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ε-strong simulation of the convex minorants of stable processes and meanders

Jorge I. González Cázares* Aleksandar Mijatović*
Gerónimo Uribe Bravo†

Abstract

Using marked Dirichlet processes we characterise the law of the convex minorant of the meander for a certain class of Lévy processes, which includes subordinated stable and symmetric Lévy processes. We apply this characterisation to construct ε-strong simulation (εSS) algorithms for the convex minorant of stable meanders, the finite dimensional distributions of stable meanders and the convex minorants of weakly stable processes. We prove that the running times of our εSS algorithms have finite exponential moments. We implement the algorithms in Julia 1.0 (available on GitHub) and present numerical examples supporting our convergence results.

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1 Introduction

1.1 Setting and motivation

The universality of stable laws, processes and their path transformations makes them ubiquitous in probability theory and many areas of statistics and natural and social sciences (see e.g. [UZ99, CPR13] and the references therein). Brownian meanders, for instance, have been used in applications, ranging from stochastic partial differential equations [BZ04] to the pricing of derivatives [FKY14] and unbiased and exact simulation of the solutions of stochastic differential equations [CH12, CH13]. Analytic information is generally hard to obtain for either the maximum [Cha13] and its temporal location [AI18, p. 2] or for the related path transformations [CD05], even in the case of the Brownian motion with drift [IO19]. Moreover, except in the case of Brownian motion

*Department of Statistics, University of Warwick, & The Alan Turing Institute, UK.
E-mail: jorge.gonzalez-cazares@warwick.ac.uk,a.mijatovic@warwick.ac.uk
†Universidad Nacional Autónoma de México, México.
E-mail: geronimo@matem.unam.mx

with drift [BS02, Dev10], exact simulation of path-functionals of weakly stable processes is rarely available. In particular, exact simulation of functionals of stable meanders, which arise in numerous path transformations (see [Ber96, Sec. VIII] and [UB14]), appears currently to be out of reach, as even the maximum of a stable processes can only be simulated as a univariate random variable in the strictly stable case [GCMUB19]. A natural question (Q1) arises: *does there exist a simulation algorithm with almost sure control of the error for stable meanders and path-functionals related to the extrema of weakly stable processes?*

A complete description of the law of the convex minorant of Lévy processes is given in [PUB12]. Its relevance in the theory of simulation was highlighted in recent contributions [GCMUB19, GCMUB18a], which developed sampling algorithms for certain path-functionals related to the extrema of Lévy processes (see also Subsection 1.3.3 below). Thus, as Lévy meanders arise in numerous path transformations and functionals of Lévy processes [DI77, Cha97, AC01, UB14, CM16, IO19], it is natural to investigate their simulation problem via their convex minorants, leading to question (Q2): *does there exist a tractable characterisation of the law of convex minorants of Lévy meanders given in terms of the marginals of the corresponding Lévy process?* This question is not trivial for the following two reasons. (I) A description of the convex minorant of a Lévy meander is known only for a Brownian meander [PR12] and is given in terms of the marginals of the meander, not the marginals of the Brownian motion (cf. Subsection 1.3.1 below). (II) Tractable descriptions of the convex minorant of a process X typically rely on the exchangeability of the increments of X in a fundamental way [PUB12, AHUB19], a property clearly not satisfied when X is a Lévy meander.

1.2 Contributions

In this paper we answer affirmatively both questions (Q1) & (Q2) stated above. More precisely, in Theorem 2.7 below, we establish a characterisation of the law of the convex minorant of Lévy meanders, based on marked Dirichlet processes, for Lévy processes with constant probability of being positive (e.g. subordinated stable and symmetric Lévy processes). In particular, Theorem 2.7 gives an alternative description of the law of the convex minorant of a Brownian meander to the one in [PR12]. Our description holds for the meanders of the aforementioned class of Lévy processes, while [PR12] is valid for Brownian motion only (see Subsection 1.3.1 below for more details).

The description in Theorem 2.7 yields a Markovian structure (see Theorem 3.4 below) used to construct ε-strong simulation (εSS) algorithms for the convex minorants of weakly stable processes and stable meanders, as well as for the finite-dimensional distributions of stable meanders. We apply our algorithms to the following problems: exact simulation of barrier crossing events; unbiased simulation of certain path-functionals of stable processes such as the moments of the crossing times of weakly stable processes; estimation of the moments of the normalised stable excursion. We report on the numerical performance in Section 4. Finally, we establish Theorem 3.6 below stating that the running times of all of these algorithms have exponential moments, a property not seen before in the context of εSS (cf. discussion in Subsection 1.3). Moreover, to the best of our knowledge, our results constitute the first simulation algorithms for stable meanders to appear in the literature. Due to the analytical intractability of their law, no simulation algorithms have been proposed so far.

1.3 Connections with the literature

Our results are linked to seemingly disparate areas in pure and applied probability. We discuss connections to each of the areas separately.

1.3.1 Convex minorants of Lévy meanders

The convex minorant of the path of a process on a fixed time interval is the (pointwise) largest convex function dominated by the path. Typically, the convex minorant is a piecewise linear function with a countably infinite number of linear segments known as *faces*, see Subsection 5.1 for definition of such functions. Note that the chronological ordering of its faces coincides with the ordering by increasing slope.

