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Exponential ergodicity for general continuous-state nonlinear branching processes

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Abstract

By combining the coupling by reflection for Brownian motion with the refined basic coupling for Poisson random measure, we present sufficient conditions for the exponential ergodicity of general continuous-state nonlinear branching processes in both the L^1-Wasserstein distance and the total variation norm, where the drift term is dissipative only for large distance, and either diffusion noise or jump noise is allowed to be vanished. Sufficient conditions for the corresponding strong ergodicity are also established.

Keywords: continuous-state nonlinear branching process; exponential ergodicity; coupling; strong ergodicity.
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1 Introduction

In this paper we will study the exponential ergodicity and the strong ergodicity for general continuous-state *nonlinear branching processes*, which will be introduced below. Consider a filtered probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ satisfying the usual hypotheses. Let $\{B_t\}_{t\ge 0}$ be an $(\mathscr{F}_t)$-Brownian motion. Throughout this paper, we write ν (which is allowed to be zero) for a σ-finite nonnegative measure on $(0,\infty)$ such that $\int_0^\infty (z\wedge z^2)\,\nu(\mathrm{d}z)<\infty$. Let $\{N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) : s,z,u>0\}$ be an independent $(\mathscr{F}_t)$-Poisson random measure on $(0,\infty)^3$ with intensity $\mathrm{d}s\,\nu(\mathrm{d}z)\,\mathrm{d}u$, and $\{\tilde N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) : s,z,u>0\}$ be a corresponding compensated measure, i.e., $\tilde N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) = N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) - \mathrm{d}s\,\nu(\mathrm{d}z)\,\mathrm{d}u$. We are interested in a general continuous-state nonlinear branching process, which is described as the

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pathwise unique nonnegative solution to the following stochastic differential equation (SDE):

$$\begin{aligned} X_t =& X_0 + \int_0^t \gamma_0(X_s)\,\mathrm{d}s + \int_0^t \sqrt{\gamma_1(X_s)}\,\mathrm{d}B_s \\ &+ \int_0^t \int_0^\infty \int_0^{\gamma_2(X_{s-})} z\,\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned} \tag{1.1}$$

Here,

- $x \mapsto \gamma_0(x)$ is a continuous function on $\mathbb{R}_+ := [0, \infty)$ such that $\gamma_0(0) \geq 0$;
- $x \mapsto \gamma_1(x)$ is a continuous function on $\mathbb{R}_+$ such that $\gamma_1(0) = 0$ and $\gamma_1(x) \geq 0$ for $x > 0$;
- $x \mapsto \gamma_2(x)$ is a continuous and non-decreasing function on $\mathbb{R}_+$ such that $\gamma_2(0) = 0$.

Intuitively, such a process can be identified as a continuous-state branching process with population-size-dependent branching rates and with competition.

If $\gamma_0(x) = a + bx$ for some $a \geq 0$ and $b \in \mathbb{R}$ and $\gamma_i(x) = c_i x$ $(i = 1, 2)$ for some $c_1, c_2 \geq 0$, then the solution to (1.1) is reduced to the classical continuous-state branching process (with constant immigration), see [1, 8, 9, 11, 13] and references therein. We would mention that, if and only if in this particular case with $a = 0$, the solution satisfies the so-called branching property, which means that different individuals act independently with each other. If $\gamma_i(x) = c_i x$ $(i = 1, 2)$ for some $c_i \geq 0$ and $\gamma_0(x) = b_1 x - b_2 x^2$ with some $b_1, b_2 > 0$, then the solution to (1.1) is called the logistic branching process in the literature and can be used to model the population dynamics with competition, see [5, 10] for more details. The quadratic regulatory term in the coefficient $\gamma_0(x)$ has an ecological interpretation, as it describes negative interactions between each pair of individuals in the population. Similar equations with general coefficients $\gamma_0(x)$ to model more general competitions were considered in [22].

Throughout this paper we always assume that (1.1) *has a unique non-explosive strong solution*, which is denoted by $(X_t)_{t\geq 0}$; see Subsection 2.1 for related discussions. Let $P_t(x, \cdot)$ and $(P_t)_{t\geq 0}$ be the transition function and the transition semigroup of the process $(X_t)_{t\geq 0}$, respectively. We are going to study the asymptotic behavior of the L^1-Wasserstein distance and the total variation distance between $P_t(x, \cdot)$ and $P_t(y, \cdot)$ for any $x, y \in \mathbb{R}_+$. As a direct consequence, we will establish sufficient conditions for the exponential ergodicity and the strong ergodicity of the process $(X_t)_{t\geq 0}$.

To the best of our knowledge, there are few known results on this topic. For the classical branching process (i.e. $\gamma_0(x) = a - bx$ and $\gamma_i(x) = c_i x$ $(i = 1, 2)$ for some $b > 0$ and $a, c_i \geq 0$), by the branching property, [14, Theorem 2.4] proved that the total variation distance between $P_t(x, \cdot)$ and $P_t(y, \cdot)$ decays exponentially fast. Recently, under uniformly dissipative condition on $\gamma_0(x)$ (see (3.7) in Remark 3.4(1) below) and finite second moment condition on the measure ν (i.e. $\int_{\mathbb{R}_+} z^2\,\nu(\mathrm{d}z) < \infty$), [6, Theorem 4.2] established the exponential decay between $P_t(x, \cdot)$ and $P_t(y, \cdot)$ with respect to the L^1-Wasserstein distance. All known results above are concerned on the case that the drift term $\gamma_0(x)$ satisfies the uniformly dissipative condition, so the approaches used in [14] and [6] are essentially based on the synchronous coupling (i.e., two marginal processes of the coupling are driven by the same noises). Instead, in the present paper we will use the combination of the coupling by reflection for Brownian motion and the refined basic coupling for Poisson random measure to establish the exponential ergodicity and the strong ergodicity for general nonlinear branching processes, where the drift term is dissipative only for large distances and the associated coefficients for driven noises are much more general.

To illustrate our main contributions, we present the following statement for the exponential ergodicity and the strong ergodicity of the process $(X_t)_{t\geq 0}$. The reader is refered to Section 3 for general results. For any probability measures μ_1, μ_2 on $\mathbb{R}_+$, the L^1-Wasserstein distance W_1 between μ_1 and μ_2 is defined by

$$W_1(\mu_1,\mu_2) = \inf_{\Pi\in\mathscr{C}(\mu_1,\mu_2)} \int_{\mathbb{R}_+\times\mathbb{R}_+} |x-y|\,\Pi(\mathrm{d}x,\mathrm{d}y),$$

where $\mathscr{C}(\mu_1,\mu_2)$ is the family of all probability measures on $\mathbb{R}_+\times\mathbb{R}_+$ having μ_1 and μ_2 as marginals. We denote by $\|\mu_1-\mu_2\|_{\text{Var}}$ the total variation norm between probability measures μ_1 and μ_2. The process $(X_t)_{t\geq 0}$ is called exponentially ergodic both in the W_1-distance and the total variation norm, if there are a unique probability measure μ on $\mathbb{R}_+$ and a constant $\lambda>0$ such that for all $x\in\mathbb{R}_+$ and $t>0$,

$$W_1(P_t(x,\cdot),\mu) \leq c(x)e^{-\lambda t}$$

and

$$\|P_t(x,\cdot)-\mu\|_{\text{Var}} \leq c(x)e^{-\lambda t},$$

where $c(x)$ is a nonnegative measurable function on $\mathbb{R}_+$. The process $(X_t)_{t\geq 0}$ is called strongly ergodic, if there are a unique probability measure μ on $\mathbb{R}_+$ and constants $\lambda, C>0$ so that for all $t>0$,

$$\sup_{x\in\mathbb{R}_+} \|P_t(x,\cdot)-\mu\|_{\text{Var}} \leq Ce^{-\lambda t}.$$

It is obvious that the strong ergodicity implies the exponential ergodicity in the total variation norm.

Theorem 1.1. *Let $(X_t)_{t\geq 0}$ be a unique strong solution to the SDE* (1.1) *such that assumptions below* (1.1) *on the coefficients are satisfied. Suppose that there are constants $l_0, k_1\geq 0$ and $k_2>0$ such that*

$$\gamma_0(x)-\gamma_0(y) \leq \begin{cases} k_1(x-y)\log\left(\frac{4l_0}{x-y}\right), & 0\leq x-y\leq l_0, \\ -k_2(x-y), & x-y>l_0. \end{cases} \tag{1.2}$$

Then the process $(X_t)_{t\geq 0}$ is exponentially ergodic both in the W_1-distance and the total variation norm if one of the following three assumptions holds:

(1) *the function $\gamma_1(x)$ is continuous and strictly positive on $(0,\infty)$, and satisfies*

$$\liminf_{x\to 0} \frac{\gamma_1(x)}{x^\beta} > 0 \tag{1.3}$$

for some $\beta\in[1,2)$;

(2) *there are constants $\alpha\in(0,2)$ and $c_0>0$ such that*

$$\nu(\mathrm{d}z) \geq c_0 \mathbb{1}_{\{0<z\leq 1\}} z^{-1-\alpha}\,\mathrm{d}z,$$

and the function $\gamma_2(x)$ is continuous and strictly positive on $(0,\infty)$, and satisfies

$$\liminf_{x\to 0} \frac{\gamma_2(x)}{x^\beta} > 0$$

for some $\beta\in[\alpha-1,\alpha)\cap(0,\infty)$;

(3) *there are constants* $\alpha \in (1,2)$ *and* $c_0 > 0$ *such that*

$$\int_0^r z^2 \, \nu(\mathrm{d}z) \geq c_0 r^{2-\alpha}, \quad 0 < r \leq 1, \tag{1.4}$$

and the function

$$\gamma_2(x) = b_2 x^{r_2} + \gamma_{2,2}(x),$$

where $b_2 > 0$*,* $r_2 \in [1, \alpha)$ *and* $\gamma_{2,2}(x)$ *is a non-decreasing function on* $\mathbb{R}_+$*.*

Furthermore, if (1.2) *is replaced by*

$$\gamma_0(x) - \gamma_0(y) \leq \begin{cases} k_1(x-y)\log\left(\frac{4l_0}{x-y}\right), & 0 \leq x-y \leq l_0, \\ -k_2(x-y)^\delta, & x-y > l_0 \end{cases} \tag{1.5}$$

for some $\delta > 1$*, then, under one of the three assumptions* (1)–(3) *above, the process* $(X_t)_{t\geq 0}$ *is strongly ergodic.*

(1.2) on the drift term $\gamma_0(x)$ for $0 < x - y \leq l_0$ is the standard one-sided non- Lipschitz continuous condition, while that for $x - y \geq l_0$ means that $\gamma_0(x)$ satisfies the dissipative condition for large distances (since l_0 is allowed to be any positive constant). By taking $\nu(\mathrm{d}z) = c_0|z|^{-1-\alpha}\mathbb{1}_{\{z>0\}}\,\mathrm{d}z$ for some $c_0 > 0$ and $\alpha \in (0,2)$, one can regard condition (2) as the extension of (1) from the Brownian motion case to the one-sided α-stable noise case. When $\alpha \in (1,2)$, condition (3) on the measure ν is much weaker than condition (2); for example, (1.4) is satisfied for the singular measure $\nu(\mathrm{d}z) := \sum_{j=0}^\infty 2^{\alpha j}\delta_{2^{-j}}(\mathrm{d}z)$ with $\alpha \in (1,2)$. In this case, it is at price of requiring a stronger assumption on the coefficient $\gamma_2(x)$. According to Theorem 1.1, we can see that the logistic branching process (i.e., $\gamma_i(x) = c_i x$ $(i = 1,2)$ for some $c_i \geq 0$ and $\gamma_0(x) = b_1 x - b_2 x^2$ with some $b_1, b_2 > 0$) is strongly ergodic; see Example 3.6 below for more general coefficients $\gamma_0(x)$ satisfying (1.5).

In the following, we will remark that conditions (1.3) and (1.5) are sharp in some concrete examples.

Remark 1.2. (1) Let $\gamma_0(x) = -x^2$, $\gamma_1(x) = 2x^2$ and $\gamma_2(x) = 0$; that is,

$$L = x^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} - x^2 \frac{\mathrm{d}}{\mathrm{d}x}.$$

Let $(X_t)_{t\geq 0}$ be the corresponding diffusion process. According to [12, the case (i)-(ib) after Example 2.18, p. 14], we know that $\mathbb{P}^x(\tau_0 = \infty) = 1$ for all $x > 0$, where $\mathbb{P}^x(\cdot) = \mathbb{P}(\cdot|X_0 = x)$ and $\tau_0 = \inf\{t > 0 : X_t = 0\}$. This is, the point 0 can be seen as the reflection boundary for the diffusion process $(X_t)_{t\geq 0}$ associated with the operator L on $[0,\infty)$. On the other hand, define $\mu(\mathrm{d}x) = x^{-2}e^{-x}\,\mathrm{d}x$. One can verify that for any $f \in C_b^2(\mathbb{R}_+)$ with $f'(0) = 0$, we have $\mu(Lf) = 0$, which implies that $\mu(\mathrm{d}x)$ is an invariant measure for the operator L. However,

$$\mu(\mathbb{R}_+) = \int_0^\infty x^{-2}e^{-x}\,\mathrm{d}x \geq e^{-1}\int_0^1 x^{-2}\,\mathrm{d}x = \infty.$$

Therefore, the process $(X_t)_{t\geq 0}$ is not ergodic, see e.g. [2, Table 5.1, p. 100]. Note that, for this example, (1.3) is satisfied with $\beta = 2$, and so this implies that (1.3) with $\beta < 2$ in Theorem 1.1 is optimal.

