(\boldsymbol{X} \geq c_n)\\
&= \int_0^1 f_{\vartheta_n,c_n}(u)\,du - \int_0^1 f_{0.c_n}(u)\,du\\
&= |c_n| \int_0^m z \int_0^1 h_{\vartheta_n}(|c_n|\,zu) - 1\,du \left(P * \inf_{0\leq t\leq 1} Z_t\right)(dz)\\
&= |c_n| \int_0^m z \int_0^1 C(\vartheta_n)\exp\left(\vartheta_n T(|c_n|\,zu)\right) - 1\ du \left(P * \inf_{0\leq t\leq 1} Z_t\right)(dz)\\
&= |c_n| \int_0^m z \int_0^1 \left(1 - \vartheta_n \int_0^1 T(x)\,dx + o(\vartheta_n)\right)(1 + \vartheta_n C + o(\vartheta_n)) - 1\ du\\
&\qquad\qquad \left(P * \inf_{0\leq t\leq 1} Z_t\right)(dz)\\
&= \vartheta_n\,|c_n|\,A\left(C - \int_0^1 T(x)\,dx\right) + o(\vartheta_n\,|c_n|),
\end{aligned}$$

which is Fact 6.

Fact 6 together with Fact 1 yields

$$\frac{P_{\vartheta_n}(\boldsymbol{X} \geq c_n) - P_0(\boldsymbol{X} \geq c_n)}{P_0(\boldsymbol{X} \geq c_n)} = \frac{1}{(n\,|c_n|)^{1/2}}\frac{\xi}{A^{1/2}} + o\left(\frac{1}{(n\,|c_n|)^{1/2}}\right). \qquad \text{(Fact 7)}$$

Repeating the arguments in the proof of Theorem 2.3 one shows that

$$\begin{aligned}
&\tau(n)\log\left(\frac{P_{\vartheta_n}(\boldsymbol{X} \geq c_n)}{P_0(\boldsymbol{X} \geq c_n)}\right) + (n - \tau(n))\log\left(\frac{1 - P_{\vartheta_n}(\boldsymbol{X} \geq c_n)}{1 - P_0(\boldsymbol{X} \geq c_n)}\right)\\
&= \xi\frac{\tau(n) - nP_0(\boldsymbol{X} \geq c_n)}{(n\,|c_n|\,A)^{1/2}} - \frac{\xi^2}{2} + o_{P_0}(1)\\
&= \xi Z_{n1} - \frac{\xi^2}{2} + o_{P_0}(1),
\end{aligned}$$

i.e., it is sufficient to prove

$$\sum_{k\leq\tau(n)} \log\left(\frac{f_{\vartheta_n,c_n}(Y_k)}{f_{0,c_n}(Y_k)}\frac{P_0(\boldsymbol{X} \geq c_n)}{P_{\vartheta_n}(\boldsymbol{X} \geq c_n)}\right) = o_{P_0}(1). \qquad (17)$$

Repeating the arguments in the proof of Fact 6 we obtain

$$\begin{aligned}
&\frac{f_{\vartheta_n,c_n}(u) - f_{0,c_n}(u)}{f_{0,c_n}(u)}\\
&= \frac{1}{A}\int_0^m z\,(h_\vartheta(|c_n|\,zu) - 1)\left(P * \inf_{0\leq t\leq 1} Z_t\right)(dz)
\end{aligned}$$

$$= \frac{1}{A}\int_0^m z\left(C(\vartheta_n)\exp(\vartheta_n T(|c_n|\,zu))-1\right)\left(P*\inf_{0\le t\le 1} Z_t\right)(dz)$$
$$= \frac{1}{A}\int_0^m z\left(\vartheta_n C-\vartheta_n\int_0^1 T(x)\,dx+o(\vartheta_n)\right)\left(P*\inf_{0\le t\le 1} Z_t\right)(dz)$$
$$= O(\vartheta_n)$$