A description of the convex minorant of a Brownian meander is given in [PR12]. To the best of our knowledge, the convex minorant of no other Lévy meander has been characterised prior to the results presented below. The description in [PR12] of the faces of the convex minorant of a Brownian meander depends in a fundamental way on the analytical tractability of the density of the marginal of a Brownian meander at the final time, a quantity not available for other Lévy processes. Furthermore, [PR12] describes the faces of the convex minorant of Brownian meanders in chronological order, a strategy feasible in the Brownian case because the right end of the interval is the only accumulation point for the faces, but infeasible in general. For example, the convex minorant of a Cauchy meander has infinitely many faces in any neighborhood of the origin since the set of its slopes is a.s. dense in $\mathbb{R}$. Hence, if a generalisation of the description in [PR12] to other Lévy meanders existed, it could work only if the sole accumulation point is the right end of the interval. Moreover, the *scaling* and *time inversion* properties of Brownian motion, not exhibited by other Lévy processes [GY05, ACGZ19], are central in the description of [PR12].

In contrast, the description in Theorem 2.7 holds for the Lévy processes with constant probability of being positive, including Brownian motion, and does not require any explicit knowledge of the transition probabilities of the Lévy meander. Moreover, to the best of our knowledge, ours is the only characterisation of the faces of the convex minorant in size-biased order where the underlying process does *not* possess exchangeable increments (a key property in all such descriptions [AP11, PUB12, AHUB19]).

1.3.2 ε-strong simulation algorithms

εSS is a procedure that generates a random element whose distance to the target random element is at most ε almost surely. The tolerance level $\varepsilon > 0$ is given *a priori* and can be refined (see Section 3 for details). The notion of εSS was introduced in [BPR12] in the context of the simulation of a Brownian path on a finite interval. This framework was extended to the reflected Brownian motion in [BC15], jump diffusions in [PJR16], multivariate Itô diffusions in [BCD17], max-stable random fields in [LBDM18] and the fractional Brownian motion in [CDN19]. In general, an εSS algorithm is required to terminate almost surely, but might have infinite expected complexity as is the case in [BPR12, PJR16]. The termination times of the algorithms in [BC15, BCD17, CDN19] are shown to have finite means. In contrast, the running times of the εSS algorithms in the present paper have finite exponential moments (see Theorem 3.6 below), making them efficient in applications (see Subsection 4.2 below).

In addition to the strong control on the error, εSS algorithms have been used in the literature as auxiliary procedures yielding exact and unbiased simulation algorithms [BPR12, CH13, BC15, BZ17, BM18, LBDM18]. We apply our εSS algorithms to obtain exact samples of indicator functions of the form $\mathbb{1}_A(\Lambda)$ for certain random elements Λ and suitable sets A (see Subsection 4.1.1 below). The exact simulation of these indicators in turn yields unbiased samples of other functionals of Λ, including those of the (analytically intractable) first passage times of weakly stable processes (see Subsection 4.1 below).

1.3.3 Simulation algorithms based on convex minorants

Papers [GCMUB19, GCMUB18a] developed simulation algorithms for the extrema of Lévy processes in various settings. We stress that algorithms and results in [GCMUB19, GCMUB18a] cannot be applied to the simulation of the path-functionals of Lévy meanders considered in this paper. There are a number of reasons for this. First, the law of a Lévy meander on a fixed time interval $[0,T]$ is given by the law of the original process X *conditioned* on X being positive on $(0,T]$, an event of probability zero if, for instance, X has infinite variation [Sat13, Thm 47.1]. Since the algorithms in [GCMUB19, GCMUB18a] apply to the *unconditioned* process, they are clearly of little direct use here. Second, the theoretical tools developed in [GCMUB19, GCMUB18a] are *not* applicable to the problems considered in the present paper. Specifically, [GCMUB18a] proposes a new simulation algorithm for the state X_T, the infimum and the time the infimum is attained on $[0,T]$ for a general (*unconditioned*) Lévy process X and establishes the geometric decay of the error in L^p of the corresponding Monte Carlo algorithm. In contrast to the almost sure control of the simulation error for various path-functionals of X *conditioned* on $\{X_t > 0 : t \in (0,T]\}$ established in the present paper, the results in [GCMUB18a] imply that the random error in [GCMUB18a], albeit very small in expectation, can take arbitrarily large values with positive probability. This makes the methods of [GCMUB18a] completely unsuitable for the analysis of algorithms requiring an almost sure control of the error, such as the ones in the present paper.

Paper [GCMUB19] develops an exact simulation algorithm for the infimum of X over the time interval $[0,T]$, where X is an (unconditioned) strictly stable process. The scaling property of X is crucial for the results in [GCMUB19]. Thus, the results in [GCMUB19] do *not* apply to the simulation of the convex minorant (and cosequently the infimum) of the weakly stable processes considered here. This problem is solved in the present paper via a novel method based on tilting X and then sampling the convex minorants of the corresponding meanders, see Subsection 3.1 for details.

