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# Asymptotic properties of an estimator of the drift coefficients of multidimensional Ornstein-Uhlenbeck processes that are not necessarily stable*

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**Abstract:** In this paper, we investigate the consistency and asymptotic efficiency of an estimator of the drift matrix, $F$, of Ornstein-Uhlenbeck processes that are not necessarily stable. We consider all the cases. (1) The eigenvalues of $F$ are in the right half space (i.e., eigenvalues with positive real parts). In this case the process grows exponentially fast. (2) The eigenvalues of $F$ are on the left half space (i.e., the eigenvalues with negative or zero real parts). The process where all eigenvalues of $F$ have negative real parts is called a stable process and has a unique invariant (i.e., stationary) distribution. In this case the process does not grow. When the eigenvalues of $F$ have zero real parts (i.e., the case of zero eigenvalues and purely imaginary eigenvalues) the process grows polynomially fast. Considering (1) and (2) separately, we first show that an estimator, $\hat{F}$, of $F$ is consistent. We then combine them to present results for the general Ornstein-Uhlenbeck processes. We adopt similar procedure to show the asymptotic efficiency of the estimator.

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## 1. Introduction

Multidimensional processes with linear drift parameter have been used for modelling various physical phenomena. Among recent papers, works by Jankunas and Khasminskii [13] and Khasminskii, Krylov and Moshchuk [16] on the estimation of the drift parameters of linear stochastic differential equations (of the form, $dX_t = AX_t dt + \sum_{i=1}^{n} \sigma_i X_t dw_i(t)$ and $dX_t = A_\theta X_t dt + \sum_{i=1}^{m} \sigma_i X_t dw_i(t)$) can be mentioned. It should be noted that our work on Ornstein-Uhlenbeck

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(OU) processes does not follow from theirs and that the methodology used in our paper is also quite different from theirs.

The motivation for this work comes from Lai and Wei's paper [23], in which the authors have shown the strong consistency of the least square estimators of the coefficients of the discrete univariate general AR(p) processes. In this paper, we not only show that an estimator (which is the maximum likelihood estimator in the special case when $A$ is nonsingular) of the drift parameter of the general multidimensional OU process is consistent but also show that it is asymptotically efficient. We consider the following SDE representation of the OU process:

$$dY_t = FY_t dt + AdW_t \tag{1.1}$$

with any starting point $Y_0$ independent of the Brownian motion $\{W_t, \quad t \geq 0\}$. Here $Y$ is a $p$-dimensional process, $A$ is a constant matrix of $p \times r$ dimesnion and $W_t$ is a $r$-dimensional standard Brownian motion. Notice that it is always easier to estimate $A$ through quadratic variation of the process by using Itô's rule. But, estimating $F$ is usually the more difficult task. It is generally believed that one needs stationarity of the process to estimate $F$. However, one may observe, $\int_0^T dY_t Y_t' = F(\int_0^T Y_t Y_t' dt) + A(\int_0^T dW_t Y_t')$. Thus, we define, $\hat{F}_T = (\int_0^T dY_t Y_t')(\int_0^T Y_t Y_t' dt)^{-1} = F + A(\int_0^T dW_t Y_t')(\int_0^T Y_t Y_t' dt)^{-1}$ when $(\int_0^T Y_t Y_t' dt)$ is invertible and, in this case, the estimator is unbiased (as the expectation of the second term is zero). We show here that $\hat{F}_T$ is a consistent and an asymptotically efficient estimator of $F$, irrespective of the stationarity (or stability) of the process, provided $F$ and $A$ together satisfy a *RANK* condition (a), given in Section 2. This RANK condition is essential to prove that $(\int_0^T Y_t Y_t' dt)$ is invertible. We note here, if $A$ is a nonsingular matrix, the *RANK* condition automatically holds. In fact, it is also easy to see that for a continuous autoregressive process (i.e., CAR(p)), the *RANK* condition holds.

We also make another assumption, condition (b). It is the distinctness of the eigenvalues with positive real parts. However, we point out that this condition can be relaxed with a condition (b') and also that if none of the conditions (b) or (b') hold it is still possible to proceed with the estimation (see the discussion after Remark 3.2). Notice that the condition (b') holds for the drift $F$ in CAR(p) processes.

An early basic work, proving weak consistency for the stable discrete-time case, was done by Mann and Wald [26]. Later, many authors extended their results in many directions, such as, strong consistency, unstable cases, mixed cases etc. (for details, see [23] and references therein). Most of the works of Konev and Pergamenshchikov in this direction (discrete and continuous time), including [17, 18] for the continuous time case, have been done in the context of sequential estimation, to get fixed accuracy estimators or to derive asymptotic or other desirable properties of such estimators. So, their aim, techniques and results somewhat differs from those of ours. The estimation of parameters for the continuous time stochastic processes have been extensively studied as well (see for example, Feigin [9], Basawa, Feigin and Heyde [7], Basawa and Prakasa Rao

[6], Arató [1], Dietz and Kutoyants [8], Kutoyants [20, 21], Barndorff-Nielson and Sorensen [3], Kutoyants and Pilibossian [22], Jankunas and Khasminskii [13], Khasminskii, Krylov and Moshchuk [16] Prakasa Rao [27, 28] and references therein). Therefore, the estimation of the paramater and its asymptotic studies have not been new. However, as far as we know, full study of multidimensional OU processes parameter estimation and the study of its asymptotics have not been done for the mixed model. In the present paper, while showing consistency and asymptotic efficiency for the multidimensional matrix valued parameter, which do not follow from that of univariate or vector valued case (see, for example, Kaufmann [15], Wei [29], Basawa and Prakasa Rao [6], Dietz and Kutoyants [8], Kutoyants [20, 21], Barndorff-Nielson and Sorensen [3], Prakasa Rao [27, 28] and references therein), it develops new methodology to deal with various cases as is done in Kaufmann [15] and Wei [29].

Our paper is organized as follows. In Section 2, we present the basic assumptions and the main theorems. In Section 3, we describe the case in which the eigenvalues of $F$ have positive real parts. Methodology used here is similar to that of Lai and Wei's paper [23]. However, it may be noted that the case, in which the eigenvalues of $F$ have negative or zero real parts, is quite different from either of them and it is discussed in Section 4. This case, in fact, combines the three cases, zero eigenvalues, purely imaginary eigenvalues and the eigenvalues with negative real parts. Details on the rates of growth and so forth for zero eigenvalues and imaginary eigenvalues are given in the Appendix. Section 5 examines the mixed case for consistency. The section 6 presents the results on asymptotic efficiency and Section 7 has some concluding remarks.

## 2. Basic assumptions and the main theorem

We can decompose any $p \times p$ matrix $F$ into the rational canonical form

$$MF = GM = \begin{pmatrix} G_0 & 0 \\ 0 & G_1 \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \end{pmatrix}$$

where $G_i$ are $p_i \times p_i$ matrices and $M_i$ are $p_i \times p$ matrices for $i = 0, 1$ and $p_0 + p_1 = p$. Rows of $M_i$ and rows of $M_j$ are orthogonal for $i \neq j$.

All roots of $G_0$ lie in the right half space; all roots of $G_1$ lie on the left half space.

**Example** Let

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 & 0 \\ 0 & -8 & 6 & 14 & 1 \\ 0 & 10 & -4 & -14 & -1 \\ 0 & -10 & 6 & 16 & 1 \\ 0 & -5 & 3 & 7 & 0 \end{pmatrix}.$$

Then the characteristic polynomial of A is

$$f(t) = (t-2)^3(t^2+1).$$

Thus $\phi_1(t) = t-2$ and $\phi_2(t) = t^2+1$ are the distinct irreducible monic divisors of $f(t)$. After computation, we find that $g(t) = \phi_1(t)^2\phi_2(t) = (t-2)^2(t^2+1)$ is the minimal polynomial of $A$ and thus the companion matrices for $\phi_1^2(t) = (t-2)^2$ and $\phi_1(t) = t-2$ are given by

$$\begin{pmatrix} 0 & -4 \\ 1 & 4 \end{pmatrix} \qquad \text{and} \qquad 2.$$

Similarly, the companion matrix for $\phi_2(t) = t^2+1$ is

$$\begin{pmatrix} 0 & -1 \\ 1 & 0. \end{pmatrix}$$

The rational canonical form of $A$ is thus

$$H_A = \begin{pmatrix} 0 & -4 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \square$$

In the example above, the rational canonical form of $A$ is formed by 3 blocks: $\begin{pmatrix} 0 & -4 \\ 1 & 4 \end{pmatrix}$, 2 and $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Therefore the dimensions of the 3 blocks are 2, 1 and 2 respectively.

**Assumption**

$$(a) \qquad \text{RANK}\left(\left[A : FA : \cdots : F^{p-1}A\right]\right) = p. \tag{2.1}$$

(b) The eigenvalues of $F$, which have positive real parts, are all distinct.

Observe that, from (1.1) $Y_t = e^{Ft}Y_0 + \int_0^t e^{F(t-s)}AdW_s$ and thus have a multivariate Gaussian distribution with the mean $e^{Ft}$ and the covariance matrix $\int_0^t e^{Ft}AA'e^{F't}$. Since $Y_t$ is Gaussian it has a positive density if and only if the covariance matrix is nonsingular. The RANK assumption which is the special case of Hörmander's hypoellipticity condition ensures the positive density of $Y_t$ (for details, see [12]), and hence the nonsingularity of covariance matrix.

Following Basawa and Rao ([6], pp.) it is clear that $\int_0^T Y_tY_t'dt$ is nonsingular under the RANK assumption.

Let $F_A = [A : FA : \cdots : F^{p-1}A]$. Then RANK $(F_A) = p$ by the RANK assumption. Consider for $i = 0, 1$,

$$p_i = \text{RANK}(M_iF_AF_A^{-1}) \le \text{RANK}(M_iF_A) \le p_i$$

where $F_A^{-1}$ is the right inverse of $F_A$. Therefore, RANK $(M_iF_A) = p_i$ for $i = 0, 1$. Since

$$\begin{aligned} M_iF_A &= \left[M_i\left[A : FA : \cdots : F^{p-1}A\right]\right] \\ &= \left[M_iA : M_iFA : \cdots : M_iF^{p-1}A\right] \\ &= \left[M_iA : G_iM_iA : \cdots : G_i^{p-1}M_iA\right], \end{aligned}$$

and as the higher power of $G_i$ can be expressed as a linear combination of $I$, $G_i, \ldots, G_i^{p_i-1}$,

$$\text{RANK}\left[M_iA : G_iM_iA : \cdots : G_i^{p_i-1}M_iA\right] = \text{RANK}\left[M_iF_A\right] = p_i. \qquad (2.2)$$

If we transform the process $Y_t$ to $U_{it} = M_iY_t$ for $i = 0, 1$,

$$\begin{aligned} M_i dY_t &= M_iFY_t dt + M_iAdW_t, \\ \textit{i.e.,} \quad dU_{it} &= G_iU_{it}dt + (M_iA)\,dW_t. \end{aligned}$$

From (2.2) and the argument given above, we conclude that $\int_0^T U_{it}U_{it}'dt$ is positive definite a.s. for $i = 0, 1$.

We now present our main theorems whose proofs are given in Section 5 and in Section 6, respectively. Throughout the paper, we use $\lambda_{\min}(C)$ and $\lambda_{\max}(C)$ to denote the minimum and maximum eigenvalues of a matrix $C$.

**Theorem 2.1** *Suppose, for the Ornstein-Uhlenbeck process defined in (1.1), the assumptions (a) and (b) hold. Define* $\hat{F}_T = (\int_0^T dY_tY_t')(\int_0^T Y_tY_t'dt)^{-1}$. *Then*

$$\liminf_{T\to\infty} \frac{1}{T}\lambda_{\min}\left(\int_0^T Y_tY_t'dt\right) > 0 \qquad \text{a.s.} \qquad (2.3)$$

*and*

$$\lim_{T\to\infty} \hat{F}_T = F \qquad \text{a.s.}$$

**Theorem 2.2** *Under the assumptions of Theorem 2.1, it follows that* $E(\text{Tr}[(\hat{F}_T - F)E(C_T)(\hat{F}_T - F)'])^{1/2} = O(1)$ *as* $T \to \infty$, *where* $\hat{F}_T$ *is as defined in Theorem 2.1 and* $C_T = \left(\int_0^T Y_tY_t'dt\right)$.

## 3. Eigenvalues in the right half space

We consider the case where all the eigenvalues of $F$ have positive real parts. In this case, it can be seen that $\|Y_t\| \to \infty$ exponentially fast as $t \to \infty$. To introduce the main result of this section we define a Gaussian random variable

$$Z = Y_0 + \int_0^\infty e^{-Fs}AdW_s.$$

Since all the eigenvalues of $F$ have positive real parts, it is clear that, $e^{-Ft}Y_t = Y_0 + \int_0^t e^{-Fs}AdW_s$ converges a.s. to $Z$ as $t \to \infty$. We now derive the following results.

**Theorem 3.1** *In addition to the assumptions and notations of Theorem 2.1, assume further that real parts of all the eigenvalues of $F$ are positive. Then,*

$$e^{-FT}\left(\int_0^T Y_tY_t'dt\right)e^{-F'T} \quad \text{converges a.s. to} \quad B = \int_0^\infty e^{-Ft}(ZZ')e^{-F't}dt.$$

*Moreover, $B$ is positive definite with probability 1. Consequently,*

$$\lim_{T\to\infty} T^{-1}\log\lambda_{\min}\left(\int_0^T Y_t Y_t' dt\right) = 2\lambda_0 \qquad \text{a.s.}$$

$$\lim_{T\to\infty} T^{-1}\log\lambda_{\max}\left(\int_0^T Y_t Y_t' dt\right) = 2\Lambda_0 \qquad \text{a.s.} \tag{3.1}$$

Here and throughout the paper, $\log x$ means the natural logarithm of $x$. Also, in the sequel we shall let $||x||$ denote the Euclidean norm of a $p$-dimensional vector $x = (x_1, \ldots, x_p)'$, i.e., $||x||^2 = x'x$. Moreover, by viewing a $p\times p$ matrix $A_0$ as linear operator, we define $||A_0|| = \sup_{||x||=1}||A_0 x||$. Thus, $||A_0||^2$ is equal to the maximum eigenvalue of $A_0' A_0$. Moreover, if $A_0$ is symmetric and non-negative definite, then $||A_0|| = \lambda_{\max}(A_0)$. In particular, for the companion matrix $e^{-FT}$ in Theorem 3.1, we have the following Lemma.