(2) Let $\gamma_0(x) = d - bx$ with $b, d > 0$, $\gamma_1(x) = \sqrt{2cx}$ with $c > 0$ and $\gamma_2(x) = 0$. Then, the solution to (1.1) is reduced into the famous Cox-Ingersoll-Ross (CIR) model. In this case, one can easily see that (1.2) and (1) in Theorem 1.1 hold. Therefore, the CIR model is exponentially ergodic in both the W_1-distance and the total variation distance. On the

other hand, denote by $\tau_1 = \inf\{t \ge 0 : X_t = 1\}$. According to [4, Corollary 9], for any $x > 1$,

$$\mathbb{E}^x[\tau_1] = \int_0^\infty \frac{e^{-z} - e^{-xz}}{bz + cz^2} \exp\left(\int_0^z \frac{d}{bu + cu^2}\,\mathrm{d}u\right)\mathrm{d}z.$$

By letting $x \to \infty$ in the above equality, we can conclude that $\sup_{x>1}\mathbb{E}^x[\tau_1] = \infty$. This together with [21, Lemma 2.1] yields that the CIR model is not strongly ergodic. In particular, this implies that (1.5) with $\delta > 1$ for the strong ergodicity in some sense is sharp.

The approach of our paper is based on recent developments of the couplings for SDEs with Lévy noises via coupling operators, see [15, 16, 19, 20, 23] for more details. However, there are a few essential differences between continuous-state nonlinear branching processes and the settings of [15, 16, 19, 20, 23]. For example, all the quoted papers above are restricted to the case that the driven noises are pure-jump processes and the coefficients for driven noises are non-degenerate, while in the present setting, the diffusion term and the jump noise are allowed to appear simultaneously in the SDE (1.1), and moreover both coefficients $\gamma_1(x)$ and $\gamma_2(x)$ are degenerate on $\mathbb{R}_+$ (since $\gamma_1(0) = \gamma_2(0) = 0$). The differences bring out much more difficulties in the present paper to efficiently apply the coupling techniques as these in [15, 16, 19, 20, 23]. For instance, due to the presence of non-degenerate diffusion term, to construct the coupling process here we will take care of couplings for both Brownian motion and Poisson random measure. Thus, we need to consider the coupling operator that contains both local part and non-local part of the associated generator (2.1). Because of the degenerate property of the coefficients, the coupling function (e.g., see (4.2) and (4.19)) and its estimates (e.g., see the proof of Theorems 3.1) in the applications of coupling process here are more complex and delicate than those in [15, 16, 19, 20, 23].

The remainder of this paper is arranged as follows. In Section 2, we recall some results from [7] on the strong solution to the SDE (1.1), and then present a Markovian coupling of the solution through the construction of a new coupling operator. General results on the exponential ergodiciy and the strong ergodicity for the SDE (1.1) are stated in Section 3. The proofs of all main results in Section 3 and Theorem 1.1 are given in the last section.

2 Unique strong solution and its coupling process

This section consists of two parts. We first recall results from [7] on the existence and the uniqueness of the strong solution to the SDE (1.1), and then construct a new Markovian coupling of the solution.

2.1 Existence and uniqueness of strong solution

The statement is taken from [7, Theorem 5.6].

Theorem 2.1 ([7, Theorem 5.6]). *Suppose that the coefficients γ_i, $i = 0, 1, 2$, satisfy the following conditions:*

(1) *there is a constant $K > 0$ so that*

$$\gamma_0(x) \le K(1 + x), \quad x \ge 0;$$

(2) *there exists a non-decreasing function $H(x)$ on $\mathbb{R}_+$ such that*

$$\gamma_1(x) \le H(x), \quad x \ge 0;$$

(3) *the function $\gamma_2(x)$ is nonnegative and non-decreasing on $\mathbb{R}_+$;*

(4) *$\gamma_0(x) = \gamma_{0,1}(x) - \gamma_{0,2}(x)$, where $\gamma_{0,1}(x)$ is continuous on $\mathbb{R}_+$, and $\gamma_{0,2}(x)$ is continuous and non-decreasing on $\mathbb{R}_+$. For each integer $m \geq 1$ there is a non-decreasing concave function $r_m(x)$ on $\mathbb{R}_+$ such that $\int_0^1 r_m(z)^{-1}\,\mathrm{d}z = \infty$, and for all $0 \leq x, y \leq m$,*

$$|\gamma_{0,1}(x) - \gamma_{0,1}(y)| \leq r_m(|x-y|);$$

(5) *for each integer $m \geq 1$ there is a nonnegative and non-decreasing function $\rho_m(x)$ on $\mathbb{R}_+$ such that $\int_0^1 \rho_m(z)^{-2}\,\mathrm{d}z = \infty$, and for all $0 \leq x, y \leq m$,*

$$|\sqrt{\gamma_1(x)} - \sqrt{\gamma_1(y)}|^2 + |\gamma_2(x) - \gamma_2(y)| \leq \rho_m(|x-y|)^2.$$

Then, for any initial value $X_0 = x \geq 0$, there exists a unique strong solution to the SDE (1.1)*, and the solution is a strong Markov process $(X_t)_{t\geq 0}$ with the generator given by*

$$Lf(x) = \gamma_0(x) f'(x) + \frac{\gamma_1(x)}{2} f''(x) + \gamma_2(x) \int_0^\infty \big(f(x+z) - f(x) - z f'(x)\big)\,\nu(\mathrm{d}z) \tag{2.1}$$

for any $f \in C_b^2(\mathbb{R}_+)$.

To investigate the exponential ergodicity of the process $(X_t)_{t\geq 0}$, we will assume that the drift term $\gamma_0(x)$ is dissipative for large distance, see (3.1) below. One can see that condition (1) in Theorem 2.1 holds with $K = \sup_{0\leq r\leq l_0} \Phi_1(r)$ under (3.1). On the other hand, we suppose that the function $\gamma_1(x)$ is continuous on $\mathbb{R}_+$ such that $\gamma_1(0) = 0$. Hence, condition (2) in Theorem 2.1 holds with $H(x) := \sup_{0\leq y\leq x} \gamma_1(y)$. We have already supposed that condition (3) is satisfied, see assumptions below the SDE (1.1). Therefore, in the setting of our paper, the SDE (1.1) has a unique strong solution under assumptions of Theorem 3.1 (or Theorem 3.2), and some locally continuous assumptions on the coefficients $\gamma_i(x)$ for all $i = 0, 1, 2$ (e.g. conditions (4) and (5) in Theorem 2.1).

2.2 Markovian coupling for continuous-state nonlinear branching process

To study the coupling of the process $(X_t)_{t\geq 0}$ determined by (1.1), we begin with the construction of a new coupling operator for its generator L given by (2.1). Recall that $(X_t, Y_t)_{t\geq 0}$ is a Markov coupling of the process $(X_t)_{t\geq 0}$ given by (1.1), if $(X_t, Y_t)_{t\geq 0}$ is a Markov process on $\mathbb{R}_+^2$ such that the marginal process $(Y_t)_{t\geq 0}$ has the same transition probability as $(X_t)_{t\geq 0}$. Denote by $\widetilde{L}$ the infinitesimal generator of the Markov coupling process $(X_t, Y_t)_{t\geq 0}$. Then, the operator $\widetilde{L}$ satisfies the following marginal property, i.e., if for any $f, g \in C^2(\mathbb{R}_+)$,

$$\tilde{L}h(x,y) = Lf(x) + Lg(y),$$

where $h(x,y) = f(x) + g(y)$ for $x, y \in \mathbb{R}_+$, and L is given by (2.1). We call $\tilde{L}$ is a *coupling operator* of L. For example, one of the standard couplings, named the synchronous coupling, is that the driven noises for the marginal process $(Y_t)_{t\geq 0}$ are the same as those of $(X_t)_{t\geq 0}$, i.e., $(Y_t)_{t\geq 0}$ is given by

$$\begin{aligned} Y_t =& y + \int_0^t \gamma_0(Y_s)\,\mathrm{d}s + \int_0^t \sqrt{\gamma_1(Y_s)}\,\mathrm{d}B_s \\ &+ \int_0^t \int_0^\infty \int_0^{\gamma_2(Y_{s-})} z\,\tilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u), \quad Y_0 = y \in \mathbb{R}_+, t \geq 0. \end{aligned}$$

In this paper, we will combine the coupling by reflection for Brownian motion and the refined basic coupling for Poisson random measure. Here the coupling by reflection for Brownian motion means that we will take $(-B_t)_{t\geq 0}$ (which is regarded as a reflection of $(B_t)_{t\geq 0}$) for the process $(Y_t)_{t\geq 0}$ before two marginal processes meet, see (2.7) below for the full expression. The readers are referred to [17, 3] for more details on the coupling by reflection for diffusion processes. To explain the meaning of the refined basic coupling for Poisson random measure, we would like to use the viewpoint from the coupling operator. Note that the function $\gamma_2(x)$ is non-decreasing on $\mathbb{R}_+$. For a given parameter $\kappa > 0$, set $x_\kappa = x \wedge \kappa$ for $x > 0$. Roughly speaking, when $x > y \geq 0$, the jumping system corresponding to the refined basic coupling of the non-local part for the operator L is given by

$$(x,y) \longrightarrow \begin{cases} (x+z, y+z+(x-y)_\kappa), & \frac{1}{2}\gamma_2(y)\mu_{-(x-y)_\kappa}(\mathrm{d}z), \\ (x+z, y+z-(x-y)_\kappa), & \frac{1}{2}\gamma_2(y)\mu_{(x-y)_\kappa}(\mathrm{d}z), \\ (x+z, y+z), & \gamma_2(y)\Big[\nu(\mathrm{d}z) - \frac{1}{2}\mu_{-(x-y)_\kappa}(\mathrm{d}z) - \frac{1}{2}\mu_{(x-y)_\kappa}(\mathrm{d}z)\Big], \\ (x+z, y), & [\gamma_2(x)-\gamma_2(y)]\nu(\mathrm{d}z), \end{cases} \tag{2.2}$$

where

$$\mu_x(\mathrm{d}z) = (\nu \wedge (\delta_x * \nu))(\mathrm{d}z) \tag{2.3}$$

for all $x \in \mathbb{R}$. Similarly, we can define the case that $0 \leq x < y$. We briefly explain the meaning of each row in (2.2) in the spirit of [20, Section 2]. Suppose that $0 \leq x - y \leq \kappa$. Then, in the first row of (2.2), the distance of the two marginals decreases from $|x-y|$ to $|(x+z) - (y+z+(x-y))| = 0$, and this reflects the idea of the basic coupling – but only with half of the common jump intensity

$$\frac{1}{2}\gamma_2(y)\mu_{-(x-y)}(\mathrm{d}z) \leq \frac{1}{2}[(\gamma_2(x)\nu(\mathrm{d}z)) \wedge (\gamma_2(y)(\delta_{y-x} * \nu)(\mathrm{d}z))]$$

from x to $x+z$ and y to $y+z+(x-y)$ (due to the increasing property of $\gamma_2(x)$). In the second row of (2.2), the distance is doubled after jumping, with the remaining half of that common jump intensity. We then divide the remaining mass into two parts. One is to couple synchronously as indicated in the third row of (2.2) with the maximum remaining mass of y-component, and the other is to couple independently on only x-component as shown in the last row of (2.2) (also thanks to the increasing property of $\gamma_2(x)$). If $|x-y| > \kappa$, then the first row of (2.2) shows that the distance after the jump is $|x-y| - \kappa$. Therefore, the parameter κ is the threshold to determine whether the marginal processes jump to the same point, or become slightly closer to each other. This is a technical point, but is crucial for our argument to make the refined basic coupling efficient for the Lévy measure ν with finite-range jumps. See [16, Subsection 3.2] and [15, Section 2] for more details on the refined basic coupling for SDEs with Lévy jumps.