The dominated-coupling-from-the-past (DCFTP) method in [GCMUB19] is based on a perpetuity equation $\mathcal{X} \overset{d}{=} V(U^{1/\alpha}\mathcal{X} + (1-U)^{1/\alpha}S)$ established therein, where $\mathcal{X}$ denotes the law of the supremum of a strictly stable process X. This perpetuity appears similar to the one in Theorem 3.4(c) below, characterising the law of X_1 conditioned on $\{X_t > 0 : t \in (0,1]\}$. However, the analysis in [GCMUB19] *cannot* be applied to the perpetuity in Theorem 3.4(c) for the following reason: the "nearly" uniform factor V in the perpetuity above (U is uniform on $[0,1]$ and S is as in Theorem 3.4(c)) is used in [GCMUB19] to modify it so that the resulting Markov chain exhibits coalescence with positive probability, a necessary feature for the DCFTP to work. Such a modification appears to be out of reach for the perpetuity in Theorem 3.4(c) due to the absence of the multiplying factor, making exact simulation of stable meanders infeasible. However, even though the coefficients of the perpetuity in Theorem 3.4(c) are dependent and have heavy tails, the Markovian structure for the error based on Theorem 3.4 allows us to define, in the present paper, a dominating process for the error whose return times to a neighbourhood of zero possess exponential moments. Since the dominating process can be simulated backwards in time, this leads to fast εSS algorithms for the convex minorant of the stable meander and, consequently, of a weakly stable process.

1.4 Organisation

The remainder of the paper is structured as follows. In Section 2 we state and prove Theorem 2.7, which identifies the distribution of the convex minorant of a Lévy meander in a certain class of Lévy processes. In Section 3 we define εSS and construct the main algorithms for the simulation from the laws of the convex minorants of both stable meanders and weakly stable processes, as well as from the finite dimensional distributions of stable meanders. Numerical examples illustrating the methodology, its speed and stability are in Section 4. Section 5 contains the analysis of the computational complexity (i.e. the proof of Theorem 3.6), the technical tools required in Section 3 and the proof of Theorem 3.4 and its Corollary 5.9 (on the moments of stable meanders) used in Section 4.

2 The law of the convex minorants of Lévy meanders

2.1 Convex minorants and splitting at the minimum

Let $X = (X_t)_{t\in[0,T]}$ be a Lévy process on $[0,T]$, where $T > 0$ is a fixed time horizon, started at zero $\mathbb{P}(X_0 = 0) = 1$. If X is a compound Poisson process with drift, exact simulation of the entire path of X is typically available. We hence work with processes that are not compound Poisson process with drift. By Doeblin's diffuseness lemma [Kal02, Lem. 13.22], this assumption is equivalent to

Assumption 2.1 (D). *$\mathbb{P}(X_t = x) = 0$ for all $x \in \mathbb{R}$ and for some (and then all) $t > 0$.*

The convex minorant of a function $f : [a,b] \to \mathbb{R}$ is the pointwise largest convex function $C(f) : [a,b] \to \mathbb{R}$ such that $C(f)(t) \le f(t)$ for all $t \in [a,b]$. Under (D), the convex minorant $C = C(X)$ of a path of X turns out to be piecewise linear with infinitely many faces (i.e. linear segments). By convexity, sorting the faces by increasing slope coincides with their chronological ordering [PUB12]. However, the ordering by increasing slopes is not helpful in determining the law of C. Instead, in the description of the law of C in [PUB12], the faces are selected using size-biased sampling (see e.g. [GCMUB18a, Sec. 4.1]).

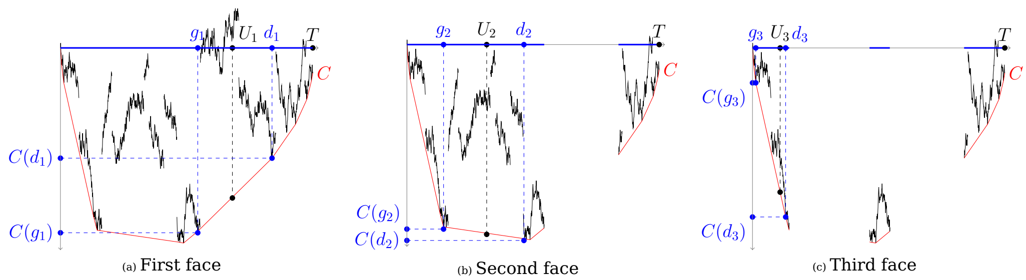


Figure 1: Selecting the first three faces of the concave majorant: the total length of the thick blue segment(s) on the abscissa equal the stick sizes T, $T-(d_1-g_1)$ and $T-(d_1-g_1)-(d_2-g_2)$, respectively. The independent random variables U_1, U_2, U_3 are uniform on the sets $[0,T]$, $[0,T]\setminus(g_1,d_1)$, $[0,T]\setminus\bigcup_{i=1}^2(g_i,d_i)$, respectively. Note that the residual length after n samples is L_n.

Put differently, choose the faces of C *independently at random uniformly on lengths*, as shown in Figure 1, and let g_n and d_n be the left and right ends of the n-th face, respectively. One way of inductively constructing the variables $(U_n)_{n\in\mathbb{N}}$ (and hence the sequence of the faces of C) in Figure 1 is from an independent identically distributed (iid) sequence $\mathcal{V}$ of uniforms on $[0,T]$, which is independent of X: U_1 is the first value in $\mathcal{V}$ and, for any $n \in \mathbb{N} = \{1,2,\dots\}$, U_{n+1} is the first value in $\mathcal{V}$ after U_n not contained in the union of intervals $\bigcup_{i=1}^n(g_i,d_i)$. Then, for any $n \in \mathbb{N}$, the length of the n-th face is $\ell_n = d_n - g_n$ and its height is $\xi_n = C(d_n) - C(g_n)$. In [PUB12, Thm 1], a complete description of the law of the sequence $((\ell_n,\xi_n))_{n\in\mathbb{N}}$ is given. In order to generalise this results to Lévy meanders, it is helpful to state the characterisation in terms of Dirichlet processes, see (2.3) in Section 2.2 below.