uniformly for $u\in[0,1]$ and $n\in\mathbb{N}$. The expansion $\log(1+\varepsilon)=\varepsilon-\varepsilon^2/2+O\left(\varepsilon^2\right)$ for $\varepsilon\to 0$ together with Fact 7, thus, yields,

$$\begin{aligned}
&\sum_{k\le\tau(n)}\log\left(\frac{f_{\vartheta_n,c_n}(Y_k)}{f_{0,c_n}(Y_k)}\frac{P_0(\boldsymbol{X}\ge c_n)}{P_{\vartheta_n}(\boldsymbol{X}\ge c_n)}\right)\\
&=\sum_{k\le\tau(n)}\log\left(1+\frac{f_{\vartheta_n,c_n}(Y_k)-f_{0,c_n}(Y_k)}{f_{0,c_n}(Y_k)}\right)\\
&\qquad-\tau(n)\log\left(1+\frac{P_{\vartheta_n}(\boldsymbol{X}\ge c_n)-P_0(\boldsymbol{X}\ge c_n)}{P_0(\boldsymbol{X}\ge c_n)}\right)\\
&=\sum_{k\le\tau(n)}\left(\frac{f_{\vartheta_n,c_n}(Y_k)-f_{0,c_n}(Y_k)}{f_{0,c_n}(Y_k)}-\frac{1}{2}\left(\frac{f_{\vartheta_n,c_n}(Y_k)-f_{0,c_n}(Y_k)}{f_{0,c_n}(Y_k)}\right)^2\right)\\
&\qquad-\tau(n)\frac{P_{\vartheta_n}(\boldsymbol{X}\ge c_n)-P_0(\boldsymbol{X}\ge c_n)}{P_0(\boldsymbol{X}\ge c_n)}\\
&\qquad+\frac{\tau(n)}{2}\left(\frac{P_{\vartheta_n}(\boldsymbol{X}\ge c_n)-P_0(\boldsymbol{X}\ge c_n)}{P_0(\boldsymbol{X}\ge c_n)}\right)^2\\
&\qquad+O_{P_0}\left(\frac{1}{n\,|c_n|^{1/2}}\right).
\end{aligned}$$

Note that

$$E_{P_0}\left(f_{\vartheta_n,c_n}(Y)\right)=\int_0^1 f_{\vartheta_n,c_n}(u)\,du=P_{\vartheta_n}(\boldsymbol{X}\ge c_n)$$

and

$$f_{0,c_n}(u)=|c_n|\,A=P_0(\boldsymbol{X}\ge c_n).$$

We, thus, obtain

$$\begin{aligned}
&\sum_{k\le\tau(n)}\log\left(\frac{f_{\vartheta_n,c_n}(Y_k)}{f_{0,c_n}(Y_k)}\frac{P_0(\boldsymbol{X}\ge c_n)}{P_{\vartheta_n}(\boldsymbol{X}\ge c_n)}\right)\\
&=\sum_{k\le\tau(n)}\frac{f_{\vartheta_n,c_n}(Y_k)-P_{\vartheta_n}(\boldsymbol{X}\ge c_n)}{|c_n|\,A}-\frac{1}{2}\sum_{k\le\tau(n)}\left(\frac{f_{\vartheta_n,c_n}(Y_k)-P_{\vartheta_n}(\boldsymbol{X}\ge c_n)}{|c_n|\,A}\right)^2\\
&\quad+\frac{\tau(n)}{2}\left(\frac{P_{\vartheta_n}(\boldsymbol{X}\ge c_n)-P_0(\boldsymbol{X}\ge c_n)}{P_0(\boldsymbol{X}\ge c_n)}\right)^2+O_{P_0}\left(\frac{1}{n\,|c_n|^{1/2}}\right)
\end{aligned}$$

$$=: \mathrm{I}_n - \mathrm{II}_n + \mathrm{III}_n + O_{P_0}\left(\frac{1}{n\left|c_n\right|^{1/2}}\right).$$