**Lemma 3.1** *Under the hypothesis of Theorem 3.1*

$$\begin{aligned} \log||e^{FT}|| &\sim \log||e^{F'T}|| \sim \Lambda_0 T, \\ \text{and} \qquad \log||e^{-FT}|| &\sim \log||e^{-F'T}|| \sim -\lambda_0 T \end{aligned} \tag{3.2}$$

*where we use the notation $f(T) \sim CT^k$ to denote $\lim_{T\to\infty} T^{-k} f(T) = C$.*

*Proof.* Suppose $\text{Re}[\lambda_k(F)] > 0$ for $k = 1, 2, \ldots, p$. Then

$$|e^{\lambda_k(F)}| = e^{\text{Re}[\lambda_k(F)]} > 1 \qquad \text{for} \quad k = 1, 2, \ldots, p.$$

Let $\lambda_0 = \min_{1\le k\le p}\text{Re}[\lambda_k(F)]$, $\Lambda_0 = \max_{1\le k\le p}\text{Re}[\lambda_k(F)]$. Denote the spectral radius of $F$ by $r_\sigma(F)$ (cf. [19]). Then

$$\lim_{T\to\infty}||e^{FT}||^{\frac{1}{T}} = r_\sigma(F) = \sup_{\lambda\in\sigma(e^F)}|\lambda| = \exp\left[\sup_{\lambda\in\sigma(F)}\text{Re}(\lambda)\right] = e^{\Lambda_0}$$

and so $\log||e^{FT}|| \sim \log||e^{F'T}|| \sim \Lambda_0 T$. Similarly, $\log||e^{-FT}|| \sim \log||e^{-F'T}|| \sim -\lambda_0 T$ since

$$\lim_{T\to\infty}||e^{-FT}||^{\frac{1}{T}} = \sup_{\lambda\in\sigma(e^{-F})}|\lambda| = \exp\left[\sup_{\lambda\in\sigma(-F)}\text{Re}(\lambda)\right] = e^{-\lambda_0}.$$

Thus, we have the proof of Lemma 3.1 □

*Proof of Theorem 3.1.* Let $Z_t = Y_0 + \int_0^t e^{-Fs} A dW_s$, then $Y_t = e^{Ft} Z_t$ and

$$Z_t \text{ converges a.s. to } Z = Y_0 + \int_0^\infty e^{-Fs} A dW_s.$$

Let $B_T = \int_0^T e^{-Ft} Z_T Z_T' e^{-F't} dt$,

$$\left|\left| e^{-FT}\left(\int_0^T Y_t Y_t' dt\right) e^{-F'T} - B_T \right|\right|$$

$$\begin{aligned}
&= \left|\left|\int_0^T e^{-F(T-t)} Z_t Z_t' e^{-F(T-t)} dt - \int_0^T e^{-Ft} Z_T Z_T' e^{-F't} dt \right|\right| \\
&= \left|\left|\int_0^T e^{-Ft} \left(Z_{T-t} Z_{T-t}' - Z_T Z_T'\right) e^{-F't} dt\right|\right| \\
&\leq \int_0^T ||e^{-Ft}||\, ||e^{-F't}||\, (||Z_{T-t}|| + ||Z_T||)\, ||Z_T - Z_{T-t}|| dt \\
&= \int_0^{T/2} ||e^{-Ft}||^2 (||Z_{T-t}|| + ||Z_T||) ||Z_T - Z_{T-t}|| dt \\
&\quad + \int_{T/2}^T ||e^{-Ft}||^2 (||Z_{T-t}|| + ||Z_T||) ||Z_T - Z_{T-t}|| dt. \qquad (3.3)
\end{aligned}$$

Since $Z_t$ converges almost surely to a finite random variable $Z$, $\sup_{\{t\geq 0\}} \|Z_t\|$ is finite almost surely and for each $t \geq T/2$, $||Z_T - Z_{T-t}||$, being a cauchy sequence, converges to zero, almost surely, as $T \to \infty$. Also, by Lemma 3.1, $\int_0^\infty ||e^{-Ft}||^2 dt < \infty$. Thus, we get, $\forall \omega$ outside a null set, $\forall \epsilon > 0$, there exists a $T_0(\omega)$ such that $\|Z_t(\omega) - Z(\omega)\| < \epsilon/(1 + \int_0^\infty ||e^{-Ft}||^2 dt + 2 \sup_{\{t\geq 0\}} \|Z_t(\omega)\|)$ for all $t \geq T_0(\omega)$. Fixing one such $\omega$, for $T \geq 2T_0(\omega)$ we have the first integral of (3.3), which is less than $\epsilon$ and the second integral goes to zero as $\sup_{\{t\geq 0\}} \|Z_t(\omega)\|$ is finite and $\int_{T/2}^T ||e^{-Ft}||^2 dt \to 0$ as $T \to \infty$.

Let $B = \int_0^\infty e^{-Ft} Z Z' e^{-F't} dt$, then with probability 1,

$$\begin{aligned}
& ||B_T - B|| \\
\leq\ & \int_T^\infty ||e^{-Ft} Z Z' e^{-F't}|| dt + \int_0^T ||e^{-Ft}(ZZ' - Z_T Z_T') e^{-F't}|| dt \\
\leq\ & ||ZZ'|| \int_T^\infty ||e^{-F't}||\, ||e^{-Ft}|| dt + ||ZZ' - Z_T Z_T'|| \int_0^T ||e^{-Ft}||\, ||e^{-F't}|| dt \\
\to\ & 0 \quad \text{a.s.,} \quad \text{as} \quad T \to \infty. \qquad (3.4)
\end{aligned}$$

Therefore,

$$e^{-FT} \left(\int_0^T Y_t Y_t' dt\right) e^{-F'T} \ \text{ converges a.s. to } \ B = \int_0^\infty e^{-Ft} Z Z' e^{-F't} dt. \quad (3.5)$$

To show $B = \int_0^\infty e^{-Ft} Z Z' e^{-F't} dt$ is positive definite with probability 1, observe that $Z$ has positive Gaussian density. Hence $P(Z \neq 0) = 1$. Fix an $\omega$, such that $Z(\omega) \neq 0$. Suppose, if possible,

$$x' \left(\int_0^\infty e^{-Ft} Z(\omega) Z(\omega)' e^{-F't} dt\right) x = 0 \quad \text{for some nonzero vector} \quad x \in \mathcal{R}^p.$$

Then, for almost all $t \in (0, T)$, $x' e^{-Ft} Z(\omega) = 0$, i.e., for almost all $t \in (0, T)$, $\sum_{k=0}^\infty \frac{1}{k!} (-1)^k x' F^k t^k Z(\omega) = 0$. This implies $x' F^k Z(\omega) = 0$, for $k = 0, 1, \ldots, p-$

1. By the assumption (b), $\sum_{k=0}^{p-1} a_k F^k$ is nonsingular for any real number $a_k$ with not all of them being zero. Hence, for any nonzero vector in $\mathcal{R}^p$, in particular for $x$, $x' \sum_{k=0}^{p-1} a_k F^k$ is a nonzero vector. In other words, for nonzero vector $x$, $\sum_{k=0}^{p-1} a_k (x' F^k)$ is nonzero for any nonzero vector $(a_0, \ldots, a_{p-1})$. Thus $\begin{pmatrix} x' \\ x'F \\ \vdots \\ x'F^{p-1} \end{pmatrix}$ is a nonsingular matrix. Hence, $\begin{pmatrix} x' \\ x'F \\ \vdots \\ x'F^{p-1} \end{pmatrix} Z(\omega) = 0$ implies $Z(\omega) = 0$, which is a contradiction. Thus, we arrive at a contradiction since $Z$ has a positive Gaussian density and hence $Z$ cannot be equal to zero on a set of positive measures. Therefore, we conclude that $B$ is positive definite with probability one.

To prove (3.1), we state the following elementary results (for the proof, see Lemma 2 of [23]): □

**Lemma 3.2** *Let $A$, $C$ be $p \times p$ matrices such that $C$ is symmetric and non-negative definite. Then*

$$\lambda_{\max}(C)\lambda_{\max}(AA') \geq \lambda_{\max}(ACA') \geq \lambda_{\min}(C)\lambda_{\max}(AA'),$$
$$\lambda_{\max}(C)\lambda_{\min}(AA') \geq \lambda_{\min}(ACA') \geq \lambda_{\min}(C)\lambda_{\min}(AA').$$

We continue the proof of (3.1) of Theorem 3.1. From Lemma 3.2 we get,

$$\begin{aligned} \log \lambda_{\min}\left(\int_0^T Y_t Y_t' dt\right) &\leq \log \lambda_{\max}\left[e^{-FT}\left(\int_0^T Y_t Y_t' dt\right) e^{-F'T}\right] \\ &\quad - \log \lambda_{\max}\left(e^{-FT} e^{-F'T}\right) \\ &\sim 2\lambda_0 T. \end{aligned}$$

Also,

$$\begin{aligned} \log \lambda_{\min}\left(\int_0^T Y_t Y_t' dt\right) &\geq \log \lambda_{min}\left[e^{-FT}\left(\int_0^T Y_t Y_t' dt\right) e^{-F'T}\right] \\ &\quad + \log \lambda_{\min}\left(e^{FT} e^{F'T}\right) \\ &\sim 2\lambda_0 T. \end{aligned}$$

Therefore

$$\lim_{T\to\infty} \frac{1}{T} \log \lambda_{\min}\left(\int_0^T Y_t Y_t' dt\right) = 2\lambda_0 \qquad \text{a.s.}$$

On the other hand,

$$\begin{aligned} \log \lambda_{\max}\left(\int_0^T Y_t Y_t' dt\right) &\leq \log \lambda_{\max}\left[e^{-FT}\left(\int_0^T Y_t Y_t' dt\right) e^{-F'T}\right] \\ &\quad + \log \lambda_{\max}\left(e^{FT} e^{F'T}\right) \\ &\sim 2\Lambda_0 T. \end{aligned}$$

Also,

$$\begin{aligned} \log \lambda_{\max}\left(\int_0^T Y_t Y_t' dt\right) &\geq \log \lambda_{\min}\left[e^{-FT}\left(\int_0^T Y_t Y_t' dt\right) e^{-F'T}\right] \\ &\quad - \log \lambda_{\min}\left(e^{-FT} e^{-F'T}\right) \\ &\sim 2\Lambda_0 T. \end{aligned}$$

Therefore

$$\lim_{T\to\infty} \frac{1}{T} \log \lambda_{\max}\left(\int_0^T Y_t Y_t' dt\right) = 2\Lambda_0 \qquad \text{a.s.}$$

Hence, we have the proof of Theorem 3.1. □

**Corollary 3.1** *Under the same assumptions and notations as in Theorem 3.1,*

$$(i) \qquad \lim_{T\to\infty} \int_0^T ||e^{-FT} Y_t|| dt = \int_0^\infty ||e^{-Ft} Z|| dt < \infty \qquad \text{a.s.} \qquad (3.6)$$

$$(ii) \qquad \frac{1}{\sqrt{T}} \left(\int_0^T dW_t Y_t'\right) e^{-F'T} = O(T^{-1/2}).$$

*Proof.* (i) Given $\epsilon > 0, \forall \omega$ outside a null set, $\exists T_0(\omega)$ such that

$$||Z_t - Z|| < \epsilon \qquad \forall t \geq T_0(\omega).$$

For $T > T_0(\omega)$,

$$\begin{aligned} &\left| \int_0^T ||e^{-F(T-t)} Z_t|| dt - \int_0^T ||e^{-F(T-t)} Z|| dt \right| \\ \leq &\int_0^T ||e^{-F(T-t)} Z_t - e^{-F(T-t)} Z|| dt \\ \leq &\int_0^T ||e^{-F(T-t)}||\, ||Z_t - Z|| dt \\ \leq &\int_0^{T_0(\omega)} ||e^{-F(T-t)}||\, ||Z_t - Z|| dt + \int_{T_0(\omega)}^T ||e^{-F(T-t)}||\, ||Z_t - Z|| dt. \end{aligned}$$

As $T \to \infty$, the first term tends to 0 since $||e^{-F(T-t)}|| \to 0$. The second term also tends to 0 since $Z_t \to Z$ and $\int_{T_0(\omega)}^T ||e^{-F(T-t)}|| dt \leq \int_0^T ||e^{-F(T-t)}|| dt = \int_0^T ||e^{-Ft}|| dt \leq \int_0^\infty ||e^{-F(T-t)}|| dt$, which is finite. Therefore,

$$\begin{aligned} \lim_{T\to\infty} \int_0^T ||e^{-FT} Y_t|| dt &= \lim_{T\to\infty} \int_0^T ||e^{-F(T-t)} Z_t|| dt \\ &= \lim_{T\to\infty} \int_0^T ||e^{-F(T-t)} Z|| dt \\ &= \int_0^\infty ||e^{-Ft} Z|| dt, \end{aligned}$$

which is finite almost surely, by Lemma 3.1.

(ii) Let $M_t = \left(\int_0^t dW_s Y_s'\right) e^{-F'T}$, which is a square integrable martingale for $0 \le t \le T$, with quadratic variation,

$$< M >_t = e^{-FT} \left( \int_0^t Y_s Y_s' ds \right) e^{-F'T} = e^{-FT} C_t e^{-F'T}$$

where $C_t = \int_0^t Y_s Y_s' ds$. By Karatzas and Shreve (cf [14], p174),

$$\begin{aligned}
\left( \int_0^t dW_s Y_s' \right) e^{-F'T} &= M_t = B_{<M>_t} \\
&= O\left( \lambda_{\max}(e^{-FT} C_t e^{-F'T}) \sqrt{\ln \ln \lambda_{\max}\left(e^{-FT} C_t e^{-F'T}\right)} \right) \\
&= O(1)
\end{aligned}$$

since for $t \le T$, $||e^{-FT} C_t e^{-F'T}|| \le ||e^{-FT} C_T e^{-F'T}|| \to B$, almost surely, as $T \to \infty$ and $B = O(1)$. Therefore,

$$\frac{1}{\sqrt{T}} \left( \int_0^T dW_t Y_t' \right) e^{-F'T} = O(T^{-1/2})$$

This completes the proof of Corollary 3.1. □

**Remark 3.1** *If all the eigenvalues of $F$ have positive real parts, we can relax condition (b) by*

$$(b') \qquad \sum_{k=0}^{p-1} a_k F^k \text{ being nonsingular for any reals } a_1, \dots, a_n \text{ with at least one of them being nonzero.} \tag{3.7}$$

*Notice that (b') could hold even if all the eigenvalues of $F$ are equal (say, $\lambda_0$), but the degree of the minimal polynomial of $F$ and the degree of the characteristic polynomial of $F$ are equal.*