The behaviour of certain statistics of the path of X, such as the infimum $\underline{X}_T = \inf_{t\in[0,T]} X_t$ and its time location $\tau_T = \tau_{[0,T]}(X) = \inf\{t > 0 : \min\{X_t, X_{t-}\} = \underline{X}_T\}$, is determined by that of the faces of C whose heights are negative (we assume throughout that X is right-continuous with left limits (càdlàg) and denote $X_{t-} = \lim_{s\uparrow t} X_s$ for $t > 0$ and $X_{0-} = 0$). Analysis of their behaviour amounts to the analysis of the convex minorants of the pre- and post-minimum processes $X^{\leftarrow} = (X^{\leftarrow}_t)_{t\in[0,T]}$ and $X^{\rightarrow} = (X^{\rightarrow}_t)_{t\in[0,T]}$, where

$$X^{\leftarrow}_t = \begin{cases} X_{(\tau_T - t)-} - \underline{X}_T, & t \in [0, \tau_T], \\ \dagger, & t \in (\tau_T, T], \end{cases} \quad \text{and} \quad X^{\rightarrow}_t = \begin{cases} X_{\tau_T + t} - \underline{X}_T, & t \in [0, T - \tau_T], \\ \dagger, & t \in (\tau_T, T], \end{cases} \tag{2.1}$$

respectively ($\dagger$ denotes a cemetery state, required only to define the processes on $[0, T]$). Clearly, as indicated by Figure 2, C may be recovered from the convex minorants $C^{\leftarrow} = C(X^{\leftarrow})$ and $C^{\rightarrow} = C(X^{\rightarrow})$ of $X^{\leftarrow}|_{[0,\tau_T]}$ and $X^{\rightarrow}|_{[0,T-\tau_T]}$, respectively. For convenience, we suppress the time interval in the notation for $C^{\leftarrow} = C(X^{\leftarrow})$ and $C^{\rightarrow} = C(X^{\rightarrow})$. In particular, throughout the paper, $C^{\leftarrow}$ and $C^{\rightarrow}$ are the convex minorants of $X^{\leftarrow}$ and $X^{\rightarrow}$, respectively, only while the processes are "alive".

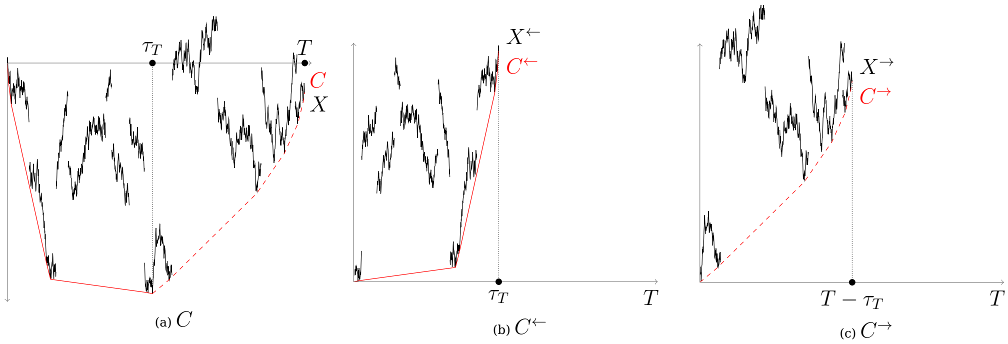


Figure 2: Decomposing (X, C) into $(X^{\leftarrow}, C^{\leftarrow})$ and $(X^{\rightarrow}, C^{\rightarrow})$.

2.2 Convex minorants as marked Dirichlet processes

Our objective now is to obtain a description of the law of the convex minorants $C^{\leftarrow}$ and $C^{\rightarrow}$. For any $n \in \mathbb{N}$ and positive reals $\theta_0, \dots, \theta_n > 0$, the *Dirichlet distribution* with parameter $(\theta_0, \dots, \theta_n)$ is given by a density proportional to $x \mapsto \prod_{i=0}^n x_i^{\theta_i - 1}$, supported on the standard n-dimensional symplex in $\mathbb{R}^{n+1}$ (i.e. the set of points $x = (x_0, \dots, x_n)$ satisfying $\sum_{i=0}^n x_i = 1$ and $x_i \in (0, 1)$ for $i \in \{0, \dots, n\}$). In the special case $n = 1$, we get the beta distribution $\mathrm{Beta}(\theta_0, \theta_1)$ on $[0, 1]$. In particular, the uniform distribution equals $U(0, 1) = \mathrm{Beta}(1, 1)$ and, for any $\theta > 0$, we append the limiting cases $\mathrm{Beta}(\theta, 0) = \delta_1$ and $\mathrm{Beta}(0, \theta) = \delta_0$, where δ_x is the Dirac measure at $x \in \mathbb{R}$.