With the idea above in mind, we then define for any $f \in C^2(\mathbb{R}_+^2)$ and $x > y \geq 0$ that

$$\begin{aligned} \tilde{L}f(x,y) =& \gamma_0(x)f'_x(x,y) + \gamma_0(y)f'_y(x,y) \\ &+ \frac{1}{2}\gamma_1(x)f''_{xx}(x,y) + \frac{1}{2}\gamma_1(y)f''_{yy}(x,y) - \sqrt{\gamma_1(x)\gamma_1(y)}f''_{xy}(x,y) \\ &+ \frac{1}{2}\gamma_2(y)\int_0^\infty (f(x+z, y+z+(x-y)_\kappa) - f(x,y) \\ &\qquad - f'_x(x,y)z - f'_y(x,y)(z+(x-y)_\kappa))\,\mu_{-(x-y)_\kappa}(\mathrm{d}z) \\ &+ \frac{1}{2}\gamma_2(y)\int_0^\infty (f(x+z, y+z-(x-y)_\kappa) - f(x,y) \\ &\qquad - f'_x(x,y)z - f'_y(x,y)(z-(x-y)_\kappa))\,\mu_{(x-y)_\kappa}(\mathrm{d}z) \\ &+ \gamma_2(y)\int_0^\infty (f(x+z, y+z) - f(x,y) - f'_x(x,y)z \\ &\qquad - f'_y(x,y)z)\left(\nu - \frac{1}{2}\mu_{-(x-y)_\kappa} - \frac{1}{2}\mu_{(x-y)_\kappa}\right)(\mathrm{d}z) \\ &+ (\gamma_2(x) - \gamma_2(y))\int_0^\infty (f(x+z,y) - f(x,y) - f'_x(x,y)z)\,\nu(\mathrm{d}z). \end{aligned} \tag{2.4}$$

Here and in what follows, $f'_x(x,y) = \frac{\partial f(x,y)}{\partial x}$, $f''_{xx}(x,y) = \frac{\partial^2 f(x,y)}{\partial x^2}$ and $f''_{xy}(x,y) = \frac{\partial^2 f(x,y)}{\partial x \partial y}$, and so on. Similarly, we can define $\tilde{L}f(x,y)$ for the case that $0 \le x < y$. By using the fact that $\mu_x = \delta_x * \mu_{-x}$ for any $x \in \mathbb{R}$ (see [20, Corollary A.2]), one can check that the generator $\tilde{L}$ constructed above is a coupling operator of L given by (2.1); see [20, Subsection 2.1].

Next, we will construct the SDE on $\mathbb{R}_+^2$ associated with the coupling operator $\tilde{L}$ defined above, and prove the existence of the strong solution to the corresponding SDE. The idea below is partly motivated by [20, Subsection 2.2]. According to [20, Corollary A.2 and Remark 2.1], $\mu_x = \delta_x * \mu_{-x}$, and

$$\mu_x(\mathbb{R}_+) = \mu_{-x}(\mathbb{R}_+) \le 2\nu(\{z \in \mathbb{R}_+ : z > |x|/2\}) < \infty \tag{2.5}$$

for all $x \in \mathbb{R}$. Recalling $\mu_x = \nu \wedge (\delta_x * \nu)$, we define the following control function

$$\rho(x,z) = \frac{\mu_x(\mathrm{d}z)}{\nu(\mathrm{d}z)} \in [0,1], \quad x \in \mathbb{R},\ z \in \mathbb{R}_+$$

with $\rho(0,z) = 1$ by convention. Consider the following SDE:

$$\begin{cases} X_t = & x + \int_0^t \gamma_0(X_s)\,\mathrm{d}s + \int_0^t \sqrt{\gamma_1(X_s)}\,\mathrm{d}B_s \\ & + \int_0^t \int_0^\infty \int_0^{\gamma_2(X_{s-})} z\,\tilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \\ Y_t = & y + \int_0^t \gamma_0(Y_s)\,\mathrm{d}s + \int_0^t \sqrt{\gamma_1(Y_s)}\,\mathrm{d}B_s^* \\ & + \int_0^t \int_0^\infty \int_0^{\frac{1}{2}\gamma_2(Y_{s-})\rho(-(U_{s-})_\kappa, z)} [z + (U_{s-})_\kappa]\,\tilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \\ & + \int_0^t \int_0^\infty \int_{\frac{1}{2}\gamma_2(Y_{s-})\rho(-(U_{s-})_\kappa, z)}^{\frac{1}{2}\gamma_2(Y_{s-})[\rho(-(U_{s-})_\kappa, z) + \rho((U_{s-})_\kappa, z)]} [z - (U_{s-})_\kappa]\,\tilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u) \\ & + \int_0^t \int_0^\infty \int_{\frac{1}{2}\gamma_2(Y_{s-})[\rho(-(U_{s-})_\kappa, z) + \rho((U_{s-})_\kappa, z)]}^{\gamma_2(Y_{s-})} z\,\tilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u), \end{cases} \tag{2.6}$$

where

$$B_t^* = \begin{cases} -B_t, & t \le T, \\ -2B_T + B_t, & t > T, \end{cases} \tag{2.7}$$

$T = \inf\{t \ge 0 : X_t = Y_t\}$, and $U_t = X_t - Y_t$.

Proposition 2.2. *For any $(x,y) \in \mathbb{R}_+^2$, the system of equations (2.6) is well defined, and has a unique strong solution $(X_t, Y_t)_{t \ge 0}$. Moreover, we have*

(1) *the infinitesimal generator of the process $(X_t, Y_t)_{t\ge 0}$ is just the coupling operator $\tilde{L}$ defined by (2.4).*
(2) *$X_t = Y_t$ for all $t \ge T$, where $T = \inf\{t > 0 : X_t = Y_t\}$.*

Proof. Recall that in the setting of our paper, we always assume that (1.1) has a non-explosive and pathwise unique strong solution $(X_t)_{t\ge 0}$. We are going to show that the sample paths of $(Y_t)_{t \ge 0}$ given in (2.6) can be obtained by repeatedly modifying those of the strong solution to the following equation:

$$Z_t = y + \int_0^t \gamma_0(Z_s)\,\mathrm{d}s + \int_0^t \sqrt{\gamma_1(Z_s)}\,\mathrm{d}B_s^* + \int_0^t \int_0^\infty \int_0^{\gamma_2(Z_{s-})} z\,\tilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u). \tag{2.8}$$

By the definition of $(B_t^*)_{t\ge 0}$, we can verify that $(B_t^*)_{t \ge 0}$ is still an $(\mathscr{F}_t)$-Brownian motion. Since the driving Poisson random measure for (1.1) and (2.8) is the same, the existence of the strong solution $(Z_t)_{t\geq 0}$ to the equation (2.8) is guaranteed by the pathwise unique strong solution to (1.1).

We first claim that the process $(Y_t)_{t\geq 0}$ given in (2.6) is the same as

$$\begin{aligned}Y_t =&y+\int_0^t \gamma_0(Y_s)\,\mathrm{d}s+\int_0^t \sqrt{\gamma_1(Y_s)}\,\mathrm{d}B_s^*\\&+\int_0^t\int_0^\infty\int_0^{\gamma_2(Y_{s-})} z\,\tilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\\&+\int_0^t (U_{s-})_\kappa\int_0^\infty\int_0^{\frac{1}{2}\gamma_2(Y_{s-})\rho(-(U_{s-})_\kappa,z)} N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)\\&-\int_0^t (U_{s-})_\kappa\int_0^\infty\int_{\frac{1}{2}\gamma_2(Y_{s-})\rho(-(U_{s-})_\kappa,z)}^{\frac{1}{2}\gamma_2(Y_{s-})[\rho(-(U_{t-})_\kappa,z)+\rho((U_{t-})_\kappa,z)]} N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u).\end{aligned}\tag{2.9}$$

Indeed, this immediately follows from the fact that for all $z>0$, $\mu_z(\mathbb{R}_+)=\mu_{-z}(\mathbb{R}_+)<\infty$, and the identity that for any $x,y\in\mathbb{R}_+$ with $x\neq y$,

$$\begin{aligned}\int_0^\infty\int_0^{\frac{1}{2}\gamma_2(y)\rho(-(x-y)_\kappa,z)} \mathrm{d}u\,\nu(\mathrm{d}z) =&\frac{1}{2}\gamma_2(y)\mu_{-(x-y)_\kappa}(\mathbb{R}_+)=\frac{1}{2}\gamma_2(y)\mu_{(x-y)_\kappa}(\mathbb{R}_+)\\=&\int_0^\infty\int_{\frac{1}{2}\gamma_2(y)\rho(-(x-y)_\kappa,z)}^{\frac{1}{2}\gamma_2(y)[\rho(-(x-y)_\kappa,z)+\rho((x-y)_\kappa z)]}\mathrm{d}u\,\nu(\mathrm{d}z).\end{aligned}$$

Hence, we next turn to construct the sample paths of $(Y_t)_{t\geq 0}$ given in (2.9).

Let $(Z_t^{(1)})_{t\geq 0}$ be the solution to (2.8) with $Z_0^{(1)}=y$. Denote by $(p_t)_{t\in\mathbb{D}_p}$ the Poisson point process associated with the Poisson random measure $N(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u)$ on $(0,\infty)^2$, and by $\Delta X_t=X_t-X_{t-}$. Let $R_t^{(1)}=R_{1,t}^{(1)}+R_{2,t}^{(1)}$, where

$$R_{1,t}^{(1)}:=\frac{1}{2}\gamma_2(Z_t^{(1)})\rho(-(X_t-Z_t^{(1)})_\kappa,\Delta X_t),\quad R_{2,t}^{(1)}:=\frac{1}{2}\gamma_2(Z_t^{(1)})\rho((X_t-Z_t^{(1)})_\kappa,\Delta X_t).$$

Define the stopping times $T_1=\inf\{t>0: Z_t^{(1)}=X_t\}$, and

$$\sigma_1=\inf\left\{t\in\mathbb{D}_p: p_t\in(0,\infty)\times(0,R_t^{(1)}]\right\}.$$

We consider two cases:

(i) On the event $\{T_1\leq\sigma_1\}$, we set $Y_t=Z_t^{(1)}$ for all $t<T_1$; moreover, by the pathwise uniqueness of (1.1), we can define $Y_t=X_t$ for $t\geq T_1$.

(ii) On the event $\{T_1>\sigma_1\}$, we define $Y_t=Z_t^{(1)}$ for all $t<\sigma_1$ and

$$Y_{\sigma_1}=Z_{\sigma_1-}^{(1)}+\Delta X_{\sigma_1}+\begin{cases}(X_{\sigma_1-}-Y_{\sigma_1-})_\kappa, & p_{\sigma_1}\in(0,\infty)\times(0,R_{1,t}^{(1)}],\\-(X_{\sigma_1-}-Y_{\sigma_1-})_\kappa, & p_{\sigma_1}\in(0,\infty)\times(R_{1,t}^{(1)},R_t^{(1)}].\end{cases}$$

In the following, we will restrict on the event $\{T_1>\sigma_1\}$ and consider the following SDE:

$$\begin{aligned}Z_t =&Y_{\sigma_1}+\int_{\sigma_1}^t\gamma_0(Z_s)\,\mathrm{d}s+\int_{\sigma_1}^t\sqrt{\gamma_1(Z_s)}\,\mathrm{d}B_s^*\\&+\int_{\sigma_1}^t\int_0^\infty\int_0^{\gamma_2(Z_{s-})} z\,\tilde{N}(\mathrm{d}s,\mathrm{d}z,\mathrm{d}u),\qquad t>\sigma_1.\end{aligned}$$

Denote its solution by $(Z_t^{(2)})_{t\geq\sigma_1}$. Similarly, we set $R_t^{(2)}=R_{1,t}^{(2)}+R_{2,t}^{(2)}$ with

$$R_{1,t}^{(2)}:=\frac{1}{2}\gamma_2(Z_{t-}^{(2)})\rho(-(X_{t-}-Z_{t-}^{(2)})_\kappa,\Delta X_t),\quad R_{2,t}^{(2)}:=\frac{1}{2}\gamma_2(Z_{t-}^{(2)})\rho((X_{t-}-Z_{t-}^{(2)})_\kappa,\Delta X_t)]$$

for all $t > \sigma_1$. We further define $T_2 = \inf\{t > 0 : Z_t^{(2)} = X_t\}$, and

$$\sigma_2 = \inf\left\{t \in \mathbb{D}_p \cap (\sigma_1, \infty) : p_t \in (0, \infty) \times (0, R_t^{(2)}]\right\}.$$

In the same way, we can define Y_t for $t \leq \sigma_2$. We then repeat this procedure. Note that

$$\frac{1}{2}\gamma_2(Y_{t-})[\mu_{-(X_{t-}-Y_{t-})_\kappa(\mathbb{R}_+)} + \mu_{(X_{t-}-Y_{t-})_\kappa(\mathbb{R}_+)}]$$

is uniformly bounded (thanks to (2.5)) for any $t < \tau_m$ with $m = 1, 2, \ldots$, where

$$\tau_m = \inf\{t \geq 0 : Y_t > m \text{ or } |X_t - Y_t| < 1/m\}.$$

Then only finitely many modifications have to be made in the interval $(0, t \wedge \tau_m)$. Finally, by letting $m \to \infty$, we can determine the unique strong solution $(Y_t)_{t\geq 0}$ to the SDE (2.9) globally. See also the proof of [15, Proposition 2.6] for the details.