**Remark 3.2** *Suppose, assumption (b) does not hold. One can still estimate the eigenvalues of $F$.*

Let the characteristic polynomial of $F$ be given as $\phi_F(x) = a_0 \Pi_{i=1}^k (x - \lambda_i)^{p_i} \Pi_{j=1}^l (x^2 + b_j x + c_j)^{q_j}$ where $\lambda_i$ are the real roots of multiplicity $p_i$ and $x^2 + b_j x + c_j$ are the irreducible polynomials giving the complex roots with multiplicity $q_j$ and $a_0$ is a constant. Let the minimal polynomial of $F$ be given by $\psi_F(x) = \Pi_{i=1}^k (x - \lambda_i)^{r_i} \Pi_{j=1}^l (x^2 + b_j x + c_j)^{s_j}$ with $r_i \le p_i$ and $s_j \le q_j$. If $r_i = p_i$ and $s_j = q_j$ for all $i$, $j$, then the degree of the minimal polynomial of $F$ and the degree of the characteristic polynomial of $F$ are the same and the assumption (b') holds and our results follow. If some of the $r_i$s are less than $p_i$s and/or $s_j$s are less than $q_j$, then, (b') does not hold for $F$. However, in that case, one can transform $F$ in the rational canonoical form as

$$\begin{pmatrix} J_1 \\ \vdots \\ J_k \\ K_1 \\ \vdots \\ K_l \\ L \end{pmatrix} F = \begin{pmatrix} B_1 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & B_k & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & C_1 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 0 & \cdots & C_l & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & D \end{pmatrix} \begin{pmatrix} J_1 \\ \vdots \\ J_k \\ K_1 \\ \vdots \\ K_l \\ L \end{pmatrix} = \begin{pmatrix} B_1 J_1 \\ \vdots \\ B_k J_k \\ C_1 K_1 \\ \vdots \\ C_l K_l \\ DL \end{pmatrix}$$

where $J_i$, $K_j$ and $L$ are rectangular matrices of full row rank, $(p_i - r_i)$, $(q_j - s_j)$, $(\sum_i r_i + \sum_j s_j)$, respectively, and $D$ is a square matrix of the dimension the same as the degree of the minimal polynomial of $F$ (i.e., same as $(\sum_i r_i + \sum_j s_j)$). For each $j$, $C_j$ is a partitioned diagonal matrix (i.e., only the diagonal blocks are nonzero blocks), each block is of dimension $2 \times 2$, and its diagonal block matrices are identical and repeating exactly $(q_j - s_j)$ times and have the characteristic polynomial $x^2 + b_j x + c_j$, and, for each $i$, $B_i$ is a diagonal matrix with diagonal entries consisting of the real characteristic root $\lambda_i$ repeating exactly $(p_i - r_i)$ times. Thus, we can work with $D$ instead of $F$. For $D$ the assumption (b') holds, since the degree of minimal polynomial of $D$ is same as that of $F$ and, consequently, the degree of the minimal polynomial of $D$ is the same as the degree of the characteristic polynomial of $D$. Estimation of $D$ can be done using the SDE of $LY_t$. For $B_i$ and $C_j$, one can consider each one separately and transform $Y_t$ to $J_i Y_t$ and $K_j Y_t$ and use the SDE of any component of $J_i Y_t$ (as it has the Markov property) to estimate $\lambda_i$ and the SDE of the first two (or, any (2m-1)th and 2mth) components of $K_j Y_t$ together, as they have the Markov property, to estimate a diagonal block of $C_j$. Hence the assertion in the last remark.

## 4. Eigenvalues on the left half space

In this Section, we study the asymptotic behavior of OU processes where the real parts of all the eigenvalues of $F$ are either zero or negative. Unlike the exponential rate of growth for $||Y_T||$, $\lambda_{\max}(\int_0^T Y_t Y_t' dt)$, $\lambda_{\min}(\int_0^T Y_t Y_t' dt)$ in Theorem 3.1 and Corollary 3.1 for the the process where all the eigenvalues of $F$ have positive real parts, the following theorem shows that these quantities grow at most polynomially fast in $t$ for these processes.

For stable processes $Y_t$ (i.e., eigenvalues of $F$ with negative real parts), we know from Basak and Bhattacharya [5] that

$$|Y_t^x - Y_t^0| \to 0 \quad \text{a.s.} \quad \text{as} \quad t \to \infty.$$

Therefore, the property of $Y_t$ starting at $x$ is the same as that from 0. Hence, without loss of generality, we can assume that $Y_0 = 0$.

**Theorem 4.1** *Suppose, for the Ornstein-Uhlenbeck process defined in (1.1), the RANK condition (2.1) holds and all the eigenvalues of $F$ have negative real parts. Then*

$$\liminf_{T\to\infty} \frac{1}{T}\lambda_{\min}\left(\int_0^T Y_t Y_t' dt\right) > 0 \qquad \text{a.s.} \tag{4.1}$$

*Moreover,*

$$\lambda_{\max}\left(\int_0^T Y_t Y_t' dt\right) \quad = \quad O(T) \qquad \text{a.s.} \tag{4.2}$$

*Proof.* To prove (4.1) and (4.2), consider each component $Y_t^i, Y_t^j$ of $Y_t$, $i, j = 1, \dots, p$. Let $\pi$ be the invariant distribution of $Y$. Then by the Strong Law of Large Numbers,

$$\frac{1}{T}\int_0^T Y_t^i Y_t^j dt \to E_\pi(Y^i Y^j) < \infty \quad \text{as} \quad T \to \infty,$$

which follows, afortiori, by the Law of the Iterated Logarithm by Basak [4]. Therefore,

$$\frac{1}{T}\int_0^T Y_t Y_t' dt \to E_\pi(YY') = \int_0^\infty e^{Fu} AA' e^{F'u} du,$$

which is positive definite a.s. Therefore,

$$\liminf_{T\to\infty} \frac{1}{T}\lambda_{\min}\left(\int_0^T Y_t Y_t' dt\right) > 0 \qquad \text{a.s.}$$

$$\text{and} \qquad \lambda_{\max}\left(\int_0^T Y_t Y_t' dt\right) = O(T) \qquad \text{a.s.}$$

Hence, the proof. □

**Remark 4.1** *(i) It is not difficult to see that for stable $Y_t$, for any $m \geq 1$, $E\left[\sup_{k-1\leq t\leq k}(Y_t' P Y_t)^m\right]$ is bounded uniformly over $k$. Hence, it would follow, for any $\delta > 0$, $||Y_t|| = o(t^{\frac{1}{2m}+\delta})$ a.s.*
*(ii) On the other hand, since $Y_t \to Y$ in distribution and $Y$ is finite with probability one, one obtains $Y_t = O_p(1)$.*

**Corollary 4.1** *With the same notations and assumptions as in Theorem 4.1, let $C_T = \int_0^T Y_t Y_t' dt$. Then*

$$(i) \qquad ||C_T^{-1/2}|| = O(T^{-1/2}), \quad \text{a.s.}$$
$$(ii) \qquad \lim_{T\to\infty} Y_T' C_T^{-1} Y_T = 0 \qquad \text{a.s.}$$

*Proof.* (i) Since $\liminf_{T\to\infty}\frac{1}{T}\lambda_{\min}(C_T) > 0$ a.s. from (4.1), therefore

$$||C_T^{-1/2}||^2 = \lambda_{\max}(C_T^{-1}) = \frac{1}{\lambda_{\min}(C_T)} = O(T^{-1}) \quad \text{a.s.}$$

(ii) By the previous remark 4.1 (i), we note that,

$$\begin{aligned} ||Y_T' C_T^{-1} Y_T|| &\leq ||Y_T||^2 ||C_T^{-1}|| \\ &= o(T^{1/2+2\delta})O(T^{-1}) \quad \text{a.s., for some } \delta > 0, \text{ small} \\ &= O(T^{-1/2+2\delta}) \end{aligned}$$

Hence, the proof. □

**Theorem 4.2** *Suppose eigenvalues of $F$ have either negative or zero real parts (i.e., the eigenvalues are on the Left Half Space, which includes zero eigenvalues, purely imaginary eigenvalues, eigenvalues with negative real parts). Then,*

$$\lim_{T\to\infty} Y_T' \left( \int_0^T Y_t Y_t' dt \right)^{-1} Y_T = 0 \quad \text{a.s.}$$

To prove Theorem 4.2, we need the following lemma:

**Lemma 4.1** *Let $\epsilon > 0$; define $F^\epsilon = F - \epsilon I$ and $dY_t^\epsilon = F^\epsilon Y_t^\epsilon dt + AdW_t$. Then $\frac{\partial}{\partial\epsilon}\ln\left[(Y_T^\epsilon)'(C_T^\epsilon)^{-1}(Y_T^\epsilon)\right]$ is bounded below, almost surely, uniformly for large values of $T$.*

*Proof.* Let $\dot{Y}_t^\epsilon = \frac{\partial}{\partial\epsilon} Y_t^\epsilon$. Then we have

$$d\dot{Y}_t^\epsilon = \left(-Y_t^\epsilon + F^\epsilon \dot{Y}_t^\epsilon\right)dt,$$

or jointly,

$$d\begin{pmatrix} Y_t^\epsilon \\ \dot{Y}_t^\epsilon \end{pmatrix} = \begin{pmatrix} F^\epsilon & 0 \\ -I & F^\epsilon \end{pmatrix}\begin{pmatrix} Y_t^\epsilon \\ \dot{Y}_t^\epsilon \end{pmatrix} dt + \begin{pmatrix} A \\ 0 \end{pmatrix} dW_t.$$

Since all eigenvalues of $\begin{pmatrix} F^\epsilon & 0 \\ -I & F^\epsilon \end{pmatrix}$ have negative real parts, $\begin{pmatrix} Y_t^\epsilon \\ \dot{Y}_t^\epsilon \end{pmatrix}$ is stable. Therefore,

$$\left|\left|\begin{pmatrix} Y_t^\epsilon \\ \dot{Y}_t^\epsilon \end{pmatrix}\right|\right| = o(t^{\frac{1}{4}+\delta}) \quad \text{a.s. for some} \quad \delta > 0$$

and

$$\frac{1}{T}\int_0^T \begin{pmatrix} Y_t^\epsilon \\ \dot{Y}_t^\epsilon \end{pmatrix} \begin{pmatrix} Y_t^\epsilon & \dot{Y}_t^\epsilon \end{pmatrix} dt$$

is positive definite (since the RANK condition holds here as well) and it converges almost surely to some positive definite constant matrix as $T \to \infty$.

Therefore, $(C_T^\epsilon)$ and $(\dot{C}_T^\epsilon)$ have the same order where $C_T^\epsilon = \int_0^T Y_t^\epsilon Y_t^\epsilon dt$ and $\dot{C}_T^\epsilon = \int_0^T \dot{Y}_t^\epsilon \dot{Y}_t^\epsilon dt$. Hence

$$(\dot{C}_T^\epsilon)(C_T^\epsilon)^{-1} = O(1) \quad \text{a.s. as} \quad T \to \infty. \tag{4.3}$$

By Corollary 4.1,

$$\begin{aligned}
\lim_{T\to\infty}(Y_T^\epsilon)'(C_T^\epsilon)^{-1}(Y_T^\epsilon) &= 0 \quad \text{a.s. and} \\
\lim_{T\to\infty}(\dot{Y}_T^\epsilon)'(C_T^\epsilon)^{-1}(\dot{Y}_T^\epsilon) &= \lim_{T\to\infty}(\dot{Y}_T^\epsilon)'(\dot{C}_T^\epsilon)^{-1}(\dot{Y}_T^\epsilon) = 0 \quad \text{a.s.}
\end{aligned}$$

Consider

$$\begin{aligned}
&\frac{\partial}{\partial\epsilon}(Y_T^\epsilon)'(C_T^\epsilon)^{-1}(Y_T^\epsilon) \\
= \;& 2(\dot{Y}_T^\epsilon)'(C_T^\epsilon)^{-1}Y_T^\epsilon + (Y_T^\epsilon)'\frac{\partial}{\partial\epsilon}(C_T^\epsilon)^{-1}Y_T^\epsilon \\
= \;& 2(\dot{Y}_T^\epsilon)'(C_T^\epsilon)^{-1}Y_T^\epsilon - (Y_T^\epsilon)'(C_T^\epsilon)^{-1}\left[\frac{\partial}{\partial\epsilon}C_T^\epsilon\right](C_T^\epsilon)^{-1}Y_T^\epsilon \\
\geq \;& -2\left[(\dot{Y}_T^\epsilon)'(C_T^\epsilon)^{-1}(\dot{Y}_T^\epsilon)\right]^{1/2}\left[(Y_T^\epsilon)'(C_T^\epsilon)^{-1}(Y_T^\epsilon)\right]^{1/2} \\
& - (Y_T^\epsilon)'(C_T^\epsilon)^{-1}\left[\int_0^T (Y_u^\epsilon)(\dot{Y}_u^\epsilon)'du + \int_0^T (\dot{Y}_u^\epsilon)(Y_u^\epsilon)'du\right](C_T^\epsilon)^{-1}Y_T^\epsilon \\
\geq \;& -2\left[(\dot{Y}_T^\epsilon)'(C_T^\epsilon)^{-1}(\dot{Y}_T^\epsilon)\right]^{1/2}\left[(Y_T^\epsilon)'(C_T^\epsilon)^{-1}(Y_T^\epsilon)\right]^{1/2} \\
& - 2\left[(Y_T^\epsilon)'(C_T^\epsilon)^{-1}(Y_T^\epsilon)\right]\int_0^T \left[(Y_u^\epsilon)'(C_T^\epsilon)^{-1}(Y_u^\epsilon)\right]^{1/2}\left[(\dot{Y}_u^\epsilon)'(C_T^\epsilon)^{-1}(\dot{Y}_u^\epsilon)\right]^{1/2}du \\
\geq \;& -2\left[(\dot{Y}_T^\epsilon)'(C_T^\epsilon)^{-1}(\dot{Y}_T^\epsilon)\right]^{1/2}\left[(Y_T^\epsilon)'(C_T^\epsilon)^{-1}(Y_T^\epsilon)\right]^{1/2} \\
& - 2\left[(Y_T^\epsilon)'(C_T^\epsilon)^{-1}(Y_T^\epsilon)\right]\left[\int_0^T (Y_u^\epsilon)'(C_T^\epsilon)^{-1}(Y_u^\epsilon)du + \int_0^T (\dot{Y}_u^\epsilon)'(C_T^\epsilon)^{-1}(\dot{Y}_u^\epsilon)du\right] \\
= \;& -2\left[(\dot{Y}_T^\epsilon)'(C_T^\epsilon)^{-1}(\dot{Y}_T^\epsilon)\right]^{1/2}\left[(Y_T^\epsilon)'(C_T^\epsilon)^{-1}(Y_T^\epsilon)\right]^{1/2} \\
& - 2\left[(Y_T^\epsilon)'(C_T^\epsilon)^{-1}(Y_T^\epsilon)\right]\left[p + \mathrm{Tr}[(\dot{C}_T^\epsilon)(C_T^\epsilon)^{-1}]\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\partial}{\partial\epsilon}\ln\left[(Y_T^\epsilon)'(C_T^\epsilon)^{-1}(Y_T^\epsilon)\right] \\
= \;& \left[(Y_T^\epsilon)'(C_T^\epsilon)^{-1}(Y_T^\epsilon)\right]^{-1}\frac{\partial}{\partial\epsilon}\left[(Y_T^\epsilon)'(C_T^\epsilon)^{-1}(Y_T^\epsilon)\right] \\
\geq \;& -2\left[\frac{(\dot{Y}_T^\epsilon)'(C_T^\epsilon)^{-1}(\dot{Y}_T^\epsilon)}{(Y_T^\epsilon)'(C_T^\epsilon)^{-1}(Y_T^\epsilon)}\right]^{1/2} - 2\left[p + \mathrm{Tr}\left[(\dot{C}_T^\epsilon)(C_T^\epsilon)^{-1}\right]\right],
\end{aligned}$$

which is bounded below (by a negative number possibly depending on $\epsilon$) uniformly for large values of $T$ by (4.3) and using the fact that both $(\dot{Y}_T^\epsilon)'(C_T^\epsilon)^{-1}(\dot{Y}_T^\epsilon)$ and $(\dot{Y}_T^\epsilon)'(\dot{C}_T^\epsilon)^{-1}(\dot{Y}_T^\epsilon)$ have the same order and the latter has the order as that of $(Y_T^\epsilon)'(C_T^\epsilon)^{-1}(Y_T^\epsilon)$. Hence the proof of Lemma 4.1. □