From Fact 7 we obtain

$$\mathrm{III}_n \sim \frac{n\left|c_n\right| A}{2}\left(\frac{1}{(n\left|c_n\right|)^{1/2}}\frac{\xi}{A^{1/2}} + o\left(\frac{1}{(n\left|c_n\right|)^{1/2}}\right)\right)^2 \sim \frac{\xi^2}{2}. \tag{18}$$

Next we show that $\mathrm{I}_n = o_{P_0}(1)$. This assertion follows, if we show that

$$E_{P_0}\left(\left(\frac{f_{\vartheta_n,c_n}(Y_k) - P_{\vartheta_n}(\boldsymbol{X} \geq c_n)}{\left|c_n\right| A}\right)^2\right) = o\left(\frac{1}{n\left|c_n\right|}\right). \tag{19}$$

By elementary arguments we obtain

$$\begin{aligned}
&E_{P_0}\left(\left(\frac{f_{\vartheta_n,c_n}(Y_k) - P_{\vartheta_n}(\boldsymbol{X} \geq c_n)}{\left|c_n\right| A}\right)^2\right)\\
&= \frac{1}{c_n^2 A^2} E_{P_0}\left(\left(\int_0^m z\int_0^1 h_{\vartheta_n}(\left|c_n\right| zY) - h_{\vartheta_n}(\left|c_n\right| zu)\left(P * \inf_{0\leq t\leq 1} Z_t\right)(dz)\right)^2\right)\\
&= \frac{C(\vartheta_n)^2}{A^2} E_{P_0}\left(\left(\int_0^m z\int_0^1 \exp(\vartheta_n T(\left|c_n\right| zY)) - \exp(\vartheta_n T(\left|c_n\right| zu))\, du\right.\right.\\
&\qquad\qquad \left.\left.\left(P * \inf_{0\leq t\leq 1} Z_t\right)(dz)\right)^2\right)\\
&= o(\vartheta_n^2)
\end{aligned}$$

which is (19).

Finally we have

$$\begin{aligned}
&E_{P_0}\left(\left(\frac{f_{\vartheta_n,c_n}(Y) - \left|c_n\right| A}{\left|c_n\right| A}\right)^2\right)\\
&= \frac{1}{A^2} E_{P_0}\left(\left(\int_0^m z(h_{\vartheta_n}(\left|c_n\right| zY) - 1)\left(P * \inf_{0\leq t\leq 1} Z_t\right)(dz)\right)^2\right)\\
&= \frac{1}{A^2} E_{P_0}\left(\left(\int_0^m z(C(\vartheta_n)\exp(\vartheta_n T(\left|c_n\right| zY)) - 1)\left(P * \inf_{0\leq t\leq 1} Z_t\right)(dz)\right)^2\right)\\
&= \frac{1}{A^2} E_{P_0}\left(\left(\int_0^m z\left(\left(1 - \vartheta_n\int_0^1 T(u)\, du + o(\vartheta_n)\right)(1 + \vartheta_n T(\left|c_n\right| zY) + o(\vartheta_n)) - 1\right)\right.\right.\\
&\qquad\qquad \left.\left.\left(P * \inf_{0\leq t\leq 1} Z_t\right)(dz)\right)^2\right)\\
&= \vartheta_n^2\left(C - \int_0^1 T(u)\, du\right)^2 + o(\left|c_n\right|^2 \vartheta_n^2).
\end{aligned}$$