With the construction of $(Y_t)_{t\geq 0}$ above, we can apply the Itô formula to the SDE (2.6) to obtain the assertion (1). The assertion (2) immediately follows from the SDE (2.6) and the assumption that (1.1) has a non-explosive and pathwise unique strong solution $(X_t)_{t\geq 0}$. □

In the following, we call $(X_t, Y_t)_{t\geq 0}$ determined by (2.6) a (Markovian) coupling process of $(X_t)_{t\geq 0}$. To conclude this part, we will give the preserving order property of the coupling process $(X_t, Y_t)_{t\geq 0}$.

Corollary 2.3. *Let $(X_t, Y_t)_{t\geq 0}$ be the coupling process determined by* (2.6) *and with the starting point (x, y). If $x > y$, then $X_t \geq Y_t$ for all $t > 0$ a.s.*

Proof. Denote by $\mathbb{P}^{(x,y)}$ and $\mathbb{E}^{(x,y)}$ the probability and the expectation of the process $(X_t, Y_t)_{t\geq 0}$ starting from (x, y), respectively. Let

$$\tilde{T} := \inf\{t > 0 : Y_t > X_t\},$$

and define $f_n \in C_b^2(\mathbb{R}_+^2)$ for $n \in \mathbb{N}$ such that $f_n \geq 0$, $f_n(x, y) = 1$ if $y \geq x + 1/n$, and $f_n(x, y) = 0$ if $y < x$. Then, for any $x > y$ and $t > 0$,

$$\mathbb{E}^{(x,y)} f_n(X_{t\wedge\tilde{T}}, Y_{t\wedge\tilde{T}}) = f_n(x, y) + \mathbb{E}^{(x,y)}\left(\int_0^{t\wedge\tilde{T}} \tilde{L} f_n(X_s, Y_s)\,\mathrm{d}s\right) = 0,$$

where in the last equality we used the fact that $\tilde{L} f_n(x, y) = 0$ for all $x \geq y$, thanks to the definition of the coupling operator $\tilde{L}$ given by (2.4) and the assumption that the function $\gamma_2(x)$ is non-decreasing on $\mathbb{R}_+$. Then, for any $x > y$ and $t > 0$, by the Fatou lemma,

$$\mathbb{P}^{(x,y)}(\tilde{T} < t) = \mathbb{E}^{(x,y)} \liminf_{n\to\infty} f_n(X_{t\wedge\tilde{T}}, Y_{t\wedge\tilde{T}}) \leq \liminf_{n\to\infty} \mathbb{E}^{(x,y)} f_n(X_{t\wedge\tilde{T}}, Y_{t\wedge\tilde{T}}) = 0.$$

Therefore, for any $x > y$,

$$\mathbb{P}^{(x,y)}(\tilde{T} = \infty) = 1.$$

That is, for any $x > y$, the coupling process $(X_t, Y_t)_{t\geq 0}$ associated with the coupling operator $\tilde{L}$ satisfies that $X_t \geq Y_t$ for all $t > 0$ a.s. □

Remark 2.4. One can use the coupling by reflection of the local part but apply the synchronous coupling (instead of the refined basic coupling) of the non-local part to construct another coupling operator for the operator L. For any $x > y \geq 0$, the synchronous coupling of the non-local part for the operator L given by (2.1) is given by

$$(x, y) \longrightarrow \begin{cases} (x + z, y + z), & \gamma_2(y)\,\nu(\mathrm{d}z), \\ (x + z, y), & (\gamma_2(x) - \gamma_2(y))\,\nu(\mathrm{d}z). \end{cases}$$

Then, for any $f \in C^2(\mathbb{R}_+^2)$ and $x > y \geq 0$, the corresponding coupling operator L^* is defined by

$$\begin{aligned}L^* f(x,y) =& \gamma_0(x) f'_x(x,y) + \gamma_0(y) f'_y(x,y)(x,y)\\ &+ \frac{1}{2}\gamma_1(x) f''_{xx}(x,y) + \frac{1}{2}\gamma_1(y) f''_{yy}(x,y) - \sqrt{\gamma_1(x)\gamma_1(y)} f''_{xy}(x,y)\\ &+ \gamma_2(y)\int_0^\infty (f(x+z,y+z) - f(x,y) - f'_x(x,y)z - f'_y(x,y)z)\,\nu(\mathrm{d}z)\\ &+ (\gamma_2(x) - \gamma_2(y))\int_0^\infty (f(x+z,y) - f(x,y) - f'_x(x,y)z)\,\nu(\mathrm{d}z).\end{aligned}$$

The difference between $\tilde{L}$ and L^* is that the coupling operator L^* does not involve the measures $\mu_{(x-y)_\kappa}$ and $\mu_{-(x-y)_\kappa}$. The coupling process associated with the coupling operator L^* above can be constructed directly. Actually, putting (1.1) and (2.8) together, we can check by Itô's formula that the generator of the Markov process $(X_t, Y_t)_{t\geq 0}$ defined by (1.1) and (2.8) on $\mathbb{R}_+^2$ is just the coupling operator L^*; moreover, $X_t = Y_t$ for $t \geq T$, where $T = \inf\{t > 0 : X_t = Y_t\}$. Similarly, we can see that this coupling process $(X_t, Y_t)_{t\geq 0}$ also enjoys the preserving order property as in Corollary 2.3.

3 Exponential convergence in the L^1-Wasserstein distance and the total variation distance

In this section, we shall give general results about the exponential ergodicity of the process $(X_t)_{t\geq 0}$ determined by the SDE (1.1), in terms of both the L^1-Wasserstein distance and the total variation norm. To present our main result, we first introduce some notation. For a strictly increasing function ψ on $\mathbb{R}_+$ and two probability measures μ_1 and μ_2 on $\mathbb{R}_+$, define

$$W_\psi(\mu_1,\mu_2) = \inf_{\Pi\in\mathscr{C}(\mu_1,\mu_2)} \int_{\mathbb{R}_+^2} \psi(|x-y|)\,\Pi(\mathrm{d}x,\mathrm{d}y),$$

where $\mathscr{C}(\mu_1,\mu_2)$ is the collection of probability measures on $\mathbb{R}_+^2$ with marginals μ_1 and μ_2. When ψ is concave, the above definition gives rise to a Wasserstein distance W_ψ in the space of probability measures μ on $\mathbb{R}_+$ such that $\int_{\mathbb{R}_+} \psi(z)\,\mu(\mathrm{d}z) < \infty$. If $\psi(r) = r$ for all $r \geq 0$, then W_ψ is the standard L^1-Wasserstein distance as introduced in Section 1. Another well known example for W_ψ is given by $\psi(r) = \mathbb{1}_{(0,\infty)}(r)$, which leads to the total variation distance

$$W_\psi(\mu_1,\mu_2) = \frac{1}{2}\|\mu_1-\mu_2\|_{\mathrm{Var}} := \frac{1}{2}[(\mu_1-\mu_2)^+(\mathbb{R}_+) + (\mu_1-\mu_2)^-(\mathbb{R}_+)].$$

The following two results give us the exponential convergence in the L^1-Wasserstein distance and the total variation norm for the SDE (1.1), respectively.

Theorem 3.1. *Suppose that there are constants $l_0 \geq 0$, $k_2 > 0$ and a nonnegative function $\Phi_1 \in C[0,2l_0] \cap C^3(0,2l_0]$ satisfying $\Phi_1(0) = 0$, $\Phi_1' \geq 0$, $\Phi_1'' \leq 0$ and $\Phi_1''' \geq 0$ on $(0,2l_0]$ such that*

$$\gamma_0(x) - \gamma_0(y) \leq \begin{cases} \Phi_1(x-y), & 0 \leq x-y \leq l_0,\\ -k_2(x-y), & x-y > l_0.\end{cases} \tag{3.1}$$

If one of the following two assumptions holds:

(A1) *there exist constants $\beta \in [1,2)$ and $k_3 > 0$ such that*

$$\int_0^1 \Phi_1(r) r^{-\beta}\,\mathrm{d}r < \infty$$

and

$$\gamma_1(x) + \gamma_1(y) \geq k_3(x-y)^\beta, \quad 0 \leq x-y \leq l_0; \tag{3.2}$$

(A2) *there exist constants* $\alpha \in (0,2)$*,* $\beta \in [\alpha-1, \alpha) \cap (0,\infty)$ *and* $C_*, k_3 > 0$ *such that*

$$\int_0^1 \Phi_1(r) r^{\alpha-\beta-2}\, \mathrm{d}r < \infty, \tag{3.3}$$

$$\int_0^r z^2\, \nu(\mathrm{d}z) \geq C_* r^{2-\alpha}, \quad 0 < r \leq 1 \tag{3.4}$$

and

$$(\gamma_2(x) - \gamma_2(y)) + \gamma_2(y) \mathbb{1}_{\{\inf_{0<z\leq\kappa}[z^\alpha \mu_z(\mathbb{R}_+)] \geq C_*\}} \geq k_3(x-y)^\beta, \quad 0 \leq x-y \leq l_0, \tag{3.5}$$

where μ_z *is given by* (2.3)*;*

then there exist positive constants C *and* λ *so that for all* $t > 0$ *and* $x, y \geq 0$*,*

$$W_1(P_t(x,\cdot), P_t(y,\cdot)) \leq Ce^{-\lambda t}|x-y|.$$

Theorem 3.2. *Under the assumptions of Theorem* 3.1*, if additionally the function* Φ_1 *in* (3.1) *satisfies*

$$\limsup_{r\to 0} \Phi_1(r) r^{1-\beta} = 0$$

when Assumption (A1) *holds, or satisfies*

$$\limsup_{r\to 0} \Phi_1(r) r^{\alpha-\beta-1} = 0 \tag{3.6}$$

when Assumption (A2) *holds, then there exist positive constants* C *and* λ *so that for all* $t > 0$ *and* $x, y \geq 0$*,*

$$\|P_t(x,\cdot) - P_t(y,\cdot)\|_{\mathrm{Var}} \leq Ce^{-\lambda t}(1 + |x-y|).$$

We make some comments on the assumptions of Theorems 3.1 and 3.2. First, (3.1) is the so-called dissipative condition for large distance on the drift term $\gamma_0(x)$. In applications there are a lot of choices for the function Φ_1; for example, $\Phi_1(r) = Cr$ corresponds to the standard one-sided locally Lipschitz continuous condition, and $\Phi_1(r) = Cr\log(4l_0/r)$ is the typical one-sided non-Lipschitz continuous condition. Both functions satisfy assumptions in Theorems 3.1 and 3.2. Secondly, since we assume that the function $\gamma_1(x)$ is continuous on $\mathbb{R}_+$ such that $\gamma_1(0) = 0$, (3.2) is satisfied when the function $\gamma_1(x)$ is strictly positive on $(0,\infty)$ such that $\liminf_{x\to 0} \frac{\gamma_1(x)}{x^\beta} > 0$. Thirdly, (3.4) implies that $\int_0^1 \nu(\mathrm{d}z) = \infty$. Suppose furthermore $\mu_x(\mathbb{R}_+) \geq C_* x^{-\alpha}$ for all $x \in (0,\kappa]$. This assumption is concerned on the concentration of the Lévy measure ν around zero (small jump activity), and it implies that the measure ν has a component that is absolutely continuous with respect to the Lebesgue measure, see [20, Proposition A.5]. Then (3.5) is equivalently saying that $\gamma_2(x) \geq k_3 x^\beta$ for all $0 \leq x \leq l_0$, which is also equivalent that $\gamma_2(x)$ is strictly positive on $(0,\infty)$ such that $\liminf_{x\to 0} \frac{\gamma_2(x)}{x^\beta} > 0$. On the other hand, when $\gamma_2(x) - \gamma_2(y) \geq k_3(x-y)^\beta$ for all $0 < x-y \leq l_0$ (this in particular indicates that the function γ_2 is strictly increasing on $\mathbb{R}_+$), we only require (3.5), which can be fulfilled even for singular measures ν, see the remarks below Theorem 1.1.

As direct consequences of Theorems 3.1 and 3.2, we have the following statement for the exponential ergodicity of the process $(X_t)_{t\geq 0}$ in terms of the W_1-distance and the total variation norm. Let $\mathscr{P}_1$ be the space of probability measures having a finite first moment.