*Proof of Theorem 4.2.* Let $F^\epsilon = F - \epsilon I,\ \epsilon > 0$. Since all eigenvalues of $F$ are on the left half space, the real parts of all eigenvalues of $F^\epsilon$ are negative, i.e., $Y_t^\epsilon$ is a stable process. By Corollary 4.1,

$$\lim_{T\to\infty}(Y_T^\epsilon)'(C_T^\epsilon)^{-1}(Y_T^\epsilon) = 0.$$

Let $f(\epsilon) = \ln(Y_T^\epsilon)(C_T^\epsilon)^{-1}(Y_T^\epsilon)$. Fix an $\epsilon_1 > 0$. $f$ is a continuous function on $[0, \epsilon_1]$ and is differentiable in $(0, \epsilon_1)$. Then by the Mean Value Theorem, there exists an $\epsilon_0 \in (0, \epsilon_1)$ such that

$$f(\epsilon_1) - f(0) = \epsilon_1 \frac{\partial}{\partial\epsilon} f(\epsilon)\big|_{\epsilon=\epsilon_0}.$$

That is,

$$\frac{(Y_T^{\epsilon_1})'(C_T^{\epsilon_1})^{-1}(Y_T^{\epsilon_1})}{Y_T' C_T^{-1} Y_T} \geq \exp\left\{\epsilon_1 \frac{\partial}{\partial\epsilon} f(\epsilon)\,|_{\epsilon=\epsilon_0}\right\}, \tag{4.4}$$

which is uniformly positive (i.e., bounded away from zero) for large values of $T$ by Lemma 4.1. Since

$$\lim_{T\to\infty}(Y_T^\epsilon)'(C_T^\epsilon)^{-1}(Y_T^\epsilon) = 0 \quad \text{a.s.}$$

by (4.4)

$$\lim_{T\to\infty} Y_T' C_T^{-1} Y_T = 0 \quad \text{a.s.}$$

Hence the proof of Theorem 4.2. □

**Corollary 4.2** *With the same assumptions and notations as in Lemma 4.1,*

$$||C_T^{-1/2}|| = O(T^{-1/2}) \quad \text{a.s.}$$

*Proof.* Consider

$$\begin{aligned}
& \frac{\partial}{\partial\epsilon}\mathrm{Tr}[(C_T^\epsilon)^{-1}] \\
= \;& -2\mathrm{Tr}\left[(C_T^\epsilon)^{-1}\int_0^T \left[(Y_u^\epsilon)(\dot{Y}_u^\epsilon)'dt\right](C_T^\epsilon)^{-1}\right] \\
\geq \;& -2\mathrm{Tr}(C_T^\epsilon)^{-1}\int_0^T \left[(\dot{Y}_u^\epsilon)'(C_T^\epsilon)^{-1}(\dot{Y}_u^\epsilon)\right]^{1/2}\left[(Y_u^\epsilon)'(C_T^\epsilon)^{-1}(Y_u^\epsilon)\right]^{1/2} du \\
\geq \;& -\mathrm{Tr}\,(C_T^\epsilon)^{-1}\left[\int_0^T \left(\dot{Y}_u^\epsilon\right)'(C_T^\epsilon)^{-1}(\dot{Y}_u^\epsilon)du + \int_0^T (Y_u^\epsilon)'(C_T^\epsilon)^{-1}(Y_u^\epsilon)du\right] \\
= \;& -\mathrm{Tr}(C_T^\epsilon)^{-1}\left(\mathrm{Tr}\left[(C_T^\epsilon)^{-1}\dot{C}_T^\epsilon\right] + \mathrm{Tr}\left[(C_T^\epsilon)^{-1}(C_T^\epsilon)\right]\right).
\end{aligned}$$

Hence $\frac{\partial}{\partial\epsilon}\ln \mathrm{Tr}[(C_T^\epsilon)^{-1}] \geq -\big(\mathrm{Tr}\big[(C_T^\epsilon)^{-1}\dot{C}_T^\epsilon\big] + p\big)$ which is bounded below (by a negative number possibly depending on $\epsilon$) uniformly for large values of $T$. Therefore, as in (4.4), by the Mean Value Theorem, $\frac{\mathrm{Tr}[(C_T^\epsilon)^{-1}]}{\mathrm{Tr}[(C_T)^{-1}]}$ is uniformly positive (i.e., bounded away from zero) for large values of $T$. Since $\mathrm{Tr}[(C_T^\epsilon)^{-1}] = O(T^{-1})$, we have $O(\mathrm{Tr}[(C_T)^{-1}]) \leq O(\mathrm{Tr}[(C_T^\epsilon)^{-1}]) = O(T^{-1})$. Again, as for any positive definite matrix $K_T$, $O(||K_T||) = O(\mathrm{Tr}(K_T))$, we obtain by Corollary 4.1(i), $||(C_T)^{-1/2}|| = ||(C_T^\epsilon)^{-1/2}|| = O(T^{-1/2})$. Hence the result. □

**Remark 4.2** *It is clear from the arguments in the above corollary 4.2 that, for the eigenvalues of $F$ on the left half space,*

$$\frac{1}{T}\lambda_{\min}(C_T) = \frac{1}{T\lambda_{\max}(C_T^{-1})} > 0,$$

*almost surely, uniformly for large values of $T$, since $T\lambda_{\max}(C_T^{-1}) = T||C_T^{-1}|| \leq T\ O(T^{-1}) = O(1)$* a.s.

## 5. General Ornstein-Uhlenbeck processes

For the Ornstein-Uhlenbeck process defined in (1.1) with RANK condition (2.1), we have considered the case in which all the eigenvalues of $F$ have positive real parts and the case in which all the eigenvalues of $F$ have zero or negative real parts (i.e., zero eigenvalues, purely imaginary and the eigenvalues with negative real parts). Now we combine these cases to discuss the *mixed model* in which $F$ can be decomposed into rational canonical form as follows:

$$MF = GM = \begin{pmatrix} G_0 & 0 \\ 0 & G_1 \end{pmatrix}\begin{pmatrix} M_0 \\ M_1 \end{pmatrix} = \begin{pmatrix} G_0M_0 \\ G_1M_1 \end{pmatrix},$$

where all the characteristic roots of $G_0$ lie in the right half space and all the characteristic roots of $G_1$ lie on the left half space. Let

$$\begin{pmatrix} U_{0t} \\ U_{1t} \end{pmatrix} = \begin{pmatrix} M_0 \\ M_1 \end{pmatrix} Y_t = MY_t.$$

Then

$$\begin{aligned} d\begin{pmatrix} U_{0t} \\ U_{1t} \end{pmatrix} &= MdY_t = MFY_tdt + MAdW_t \\ &= \begin{pmatrix} G_0 & 0 \\ 0 & G_1 \end{pmatrix}\begin{pmatrix} U_{0t} \\ U_{1t} \end{pmatrix} dt + MAdW_t. \end{aligned}$$

Also,

$$\begin{aligned} \left(\int_0^T dW_tY_t'\right)M' &= \begin{pmatrix} \int_0^T dW_tU_{0t}' \\ \int_0^T dW_tU_{1t}' \end{pmatrix} \\ \text{and} \qquad M\left(\int_0^T Y_tY_t'dt\right)M' &= \begin{pmatrix} \int_0^T U_{0t}U_{0t}'dt & \int_0^T U_{0t}U_{1t}'dt \\ \int_0^T U_{1t}U_{0t}'dt & \int_0^T U_{1t}U_{1t}'dt \end{pmatrix}. \end{aligned}$$

Define, $C_{1T} = \int_0^T U_{1t}U'_{1t}dt$. We now derive the following result.

**Lemma 5.1** *Suppose, for the Ornstein-Uhlenbeck process defined in (1.1), the assumptions (a) and (b) hold. Then*

$$\Sigma_T^{-1} = \left[D_T M \left(\int_0^T Y_t Y'_t dt\right) M' D'_T\right]^{-1} \to \begin{pmatrix} B^{-1} & 0 \\ 0 & I_{p_1} \end{pmatrix} \quad \text{a.s.} \tag{5.1}$$

*where $B$ is defined in Section 2 (before (3.4)), $I_{p_1}$ is a $p_1$-dimensional identity matrix and*

$$D_T = \begin{pmatrix} e^{-G_0 T} & 0 \\ 0 & C_{1T}^{-1/2} \end{pmatrix}.$$

*Proof.* Observing (5.1), we obtain, by Theorem 3.1, that

$$\lim_{T\to\infty} e^{-G_0 T}\left(\int_0^T U_{0t}U'_{0t}dt\right) e^{-G'_0 T} = B \qquad \text{is positive definite a.s.}$$

Again, $(\Sigma_T)_{11} = C_{1T}^{-1/2} C_{1T} C_{1T}^{-1/2} = I_{p_1}$. Hence, the proof is complete once we show $e^{-G_0 T}(\int_0^T U_{0t}U'_{1t}dt)C_{1T}^{-1/2} \to 0_{p_0\times p_1}$ matrix almost surely, as $T \to \infty$. Notice that, by Corollary 3.1,

$$\lim_{T\to\infty} \int_0^T ||e^{-G_0 T}U_{0t}||dt < \infty \qquad \text{a.s.}$$

and from Theorem 4.2

$$\lim_{T\to\infty} U'_{1T} C_{1T}^{-1} U_{1T} = 0 \qquad \text{a.s.}$$

Therefore, for all $\omega$ outside a null set, and for any given $\epsilon > 0$, there exists $T_0(\omega) > 0$ such that for all $t \geq T_0(\omega)$, $(U'_{1t}C_{1t}^{-1}U_{1t})^{1/2} < \epsilon/(\lim_{T\to\infty}\int_0^T ||e^{-G_0 T}U_{0t}(\omega)||dt)$. Hence

$$\begin{aligned} \left|\left|e^{-G_0 T}\left(\int_0^T U_{0t}U'_{1t}dt\right)C_{1T}^{-1/2}\right|\right| &\leq \int_0^T ||e^{-G_0 T}U_{0t}U'_{1t}C_{1T}^{-1/2}||dt \\ &\leq \int_0^{T_0(\omega)} ||e^{-G_0 T}U_{0t}||\, ||C_{1T}^{-1/2}U_{1t}||dt \\ &\quad + \int_{T_0(\omega)}^T ||e^{-G_0 T}U_{0t}||\, ||C_{1T}^{-1/2}U_{1t}||dt \end{aligned}$$

As $T \to \infty$, the first term goes to 0 since $T_0(\omega)$ is fixed. The second term is less than $\epsilon$ by the choice of $T_0(\omega)$ since $C_{1t}$ is increasing in $t$ (in the sense that $C_{1t_2} - C_{1t_1}$ is positive definite whenever $t_2 > t_1$) and $||C_{1T}^{-1/2}U_{1t}|| = (U'_{1t}C_{1T}^{-1}U_{1t})^{1/2} \leq (U'_{1t}C_{1t}^{-1}U_{1t})^{1/2}$. As $\epsilon$ is arbitrary, the proof is complete. □

We now observe that,

$$\hat{F}_T - F = \left[T^{-1/2}A\left(\int_0^T dW_tY_t'\right)M'D_T'\right]\left[D_TM\left(\int_0^T Y_tY_t'dt\right)M'D_T'\right]^{-1} \times(T^{1/2}D_TM)$$

and

$$T^{-1/2}A\left(\int_0^T dW_tY_t'\right)M'D_T' = \begin{pmatrix} T^{-1/2}e^{-G_0T}\left(\int_0^T U_{0t}dW_t'\right)A' \\ T^{-1/2}C_{1T}^{-1/2}\left(\int_0^T U_{1t}dW_t'\right)A' \end{pmatrix}'.$$

The first term $T^{-1/2}A\left(\int_0^T dW_tU_{0t}'\right)e^{-G_0'T} = O(T^{-1/2})$ by Corollary 3.1(ii). To show the remaining terms converges to 0, we prove the following Theorem. This theorem is in the spirit of Theorem 2.2 of Wei [29], which is presented for the discrete case.

**Theorem 5.1** *Suppose, for the Ornstein-Uhlenbeck process defined in (1.1), the RANK condition (2.1) holds. Then*

$$\frac{1}{\sqrt{T}}\left(\int_0^T dW_tU_{1t}'\right)C_{1T}^{-1/2} \to 0 \qquad \text{a.s. as} \quad T\to\infty.$$

To prove Theorem 5.1, we need the following lemmas.