The law of large numbers implies

$$\mathrm{II}_n \to_{n\to\infty} -\frac{\xi^2}{2}$$

in probability, and, hence, (18) yields

$$\mathrm{III}_n - \mathrm{II}_n = o_{P_0}(1).$$

We, thus, have established (17), which completes the proof of Theorem 3.2. □

Proof of Lemma 3.3. We have

$$\begin{aligned}
&E_{\vartheta_n,c_n}\left(\Phi^{-1}(Y)\right)\\
&= \int_0^1 \Phi^{-1}(u) p_{\vartheta_n,c_n}(u)\,du\\
&= \frac{|c_n|}{P_{\vartheta_n}(\boldsymbol{X} \ge c_n)} \int_0^m z \int_0^1 h_{\vartheta_n}(|c_n|\, uz)\,du \left(P * \inf_{0\le t\le 1} Z_t\right)(dz)\\
&= \frac{|c_n|}{P_{\vartheta_n}(\boldsymbol{X} \ge c_n)} \int_0^m z \int_{-\infty}^{\infty} x h_{\vartheta_n}(|c_n|\,\Phi(x)z)\varphi(x)\,dx \left(P * \inf_{0\le t\le 1} Z_t\right)(dz).
\end{aligned}$$

Fact 7 implies

$$\begin{aligned}
&\int_0^m z \int_{-\infty}^{\infty} x h_{\vartheta_n}(|c_n|\,\Phi(x)z)\varphi(x)\,dx \left(P * \inf_{0\le t\le 1} Z_t\right)(dz)\\
&= \int_0^m z \int_{-\infty}^{\infty} x C(\vartheta_n) \exp\left(\vartheta_n T\left(|c_n|\,\Phi(x)z\right)\right)\varphi(x)\,dx \left(P * \inf_{0\le t\le 1} Z_t\right)(dz)\\
&= \int_0^m z \int_{-\infty}^{\infty} x \left(1 - \vartheta_n \int_0^1 T(u)\,du + o(\vartheta_n)\right)(1 + \vartheta_n C + o(\vartheta_n))\varphi(x)\,dx\\
&\qquad\qquad \left(P * \inf_{0\le t\le 1} Z_t\right)(dz)\\
&= \int_0^m z \int_{-\infty}^{\infty} x \left(1 + \vartheta_n\left(C - \int_0^1 T(u)\,du\right) + o(\vartheta_n)\right)\varphi(x)\,dx\\
&\qquad\qquad \left(P * \inf_{0\le t\le 1} Z_t\right)(dz)\\
&= o(\vartheta_n).
\end{aligned}$$

We have, moreover,

$$\begin{aligned}
&E_{\vartheta_n,c_n}\left(\left(\Phi^{-1}(Y)\right)^2\right)\\
&= \frac{|c_n|}{P_{\vartheta_n}(\boldsymbol{X} \ge c_n)} \int_0^m z \int_{-\infty}^{\infty} x^2 h_{\vartheta_n}(|c_n|\,\Phi(x)z)\varphi(x)\,dx \left(P * \inf_{0\le t\le 1} Z_t\right)(dz),
\end{aligned}$$

where

$$\int_0^m z \int_{-\infty}^{\infty} x^2 h_{\vartheta_n}(|c_n|\,\Phi(x)z)\varphi(x)\,dx\,\left(P * \inf_{0\le t\le 1} Z_t\right)(dz)$$
$$= A + \vartheta_n\left(C - \int_0^1 T(u)\,du\right)A + o(\vartheta_n).$$

From Fact 1 and Fact 6 we obtain

$$P_{\vartheta_n}(\boldsymbol{X} \ge c_n) = |c_n|\,A + \left(\frac{|c_n|}{n}\right)^{1/2} A^{1/2}\xi + o\left(\left(\frac{|c_n|}{n}\right)^{1/2}\right)$$

and, thus,

$$E_{\vartheta_n,c_n}\left(\Phi^{-1}(Y)\right) = o(\vartheta_n)$$

and

$$Var_{\vartheta_n,c_n}\left(\Phi^{-1}(Y)\right) = E_{\vartheta_n,c_n}\left(\left(\Phi^{-1}(Y)\right)^2\right) - E_{\vartheta_n,c_n}\left(\Phi^{-1}(Y)\right)^2$$
$$= 1 + o(\vartheta_n). \qquad \square$$

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