Corollary 3.3. (1) *Under assumptions of Theorem* 3.1*, there exist a unique invariant probability measure* $\mu \in \mathscr{P}_1$ *and a constant* $\lambda > 0$ *such that for all* $t > 0$ *and* $\mu_0 \in \mathscr{P}_1$*,*

$$W_1(\mu_0 P_t, \mu) \le C_{\mu_0} e^{-\lambda t},$$

where C_{μ_0} *is a positive constant depending on* μ_0*.*

(2) *Under assumptions of Theorem* 3.2*, there exist a unique invariant probability measure* $\mu \in \mathscr{P}_1$ *and a constant* $\lambda > 0$ *such that for all* $t > 0$ *and* $\mu_0 \in \mathscr{P}_1$*,*

$$\|\mu_0 P_t - \mu\|_{\mathrm{Var}} \le C_{\mu_0} e^{-\lambda t},$$

where C_{μ_0} *is a positive constant depending on* μ_0*.*

Remark 3.4. (1) Recently, under the uniformly dissipative condition on the drift term $\gamma_0(x)$, i.e., (3.1) holds with $l_0 = 0$, which is equivalent to say that

$$\gamma_0(x) - \gamma_0(x) \le -k_2(x-y), \quad 0 \le y \le x, \tag{3.7}$$

and the finite second moment condition for the jump measure ν, as well as some growth conditions on the coefficients $\gamma_1(x)$ and $\gamma_2(x)$, [6, Theorem 4.2] establishes the exponential ergodicity in the L^1-Wasserstein distance for continuous-state nonlinear branching processes. On the other hand, Theorem 3.1 and Corollary 3.3(1) establish the exponential ergodicity under the weaker assumptions. Here the drift term is only required to be dissipative for large distances as indicated by (3.1), or the jump measure with finite first moment. In particular, Theorem 3.1 and Corollary 3.3(1) are workable for

$$\nu(\mathrm{d}z) = \left(|z|^{-1-\alpha} \mathbb{1}_{\{0<z\le 1\}} + |z|^{-1-\alpha_1} \mathbb{1}_{\{z>1\}}\right) \mathrm{d}z$$

with $\alpha \in (0,2)$ and $\alpha_1 > 1$.

(2) We mention that, by the remarks below [12, Example 2.18], one can easily give examples such that the assumptions of Theorem 3.1 (or Theorem 3.2) are satisfied, but for any $x > 0$, $\mathbb{P}^x(\tau_0 < \infty) > 0$ (or even =1), where $\tau_0 = \inf\{t > 0 : X_t = 0\}$. Therefore, under assumptions of Theorem 3.1 (or Theorem 3.2) the invariant probability measure of the process $(X_t)_{t\ge 0}$ could be allowed to have an atom at $\{0\}$.

The following assertion is furthermore concerned on the strong ergodicity of the process $(X_t)_{t\ge 0}$.

Theorem 3.5. *Under assumptions of Theorem* 3.2*, if* (3.1) *is strengthened into the condition that there are a constant* $l_0 \ge 0$ *and two nonnegative functions* Φ_1 *and* Φ_2 *such that*

$$\gamma_0(x) - \gamma_0(y) \le \begin{cases} \Phi_1(x-y), & 0 \le x-y \le l_0, \\ -\Phi_2(x-y), & x-y > l_0, \end{cases} \tag{3.8}$$

where Φ_1 *is the same as that in Theorem* 3.2*, and* $\Phi_2 \in C^2[l_0,\infty)$ *satisfies* $\Phi_2' \ge 0$ *and* $\Phi_2'' \ge 0$ *on* $[l_0,\infty)$*, as well as*

$$\int_{l_0}^{\infty} \frac{1}{\Phi_2(s)}\,\mathrm{d}s < \infty,$$

then the process $(X_t)_{t\ge 0}$ *is strongly ergodic, i.e., there exist a unique invariant probability measure* μ *and constants* $C, \lambda > 0$ *such that for all* $t > 0$ *and* $x \ge 0$*,*

$$\|\delta_x P_t - \mu\|_{\mathrm{Var}} \le C e^{-\lambda t}.$$

Note that since $\Phi_2'' \ge 0$ on $[2l_0,\infty)$, $\Phi_2(r) \ge \Phi_2(2l_0) + \Phi_2'(2l_0) r$. So, (3.8) is stronger than (3.1) (by choosing $l_0 > 0$ large enough if necessary). A typical example for the function Φ_2 in Theorem 3.5 is that $\Phi_2(r) = c_0 r^\delta$ with $c_0 > 0$ and $\delta > 1$.

We close this section with the following examples on the coefficient $\gamma_0(x)$.

Example 3.6. (1) Let $\gamma_0(x) = b_1 x \log(1+1/x) - b_2 x$ with $b_1, b_2 > 0$. Then, (3.1) holds with $\Phi_1(r) = b_1 r \log(1+1/r)$ and $k_2 = b_2/2$ for some $l_0 > 0$ large enough.

(2) Let $\gamma_0(x) = b_1 x - b_2 x^\delta$ with $\delta > 1$ and $b_1, b_2 > 0$. Then, (3.8) holds with $\Phi_1(r) = b_1 r$ and $\Phi_2(r) = b_2 r^\delta/2$ for some $l_0 > 0$.

(3) Let $\gamma_0(x) = b_1 x - b_2 e^{cx^\delta}$ with $c, \delta, b_1, b_2 > 0$. Then, (3.8) holds with $\Phi_1(r) = b_1 r$, and $\Phi_2(r) = cr^\theta$ with any $\theta > 1$ and some $l_0, c > 0$.

4 Proofs

4.1 Lemmas

To prove the main results in this paper, we need the following elementary lemmas.

Lemma 4.1. *For fixed $l_0 > 0$, let $g \in C[0, 2l_0] \cap C^3(0, 2l_0]$ be satisfying $g(0) = 0$ and*

$$g'(r) \geq 0,\ g''(r) \leq 0 \text{ and } g'''(r) \geq 0 \quad \text{for any } r \in (0, 2l_0]. \tag{4.1}$$

Then for all $c_1, c_2 > 0$ the function

$$\psi(r) = \begin{cases} c_1 r + \int_0^r e^{-c_2 g(s)}\,\mathrm{d}s, & r \in [0, 2l_0], \\ \psi(2l_0) + \frac{\psi'(2l_0)}{2} \int_0^{r-2l_0} \left[1 + \exp\left(\frac{2\psi''(2l_0)}{\psi'(2l_0)} s\right)\right] \mathrm{d}s, & r \in (2l_0, \infty) \end{cases} \tag{4.2}$$

satisfies

(1) *$\psi \in C^2(\mathbb{R}_+)$ such that $\psi' > 0$ and $\psi'' < 0$ on $\mathbb{R}_+$;*

(2) *$\psi''' \geq 0$ and $\psi^{(4)} \leq 0$ on $(0, 2l_0]$. In particular, for any $0 \leq \delta \leq r \leq l_0$,*

$$\psi(r+\delta) + \psi(r-\delta) - 2\psi(r) \leq \psi''(r)\delta^2;$$

(3) *for all $r > 0$,*

$$\min\left\{c_1, \frac{\psi(2l_0)}{4l_0}, \frac{\psi'(2l_0)}{4}\right\} r \leq \psi(r) \leq (1+c_1)r.$$

Proof. The assertion (1) follows from the definition of ψ. The assertion (2) has been proven in [20, Lemma 4.1]. Since $\|\psi'\|_\infty = 1 + c_1$ and $\psi(0) = 0$, the second inequality in the assertion (3) holds. On the other hand, for any $r \in [0, 2l_0]$, $\psi(r) \geq c_1 r$; for any $r \in [4l_0, \infty)$, $\psi(r) \geq \frac{\psi'(2l_0)}{2}(r - 2l_0) \geq \frac{\psi'(2l_0)}{4} r$; for any $r \in (2l_0, 4l_0]$, $\psi(r) \geq \psi(2l_0) \geq \frac{\psi(2l_0)}{4l_0} r$. Combining with all the estimates above, we can prove the first inequality in the assertion (3). □

We have the following typical choice of functions g in the definition (4.2) for ψ.

Lemma 4.2. *For fixed $l_0 > 0$, let $\Phi_1 \in C[0, 2l_0] \cap C^3(0, 2l_0]$ be a nonnegative function such that $\Phi_1(0) = 0$, $\Phi_1' \geq 0$, $\Phi_1'' \leq 0$ and $\Phi_1''' \geq 0$ on $(0, 2l_0]$. Suppose that for some $\theta \in (0, 1]$,*

$$\int_0^r \Phi_1(z) z^{\theta-2}\,\mathrm{d}z < \infty, \quad r \in [0, 2l_0].$$

For any $c_0 > 0$, set

$$g(r) := r^\theta + c_0 \int_0^r \Phi_1(z) z^{\theta-2}\,\mathrm{d}z.$$

Then $g \in C[0, 2l_0] \cap C^3(0, 2l_0]$ such that $g(0) = 0$, and (4.1) holds; moreover,

$$\sup_{0 < r \leq 2l_0} \left(r g'(r) - \frac{r g''(r)}{g'(r)} \right) < \infty.$$

Proof. Let $g_1(r) = r^\theta$ and $g_2(r) = c_0 \int_0^r \Phi_1(z) z^{\theta-2}\, \mathrm{d}z$ for some $\theta \in (0,1]$ and $c_0 > 0$. It is clear that $g_1(0) = 0$, and g_1 satisfies (4.1). We next claim that g_2 also enjoys the property (4.1). Indeed, by assumptions, it is clear that $g_2(0) = 0$, $g_2'(r) = c_0 \Phi_1(r) r^{\theta-2} \geq 0$, and

$$\begin{aligned} g_2''(r) =& c_0(\theta-2)\Phi_1(r) r^{\theta-3} + c_0 \Phi_1'(r) r^{\theta-2} = c_0 r^{\theta-3}\left((\theta-2)\Phi_1(r) + \Phi_1'(r) r\right) \\ \leq& c_0 r^{\theta-3}\left(-\Phi_1(r) + \Phi_1'(r) r\right) \leq 0 \end{aligned}$$

for $r \in (0, 2l_0]$, where in the first inequality we used the facts that $\theta \in (0,1]$ and $\Phi_1(r) \geq 0$ for all $r \in (0, 2l_0]$, and the last inequality follows from the facts that $\Phi_1(0) = 0$ and $\Phi_1''(r) \leq 0$ for $r \in (0, 2l_0]$. Furthermore, for $r \in (0, 2l_0]$,

$$\begin{aligned} g_2'''(r) =& c_0 \Phi_1''(r) r^{\theta-2} + 2c_0(\theta-2)\Phi_1'(r) r^{\theta-3} + c_0(\theta-2)(\theta-3)\Phi_1(r) r^{\theta-4} \\ =& c_0 r^{\theta-4}\left[(\theta-2)(\theta-3)\Phi_1(r) + 2(\theta-2)\Phi_1'(r) r + \Phi_1''(r) r^2\right]. \end{aligned}$$

By the facts that $\Phi_1(0) = 0$ and $\Phi_1''' \geq 0$ on $(0, 2l_0]$, and the mean value theorem,

$$\begin{aligned} 0 =& 2(2-\theta)\Phi_1(0) \leq 2(2-\theta)\Phi_1(r) - 2(2-\theta)\Phi_1'(r) r + (2-\theta)\Phi_1''(r) r^2 \\ \leq& (3-\theta)(2-\theta)\Phi_1(r) - 2(2-\theta)\Phi_1'(r) r + \Phi_1''(r) r^2 \end{aligned}$$

for all $r \in (0, 2l_0]$, where in the second inequality above we used the facts that $\theta \in (0,1]$, $\Phi_1 \geq 0$ and $\Phi_1'' \leq 0$ on $(0, 2l_0]$. This implies that $g_2''' \geq 0$ on $(0, 2l_0]$. Combining with all the estimates above, we prove the desired assertion for g_2. Since $g = g_1 + g_2$, we show that g satisfies (4.1).

Since $g(0) = 0$ and $g''(r) \leq 0$ for all $r \in (0, 2l_0]$, $rg'(r) \leq g(r)$ for all $r \in (0, 2l_0]$ and so

$$\sup_{r \in (0, 2l_0]} rg'(r) \leq \sup_{r \in (0, 2l_0]} g(r) = g(2l_0).$$

On the other hand,

$$g'(r) = \theta r^{\theta-1} + c_0 \Phi_1(r) r^{\theta-2}$$

and

$$\begin{aligned} -rg''(r) &= \theta(1-\theta) r^{\theta-1} + c_0(2-\theta) r^{\theta-2}\Phi_1(r) - c_0 r^{\theta-1}\Phi_1'(r) \\ &\leq \theta(1-\theta) r^{\theta-1} + c_0(2-\theta) r^{\theta-2}\Phi_1(r), \end{aligned}$$

where we used the fact that $\Phi_1' \geq 0$ on $(0, 2l_0]$. Thus, by $\theta \in (0,1]$,

$$\sup_{r \in (0, 2l_0]} \frac{-rg''(r)}{g'(r)} \leq \sup_{r \in (0, 2l_0]} \frac{\theta(1-\theta) r^{\theta-1} + c_0(2-\theta) r^{\theta-2}\Phi_1(r)}{\theta r^{\theta-1} + c_0 \Phi_1(r) r^{\theta-2}} \leq 2-\theta.$$

Therefore, the proof is complete. □

In the following, for any $f \in C^2(\mathbb{R}_+)$, $\tilde{L} f(x-y) := \tilde{L} F(x,y)$, where $F(x,y) = f(x-y)$.