**Lemma 5.2** *Fix $t_0 > 0$. Then,*

$$\int_{t_0}^T U_{1t}'C_{1t}^{-1}U_{1t}dt = O(\log T) \qquad \text{a.s. as} \quad T\to\infty.$$

*Proof.* Notice that,

$$\frac{d}{dt}\log|C_{1t}| = \mathrm{Tr}\left(C_{1t}^{-1}\frac{d}{dt}C_{1t}\right) = \mathrm{Tr}\left(C_{1t}^{-1}U_{1t}U_{1t}'\right) = U_{1t}'C_{1t}^{-1}U_{1t},$$

where $|C_{1t}|$ is the determinant of $C_{1t}$. Observe that, $G_1$ can be further decomposed into a rational canonical form as follows:

$$\begin{pmatrix} M_{11} \\ M_{12} \\ M_{13} \end{pmatrix} G_1 = \begin{pmatrix} G_{11} & 0 & 0 \\ 0 & G_{12} & 0 \\ 0 & 0 & G_{13} \end{pmatrix}\begin{pmatrix} M_{11} \\ M_{12} \\ M_{13} \end{pmatrix} = \begin{pmatrix} G_{11}M_{11} \\ G_{12}M_{12} \\ G_{13}M_{13} \end{pmatrix},$$

where all the characteristic roots of $G_{11}$ have negative real parts, those of $G_{12}$ are purely imaginary and those of $G_{13}$ are zero. For $i,j = 1,2,3$, define $C_{1tij} = \int_0^t U_{1is}U_{1js}'ds$, where

$$\begin{pmatrix} U_{11s} \\ U_{12s} \\ U_{13s} \end{pmatrix} = \begin{pmatrix} M_{11} \\ M_{12} \\ M_{13} \end{pmatrix} U_{1s}.$$

Thus $C_{1t} = ((C_{1tij}))_{i,j=1,2,3}$, and hence $|C_{1t}| \leq |C_{1t11}|\,|C_{1t22}|\,|C_{1t33}|$. Therefore, by Theorem 4.1 in Section 4 and Theorems 8.1 and 8.2 in the Appendix, one obtains

$$\int_{t_0}^{T} U_{1t}' C_{1t}^{-1} U_{1t} dt = \log \frac{|C_{1T}|}{|C_{1t_0}|} = O(\log T) \qquad \text{a.s. as} \quad T \to \infty.$$

Hence, the proof. □

We observe that, from Lemma 5.2, if we let $g(T) = \int_{t_0}^{T} U_{1t}' C_{1t}^{-1} U_{1t} dt$, then $g(T) \uparrow \infty$ as $T \uparrow \infty$ almost surely. Also, $E(\log|C_{1T}|) = E(\sum_i \log(\lambda_i(C_{1T}))) = \sum_i E(\log(\lambda_i(C_{1T}))) \leq \sum_i \log(E(\lambda_i(C_{1T}))) \leq p_1 \log(E(\lambda_{max}(C_{1T}))) \leq p_1 \log \int_0^T E(\|U_{1t}\|^2) dt$. It is clear that, for the eigenvalues on the left half space, $E(\|U_{1t}\|^2)$ is at most $O(t^k)$, i.e., it grows at most like a polynomial in $t$. Thus, $E(\log|C_{1T}|) = O(\log T)$ as well. Hence, using integration by parts, we obtain,

$$E\left( \int_{t_0}^{\infty} \frac{U_{1t}' C_{1t}^{-1} U_{1t}}{t} dt \right) < \infty. \tag{5.2}$$

**Lemma 5.3** *Let $M_{1T} = \int_0^T dW_t U_{1t}'$. Then, under the hypothesis of Theorem 5.1,*

$$\frac{1}{T^{1/2}} M_{1T} C_T^{-1/2} \to 0 \qquad \text{in probability.}$$

*Proof.* Notice that $M_{1t}$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ where $\mathcal{F}_t = \sigma\{W_s \,:\, 0 \leq s \leq t\}$. Define $N_{1T} = \int_{t_1}^{T} dW_t U_{1t}' = M_{1T} - M_{1t_1}$. Then, for $T > t_1$, $N_{1T}$ is also a martingale. Define $V_t = \mathrm{Tr}[C_{1t}^{-1} M_{1t}' M_{1t}]/t$ and $\tilde{V}_t = \mathrm{Tr}[C_{1t}^{-1} N_{1t}' N_{1t}]/t$. Since $\|\frac{1}{T^{1/2}} M_{1T} C_T^{-1/2}\|^2 \leq V_T \leq 2\tilde{V}_T + 2\mathrm{Tr}(C_{1T}^{-1} M_{1t_1}' M_{1t_1}]/T$ and $\mathrm{Tr}(C_{1T}^{-1} M_{1t_1}' M_{1t_1}]/T \to 0$, almost surely, as $T \to \infty$, it is enough to show that $\tilde{V}_T \to 0$, in probability, as $T \to \infty$ and this would be immediate once one shows $E(\tilde{V}_T) \to 0$ as $T \to \infty$. Now use Itô's Lemma to get

$$d\tilde{V}_t \;=\; \frac{\left[\mathrm{Tr}\left(C_{1t}^{-1} d(N_{1t}' N_{1t})\right) + \mathrm{Tr}\left[(\dot{C}_{1t}^{-1}) N_{1t}' N_{1t}\right] dt\right]}{t} - \frac{\tilde{V}_t}{t} dt \tag{5.3}$$

where $\dot{C}_{1t}^{-1} = -C_{1t}^{-1} (\dot{C}_{1t}) C_{1t}^{-1} = -C_{1t}^{-1} U_{1t} U_{1t}' C_{1t}^{-1}$ which is non-positive definite. Thus, $\mathrm{Tr}\big[(\dot{C}_{1t}^{-1}) N_{1t}' N_{1t}\big] = -U_{1t}' C_{1t}^{-1} N_{1t}' N_{1t} C_{1t}^{-1} U_{1t} \leq 0$. Therefore, by (5.3) and applying the Itô's Lemma again, one obtains

$$\begin{aligned} \tilde{V}_T &\leq \int_{t_1}^{T} \mathrm{Tr}\left(C_{1t}^{-1} d(N_{1t}' N_{1t})\right)/t \\ &= \int_{t_1}^{T} \mathrm{Tr}\left(C_{1t}^{-1} \left[(dN_{1t}') N_{1t} + N_{1t}'(dN_{1t}) + (dN_{1t}')(dN_{1t})\right]\right)/t. \end{aligned}$$

Define $\tau_n = \inf\{t > t_1 : |\tilde{V}_t| \geq n\}$, then

$$\begin{aligned} E\tilde{V}_{T\wedge\tau_n} &\leq E\int_{t_1}^{T\wedge\tau_n} \mathrm{Tr}\left(C_{1t}^{-1}(dN'_{1t})(dN_{1t})\right)/t \qquad (5.4) \\ &= E\int_{t_1}^{T\wedge\tau_n} \frac{U'_{1t}C_{1t}^{-1}U_{1t}}{t}dt. \end{aligned}$$

Since $V_{T\wedge\tau_n}$ and $U'_{1t}C_{1t}^{-1}U_{1t}$ are non-negative, by Fatou's Lemma and the Monotone Convergence Theorem,

$$E\tilde{V}_T \leq E\int_{t_1}^{T} \frac{U'_{1t}C_{1t}^{-1}U_{1t}}{(\log t)^{1+\alpha}}dt.$$

Now, by the argument in (5.2), one has $\limsup_{\{T\to\infty\}} E\tilde{V}_T \leq \alpha C t_1^{-\alpha}$. As $t_1$ can be taken to be arbitrarily large, we have the result. ☐

**Lemma 5.4** *Let $V_t = \mathrm{Tr}[C_{1t}^{-1}M'_{1t}M_{1t}]/t$. Then, with the same assumptions and notations as in Lemma 5.3,*

$$\int_{t_1}^{\infty} E\left[E(dV_t|\mathcal{F}_t)\right]^+ < \infty.$$

*Proof.* Applying Itô's Lemma on $V_t$,

$$dV_t = \frac{\left(\mathrm{Tr}\left[C_{1t}^{-1}d(M'_{1t}M_{1t})\right] + \mathrm{Tr}\left[\dot{C}_{1t}^{-1}(M'_{1t}M_{1t})\right]dt\right)}{t} - \frac{V_t}{t}dt$$

where $\dot{C}_{1t}^{-1} = -C_{1t}^{-1}(\dot{C}_{1t})C_{1t}^{-1} = -C_{1t}^{-1}U_{1t}U'_{1t}C_{1t}^{-1}$ and

$$\mathrm{Tr}\left[\dot{C}_{1t}^{-1}(M'_{1t}M_{1t})\right] = -U'_{1t}C_{1t}^{-1}M'_{1t}M_{1t}C_{1t}^{-1}U_{1t} \leq 0.$$

Therefore,

$$\begin{aligned} & E(dV_t|\mathcal{F}_t) \\ \leq\ & E\left(\left[\mathrm{Tr}(C_{1t}^{-1}d(M'_{1t}M_{1t}))\right]/t \mid \mathcal{F}_t\right) \\ =\ & E\left(\left[\mathrm{Tr}\left(C_{1t}^{-1}[(dM'_{1t})M_{1t} + M'_{1t}(dM_{1t}) + (dM'_{1t})(dM_{1t})]\right)\right]/t \mid \mathcal{F}_t\right) \\ =\ & E\left(\left[\mathrm{Tr}\left(C_{1t}^{-1}(dM'_{1t})(dM_{1t})\right)\right]/t \mid \mathcal{F}_t\right) \\ =\ & E\left(\frac{U'_{1t}C_{1t}^{-1}U_{1t}}{t}dt \mid \mathcal{F}_t\right) \\ =\ & \frac{U'_{1t}C_{1t}^{-1}U_{1t}}{t}dt. \end{aligned}$$

Thus,

$$\left[E(dV_t|\mathcal{F}_t)\right]^+ \leq \frac{U'_{1t}C_{1t}^{-1}U_{1t}}{t}dt.$$

Since $U_{1t}'C_{1t}^{-1}U_{1t} \geq 0$, by Fubini's theorem and by (5.2)

$$\int_{t_1}^{\infty} E\left[E(dV_t|\mathcal{F}_t)\right]^+ = E\int_{t_1}^{\infty}\left[E(dV_t|\mathcal{F}_t)\right]^+ \leq E\int_{t_1}^{\infty}\frac{U_{1t}'C_{1t}^{-1}U_{1t}}{t}dt < \infty.$$

Hence, the proof. □

*Proof of Theorem 5.1.* Define $A_\delta^{t_1,T} = \{\max_{t_1<t<T} V_t > \delta\}$ and $H_{t_1} = \{V_{t_1} \leq \epsilon\}$ for any $\epsilon > 0$. Then, using the Lenglart Inequality (cf. Karatzas and Shreve [14] p30 or Lenglart [25]),

$$P\big(A_\delta^{t_1,T} \cap H_{t_1}\big) \leq \frac{1}{\delta}EV_{t_1}I_{H_{t_1}} + \frac{1}{\delta}\int_{t_1}^{T} E\left([E(dV_t|\mathcal{F}_t)]^+ I_{H_{t_1}}\right).$$

Therefore,

$$\begin{aligned}
P\big(A_\delta^{t_1,T}\big) &= P\big(A_\delta^{t_1,T} \cap H_{t_1}^c\big) + P\big(A_\delta^{t_1,T} \cap H_{t_1}\big)\\
&\leq P\left(H_{t_1}^c\right) + P\big(A_\delta^{t_1,T} \cap H_{t_1}\big)\\
&\leq P\left(H_{t_1}^c\right) + \frac{1}{\delta}EV_{t_1}I_{H_{t_1}} + \frac{1}{\delta}\int_{t_1}^{T} E\left([E(dV_t|\mathcal{F}_t)]^+ I_{H_{t_1}}\right)\\
&\leq P\left(H_{t_1}^c\right) + \frac{\epsilon}{\delta} + \frac{1}{\delta}\int_{t_1}^{\infty} E\left[E(dV_t|\mathcal{F}_t)\right]^+,
\end{aligned}$$

which is finite since $\int_{t_1}^{\infty} E[E(dV_t|\mathcal{F}_t)]^+ < \infty$ by Lemma 5.4. Therefore, as $T \to \infty$,

$$\begin{aligned}
P\big(\lim_{T\to\infty} A_\delta^{t_1,T}\big) &= \lim_{T\to\infty} P\big(A_\delta^{t_1,T}\big)\\
&\leq P\left(H_{t_1}^c\right) + \frac{\epsilon}{\delta} + \frac{1}{\delta}\int_{t_1}^{\infty} E\left[E(dV_t|\mathcal{F}_t)\right]^+.
\end{aligned}$$

Thus,

$$\limsup_{t_1\to\infty}\ P\big(\lim_{T\to\infty} A_\delta^{t_1,T}\big) \leq \frac{\epsilon}{\delta}.$$

Since this is true for all $\epsilon > 0$,

$$\limsup_{t_1\to\infty}\ P\big(\lim_{T\to\infty} A_\delta^{t_1,T}\big) = 0.$$

This implies,

$$\frac{1}{T^{1/2}}\left(\int_0^T dW_t U_{1t}'\right) C_{1T}^{-1/2} \to 0 \qquad \text{a.s.}$$

Hence, the Theorem. □

*Proof of Theorem 2.1.* From Lemma 3.1, we have $||e^{-G_0T}|| = O(e^{-\lambda_0 T})$ and, from Corollary 4.2, we have $||C_{1T}^{-1/2}|| = O(T^{-1/2})$ almost surely, as $T \to \infty$. Thus,

$$||T^{1/2}D_T M|| = T^{1/2}||M||\left(||e^{-G_0T}|| + ||C_{1T}^{-1/2}||\right) = O(1) \quad \text{a.s. as } T \to \infty.$$

Therefore, from (5.1), Corollary 3.1(ii) and Theorem 5.1, we have $\lim_{T\to\infty} \hat{F}_T = F$ a.s.

To show that (2.3) holds, we observe that, for the eigenvalues of $F$ in the right half space (2.3) follows from Theorem 3.1 and, for the eigenvalues of $F$ on the left half space (2.3) follows from arguments in Corollary 4.2 and Remark 4.2. For the mixed model, we observe

$$\left(\int_0^T Y_t Y_t' dt\right)^{-1} = D_T M \Sigma_T M' D_T'$$

where $\lim_{T\to\infty} \Sigma_T$ is a.s. positive definite. Thus, by Lemma 3.2,

$$\lambda_{\max}\left[\left(\int_0^T Y_t Y_t' dt\right)^{-1}\right] = O(\lambda_{\max}(D_T D_T')) = O(T^{-1}).$$

Therefore, the Theorem follows. □

## 6. Asymptotic efficiency

In this section we would like to show that our estimator for the drift matrix $F$ is asymptotically efficient even if the underlying process is not necessarily stationary (stable). For matrix-valued estimator there several ways to define asymptotic efficiency (see Barndorff-Nielson and Sorensen [3], for details).

The result is already known in one-dimensional case and for vector-valued parameters (e.g., [6, 8, 21, 27] and references therein) when the processes are not necessarily stationary. For multi-dimensional matrix-valued case, similar things can be proved once the asymptotic efficiency is properly defined for the matrix valued estimator.

Observe that, when $AA'$ is nonsingular, the log-likelihood of $F$, (see [6], pp. 213–214), on $[0, T]$ is defined by,

$$L_A(F) = \int_0^T (Y_t' F' (AA')^{-1} dY_t) - (1/2) \int_0^T (Y_t' F' (AA')^{-1} F Y_t) dt.$$

Thus,

$$dL_A(F) = \mathrm{Tr}\left[dF\left(\int_0^T Y_t dY_t'\right)(AA')^{-1} - dF\left(\int_0^T Y_t Y_t' dt\right) F'(AA')^{-1}\right].$$

Therefore, $dL_A(F)/dF = \left(\int_0^T dY_t Y_t'\right)\left(\int_0^T Y_t Y_t' dt\right)^{-1}$. When $AA'$ is not nonsingular, the log-likelihood of $F$ cannot be written explicitly. Therefore, M.L.E. of $F$ could not be achieved. However, we would show that the above estimator is asymptotically efficient under the assumptions of the section 2.

We show that $E(\mathrm{Tr}[(\hat{F}_T - F)E(C_T)(\hat{F}_T - F)'])^{1/2} = O(1)$ as $T \to \infty$.