Let $(\mathbb{X}, \mathcal{X}, \mu)$ be a measure space with $\mu(\mathbb{X}) \in (0, \infty)$. A random probability measure Ξ (i.e. a stochastic process indexed by the sets in $\mathcal{X}$) is a *Dirichlet process* on $(\mathbb{X}, \mathcal{X})$ based on the finite measure μ if for any measurable partition $\{B_0, \dots, B_n\} \subset \mathcal{X}$ (i.e. $n \in \mathbb{N}$, $B_i \cap B_j = \emptyset$ for all distinct $i, j \in \{0, \dots, n\}$, $\bigcup_{i=0}^n B_i = \mathbb{X}$ and $\mu(B_i) > 0$ for all $i \in \{0, \dots, n\}$), the vector $(\Xi(B_0), \dots, \Xi(B_n))$ is a Dirichlet random vector with parameters $(\mu(B_0), \dots, \mu(B_n))$. We use the notation $\Xi \sim \mathcal{D}_\mu$ throughout.

Define the sets $\mathcal{Z}^n = \{k \in \mathbb{Z} : k < n\}$ and $\mathcal{Z}^n_m = \mathcal{Z}^n \setminus \mathcal{Z}^m$ for $n, m \in \mathbb{Z}$ and adopt the convention $\prod_{k\in\emptyset} = 1$ and $\sum_{k\in\emptyset} = 0$ ($\mathbb{Z}$ denotes the integers). Sethuraman [Set94] introduced the construction

$$\Xi = \sum_{n=1}^{\infty} \pi_n \delta_{x_n} \sim \mathcal{D}_\mu, \tag{2.2}$$

where $(x_n)_{n\in\mathbb{N}}$ is an iid sequence with distribution $\mu(\mathbb{X})^{-1}\mu$ and $(\pi_n)_{n\in\mathbb{N}}$ is a stick-breaking process based on $\mathrm{Beta}(1,\mu(\mathbb{X}))$ constructed as follows: $\pi_n=\beta_n\prod_{k\in\mathcal{Z}_1^n}(1-\beta_k)$ where $(\beta_n)_{n\in\mathbb{N}}$ is an iid sequence with distribution $\mathrm{Beta}(1,\mu(\mathbb{X}))$.

Consider a further measurable space $(\mathbb{Y},\mathcal{Y})$ and a triple (θ,μ,κ), where $\theta>0$, μ is a finite measure on $(\mathbb{X},\mathcal{X})$ and $\kappa:[0,\theta]\times\mathbb{X}\to\mathbb{Y}$ is a measurable function. Let Ξ be as in (2.2). A *marked Dirichlet process* on $(\mathbb{Y},\mathcal{Y})$ is given by the random probability measure $\sum_{n=1}^\infty\pi_n\delta_{\kappa(\theta\pi_n,x_n)}$ on $(\mathbb{Y},\mathcal{Y})$. We denote its distribution by $\mathcal{D}_{(\theta,\mu,\kappa)}$.

Let $F(t,x)=\mathbb{P}(X_t\le x)$, $x\in\mathbb{R}$, be the distribution function of X_t for $t\ge0$ and let $G(t,\cdot)$ be the generalised right inverse of $F(t,\cdot)$. Hence $G(t,U)$ follows the law $F(t,\cdot)$ for any uniform random variable $U\sim U(0,1)$. Given these definitions, [PUB12, Thm 1] can be rephrased as

$$T^{-1}\sum_{n=1}^\infty\ell_n\delta_{\xi_n}=T^{-1}\sum_{n=1}^\infty(d_n-g_n)\delta_{C(d_n)-C(g_n)}\sim\mathcal{D}_{(T,U(0,1),G)},\tag{2.3}$$

where C is the convex minorant of X over the interval $[0,T]$, with the length and height of the n-th face given by $\ell_n=d_n-g_n$ and $\xi_n=C(d_n)-C(g_n)$, respectively, as defined in Section 2.1 above. Consequently, the faces of C are easy to simulate if one can sample from $F(t,\cdot)$. Indeed, $(\ell_n/T)_{n\in\mathbb{N}}$ has the law of the stick-breaking process with uniform sticks and, given ℓ_n, we have $\xi_n\sim F(\ell_n,\cdot)$ for all $n\in\mathbb{N}$.

It is evident that the size-biased sampling of the faces of $C^\leftarrow$ and $C^\rightarrow$, analogous to the one described in the second paragraph of Section 2.1 for the faces of C (see also Figure 2), can be applied on the intervals $[0,\tau_T]$ and $[0,T-\tau_T]$, respectively. However, in order to characterise the respective laws of the two sequences of lengths and heights, we need to restrict to the following class of Lévy processes.

Assumption 2.2 (P)**.** *The probability $\mathbb{P}(X_t>0)$ equals some $\rho\in[0,1]$ for all $t>0$.*

The family of Lévy processes that satisfy (P) has, to the best of our knowledge, not been characterised in terms of the characteristics of the process X (e.g. its Lévy measure or characteristic exponent). However, it is easily seen that it includes the following wide variety of examples: symmetric Lévy processes with $\rho=1/2$, stable processes with ρ given by its positivity parameter (see e.g. [GCMUB19, App. A]) and subordinated stable processes with ρ equal to the positivity parameter of the stable process. Note also that under (P), the random variable $G(t,U)$ (with U uniform on $[0,1]$) is negative if and only if $U\le1-\rho$.