Lemma 4.3. *For any $n \geq 1$, let $\psi_n \in C^2(\mathbb{R}_+)$ be satisfying $\psi_n(0) = 0$ and*

$$\tilde{L}\psi_n(x-y) \leq -\lambda \psi_n(x-y), \quad 1/n \leq x-y \leq n,$$

where $\lambda > 0$ is independent of n, x and y. Then for any $t > 0$ and $x, y \in \mathbb{R}_+$,

$$W_\psi(P_t(x,\cdot), P_t(y,\cdot)) \leq \psi(|x-y|) e^{-\lambda t},$$

where $\psi := \liminf_{n\to\infty} \psi_n$.

Proof. This lemma follows from the preserving order property for the coupling process $(X_t, Y_t)_{t\geq 0}$ associated with the coupling operator $\tilde{L}$ proved in Corollary 2.3, and the arguments in part (2) of the proof for [20, Theorem 3.1]. So, we omit the details here. □

4.2 Proofs of the main results

Now, we are in a position to prove Theorems 3.1 and 3.2.

Proof of Theorem 3.1. (1) We first verify the assertion when (A2) is satisfied. At the beginning we will prove it under the assumption that

$$\mu_z(\mathbb{R}_+) \geq C_* z^{-\alpha}, \quad z \in (0, \kappa]. \tag{4.3}$$

In this case, (3.5) is reduced to

$$\gamma_2(x) \geq k_3(x-y)^\beta, \quad 0 \leq x-y \leq l_0,$$

which is equivalent to

$$\gamma_2(x) \geq k_3 x^\beta, \quad 0 \leq x \leq l_0. \tag{4.4}$$

Throughout the proof, without loss of generality we can assume that $l_0 \geq 1$ and $\kappa \in (0,1]$. According to the definition (2.4) of the coupling operator $\tilde{L}$, we know that for any $f \in C^2(\mathbb{R}_+)$ and any $x > y \geq 0$,

$$\begin{aligned}
\tilde{L}f(x-y) =& \frac{1}{2}\gamma_2(y)\big[f((x-y)+(x-y)_\kappa) + f((x-y)-(x-y)_\kappa) - 2f(x-y)\big] \\
&\times \mu_{(x-y)_\kappa}(\mathbb{R}_+) \\
&+ (\gamma_2(x)-\gamma_2(y))\int_0^\infty (f(x-y+z) - f(x-y) - f'(x-y)z)\,\nu(\mathrm{d}z) \\
&+ (\gamma_0(x)-\gamma_0(y))\,f'(x-y) + \frac{1}{2}(\sqrt{\gamma_1(x)}+\sqrt{\gamma_1(y)})^2 f''(x-y).
\end{aligned} \tag{4.5}$$

In the following, we will take f to be the function ψ defined by (4.2), where the constants $c_1, c_2 > 0$ and the function g will be determined later. According to Lemma 4.1(1), $\psi''(r) \leq 0$ for all $r > 0$, and so by the mean value theorem, for any $x > y \geq 0$ and $z > 0$,

$$\psi(x-y+z) - \psi(x-y) - \psi'(x-y)z \leq 0; \tag{4.6}$$

moreover, thanks to Lemma 4.1(2), for $0 < x-y \leq l_0$ and $0 < z \leq l_0$, we see

$$\psi(x-y+z) - \psi(x-y) - \psi'(x-y)z \leq \frac{1}{2}\psi''(x-y)z^2 + \frac{1}{6}\psi'''(x-y)z^3 \tag{4.7}$$

and

$$\psi((x-y)+(x-y)_\kappa) + \psi((x-y)-(x-y)_\kappa) - 2\psi(x-y) \leq \psi''(x-y)(x-y)_\kappa^2. \tag{4.8}$$

Note that for any $0 < r \leq l_0$,

$$\psi'(r) = c_1 + e^{-c_2 g(r)}, \psi''(r) = -c_2 g'(r) e^{-c_2 g(r)}, \psi'''(r) = -c_2 g'(r) e^{-c_2 g(r)}\left[c_2 g'(r) + \frac{g''(r)}{g'(r)}\right].$$

Then, by (4.7), (3.4) and the increasing property of $\gamma_2(x)$, for any $c_0 \in (0, 1/l_0] \subset (0,1]$ (whose value will be chosen later), and any $x > y \geq 0$ with $0 < x-y \leq l_0$,

$$\begin{aligned}
&(\gamma_2(x)-\gamma_2(y))\int_0^{c_0(x-y)} (\psi(x-y+z) - \psi(x-y) - \psi'(x-y)z)\ \nu(\mathrm{d}z) \\
&\leq (\gamma_2(x)-\gamma_2(y))\psi''(x-y)\cdot\frac{1}{2}\int_0^{c_0(x-y)} z^2\,\nu(\mathrm{d}z) \\
&\quad + \frac{c_0(x-y)(\gamma_2(x)-\gamma_2(y))}{3}\psi'''(x-y)\cdot\frac{1}{2}\int_0^{c_0(x-y)} z^2\,\nu(\mathrm{d}z) \\
&\leq -\frac{c_2 C_* c_0^{2-\alpha}}{2}(\gamma_2(x)-\gamma_2(y))g'(x-y)e^{-c_2 g(x-y)}(x-y)^{2-\alpha} \\
&\quad \times \left[1 - \frac{c_0(x-y)}{3}\left(c_2 g'(x-y) - \frac{g''(x-y)}{g'(x-y)}\right)\right].
\end{aligned}$$

On the other hand, according to (4.8) and (4.3), for any $x > y \ge 0$ with $0 < x - y \le l_0$,

$$\begin{aligned}&\frac{1}{2}\gamma_2(y)\big[\psi((x-y)+(x-y)_\kappa)+\psi((x-y)-(x-y)_\kappa)-2\psi(x-y)\big]\mu_{(x-y)_\kappa}(\mathbb{R}_+)\\&\le \frac{C_*}{2}\gamma_2(y)\psi''(x-y)(x-y)_\kappa^{2-\alpha}\\&\le -\frac{C_*c_2}{2}\frac{\kappa^{2-\alpha}}{l_0^{2-\alpha}}\gamma_2(y)g'(x-y)e^{-c_2g(x-y)}(x-y)^{2-\alpha}.\end{aligned}$$

Therefore, putting these two estimates and (3.1) in (4.5), we arrive at that for any $c_0 \in (0, 1/l_0] \subset (0, 1]$ and any $x > y \ge 0$ with $0 < x - y \le l_0$,

$$\begin{aligned}\tilde{L}\psi(x-y) \le&\frac{1}{2}\gamma_2(y)\big[\psi((x-y)+(x-y)_\kappa)+\psi((x-y)-(x-y)_\kappa)-2\psi(x-y)\big]\\&\qquad\times\mu_{(x-y)_\kappa}(\mathbb{R}_+)\\&+(\gamma_2(x)-\gamma_2(y))\int_0^{c_0(x-y)}(\psi(x-y+z)-\psi(x-y)-\psi'(x-y)z)\ \nu(\mathrm{d}z)\\&+\Phi_1(x-y)\psi'(x-y)+\frac{1}{2}(\sqrt{\gamma_1(x)}+\sqrt{\gamma_1(y)})^2\psi''(x-y)\\ \le&-\frac{C_*c_2}{2}\frac{\kappa^{2-\alpha}}{l_0^{2-\alpha}}\gamma_2(y)g'(x-y)e^{-c_2g(x-y)}(x-y)^{2-\alpha}\\&-\frac{c_2C_*c_0^{2-\alpha}}{2}(\gamma_2(x)-\gamma_2(y))g'(x-y)e^{-c_2g(x-y)}(x-y)^{2-\alpha}\\&\qquad\times\left[1-\frac{c_0(x-y)}{3}\left(c_2g'(x-y)-\frac{g''(x-y)}{g'(x-y)}\right)\right]\\&+\Phi_1(x-y)(c_1+e^{-c_2g(x-y)}),\end{aligned}\tag{4.9}$$

where in the second inequality we used the fact that $\psi''(r) \le 0$ for all $r \in (0, l_0]$.

Next, we choose

$$g(r) = r^{\alpha-\beta} + c_3\int_0^r \Phi_1(s)s^{\alpha-\beta-2}\,\mathrm{d}s,$$

where the constant c_3 will be chosen later. Note that, by (3.3), $g(r)$ is well defined. Set

$$c_0 = \min\left\{\frac{1}{l_0}, \frac{1}{\sup_{0<r\le l_0}\frac{-rg''(r)}{g'(r)}}\right\},\quad c_2 = \frac{\sup_{0<r\le l_0}\frac{-rg''(r)}{g'(r)}}{\sup_{0<r\le l_0} rg'(r)},\quad c_1 = e^{-c_2g(l_0)}.$$

Since $0 < \alpha - \beta \le 1$, according to Lemma 4.2, we know that $c_0, c_2 \in (0, \infty)$. It follows from the definition of c_2 that for $0 < x - y \le l_0$

$$(x-y)\left(c_2g'(x-y)-\frac{g''(x-y)}{g'(x-y)}\right) \le 2\sup_{0<r\le l_0}\frac{-rg''(r)}{g'(r)}.$$

This along with the definition of c_0 in turn immediately yields

$$1-\frac{c_0(x-y)}{3}\left(c_2g'(x-y)-\frac{g''(x-y)}{g'(x-y)}\right) \le \frac{1}{3}.$$

By substituting the above inequality into (4.9) and using the definition of c_1, we see

$$\begin{aligned}\tilde{L}\psi(x-y)\le&-\frac{C_*c_2}{2}\frac{\kappa^{2-\alpha}}{l_0^{2-\alpha}}\gamma_2(y)g'(x-y)e^{-c_2g(x-y)}(x-y)^{2-\alpha}\\&-\frac{C_*c_2c_0^{2-\alpha}}{6}(\gamma_2(x)-\gamma_2(y))g'(x-y)e^{-c_2g(x-y)}(x-y)^{2-\alpha}\\&+2\Phi_1(x-y)e^{-c_2g(x-y)}\\\le&-\frac{C_*c_2}{2}\min\left\{\frac{\kappa^{2-\alpha}}{l_0^{2-\alpha}},\frac{c_0^{2-\alpha}}{3}\right\}\gamma_2(x)g'(x-y)e^{-c_2g(x-y)}(x-y)^{2-\alpha}\\&+2\Phi_1(x-y)e^{-c_2g(x-y)}\\\le&-\frac{C_*c_2k_3}{2}\min\left\{\frac{\kappa^{2-\alpha}}{l_0^{2-\alpha}},\frac{c_0^{2-\alpha}}{3}\right\}g'(x-y)e^{-c_2g(x-y)}(x-y)^{2+\beta-\alpha}\\&+2\Phi_1(x-y)e^{-c_2g(x-y)},\end{aligned}\tag{4.10}$$

where in the second inequality we used the fact that

$$\frac{\kappa^{2-\alpha}}{l_0^{2-\alpha}}\gamma_2(y)+\frac{c_0^{2-\alpha}}{3}(\gamma_2(x)-\gamma_2(y))\ge\min\left\{\frac{\kappa^{2-\alpha}}{l_0^{2-\alpha}},\frac{c_0^{2-\alpha}}{3}\right\}\gamma_2(x)$$

and in the last inequality we used (4.4). Furthermore, taking

$$c_3=2\left[\frac{C_*c_2k_3}{2}\min\left\{\frac{\kappa^{2-\alpha}}{l_0^{2-\alpha}},\frac{c_0^{2-\alpha}}{3}\right\}\right]^{-1}$$

and recalling

$$g'(r)=(\alpha-\beta)r^{\alpha-\beta-1}+c_3\Phi_1(r)r^{\alpha-\beta-2},$$

we arrive at that for any $x,y\in\mathbb{R}_+$ with $0<x-y\le l_0$,

$$\begin{aligned}\tilde{L}\psi(x-y)\le&-\frac{C_*c_2k_3(\alpha-\beta)}{2}\min\left\{\frac{\kappa^{2-\alpha}}{l_0^{2-\alpha}},\frac{c_0^{2-\alpha}}{3}\right\}e^{-c_2g(x-y)}(x-y)\\\le&-\frac{C_*c_2k_3(\alpha-\beta)}{2}\min\left\{\frac{\kappa^{2-\alpha}}{l_0^{2-\alpha}},\frac{c_0^{2-\alpha}}{3}\right\}e^{-c_2g(l_0)}(x-y).\end{aligned}\tag{4.11}$$

On the other hand, by (4.2), for any $r>2l_0$,

$$\psi'(r)=\frac{\psi'(2l_0)}{2}\left(1+\exp\left(\frac{2\psi''(2l_0)}{\psi'(2l_0)}(r-2l_0)\right)\right]\ge\frac{\psi'(2l_0)}{2}.$$

Then, it follows that $\psi'(r)\ge\frac{\psi'(2l_0)}{2}$ for any $r>0$, since $\psi'(r)$ is decreasing on $(0,\infty)$. Hence, for any $x-y>l_0$, according to (4.5), (4.6), (3.1) and the fact $\psi''\le0$,

$$\begin{aligned}\tilde{L}\psi(x-y)\le&\frac{1}{2}\gamma_2(y)\big[\psi((x-y)+(x-y)_\kappa)+\psi((x-y)-(x-y)_\kappa)-2\psi(x-y)\big]\\&\times\mu_{(x-y)_\kappa}(\mathbb{R}_+)-k_2(x-y)\psi'(x-y)\\\le&-\frac{k_2\psi'(2l_0)}{2}(x-y),\end{aligned}\tag{4.12}$$

where in the last inequality we used the fact that

$$\psi(r+\delta)+\psi(r-\delta)-2\psi(r)\le0,\quad 0<\delta\le r,\tag{4.13}$$

thanks to $\psi''\le0$ again.