Let $S_T = \left(\int_0^T AdW_tY_t'\right)$, and $C_T = \left(\int_0^T Y_tY_t'dt\right)$ as before. We use

$$\begin{aligned}\mathrm{Tr}[(\hat{F}_T - F)E(C_T)(\hat{F}_T - F)'] &= \mathrm{Tr}[S_TC_T^{-1}E(C_T)C_T^{-1}S_T'] \\ &\leq \mathrm{Tr}[S_TC_T^{-1}S_T']\mathrm{Tr}[C_T^{-1}E(C_T)]\end{aligned}$$

while proving the following theorem.

*Proof of Theorem 2.2.*

Case 1: Eigenvalues of $F$ are in the positive half space.

Observe that, $\mathrm{Tr}(S_TC_T^{-1}S_T') = \mathrm{Tr}(S_Te^{-F'T}(e^{-FT}C_Te^{-F'T})^{-1}e^{-FT}S_T')$. Since $S_Te^{-F'T}$ is a Gaussian process and its mean zero and variance $e^{-FT}E(C_T)e^{-F'T}$ converges (in fact, to $E(B)$) as $T \to \infty$, $S_Te^{-F'T}$ converges to a finite Gaussian random variable in distribution. Also, from Theorem (3.1), as $T \to \infty$, $e^{-FT}C_Te^{-F'T}$ converges almost surely to $B$ (which is positive definite with probability one). Thus, we obtain $\mathrm{Tr}(S_Te^{-F'T}(e^{-FT}C_Te^{-F'T})^{-1}e^{-FT}S_T')$ converges in distribution to finite random variable with finite expectation.

Now, $\mathrm{Tr}(C_T^{-1}E(C_T)) = \mathrm{Tr}((e^{-FT}C_Te^{-F'T})^{-1}(e^{-FT}E(C_T)e^{-F'T}))$, and from Theorem (3.1), as $T \to \infty$, $(e^{-FT}C_Te^{-F'T})^{-1}$ converges to $B^{-1}$ almost surely. Also, $e^{-FT}E(C_T)e^{-F'T} = \int_0^T e^{-Ft}Y_0Y_0'e^{-F't}dt + \int_0^T te^{-Ft}AA'e^{-F't}dt$, which is finite as $T \to \infty$. Thus, it remains to show, as $T \to \infty$, $E(e^{-FT}C_Te^{-F'T})^{-1}$ converges to $E(B^{-1})$ (which is finite). First observe that, $Z_t - Y_0 = \int_0^t e^{-Fs}AdW_s$ is a symmetric (Gaussian) martingale and with $E|Z_t - Y_0|^2 \leq E|Z - Y_0|^2 < \infty$. Thus $M_Z = \max_{0\leq t<\infty}(Z_t - Y_0)$ exists and has finite expectation. Also, (by symmetry) $m_Z = \min_{0\leq t<\infty}(Z_t - Y_0)$ exists and has finite second moment. For symmetric matrices $D_1$ and $D_2$, define, $D_1 \geq D_2$ if $D_1 - D_2$ is non-negative definite. Therefore,

$$\begin{aligned}e^{-FT}C_Te^{-F'T} &= \int_0^T e^{-Ft}Z_{T-t}Z_{T-t}'e^{-F't}dt \\ &\geq \int_0^T e^{-Ft}(m_Z + Y_0)(m_Z + Y_0)'e^{-F't}dt \\ &\geq \int_0^{T_0} e^{-Ft}(m_Z + Y_0)(m_Z + Y_0)'e^{-F't}dt\end{aligned}$$

for all $T \geq T_0$, for some $T_0 > 0$ ($T_0$ may be taken to be 1). Thus, for all $T \geq T_0$, $(e^{-FT}C_Te^{-F'T})^{-1} \leq (\int_0^{T_0} e^{-Ft}(m_Z+Y_0)(m_Z+Y_0)'e^{-F't}dt)^{-1}$. Since right hand side has finite expectation, using dominated convergence type theorem deduce $E(B^{-1}) = \lim_{T\to\infty} E(e^{-FT}C_Te^{-F'T})^{-1} \leq E(\int_0^{T_0} e^{-Ft}(m_Z + Y_0)(m_Z + Y_0)' \times e^{-F't}dt)^{-1}$. Therefore, $E(\mathrm{Tr}(C_T^{-1}E(C_T)))$ is finite and hence, $E(\mathrm{Tr}[(\hat{F}_T - F) \times E(C_T)(\hat{F}_T - F)'])^{1/2} = O(1)$.

Case 2: Eigenvalues of $F$ are on the left half space.

When all the eigenvalues have real parts negative, by ergodic theorem, $\lim_{T\to\infty} \frac{1}{T}C_T = \int_0^\infty e^{Ft}AA'e^{F't}dt = \lim_{T\to\infty} E(\frac{1}{T}C_T)$. Thus,

$$\begin{aligned}\lim_{T\to\infty} E(\mathrm{Tr}(S_T C_T^{-1} S_T')) &= \lim_{T\to\infty} E\left(\mathrm{Tr}\left(\frac{1}{T}S_T' S_T\left(\int_0^\infty e^{Ft}AA'e^{F't}dt\right)^{-1}\right)\right) \\ &= p\ \mathrm{Tr}(AA'), \quad \text{i.e., of } O(1).\end{aligned}$$

Also, $\lim_{T\to\infty} E(\mathrm{Tr}(C_T^{-1}E(C_T))) = \lim_{T\to\infty} E(\mathrm{Tr}((\frac{1}{T}C_T)^{-1}E(\frac{1}{T}C_T))) = p$. Therefore, $E(\mathrm{Tr}[(\hat{F}_T - F)E(C_T)(\hat{F}_T - F)'])^{1/2} = O(1)$.

**Zero and purely imaginary eigenvalues.**

When the eigenvalues are either all purely imaginary or all zero, replace $F$ by $F - \epsilon I = F^\epsilon$, as it is done in Section 4, get the result as above by ergodic theorem.

Now, as in Lemma 4.1, consider

$$\begin{aligned}
& \frac{\partial}{\partial\epsilon}\mathrm{Tr}E((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon)) \\
= \;& 2\mathrm{Tr}E((\dot{S}_T^\epsilon)'(C_T^\epsilon)^{-1}S_T^\epsilon) + \mathrm{Tr}E((S_T^\epsilon)'\frac{\partial}{\partial\epsilon}(C_T^\epsilon)^{-1}S_T^\epsilon) \\
= \;& 2\mathrm{Tr}E((\dot{S}_T^\epsilon)'(C_T^\epsilon)^{-1}S_T^\epsilon) - \mathrm{Tr}E((S_T^\epsilon)'(C_T^\epsilon)^{-1}\left[\frac{\partial}{\partial\epsilon}C_T^\epsilon\right](C_T^\epsilon)^{-1}S_T^\epsilon) \\
\geq \;& -2E\left(\left[\mathrm{Tr}((\dot{S}_T^\epsilon)'(C_T^\epsilon)^{-1}(\dot{S}_T^\epsilon))\right]^{1/2}\left[\mathrm{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]^{1/2}\right) \\
& - \mathrm{Tr}E\left[(S_T^\epsilon)'(C_T^\epsilon)^{-1}\left[\int_0^T (Y_u^\epsilon)(\dot{Y}_u^\epsilon)'du + \int_0^T (\dot{Y}_u^\epsilon)(Y_u^\epsilon)'du\right](C_T^\epsilon)^{-1}S_T^\epsilon\right] \\
\geq \;& -2\left(E\left[\mathrm{Tr}((\dot{S}_T^\epsilon)'(C_T^\epsilon)^{-1}(\dot{S}_T^\epsilon))\right]\right)^{1/2}\left(E\left[\mathrm{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]\right)^{1/2} \\
& - 2E\Bigg(\left[\mathrm{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right] \times \\
& \int_0^T \left[(Y_u^\epsilon)'(C_T^\epsilon)^{-1}(Y_u^\epsilon)\right]^{1/2}\left[(\dot{Y}_u^\epsilon)'(C_T^\epsilon)^{-1}(\dot{Y}_u^\epsilon)\right]^{1/2}du\Bigg) \\
\geq \;& -2\left(E\left[\mathrm{Tr}((\dot{S}_T^\epsilon)'(C_T^\epsilon)^{-1}(\dot{S}_T^\epsilon))\right]\right)^{1/2}\left(E\left[\mathrm{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]\right)^{1/2} \\
& - E\Bigg(\left[\mathrm{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right] \times \\
& \left[\int_0^T (Y_u^\epsilon)'(C_T^\epsilon)^{-1}(Y_u^\epsilon)du + \int_0^T (\dot{Y}_u^\epsilon)'(C_T^\epsilon)^{-1}(\dot{Y}_u^\epsilon)du\right]\Bigg) \\
= \;& -2\left(E\left[\mathrm{Tr}((\dot{S}_T^\epsilon)'(C_T^\epsilon)^{-1}(\dot{S}_T^\epsilon))\right]\right)^{1/2}\left(E\left[\mathrm{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]\right)^{1/2} \\
& - E\left(\left[\mathrm{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]\left[p + \mathrm{Tr}[(\dot{C}_T^\epsilon)(C_T^\epsilon)^{-1}]\right]\right).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
& \frac{\partial}{\partial \epsilon} \ln E\left[\operatorname{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right] \\
= \quad & \left[E\operatorname{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]^{-1} \frac{\partial}{\partial \epsilon} E\left[\operatorname{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right] \\
\geq \quad & -2\left[\frac{E\operatorname{Tr}((\dot{S}_T^\epsilon)'(C_T^\epsilon)^{-1}(\dot{S}_T^\epsilon))}{E\operatorname{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))}\right]^{1/2} \\
& -\frac{E\left(\left[\operatorname{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]\left[p+\operatorname{Tr}\left[(\dot{C}_T^\epsilon)(C_T^\epsilon)^{-1}\right]\right]\right)}{E\left[\operatorname{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]},
\end{aligned}
$$

which is bounded below (by a negative number possibly depending on $\epsilon$) uniformly for large values of $T$ by (4.3) and using the fact that both $\operatorname{Tr}E((\dot{S}_T^\epsilon)' \times (C_T^\epsilon)^{-1}(\dot{S}_T^\epsilon))$ and $\operatorname{Tr}E((\dot{S}_T^\epsilon)'(\dot{C}_T^\epsilon)^{-1}(\dot{S}_T^\epsilon))$ have the same order and the latter has the order as that of $\operatorname{Tr}E((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))$.

Now as in the argument in consistency part, since all eigenvalues of $F$ are on the left half space, the real parts of all eigenvalues of $F^\epsilon$ are negative, i.e., $Y_t^\epsilon$ is a stable process and

$$
\lim_{T\to\infty} \operatorname{Tr}E((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon)) = O(1).
$$

Similarly, to get a upper bound, consider

$$
\begin{aligned}
& \frac{\partial}{\partial \epsilon} \operatorname{Tr}E((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon)) \\
= \quad & 2\operatorname{Tr}E((\dot{S}_T^\epsilon)'(C_T^\epsilon)^{-1}S_T^\epsilon) - \operatorname{Tr}E((S_T^\epsilon)'(C_T^\epsilon)^{-1}\left[\frac{\partial}{\partial \epsilon}C_T^\epsilon\right](C_T^\epsilon)^{-1}S_T^\epsilon) \\
\leq \quad & 2E\left(\left[\operatorname{Tr}((\dot{S}_T^\epsilon)'(C_T^\epsilon)^{-1}(\dot{S}_T^\epsilon))\right]^{1/2}\left[\operatorname{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]^{1/2}\right) \\
& + \operatorname{Tr}E\left[(S_T^\epsilon)'(C_T^\epsilon)^{-1}\left[\int_0^T (Y_u^\epsilon)(\dot{Y}_u^\epsilon)'du + \int_0^T (\dot{Y}_u^\epsilon)(Y_u^\epsilon)'du\right](C_T^\epsilon)^{-1}S_T^\epsilon\right] \\
\leq \quad & 2\left(E\left[\operatorname{Tr}((\dot{S}_T^\epsilon)'(C_T^\epsilon)^{-1}(\dot{S}_T^\epsilon))\right]\right)^{1/2}\left(E\left[\operatorname{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]\right)^{1/2} \\
& + 2E\Bigg(\left[\operatorname{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right] \times \\
& \int_0^T \left[(Y_u^\epsilon)'(C_T^\epsilon)^{-1}(Y_u^\epsilon)\right]^{1/2}\left[(\dot{Y}_u^\epsilon)'(C_T^\epsilon)^{-1}(\dot{Y}_u^\epsilon)\right]^{1/2} du\Bigg) \\
\leq \quad & 2\left(E\left[\operatorname{Tr}((\dot{S}_T^\epsilon)'(C_T^\epsilon)^{-1}(\dot{S}_T^\epsilon))\right]\right)^{1/2}\left(E\left[\operatorname{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]\right)^{1/2} \\
& + E\Bigg(\left[\operatorname{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right] \times \\
& \left[\int_0^T (Y_u^\epsilon)'(C_T^\epsilon)^{-1}(Y_u^\epsilon)du + \int_0^T (\dot{Y}_u^\epsilon)'(C_T^\epsilon)^{-1}(\dot{Y}_u^\epsilon)du\right]\Bigg)
\end{aligned}
$$

$$= 2\left(E\left[\mathrm{Tr}((\dot{S}_T^\epsilon)'(C_T^\epsilon)^{-1}(\dot{S}_T^\epsilon))\right]\right)^{1/2}\left(E\left[\mathrm{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]\right)^{1/2}$$
$$+E\left(\left[\mathrm{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]\left[p+\mathrm{Tr}[(\dot{C}_T^\epsilon)(C_T^\epsilon)^{-1}]\right]\right).$$

Therefore,

$$\frac{\partial}{\partial\epsilon}\ln E\left[\mathrm{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]$$
$$= \left[E\mathrm{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]^{-1}\frac{\partial}{\partial\epsilon}E\left[\mathrm{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]$$
$$\leq 2\left[\frac{E\mathrm{Tr}((\dot{S}_T^\epsilon)'(C_T^\epsilon)^{-1}(\dot{S}_T^\epsilon))}{E\mathrm{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))}\right]^{1/2}$$
$$+\frac{E\left(\left[\mathrm{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]\left[p+\mathrm{Tr}\left[(\dot{C}_T^\epsilon)(C_T^\epsilon)^{-1}\right]\right]\right)}{E\left[\mathrm{Tr}((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))\right]},$$

which is bounded above (by a positive number possibly depending on $\epsilon$) uniformly for large values of $T$ by (4.3).