Proposition 2.3. *Let X be a Lévy process on $[0,T]$ satisfying (D) and (P) for $\rho\in[0,1]$ with pre- and post-minimum processes $X^\leftarrow$ and $X^\rightarrow$, respectively, defined in* (2.1)*. Let $((\ell_n^\leftarrow,-\xi_n^\leftarrow))_{n\in\mathbb{N}}$ and $((\ell_n^\rightarrow,\xi_n^\rightarrow))_{n\in\mathbb{N}}$ be the faces of $C^\leftarrow=C(X^\leftarrow)$ and $C^\rightarrow=C(X^\rightarrow)$, respectively, when sampled independently at random uniformly on lengths as described in Section 2.1. Then τ_T/T follows the law $\mathrm{Beta}(1-\rho,\rho)$, the random functions $C^\leftarrow$ and $C^\rightarrow$ are conditionally independent given τ_T and, conditional on τ_T, we have*

$$\begin{aligned}\tau_T^{-1}\sum_{n=1}^\infty\ell_n^\leftarrow\delta_{\xi_n^\leftarrow}&\sim\mathcal{D}_{(\tau_T,U(0,1)|_{[0,1-\rho]},G)},\\(T-\tau_T)^{-1}\sum_{n=1}^\infty\ell_n^\rightarrow\delta_{\xi_n^\rightarrow}&\sim\mathcal{D}_{(T-\tau_T,U(0,1)|_{[1-\rho,1]},G)}.\end{aligned}\tag{2.4}$$

Remark 2.4. (i) The measure $U(0,1)|_{[0,1-\rho]}$ (resp. $U(0,1)|_{[1-\rho,1]}$) on the interval $[0,1]$ has a density $x\mapsto\mathbb{1}_{[0,1-\rho]}(x)$ (resp. $x\mapsto\mathbb{1}_{[1-\rho,1]}(x)$).[1] In the case $\rho=1$, X is a subordinator by [Sat13, Thm 24.11]. Then $\tau_T=T$ and only the first equality in law in (2.4) makes

[1]Here and throughout $\mathbb{1}_A$ denotes the indicator function of a set A.

sense (since there is no pre-minimum process) and equals that in (2.3). The case $\rho = 0$ is analogous.
(ii) Proposition 2.3 provides a simple proof of the generalized arcsine law: under (D) and (P), we have $\tau_T/T \sim \mathrm{Beta}(1-\rho, \rho)$ (see [Ber96, Thm VI.3.13] for a classical proof of this result).
(iii) Proposition 2.3 implies that the heights $(\xi_n^{\leftarrow})_{n\in\mathbb{N}}$ (resp. $(\xi_n^{\rightarrow})_{n\in\mathbb{N}}$) of the faces of the convex minorant $C^{\leftarrow}$ (resp. $C^{\rightarrow}$) are conditionally independent given $(\ell_n^{\leftarrow})_{n\in\mathbb{N}}$ (resp. $(\ell_n^{\rightarrow})_{n\in\mathbb{N}}$). Moreover, $\xi_n^{\leftarrow}$ (resp. $\xi_n^{\rightarrow}$) is distributed as $F(\ell_n^{\leftarrow}, \cdot)$ (resp. $F(\ell_n^{\rightarrow}, \cdot)$) conditioned to the negative (resp. positive) half-line. Given τ_T, the sequence $(\ell_n^{\leftarrow}/\tau_T)_{n\in\mathbb{N}}$ (resp. $(\ell_n^{\rightarrow}/(T-\tau_T))_{n\in\mathbb{N}}$) is a stick-breaking process based on $\mathrm{Beta}(1, 1-\rho)$ (resp. $\mathrm{Beta}(1,\rho)$).
(iv) If T is an exponential random variable with mean $\theta > 0$ independent of X, the random times τ_T and $T - \tau_T$ are independent gamma random variables with common scale parameter θ and shape parameters $1-\rho$ and ρ, respectively. This is because, the distribution of τ_T/T, conditional on any value of T, is $\mathrm{Beta}(1-\rho,\rho)$ (see Proposition 2.3), making τ_T/T and T independent. Furthermore, by [PUB12, Cor. 2], the random measures $\sum_{n=1}^{\infty} \delta_{(\ell_n^{\leftarrow}, \xi_n^{\leftarrow})}$ and $\sum_{n=1}^{\infty} \delta_{(\ell_n^{\rightarrow}, \xi_n^{\rightarrow})}$ are independent Poisson point processes with intensities given by the restriction of the measure $e^{-t/\theta} t^{-1} dt \mathbb{P}(X_t \in dx)$ on $(t,x) \in [0,\infty) \times \mathbb{R}$ to the subsets $[0,\infty) \times (-\infty, 0)$ and $[0,\infty)\times[0,\infty)$, respectively.

The proof of Proposition 2.3 relies on the following property of Dirichlet processes, which is a direct consequence of the definition and [Set94, Lem. 3.1].