According to (4.11), (4.12) and Lemma 4.1(3), we know that for any $0<y<x$,

$$\tilde{L}\psi(x-y)\le -\lambda\psi(x-y),$$

where

$$\lambda=\frac{1}{1+c_1}\min\left\{\frac{C_*c_2k_3(\alpha-\beta)}{2}\min\left\{\frac{\kappa^{2-\alpha}}{l_0^{2-\alpha}},\frac{c_0^{2-\alpha}}{3}\right\}e^{-c_2g(l_0)},\frac{k_2\psi'(2l_0)}{2}\right\}.$$

This along with Lemma 4.3 yields that for any $t>0$ and $x,y\in\mathbb{R}_+$,

$$W_\psi(P_t(x,\cdot),P_t(y,\cdot))\le\psi(|x-y|)e^{-\lambda t}.$$

Hence, the required assertion follows from the inequality above and Lemma 4.1(3).

When

$$\gamma_2(x)-\gamma_2(y)\ge k_3(x-y)^\beta,\quad 0\le x-y\le l_0, \tag{4.14}$$

one can follow the arguments above to obtain the desired assertion. Indeed, in this case we can get rid of the term involving $\mu_{(x-y)_\kappa}(\mathbb{R}_+)$ in estimates for $\tilde{L}\psi(x-y)$ for any $x,y\in\mathbb{R}_+$ with $0<x-y\le l_0$, since this term is non-positive. We also note that under (4.14) we can also directly apply the coupling operator L^* and the associated coupling process mentioned in Remark 2.4; however, such a coupling can not deal with the case that (4.4) is satisfied. This explains the reason why we adopt the refined basic coupling for the non-local part of the operator L, rather than simply applying the synchronous coupling.

(2) We next verify the assertion when (A1) is satisfied. Let ψ be the function defined by (4.2). From the fact $\psi''\le 0$ and the increasing property of γ_2, we see the first two terms in the right hand side of (4.5) is non-positive. Then, we get from estimates for ψ, (3.2) and (3.1) that, for any $x>y$ with $0<x-y\le l_0$,

$$\begin{aligned}\tilde{L}\psi(x-y)\le&(\gamma_0(x)-\gamma_0(y))\psi'(x-y)+\frac{k_3}{2}(x-y)^\beta\psi''(x-y)\\ \le&\Phi_1(x-y)\psi'(x-y)+\frac{k_3}{2}(x-y)^\beta\psi''(x-y),\end{aligned}$$

where in the first inequality we used the fact that

$$(\sqrt{\gamma_1(x)}+\sqrt{\gamma_1(y)})^2\ge k_3(x-y)^\beta,\quad 0\le x-y\le l_0,$$

thanks to (3.2). Furthermore, we choose

$$g(r)=r^{2-\beta}+c_3\int_0^r\Phi_1(s)s^{-\beta}\,\mathrm{d}s.$$

Similarly, with possible choice of constants c_1, c_2 and c_3 in the definition of ψ, one can follow the argument in part (1) to verify the desired assertion. The details are omitted here. □

Proof of Theorem 3.2. For simplicity, we only verify the case that (A2) is satisfied and that $\mu_z(\mathbb{R}_+)\ge C_*z^{-\alpha}$ for all $z\in(0,\kappa]$ (i.e., (4.3) and so (4.4) holds), since one can prove the desired assertion similarly (and even easier) for other cases.

Without loss of generality, we assume that $l_0\ge 1$ and $\kappa\in(0,1]$. For any $n\ge 1$, define $f_n\in C^2(\mathbb{R}_+)$ such that

$$f_n(r)\begin{cases}=\psi(r), & 0<r\le 1/(n+1),\\ \le 1+b\left(\frac{r}{1+r}\right)^\theta+\psi(r), & 1/(n+1)<r\le 1/n,\\ =1+b\left(\frac{r}{1+r}\right)^\theta+\psi(r), & r\ge 1/n,\end{cases}$$

where $b > 0$ will be chosen later, $\theta = (\alpha - \beta)/2 \in (0,1)$, and ψ is defined by (4.2) (which is the one in part (1) of the proof of Theorem 3.1 with some modification on the associated constant c_3). We will verify that there exists a constant $\lambda > 0$ such that for any $n \geq 1$ and $x - y > 1/n$,

$$\tilde{L} f_n(x-y) \leq -\lambda f_n(x-y). \tag{4.15}$$

If (4.15) holds, then, the assertion follows from Lemma 4.3 and the fact that

$$\liminf_{n\to\infty} f_n(x,y) = \mathbb{1}_{\{x\neq y\}}(1 + b(|x-y|/(1+|x-y|))^\theta + \psi(|x-y|) \asymp \mathbb{1}_{\{x\neq y\}}(1+|x-y|),$$

where for any nonnegative functions f, g on $\mathbb{R}_+^2$, $f \asymp g$ means that there is a constant $c \geq 1$ such that $c^{-1} f(x,y) \leq g(x,y) \leq c f(x,y)$ for all $(x,y) \in \mathbb{R}_+^2$.

In the following, let $\psi_0(r) = b(r/(1+r))^\theta$. Then,

$$\psi_0'(r) = b\theta(1+r)^{-2}\left(\frac{r}{1+r}\right)^{\theta-1} \tag{4.16}$$

and

$$\psi_0''(r) = b\theta\left[(\theta-1)(1+r)^{-4}\left(\frac{r}{1+r}\right)^{\theta-2} - 2(1+r)^{-3}\left(\frac{r}{1+r}\right)^{\theta-1}\right]. \tag{4.17}$$

Moreover, it is easy to see $\psi_0''' \geq 0$ and $\psi_0^{(4)} \leq 0$. Following the argument of Lemma 4.1(2) we know that (4.8) still holds with ψ_0 in place of ψ, and can obtain that

$$\psi_0(r+s) - \psi_0(r) - \psi_0'(r)s \leq \frac{s^2}{2}\psi_0''(2r), \quad 0 < s \leq r. \tag{4.18}$$

Since $\psi_0{}'' \leq 0$, (4.6) and (4.13) also hold with ψ_0 replaced by ψ. Then, by (4.5), (4.16) and the fact that $\theta \in (0,1)$, for $l_0^* \in (0,\kappa]$ (which is determined later), $n \geq 1$ and $1/n \leq l_0^* \leq x-y \leq l_0$.

$$\tilde{L}\psi_0(x-y) \leq \Phi_1(x-y)\psi_0'(x-y) \leq b\theta\Phi_1(x-y)(1+l_0^*)^{-\theta-1}(l_0^*)^{\theta-1} \leq b\theta\Phi_1(x-y)(l_0^*)^{\theta-1}.$$

On the other hand, with the same function ψ (i.e, with the same function g and the constants c_0, c_1, c_2) in the proof of Theorem 3.1, we find that for $n \geq 1$ and $1/n \leq l_0^* \leq x-y \leq l_0$, the inequality (4.10) still holds. Combining both estimates together, we obtain that for any $n \geq 1$ and $1/n \leq l_0^* \leq x-y \leq l_0$,

$$\begin{aligned}\tilde{L} f_n(x-y) \leq &-\frac{C_* c_2 k_3}{2}\min\left\{\frac{\kappa^{2-\alpha}}{l_0^{2-\alpha}}, \frac{c_0^{2-\alpha}}{3}\right\} g'(x-y)e^{-c_2 g(x-y)}(x-y)^{2+\beta-\alpha}\\ &+ \Phi_1(x-y)e^{-c_2 g(x-y)}(2 + b\theta e^{c_2 g(l_0)} {l_0^*}^{\theta-1}).\end{aligned}$$

Furthermore, choosing $b = e^{-c_2 g(l_0)}$ and noticing that $\theta \in (0,1)$, we arrive at

$$\begin{aligned}\tilde{L} f_n(x-y) \leq &-\frac{C_* c_2 k_3}{2}\min\left\{\frac{\kappa^{2-\alpha}}{l_0^{2-\alpha}}, \frac{c_0^{2-\alpha}}{3}\right\} g'(x-y)e^{-c_2 g(x-y)}(x-y)^{2+\beta-\alpha}\\ &+ \Phi_1(x-y)e^{-c_2 g(x-y)}(2 + {l_0^*}^{\theta-1}).\end{aligned}$$

Now, replacing the constant c_3 in the proof for Theorem 3.1 by

$$c_3 = (2 + {l_0^*}^{\theta-1})\left[\frac{C_* c_2 k_3}{2}\min\left\{\frac{\kappa^{2-\alpha}}{l_0^{2-\alpha}}, \frac{c_0^{2-\alpha}}{3}\right\}\right]^{-1},$$

we can get that for any $n \geq 1$ and $1/n \leq l_0^* \leq x-y \leq l_0$,

$$\tilde{L} f_n(x-y) \leq -\frac{C_* c_2 k_3(\alpha-\beta)}{2} \min\left\{\frac{\kappa^{2-\alpha}}{l_0^{2-\alpha}}, \frac{c_0^{2-\alpha}}{3}\right\} e^{-c_2 g(l_0)}(x-y).$$

Next, we turn to the case that $1/n < x-y \leq l_0^*$ with any $n \geq 1$. According to the definition of the function ψ and the proof of Theorem 3.1, we know that (4.11) still holds true for any $0 < x-y \leq l_0^*$. On the other hand, according to (4.3) and the facts that $l_0^* \leq \kappa$ and (4.8) holds with ψ_0 in place of ψ, we find for any $0 < x-y \leq l_0^* \leq \kappa$

$$\begin{aligned}
&\frac{1}{2}\gamma_2(y)\Big[\psi_0((x-y)+(x-y)_\kappa)+\psi_0((x-y)-(x-y)_\kappa)-2\psi_0(x-y)]\mu_{x-y}(\mathbb{R}_+)\\
&\leq \frac{C_*}{2}\gamma_2(y)\psi_0''(x-y)(x-y)^{2-\alpha}.
\end{aligned}$$

By (3.4) and (4.18), we have for any $0 < x-y \leq l_0^* \leq \kappa$

$$\begin{aligned}
&(\gamma_2(x)-\gamma_2(y))\int_0^{x-y}(\psi_0(x-y+z)-\psi_0(x-y)-\psi_0'(x-y)z)\,\nu(\mathrm{d}z)\\
&\leq \frac{1}{2}\,(\gamma_2(x)-\gamma_2(y))\,\psi_0''(2(x-y))\int_0^{x-y} z^2\,\nu(\mathrm{d}z)\\
&\leq \frac{C_*}{2}\,(\gamma_2(x)-\gamma_2(y))\,\psi_0''(2(x-y))(x-y)^{2-\alpha}.
\end{aligned}$$

Note that, by $\psi_0'''(r) \geq 0$ and (4.17), for $r \in (0, l_0^*]$ and $l_0^* \leq 1$,

$$\psi_0''(r) \leq \psi_0''(2r) \leq -b\theta(1-\theta)2^{\theta-2}(1+2l_0^*)^{-\theta-2}r^{\theta-2}.$$