As in the proof of Theorem 4.2, let $f(\epsilon)=\ln\mathrm{Tr}E((S_T^\epsilon)(C_T^\epsilon)^{-1}(S_T^\epsilon))$. Fix an $\epsilon_1>0$. $f$ is a continuous function on $[0,\epsilon_1]$ and is differentiable in $(0,\epsilon_1)$. Then by the Mean Value Theorem, there exists an $\epsilon_0\in(0,\epsilon_1)$ such that

$$f(\epsilon_1)-f(0)=\epsilon_1\frac{\partial}{\partial\epsilon}f(\epsilon)\big|_{\epsilon=\epsilon_0}.$$

That is,

$$\frac{\mathrm{Tr}E((S_T^{\epsilon_1})'(C_T^{\epsilon_1})^{-1}(S_T^{\epsilon_1}))}{\mathrm{Tr}E(S_T'C_T^{-1}S_T)}=\exp\left\{\epsilon_1\frac{\partial}{\partial\epsilon}f(\epsilon)\,|_{\epsilon=\epsilon_0}\right\}, \tag{6.1}$$

which is uniformly bounded and positive (i.e., bounded away from zero and infinity) for large values of $T$ as argued above. Since

$$\lim_{T\to\infty}\mathrm{Tr}E((S_T^\epsilon)'(C_T^\epsilon)^{-1}(S_T^\epsilon))=O(1).$$

by (6.1)

$$\lim_{T\to\infty}\mathrm{Tr}E(S_T'C_T^{-1}S_T)=O(1).$$

Mimicking the above argument, find

$$\frac{\partial}{\partial\epsilon}\mathrm{Tr}\Big(E((C_T^\epsilon)^{-1})E(C_T^\epsilon)\Big)$$
$$= -\mathrm{Tr}\left(E\left[(C_T^\epsilon)^{-1}\left[\int_0^T(Y_u^\epsilon)(\dot{Y}_u^\epsilon)'du+\int_0^T(\dot{Y}_u^\epsilon)(Y_u^\epsilon)'du\right](C_T^\epsilon)^{-1}\right]E(C_T^\epsilon)\right)$$
$$+\mathrm{Tr}\left(E((C_T^\epsilon)^{-1})E\left[\int_0^T(Y_u^\epsilon)(\dot{Y}_u^\epsilon)'du+\int_0^T(\dot{Y}_u^\epsilon)(Y_u^\epsilon)'du\right]\right)$$

$$\begin{aligned}
\geq\ & -2E\Bigg(\Big[\mathrm{Tr}((C_T^\epsilon)^{-1}E(C_T^\epsilon))\Big]\times \\
& \int_0^T \Big[(Y_u^\epsilon)'(C_T^\epsilon)^{-1}(Y_u^\epsilon)\Big]^{1/2}\Big[(\dot{Y}_u^\epsilon)'(C_T^\epsilon)^{-1}(\dot{Y}_u^\epsilon)\Big]^{1/2}du\Bigg) \\
& -2E\left(\int_0^T \Big[(Y_u^\epsilon)'(E((C_T^\epsilon)^{-1}))(Y_u^\epsilon)\Big]^{1/2}\Big[(\dot{Y}_u^\epsilon)'(E((C_T^\epsilon)^{-1}))(\dot{Y}_u^\epsilon)\Big]^{1/2}du\right) \\
\geq\ & -E\left(\Big[\mathrm{Tr}((C_T^\epsilon)^{-1}E(C_T^\epsilon))\Big]\Big[p+\mathrm{Tr}[(\dot{C}_T^\epsilon)(C_T^\epsilon)^{-1}]\Big]\right) \\
& -2E\left(\Big[\mathrm{Tr}(E((C_T^\epsilon)^{-1}))(C_T^\epsilon)\Big]^{1/2}\Big[\mathrm{Tr}(E((C_T^\epsilon)^{-1}))(\dot{C}_T^\epsilon)\Big]^{1/2}\right) \\
\geq\ & -E\left(\Big[\mathrm{Tr}((C_T^\epsilon)^{-1}E(C_T^\epsilon))\Big]\Big[p+\mathrm{Tr}[(\dot{C}_T^\epsilon)(C_T^\epsilon)^{-1}]\Big]\right) \\
& -2\left(\mathrm{Tr}\left[E((C_T^\epsilon)^{-1})E(C_T^\epsilon)\right]\right)^{1/2}\left(\mathrm{Tr}\Big[E((C_T^\epsilon)^{-1})E(\dot{C}_T^\epsilon)\Big]\right)^{1/2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \frac{\partial}{\partial\epsilon}\ln\mathrm{Tr}(E((C_T^\epsilon)^{-1})E(C_T^\epsilon)) \\
=\ & \left[\mathrm{Tr}(E((C_T^\epsilon)^{-1})E(C_T^\epsilon))\right]^{-1}\frac{\partial}{\partial\epsilon}\mathrm{Tr}(E((C_T^\epsilon)^{-1})E(C_T^\epsilon)) \\
\geq\ & -\frac{E\left(\left[\mathrm{Tr}((C_T^\epsilon)^{-1}E(C_T^\epsilon))\right]\left[p+\mathrm{Tr}[(\dot{C}_T^\epsilon)(C_T^\epsilon)^{-1}]\right]\right)}{\mathrm{Tr}(E((C_T^\epsilon)^{-1})E(C_T^\epsilon))} \\
& -2\left[\frac{\mathrm{Tr}(E((C_T^\epsilon)^{-1})E(\dot{C}_T^\epsilon))}{\mathrm{Tr}(E((C_T^\epsilon)^{-1})E(C_T^\epsilon))}\right]^{1/2},
\end{aligned}$$

which is bounded below (by a negative number possibly depending on $\epsilon$) uniformly for large values of $T$ by (4.3) and from the fact that both $\mathrm{Tr}(E((C_T^\epsilon)^{-1})E(\dot{C}_T^\epsilon))$ and $\mathrm{Tr}(E((C_T^\epsilon)^{-1})E(C_T^\epsilon))$ have the same order.

Similary, to get an upper bound, consider

$$\begin{aligned}
& \frac{\partial}{\partial\epsilon}\mathrm{Tr}\left(E((C_T^\epsilon)^{-1})E(C_T^\epsilon)\right) \\
=\ & -\mathrm{Tr}\left(E\Big[(C_T^\epsilon)^{-1}\left[\int_0^T(Y_u^\epsilon)(\dot{Y}_u^\epsilon)'du+\int_0^T(\dot{Y}_u^\epsilon)(Y_u^\epsilon)'du\right](C_T^\epsilon)^{-1}\Big]E(C_T^\epsilon)\right) \\
& +\mathrm{Tr}\left(E((C_T^\epsilon)^{-1})E\left[\int_0^T(Y_u^\epsilon)(\dot{Y}_u^\epsilon)'du+\int_0^T(\dot{Y}_u^\epsilon)(Y_u^\epsilon)'du\right]\right) \\
\leq\ & 2E\Bigg(\Big[\mathrm{Tr}((C_T^\epsilon)^{-1}E(C_T^\epsilon))\Big]\times \\
& \int_0^T\left[(Y_u^\epsilon)'(C_T^\epsilon)^{-1}(Y_u^\epsilon)\right]^{1/2}\Big[(\dot{Y}_u^\epsilon)'(C_T^\epsilon)^{-1}(\dot{Y}_u^\epsilon)\Big]^{1/2}du\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+2E\left(\int_0^T \left[(Y_u^\epsilon)'(E((C_T^\epsilon)^{-1}))(Y_u^\epsilon)\right]^{1/2}\left[(\dot{Y}_u^\epsilon)'(E((C_T^\epsilon)^{-1}))(\dot{Y}_u^\epsilon)\right]^{1/2} du\right)\\
\leq\ & E\left(\left[\mathrm{Tr}((C_T^\epsilon)^{-1}E(C_T^\epsilon))\right]\left[p+\mathrm{Tr}[(\dot{C}_T^\epsilon)(C_T^\epsilon)^{-1}]\right]\right)\\
&+2E\left(\left[\mathrm{Tr}(E((C_T^\epsilon)^{-1}))(C_T^\epsilon)\right]^{1/2}\left[\mathrm{Tr}(E((C_T^\epsilon)^{-1}))(\dot{C}_T^\epsilon)\right]^{1/2}\right)\\
\leq\ & E\left(\left[\mathrm{Tr}((C_T^\epsilon)^{-1}E(C_T^\epsilon))\right]\left[p+\mathrm{Tr}[(\dot{C}_T^\epsilon)(C_T^\epsilon)^{-1}]\right]\right)\\
&+2\left(\mathrm{Tr}\left[E((C_T^\epsilon)^{-1})E(C_T^\epsilon)\right]\right)^{1/2}\left(\mathrm{Tr}\left[E((C_T^\epsilon)^{-1})E(\dot{C}_T^\epsilon)\right]\right)^{1/2}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\frac{\partial}{\partial\epsilon}\ln\mathrm{Tr}(E((C_T^\epsilon)^{-1})E(C_T^\epsilon))\\
=\ & \left[\mathrm{Tr}(E((C_T^\epsilon)^{-1})E(C_T^\epsilon))\right]^{-1}\frac{\partial}{\partial\epsilon}\mathrm{Tr}(E((C_T^\epsilon)^{-1})E(C_T^\epsilon))\\
\leq\ & \frac{E\left(\left[\mathrm{Tr}((C_T^\epsilon)^{-1}E(C_T^\epsilon))\right]\left[p+\mathrm{Tr}[(\dot{C}_T^\epsilon)(C_T^\epsilon)^{-1}]\right]\right)}{\mathrm{Tr}(E((C_T^\epsilon)^{-1})E(C_T^\epsilon))}\\
&+2\left[\frac{\mathrm{Tr}(E((C_T^\epsilon)^{-1})E(\dot{C}_T^\epsilon))}{\mathrm{Tr}(E((C_T^\epsilon)^{-1})E(C_T^\epsilon))}\right]^{1/2},
\end{aligned}
$$

which is bounded above (by a positive number possibly depending on $\epsilon$) uniformly for large values of $T$ by (4.3).

Thus, using the similar argument as in (6.1) we show, since $\lim_{T\to\infty}\mathrm{Tr}(E\times((C_T^{\epsilon_1})^{-1})E(C_T^{\epsilon_1}))=O(1)$, $\lim_{T\to\infty}\mathrm{Tr}(E(C_T^{-1})E(C_T))=O(1)$. Hence, for eigenvalues of $F$ on the left half space, we prove that $E(\mathrm{Tr}[(\hat{F}_T-F)E(C_T)(\hat{F}_T-F)'])^{1/2}=O(1)$.

Case 3: Mixed model.

In this case, use the decomposition of $F$ as in Section 5, to decompose $Y_t'M'=(U_{0t}',U_{1t}')$. Then, one gets,

$$
\begin{aligned}
\mathrm{Tr}(S_TC_T{}^{-1}S_T') &= \mathrm{Tr}(S_TM'D_T'(D_TMC_TM'D_T')^{-1}D_TMS_T')\\
&\leq \mathrm{Tr}(S_TM'D_T'D_TMS_T')\mathrm{Tr}(D_TMC_TM'D_T')^{-1}\\
&\leq \left(\mathrm{Tr}(S_{0T}e^{-G_0'T}e^{-G_0T}S_{0T}')+\mathrm{Tr}(S_{1T}C_{1T}{}^{-1}S_{1T}')\right)\times\\
&\quad \mathrm{Tr}(D_TMC_TM'D_T')^{-1}.
\end{aligned}
$$

Since for a symmetric invertible partition matrix,

$$
K=\begin{bmatrix} E & F\\ F' & H\end{bmatrix}
$$

with $E$ and $H$ invertible, $\mathrm{Tr}(K)=\mathrm{Tr}(E-FH^{-1}F')^{-1}+\mathrm{Tr}(H-F'E^{-1}F)^{-1}$. Taking $E=e^{-G_0T}C_{0T}e^{-G_0'T}$, $F=e^{-G_0T}\int_0^T U_{0t}U_{1t}'dtC_{1T}^{-1/2}$ and $H=I$, i.e., identity matrix of order $p_1$. Since $F$ converges to zero almost surely by the proof of Lemma 5.1 and by the same lemma $E$ converges to $B$ almost surely, one obtains $\mathrm{Tr}(D_T M C_T M' D'_T)^{-1} \to \mathrm{Tr}(B^{-1}) + p_1$ almost surely, as $T \to \infty$. Therefore,

$$\begin{aligned} & E\Big[\Big(\mathrm{Tr}(e^{-G_0 T} S'_{0T} S_{0T} e^{-G'_0 T}) + \mathrm{Tr}(S_{1T} C_{1T}{}^{-1} S'_{1T})\Big) \times \\ & \mathrm{Tr}(D_T M C_T M' D'_T)^{-1}\Big]^{1/2} \\ \leq \; & E(\mathrm{Tr}(e^{-G_0 T} S'_{0T} S_{0T} e^{-G'_0 T}))E(\mathrm{Tr}(D_T M C_T M' D'_T)^{-1}) \\ & + E(\mathrm{Tr}(S_{1T} C_{1T}{}^{-1} S'_{1T}))E(\mathrm{Tr}(D_T M C_T M' D'_T)^{-1}) \\ = \; & O(1) \end{aligned} \tag{6.2}$$

by the case 1, and case 2. Similarly,

$$\begin{aligned} & \mathrm{Tr}\Big((D_T M C_T M' D'_T)^{-1} D_T E(M C_T M') D'_T\Big) \\ \leq \; & \mathrm{Tr}((D_T M C_T M' D'_T)^{-1}) \mathrm{Tr}(D_T E(M C_T M') D'_T) \end{aligned}$$

and $\mathrm{Tr}(D_T E(M C_T M') D'_T) = \mathrm{Tr}(e^{-G_0 T} E(C_{0T}) e^{-G'_0 T}) + \mathrm{Tr}(C_{1T}^{-1} E(C_{1T}))$ expectation of which is finite by case 1 and case 2. Therefore one proves, for the mixed model, $E(\mathrm{Tr}[(\hat{F}_T - F)E(C_T)(\hat{F}_T - F)'])^{1/2} = O(1)$. □

## 7. Concluding remarks and discussion

It is easy to see that the state space equation of the general continuous autoregressive process (CAR(p)) of the form $dX_t^{p-1} = \alpha_p X_t + \alpha_{p-1} X_t^1 + \cdots + \alpha_1 X_t^{p-1} + \sigma dW_t$ is a special case of multidimensional OU processes where

$$F = \begin{pmatrix} \mathbf{0}_{(p-1)\times 1} & \mathbf{I}_{p-1} \\ \alpha_p & \cdots \quad \alpha_1 \end{pmatrix}, \qquad A = (0, \ldots, 0, \sigma)'$$

with $\alpha_i$ real numbers, $\sigma > 0$ and $W_t$ a one-dimensional Browian motion. Clearly, $A$ is not singular. However, the $RANK$ condition (a) holds for this $F$ and $A$ and, the condition (b') holds for this $F$. Hence, from our result, the consistency and the asymptotic efficiency of the $\hat{F}$ of general CAR(p) follows.