Lemma 2.5. *Let μ_1 and μ_2 be two non-trivial finite measures on a measurable space $(\mathbb{X}, \mathcal{X})$. Let $\Xi_i \sim \mathcal{D}_{\mu_i}$ for $i = 1, 2$ and $\beta \sim \mathrm{Beta}(\mu_1(\mathbb{X}), \mu_2(\mathbb{X}))$ be jointly independent, then*

$$\beta\Xi_1 + (1-\beta)\Xi_2 \sim \mathcal{D}_{\mu_1+\mu_2}.$$

Proof of Proposition 2.3. Recall that $\ell_n = d_n - g_n$ (resp. $\xi_n = C(d_n) - C(g_n)$) denotes the length (resp. height) of the n-th face of the convex minorant C of X (see Section 2.1 above for definition). By (2.3), the random variables $\upsilon_n = F(\ell_n, \xi_n)$ form a $U(0,1)$ distributed iid sequence $(\upsilon_n)_{n\in\mathbb{N}}$ independent of the stick-breaking process $(\ell_n)_{n\in\mathbb{N}}$. Since the faces of C are placed in a strict ascending order of slopes, by (2.2)–(2.3) the convex minorant C of a path of X is in a one-to-one correspondence with a realisation of the marked Dirichlet process $T^{-1}\sum_{n=1}^{\infty} \ell_n \delta_{\xi_n}$ and thus with the Dirichlet process $\Xi = T^{-1}\sum_{n=1}^{\infty} \ell_n \delta_{\upsilon_n} \sim \mathcal{D}_{U(0,1)}$.

Assume now that $\rho \in (0,1)$. Since $U(0,1)|_{[0,1-\rho]} + U(0,1)|_{[1-\rho,1]} = U(0,1)$ as measures on the interval $[0,1]$, Lemma 2.5 and [Kal02, Thm 5.10] imply that by possibly extending the probability space we may decompose $\Xi = \beta\Xi^{\leftarrow} + (1-\beta)\Xi^{\rightarrow}$, where the random elements $\beta \sim \mathrm{Beta}(1-\rho,\rho)$, $\Xi^{\leftarrow} \sim \mathcal{D}_{U(0,1)|_{[0,1-\rho]}}$ and $\Xi^{\rightarrow} \sim \mathcal{D}_{U(0,1)|_{[1-\rho,1]}}$ are independent (note that we can distinguish between values above and below $1-\rho$ a.s. since, with probability 1, no variable υ_n is exactly equal $1-\rho$). Since $\rho \in (0,1)$, condition (D) and [Sat13, Thm 24.10] imply that $\mathbb{P}(0 < X_t < \epsilon) > 0$ for all $\epsilon > 0$ and $t > 0$. Then (P) implies the equivalence: $F(t,x) \le 1-\rho$ if and only if $x \le 0$.

The construction of $(\upsilon_n)_{n\in\mathbb{N}}$ ensures that the faces of C with negative (resp. positive) heights correspond to the atoms of $\Xi^{\leftarrow}$ (resp. $\Xi^{\rightarrow}$). Therefore the identification between the faces of C with the Dirichlet process Ξ described above implies that $\Xi^{\leftarrow}$ (resp. $\Xi^{\rightarrow}$) is also in one-to-one correspondence with the faces of $C^{\leftarrow}$ (resp. $C^{\rightarrow}$). In particular, since $\tau_T = \sum_{n\in\mathbb{N}} \ell_n \cdot \mathbb{1}_{\{\xi_n<0\}}$ equals the sum of all the lengths of the faces of C with negative heights, this identification implies $\tau_T \sim T\beta$ and the generalised arcsine law $\tau_T/T \sim \mathrm{Beta}(1-\rho,\rho)$ follows from Lemma 2.5 applied to the measures $U(0,1)|_{[0,1-\rho]}$ and $U(0,1)|_{[1-\rho,1]}$ on $[0,1]$. Moreover, the lengths of the faces of $C^{\leftarrow}$ correspond to the masses of the atoms of $\beta\Xi^{\leftarrow}$. The independence of β and $\Xi^{\leftarrow}$ implies that the sequence of the masses of the atoms of $\beta\Xi^{\leftarrow}$ is precisely a stick-breaking process based on the distribution $\mathrm{Beta}(1, 1-\rho)$ multiplied by β. Similarly, the random variables $F(\ell_n^{\leftarrow}, \xi_n^{\leftarrow})$ can be identified with the atoms of $\Xi^{\leftarrow}$ and thus form an iid sequence of uniform random variables on the interval $[0, 1-\rho]$. Hence, conditional on τ_T, the law of $\tau_T^{-1}\sum_{n=1}^{\infty}\ell_n^{\leftarrow}\delta_{\xi_n^{\leftarrow}}$ is as stated in the proposition. An analogous argument yields the correspondence between the Dirichlet process $\Xi^{\rightarrow}$ and the faces of $C^{\rightarrow}$. The fact that the orderings correspond to size-biased samplings follows from [Pit06, Sec. 3.2].

It remains to consider the case $\rho \in \{0, 1\}$. By [Sat13, Thm 24.11], X (resp. $-X$) is a subordinator if $\rho = 1$ (resp. $\rho = 0$) satisfying (D). Then, clearly, $\rho = 1$, $\tau_T = 0$, $C^{\rightarrow} = C$ (resp. $\rho = 0$, $\tau_T = T$, $C^{\leftarrow} = C$) and the proposition follows from (2.3). □

2.3 Lévy meanders and their convex minorants

If 0 is regular for $(0, \infty)$, then it is possible to define the Lévy meander $X^{\mathrm{me},T} = (X_t^{\mathrm{me},T})_{t\in[0,T]}$ as the weak limit as $\varepsilon \downarrow 0$ of the law of X conditioned on the event $\{\underline{X}_T > -\varepsilon\}$ (see [CD05, Lem. 7] and [CD08, Cor. 1]). Condition (P) and Rogozin's criterion [Ber96, Prop. VI.3.11] readily imply that 0 is regular for $(0, \infty)$ if $\rho > 0$, in which case the respective Lévy meander is well defined. As discussed in Section 2.2, the case $\rho = 0$ corresponds to the negative of a subordinator where the meander does not exist.