Thus, combining with all the estimates above with (4.4), we arrive at for any $0 < x-y \leq l_0^* \leq \kappa$,

$$\begin{aligned}
&\frac{1}{2}\gamma_2(y)\Big[\psi_0((x-y)+(x-y)_\kappa)+\psi_0((x-y)-(x-y)_\kappa)-2\psi_0(x-y)]\mu_{x-y}(\mathbb{R}_+)\\
&+(\gamma_2(x)-\gamma_2(y))\int_0^{x-y}(\psi_0(x-y+z)-\psi_0(x-y)-\psi_0'(x-y)z)\,\nu(\mathrm{d}z)\\
&\leq -b\theta C_* k_3(1-\theta)2^{\theta-3}(1+2l_0^*)^{-\theta-2}(x-y)^{\beta-\alpha+\theta}.
\end{aligned}$$

Therefore, by (4.5), (3.1) and the fact that $\psi_0'' \leq 0$, for any $1/n < x-y \leq l_0^* \leq \kappa$, we see

$$\begin{aligned}
\tilde{L}\psi_0(x-y) \leq & \frac{1}{2}\gamma_2(y)\Big[\psi_0((x-y)+(x-y)_\kappa)+\psi_0((x-y)-(x-y)_\kappa)-2\psi_0(x-y)]\\
&\quad\times \mu_{x-y}(\mathbb{R}_+)\\
&+(\gamma_2(x)-\gamma_2(y))\int_0^{x-y}(\psi_0(x-y+z)-\psi_0(x-y)-\psi_0'(x-y)z)\,\nu(\mathrm{d}z)\\
&+(\gamma_0(x)-\gamma_0(y))\,\psi_0'(x-y)\\
\leq & -b\theta C_* k_3(1-\theta)2^{\theta-3}(1+2l_0^*)^{-\theta-2}(x-y)^{\beta-\alpha+\theta}+b\theta\Phi_1(x-y)(x-y)^{\theta-1}\\
\leq & b\theta(x-y)^{\beta-\alpha+\theta}\left[-\frac{C_* k_3(1-\theta)}{216}+\sup_{0<r\leq l_0^*}(\Phi_1(r)r^{\alpha-\beta-1})\right],
\end{aligned}$$

where in the last inequality we used the facts that $\theta, l_0^* \in (0,1)$. This along with (3.6) yields that there is $l_0^* \in (0,1]$ small enough such that for all $n \geq 1$ and $1/n < x-y \leq l_0^* \leq 1$,

$$\tilde{L}\psi_0(x-y) \leq -\frac{b\theta C_* k_3(1-\theta)}{432}(x-y)^{\beta-\alpha+\theta}.$$

Recalling that $\theta = (\alpha - \beta)/2$, we then get that for any $n \geq 1$ and $1/n < x - y \leq l_0^*$,

$$\tilde{L} f_n(x-y) \leq -\frac{b\theta C_* k_3(1-\theta)}{432} l_0^{*-(\alpha-\beta)/2}.$$

Finally, according to (4.12), (4.5) and the facts that $\psi_0' \geq 0$ and $\psi_0'' \leq 0$, we find that for any $x - y > l_0$,

$$\tilde{L} f_n(x-y) \leq \tilde{L}\psi(x-y) \leq -k_2(x-y)\psi'(x-y) \leq -\frac{k_2\psi'(2l_0)}{2}(x-y).$$

Combining all the estimates above for $\tilde{L} f_n(x-y)$, we can obtain (4.15), thanks to the fact that there exists a constant $C_0 \geq 1$ such that $C_0^{-1}(1+r) \leq f_n(r) \leq C_0(1+r)$ for all $r \in \mathbb{R}_+$ and $n \geq 1$. The proof is complete. □

Proof of Corollary 3.3. Denote by $\mathbb{E}^x$ the expectation under the probability measure $\mathbb{P}^x = \mathbb{P}(\cdot|X_0 = x)$, and $\mu P_t(A) = \int_{\mathbb{R}_+} P_t(x, A)\,\mu(dx)$ for all probability measures μ, $t > 0$ and $A \in \mathscr{B}(\mathbb{R}_+)$. By [7, Proposition 2.3], we see that under assumptions of Theorem 3.1 (or Theorem 3.2), there exist constants $C_1, K > 0$ such that for all $x \in \mathbb{R}_+$ and $t > 0$,

$$\mathbb{E}^x X_t \leq C_1(1+x)e^{Kt}.$$

This is, $\int_{\mathbb{R}_+} y\,(\delta_x P_t)(dy) \leq C_1(1+x)e^{Kt} < \infty$. So $\delta_x P_t \in \mathscr{P}_1$ for any $x \in \mathbb{R}_+$ and $t > 0$, and so $\mu P_t \in \mathscr{P}_1$ for each $\mu \in \mathscr{P}_1$, where $\mathscr{P}_1$ is the space of all probability measures on $(\mathbb{R}_+, \mathscr{B}(\mathbb{R}_+))$ with the first finite moment. With this at hand, the proof of Corollary 3.3 essentially follows from that of [18, Corollary 1.8]. We omit the details here. □

Proof of Theorem 3.5. Similar to the proof of Theorem 3.2, we only verify the case that (A2) is satisfied and that $\mu_z(\mathbb{R}_+) \geq C_* z^{-\alpha}$ for all $z \in (0, \kappa]$ with $C_*, \kappa > 0$ and $\alpha \in (0, 2)$ (i.e., (4.4) holds). To verify the desired assertion, we will use the following test function

$$\psi(r) = \begin{cases} c_1 r + \int_0^r e^{-c_2 g(s)}\,\mathrm{d}s, & r \in [0, 2l_0], \\ \psi(2l_0) + A\int_0^{r-2l_0} \frac{1}{\Phi_2(Bs+2l_0)}\,\mathrm{d}s + \delta A\int_0^{r-2l_0} \frac{1}{\Phi_2(s+2l_0)}\,\mathrm{d}s, & r \in (2l_0, \infty), \end{cases} \tag{4.19}$$

where $A = \frac{\psi'(2l_0)\Phi_2(2l_0)}{\delta+1}$, $B = -\frac{\psi''(2l_0)\Phi_2(2l_0)(\delta+1)}{\psi'(2l_0)\Phi_2'(2l_0)} - \delta$ and $\delta > 0$ is sufficient small such that $B > 0$. Note that the modification between the test function ψ given by (4.19) and the one in the the proof of Theorem 3.1 is only made for $r \in (2l_0, \infty)$. It is easy to see that $\psi \in C^2(\mathbb{R}_+)$ such that $\psi' > 0$ and $\psi'' < 0$ on $\mathbb{R}_+$; moreover, by $\int_1^\infty \frac{1}{\Phi_2(s)}\,\mathrm{d}s < \infty$, $\psi \in C_b(\mathbb{R}_+)$.

With the test function ψ above, we can define a sequence of functions $\{f_n\}_{n\geq 1} \subset C^2(\mathbb{R}_+)$ such that

$$f_n(r) \begin{cases} = \psi(r), & 0 < r \leq 1/(n+1), \\ \leq 1 + b\left(\frac{r}{1+r}\right)^\theta + \psi(r), & 1/(n+1) < r \leq 1/n, \\ = 1 + b\left(\frac{r}{1+r}\right)^\theta + \psi(r), & r \geq 1/n, \end{cases}$$

where b and θ are the same as in the proof of Theorem 3.2. In particular, $\{f_n\}_{n\geq 1}$ is uniformly bounded, i.e. $\sup_{n\geq 1} \|f_n\|_\infty < \infty$.

Following the proof of Theorem 3.2, we know (4.15) still holds true with some $\lambda > 0$ when $1/n \leq x - y \leq l_0$. Next, we consider the estimate for $x - y > l_0$. First, let $\psi_0(r) = (r/(1+r))^\theta$. By (4.5) and the facts that $\psi_0' \geq 0$, $\psi_0'' \leq 0$ and $\psi'' \leq 0$, for any $x - y \geq 2l_0$, we have

$$\tilde{L} f_n(x-y) \leq -\Phi_2(x-y)\psi'(x-y) \leq -\delta A.$$

On the other hand, for any $l_0 < x - y \le 2l_0$,

$$\tilde{L} f_n(x-y) \le -\Phi_2(x-y)\psi'(x-y) \le -c_1\Phi_2(x-y) \le -c_1\Phi_2(l_0).$$

Combining with all the estimates above, we obtain that there is a constant $\lambda > 0$ such that for all $x, y \in \mathbb{R}_+$ with $x > y$ and $n \ge 1$,

$$\tilde{L} f_n(x,y) \le -\lambda.$$

This, along with Lemma 4.3 and the fact that there is a constant $C_0 \ge 1$ such that $C_0^{-1}\mathbb{1}_{(0,\infty)}(r) \le f_n(r) \le C_0\mathbb{1}_{(0,\infty)}$ for all $n \ge 1$ and $r \in \mathbb{R}_+$, in turn yields that there exists a positive constant C so that for all $t > 0$ and $x, y \in \mathbb{R}_+$,

$$\|P_t(x,\cdot) - P_t(x,\cdot)\|_{\mathrm{Var}} \le Ce^{-\lambda t}.$$

Hence, the desired assertion follows from the proof of Corollary 3.3. □

Next, we turn to the

Proof of Theorem 1.1. Condition (1.2) means that (3.1) holds with

$$\Phi_1(r) = k_1 r \log\left(\frac{4l_0}{r}\right).$$

When (1) holds, Assumption (A1) in Theorem 3.1 is satisfied; see the remarks below Theorem 3.2.

When $\nu(\mathrm{d}z) \ge c_0\mathbb{1}_{\{0<z\le 1\}} z^{-1-\alpha}\,\mathrm{d}z$ for some $c_0 > 0$ and $\alpha \in (0,2)$, by [20, Example 1.2], we know that $\mu_z(\mathbb{R}_+) \ge C_* z^{-\alpha}$ for all $z \in (0,\kappa]$ with some $C_*, \kappa > 0$. Then, that condition (2) holds implies that Assumption (A2) in Theorem 3.1 is satisfied too; see also the remarks below Theorem 3.2.

For $\gamma_2(x)$ given in (3), we have for all $x > y \ge 0$,

$$\gamma_2(x) - \gamma_2(y) \ge b_2(x^{r_2} - y^{r_2}).$$

Since $r_2 \in [1,\alpha)$ with $\alpha \in (1,2)$, for all $x > y \ge 0$,

$$x^{r_2} - y^{r_2} \ge x^{r_2-1}(x-y) \ge (x-y)^{r_2}.$$

Hence, Assumption (A2) in Theorem 3.1 is satisfied. With all the conclusions above, we can obtain the desired assertion from Theorems 3.1, 3.2 and 3.5, as well as Corollary 3.3. □

Finally, we present the

Proof of Example 3.6. (1) For any $x > y > 0$,

$$\begin{aligned}\gamma_0(x) - \gamma_0(y) =&[b_1 x\log(1+1/x) - b_1 y\log(1+1/y)] - [b_2 x - b_2 y]\\ \le& b_1(x-y)\log(1+1/x) - b_2(x-y)\\ \le& b_1(x-y)\log(1+1/(x-y)) - b_2(x-y).\end{aligned}$$

This implies that (3.1) holds with $\Phi_1(r) = b_1 r\log(1+1/r)$ and $k_2 = b_2/2$, by setting $l_0 > 0$ large enough such that $b_1\log(1+l_0^{-1}) \le b_2/2$. We note that, by some elementary calculations, $\Phi_1(r) = b_1 r\log(1+1/r)$ satisfies all the assumptions in Theorem 3.1.

(2) Note that for all $\delta > 1$ and $x > y \ge 0$,

$$x^\delta - y^\delta \ge x^{\delta-1}(x-y) \ge (x-y)^\delta.$$

Then,

$$\begin{aligned}\gamma_0(x)-\gamma_0(y) =&[b_1x-b_1y]-[b_2x^\delta-b_2y^\delta]\\ \le& b_1(x-y)-b_2(x-y)^\delta.\end{aligned}$$

Hence, we know that (3.8) holds with $\Phi_1(r)=b_1r$, $\Phi_2(r)=b_2r^\delta/2$ and $l_0=(2b_1/b_2)^{1/(\delta-1)}$.

(3) We consider the function $x\mapsto e^{cx^\delta}$ with $c,\delta>0$ on $\mathbb{R}_+$. For any $x,y\in\mathbb{R}_+$ with $x-y\ge l_0$ and some $l_0>0$ large enough,

$$e^{cx^\delta}-e^{cy^\delta}\ge\frac{e^{cx^\delta}}{x}(x-y)\ge c_0x^\theta(x-y)\ge c_0(x-y)^{1+\theta},$$

where c_0 and θ are positive constants. Here, in the first inequality we used the fact that the function $x\mapsto\frac{e^{cx^\delta}}{x}$ is increasing for $x>0$ large enough, and the second inequality follows from the fact that $\frac{e^{cx^\delta}}{x}\ge c_0x^\theta$ for $x>0$ large enough, where $\theta>0$ can be chosen to be any positive constant. With aid of this inequality, we can prove the desired assertion by following the argument in (2). □

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