It is important to observe that this estimation procedure may be the first step in developing a test of zero roots of some $F$, which is necessary to determine whether univariate processes are co-integrated. Also, if one needs to develop a test to determine whether the model for $Y_t$ is stationary, it is often enough to test whether all eigenvalues of $F$ have negative real parts against the alternative that some of them have zero real parts. Therefore, one need not often worry about the assumption (b) or (b') for testing stationarity. Thus, a related question arises on, whether any Asymptotically Mixed Normality property holds for the estimator $\hat{F}_T$, i.e., whether $(\int_0^T Y_t Y'_t dt)^{1/2}(\hat{F}_T - F)$ follows asymptotically Normal, so that we could compute approximate confidence interval for the above testing procedures for the necessary parameters in $F$. As far as we know, these results are still unknown. Investigating the Asymptotically Mixed Normality property may be an important future direction to consider. One can look into LAMN property as well.

Besides, when the drift coefficient matrix depends on an unknown discrete paratmeter $\theta$ which follows a Markov chain (that helps the process to switch regimes), finding a consistent and asymptotically efficient estimator becomes important. Above questions can be asked in that setup as well.

In applications, we almost always use discrete sampled data. Similar questions can be asked for this model, when the data sampled are in deterministic (equal or unequal) time interval or in random interval. That can also be a focus of the future direction.

## 8. Appendix

### 8.1. Purely imaginary eigenvalues

In this Section, we study the asymptotic behavior of OU processes when the drift matrix $F$ only contains purely imaginary eigenvalues. The main results are summarized in the following:

**Theorem 8.1** *Suppose for the Ornstein-Uhlenbeck process defined in (1.1), the RANK condition (2.1) holds and all the eigenvalues of $F$ are purely imaginary. Let $2\rho$ be the dimension of the largest block of the rational canonical form of $F$ as defined in Section 2 (see the Example). Then*

$$||Y_T|| = \begin{cases} O(T^{1/2}\sqrt{\ln\ln T}) & \text{a.s.} \quad \text{if} \quad \rho = 1 \\ O(T^{2\rho-5/2}\sqrt{\ln\ln T}) & \text{a.s.} \quad \text{if} \quad \rho \geq 2. \end{cases}$$

*Moreover,*

$$\lambda_{\max}\left(\int_0^T Y_t Y_t' dt\right) = \begin{cases} O(T^2(\ln\ln T)) & \text{a.s.} \quad \text{if} \quad \rho = 1 \\ O(T^{4\rho-4}(\ln\ln T)) & \text{a.s.} \quad \text{if} \quad \rho \geq 2. \end{cases} \tag{8.1}$$

To prove Theorem 8.1, we need the following Lemmas.

**Lemma 8.1**

$$\sum_{n=j}^{\infty} \frac{(-1)^n (vt)^{2n-j}}{(2n-j)!} = \begin{cases} O(1) & \text{if} \quad j = 0, 1 \\ O(t^{j-2}) & \text{if} \quad j \geq 2. \end{cases}$$

*Proof.*

$$\begin{aligned} & \sum_{n=j}^{\infty} \frac{(-1)^n (vt)^{2n-j}}{(2n-j)!} \\ = \quad & (-1)^j \left[\frac{(vt)^j}{j!} - \frac{(vt)^{j+2}}{(j+2)!} + \frac{(vt)^{j+4}}{(j+4)!} - \cdots\right] \end{aligned}$$

$$
= \begin{cases} \cos(vt) & \text{if } \ j=0 \\ -\sin(vt) & \text{if } \ j=1 \\ (-1)^{j/2}\left\{\cos(vt) - \left[1 - \frac{(vt)^2}{2!} + \cdots + (-1)^{j/2-1}\frac{(vt)^{j-2}}{(j-2)!}\right]\right\} & \\ \quad \text{if } \ j \ \text{ is even, } j \geq 2 & \\ (-1)^{(j-3)/2}\left\{\sin(vt) - \left[vt - \frac{(vt)^3}{3!} + \cdots + (-1)^{(j-1)/2}\frac{(vt)^{j-2}}{(j-2)!}\right]\right\} & \\ \quad \text{if } \ j \ \text{ is odd, } j \geq 3 & \end{cases}
$$

$$
= \begin{cases} O(1) & \text{if } \ j = 0, 1 \\ O(t^{j-2}) & \text{if } \ j \geq 2. \end{cases}
$$

Hence, the lemma follows. □

**Lemma 8.2** *With the same assumptions as in Theorem 8.1,*

$$
||e^{Ft}|| = \begin{cases} O(1) & \text{a.s.} \quad \text{if} \quad \rho = 1 \\ O(t^{2\rho-3}) & \text{a.s.} \quad \text{if} \quad \rho \geq 2. \end{cases}.
$$

*Proof.* Suppose $F$ is a $2\rho \times 2\rho$ matrix and has $\rho$ eigenvalues of $\lambda_1 = iv$ and $\bar{\lambda}_1 = -iv$. Since the characteristic equation for $F$ is $0 = |\lambda I - F| = (\lambda - iv)^\rho(\lambda + iv)^\rho = (\lambda^2 + v^2)^\rho$, by the Cayley-Hamilton theorem,

$$
(F^2 + v^2 I)^\rho = 0. \tag{8.2}
$$

**Case 1**: When $\rho = 1$, then $F^{2n} = (-1)^n v^{2n} I$ and

$$
\begin{aligned}
e^{Ft} &= \sum_{n=0}^{\infty} \frac{F^{2n}t^{2n}}{(2n)!} + F\sum_{n=0}^{\infty} \frac{F^{2n}t^{2n+1}}{(2n+1)!} \\
&= I\sum_{n=0}^{\infty} \frac{(-1)^n (vt)^{2n}}{(2n)!} + \frac{F}{v}\sum_{n=0}^{\infty} \frac{(-1)^n (vt)^{2n+1}}{(2n+1)!} \\
&= I\cos(vt) + \frac{F}{v}\sin(vt).
\end{aligned} \tag{8.3}
$$

Therefore, $||e^{Ft}|| = O(1)$ when $\rho = 1$.

**Case 2**: When $\rho \geq 2$, then $A = F^2 + v^2 I$ is a nilpotent matrix of order $\rho$ by (8.2). Thus,

$$
\begin{aligned}
F^2 &= -v^2\left[I - \frac{A}{v^2}\right] \qquad \text{and} \\
F^{2n} &= (-1)^n v^{2n} \sum_{k=0}^{\rho-1} (-1)^k \binom{n}{k} \frac{A^k}{v^{2k}} \\
&= (-1)^n v^{2n}\left(I - \frac{nA}{v^2} + \cdots + (-1)^{\rho-1}\binom{n}{\rho-1}\frac{A^{\rho-1}}{v^{2(\rho-1)}}\right).
\end{aligned}
$$

Therefore,

$$
e^{Ft} = \sum_{n=0}^{\infty} \frac{F^{2n}t^{2n}}{(2n)!} + F\sum_{n=0}^{\infty} \frac{F^{2n}t^{2n+1}}{(2n+1)!}. \tag{8.4}
$$

Let $f_j(n) = 2n(2n-1)\cdots(2n-j+1)$ if $j \geq 1$ and $f_0(n) = 1$. Then, since $f_0(n), f_1(n), \ldots, f_k(n)$ are independent, there exist unique $C_0, C_1 \cdots C_k \in \mathcal{Z}$ such that

$$\binom{n}{k} = \sum_{j=0}^{k} C_j f_j(n).$$

Similarly, let $f_j^*(n) = (2n+1)(2n)\cdots(2n-j+2)$ if $j \geq 1$ and $f_0^*(n) = 1$. Then, there exist unique $C_0^*, C_1^*, \ldots C_k^* \in \mathcal{Z}$ such that

$$\binom{n}{k} = \sum_{j=0}^{k} C_j^* f_j^*(n).$$

By Lemma 8.1, the first term of (8.4) can be expressed as

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{(-1)^n (vt)^{2n}}{(2n!)} \left[ \sum_{k=0}^{(\rho-1)\wedge n} (-1)^k \binom{n}{k} \frac{A^k}{v^{2k}} \right] \\
= & \sum_{k=0}^{\rho-1} \left( -\frac{A}{v^2} \right)^k \left[ \sum_{n=k}^{\infty} \frac{(-1)^n (vt)^{2n}}{(2n)!} \left( \sum_{j=0}^{k} C_j f_j(n) \right) \right] \\
= & \sum_{k=0}^{\rho-1} \left( -\frac{A}{v^2} \right)^k \left[ \sum_{j=0}^{k} (vt)^j C_j \left( \sum_{n=k}^{\infty} \frac{(-1)^n (vt)^{2n-j}}{(2n-j)!} \right) \right] \\
= & \begin{cases} \sum_{k=0}^{1} \left(-\frac{A}{v^2}\right)^k \times O(t) & \text{for } \rho = 2 \\ \sum_{k=0}^{\rho-1} \left(-\frac{A}{v^2}\right)^k \times O(t^{2k-2}) & \text{for } \rho \geq 3 \end{cases} \\
= & \begin{cases} O(t) & \text{for } \rho = 2 \\ O(t^{2\rho-4}) & \text{for } \rho \geq 3. \end{cases}
\end{aligned}$$

Similarly, the second term of (8.4) can be expressed as

$$\begin{aligned}
& F \sum_{n=0}^{\infty} \frac{F^{2n} t^{2n+1}}{(2n+1)!} \\
= & \frac{F}{v} \sum_{n=0}^{\infty} \frac{(-1)^n (vt)^{2n+1}}{(2n+1)!} \sum_{k=0}^{(\rho-1)\wedge n} (-1)^k \binom{n}{k} \frac{A^k}{v^{2k}} \\
= & \frac{F}{v} \sum_{k=0}^{\rho-1} \left( -\frac{A}{v^2} \right)^k \left[ \sum_{j=0}^{k} (vt)^j C_j \left( \sum_{n=k}^{\infty} \frac{(-1)^n (vt)^{2n-j+1}}{(2n-j+1)!} \right) \right] \\
= & \frac{F}{v} \sum_{k=0}^{\rho-1} \left( -\frac{A}{v^2} \right)^k \times O(t^{2k-1}) \\
= & O(t^{2\rho-3}).
\end{aligned}$$

Hence, the lemma follows. □

**Lemma 8.3**

$$\int_0^T (T-s)^k AdW_s = O\left(T^{k+1/2}\sqrt{\ln\ln T}\right)$$

*Proof.* Let $M_u = \int_0^u (t-s)^k AdW_s$, which is a square integrable martingale for $[0 < u \leq t]$ and $< M >_u = \int_0^u (t-s)^{2k} AA' ds = [t^{2k+1} - (t-u)^{2k+1}]AA'/(2k+1)$. Since $M_u = B_{<M>_u}$ by Karatzas and Shreve ([14] p. 174),

$$\int_0^T (T-s)^k AdW_s = O(B_{T^{2k+1}}) = O(T^{k+1/2}\sqrt{\ln\ln T}).$$

Hence, the lemma follows. □

*Proof of Theorem 8.1.* If $\rho = 1$, then there exist $C \in \mathcal{R}$ such that $||e^{Ft}|| \leq C$ by (8.3). Therefore,

$$\begin{aligned} ||Y_T|| &= ||e^{FT}Y_0 + \int_0^T e^{F(T-s)} AdW_s|| \\ &\leq CY_0 + C\left[O(\sqrt{T\ln\ln T})\right] \\ &= O(\sqrt{T\ln\ln T}). \end{aligned}$$

For $\rho \geq 2$, by Lemma 8.2 and 8.3,

$$\begin{aligned} ||Y_T|| &= ||e^{FT}Y_0 + \int_0^T e^{F(T-s)} AdW_s|| \\ &\leq O\left(||e^{FT}Y_0|| + \left\|\int_0^T \sum_{k=0}^{2\rho-3} C_k (T-s)^k AdW_s\right\|\right) \\ &= O\left(||e^{FT}Y_0|| + \left\|\sum_{k=0}^{2\rho-3} C_k \int_0^T (T-s)^k AdW_s\right\|\right) \\ &\leq O\left(||e^{FT}Y_0|| + \sum_{k=0}^{2\rho-3} |C_k| \times ||O(T^{k+1/2}\sqrt{\ln\ln T})||\right) \\ &= O(T^{2\rho-5/2}\sqrt{\ln\ln T}). \end{aligned}$$

To show (8.1), we have

$$\begin{aligned} \lambda_{\max}\left(\int_0^T Y_t Y_t' dt\right) &= O\left(\mathrm{Tr}\int_0^T Y_t Y_t' dt\right) \\ &= O\left(\int_0^T ||Y_t||^2 dt\right) \\ &= \begin{cases} O(T^2(\ln\ln T)) & \text{a.s.} \quad \text{if} \quad \rho = 1 \\ O(T^{4\rho-4}(\ln\ln T)) & \text{a.s.} \quad \text{if} \quad \rho \geq 2. \end{cases} \end{aligned}$$

Hence, the proof of the theorem. □

### 8.2. Zero eigenvalues

In this Section, we study the asymptotic behavior of the OU processes when the drift matrix $F$ contains only zeros eigenvalues.(i.e., F is a nilpotent matrix.) The main results are summarized in the following:

**Theorem 8.2** *Suppose for the OU process defined in (1.1), the RANK condition (2.1) holds and, all eigenvalues of $F$ are zeros. Let $\gamma$ be the dimension of the largest block of the rational canonical form of $F$ as defined in Section 2 (i.e., $F^\gamma = 0$; see the Example). Then*

$$||Y_T|| = O(T^{\gamma-1/2}\sqrt{\ln\ln T}) \qquad \text{a.s.}$$

*Moreover,*

$$\lambda_{\max}\left(\int_0^T Y_t Y_t' dt\right) = O(T^{2\gamma}(\ln\ln T)) \qquad \text{a.s.} \tag{8.5}$$

*Proof.* Since $F$ is a $k \times k$ nilpotent matrix of order $\gamma$ $(1 \le \gamma \le k)$, then $F^\gamma = 0$ and

$$e^{Ft} = \sum_{n=0}^{\gamma-1} \frac{F^n t^n}{n!} = O(t^{\gamma-1}).$$

$$\begin{aligned}
||Y_T|| &\le O\left(||e^{FT}Y_0|| + \int_0^T \sum_{k=0}^{\gamma-1} C_k (T-s)^k A dW_s\right) \\
&= O(||e^{FT}Y_0||) + O\left(\sum_{k=0}^{\gamma-1} C_k \int_0^T (T-s)^k A dW_s\right) \\
&= O(T^{\gamma-1}) + O(T^{\gamma-1/2}\sqrt{\ln\ln T}) \\
&= O(T^{\gamma-1/2}\sqrt{\ln\ln T}).
\end{aligned}$$

To prove (8.5) observe,

$$\begin{aligned}
\lambda_{\max}\left(\int_0^T Y_t Y_t' dt\right) &= O\left(\operatorname{Tr}\int_0^T Y_t Y_t' dt\right) \\
&= O\left(\int_0^T ||Y_t||^2 dt\right) \\
&= O(T^{2\gamma}(\ln\ln T)).
\end{aligned}$$

Hence, the proof. ∎

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