In this section we will use the following assumption, which implies the existence of a density of X_t for every $t > 0$ and hence also Assumption (D).

Assumption 2.6 (K)**.** *$\int_{\mathbb{R}} \left|\mathbb{E}\left(e^{iuX_t}\right)\right| du < \infty$ for every $t > 0$.*

Lévy meanders arise under certain path transformations of Lévy processes [Ber96, Sec. VI.4]. For instance, by [UB14, Thm 2], if (K) holds and 0 is regular for both $(-\infty, 0)$ and $(0, \infty)$, then the pre- and post-minimum processes $X^{\leftarrow}$ and $X^{\rightarrow}$ are conditionally independent given τ_T and distributed as meanders of $-X$ and X on the intervals $[0, \tau_T]$ and $[0, T-\tau_T]$, respectively, generalising the result for stable processes [Ber96, Cor. VIII.4.17]. The next theorem constitutes the main result of this section.

Theorem 2.7. *Assume X satisfies (P) with $\rho \in (0, 1]$ and (K). Pick a finite time horizon $T > 0$ and let $X^{\mathrm{me},T}$ be the Lévy meander and let $((\ell_n^{\mathrm{me}}, \xi_n^{\mathrm{me}}))_{n\in\mathbb{N}}$ be the lengths and heights of the faces of $C(X^{\mathrm{me},T})$ chosen independently at random uniformly on lengths. The sequence $((\ell_n^{\mathrm{me}}, \xi_n^{\mathrm{me}}))_{n\in\mathbb{N}}$ encodes a marked Dirichlet process as follows:*

$$T^{-1}\sum_{n=1}^{\infty}\ell_n^{\mathrm{me}}\delta_{\xi_n^{\mathrm{me}}} \sim \mathcal{D}_{(T,\rho U(1-\rho,1),G)}. \tag{2.5}$$

Proof. The case $\rho = 1$ is trivial since X is then a subordinator by [Sat13, Thm 24.11], clearly equal to its meander, and (2.5) is the same as (2.3). If $\rho \in (0, 1)$, then 0 is regular for both half lines by Rogozin's criterion [Ber96, Prop. VI.3.11]. Fix $T' > T$ and consider the Lévy process X on $[0, T']$. Conditional on $\tau_{T'} = T' - T$, the post-minimum process $(X_t^{\rightarrow})_{t\in[0,T']}$ defined in (2.1) is killed at $\tau_{T'} = T' - T$ and the law of $(X_t^{\rightarrow})_{t\in[0,T]}$ prior to the killing time is the same as the law of the meander $X^{\mathrm{me},T}$ on $[0, T]$ by [UB14, Thm 2]. Hence, conditional on $\tau_{T'} = T' - T$, the law of the faces of the convex minorant $C(X^{\rightarrow})$ on $[0, T]$ agree with those of the convex minorant $C(X^{\mathrm{me},T})$. Thus, the distributional characterisation of Proposition 2.3 also applies to $T^{-1}\sum_{n=1}^{\infty}\ell_n^{\mathrm{me}}\delta_{\xi_n^{\mathrm{me}}}$. □

Remark 2.8. (i) Condition (K) is slightly stronger than (D). In fact, it holds if there is a Brownian component or if the Lévy measure has sufficient activity [Kal81, Sec. 5] (see also Lemma B.1 in Appendix B below). Hence Condition (K) is satisfied by most subordinated stable processes.
(ii) Although sufficient and simple, Condition (K) is not a necessary condition for [UB14, Thm 2]. The minimal requirement is that the density $(t, x) \mapsto \frac{\partial}{\partial x}F(t, x)$ exists and is uniformly continuous for $t > 0$ bounded away from 0.

(iii) Identity (2.3) (in the form of [PUB12, Thm 1]), applied to concave majorants, was used in [GCMUB18a] to obtain a geometrically convergent simulation algorithm of the triplet $(X_T, \overline{X}_T, \tau_T(-X))$, where $\tau_T(-X)$ is the location of the supremum $\overline{X}_T = \sup_{t\in[0,T]} X_t$. In the same manner, a geometrically convergent simulation of the marginal $X_T^{\mathrm{me},T}$ can be constructed using the identity in (2.5).
(iv) The proof of Theorem 2.7 and Remark 2.4(iv) above imply that if T is taken to be an independent gamma random variable with shape parameter ρ and scale parameter $\theta > 0$, then the random measure $\sum_{n=1}^{\infty} \delta_{(\ell_n^{\mathrm{me}}, \xi_n^{\mathrm{me}})}$ is a Poisson point process on $(t, x) \in [0, \infty) \times [0, \infty)$ with intensity $e^{-t/\theta} t^{-1} dt \mathbb{P}(X_t \in dx)$. This description and [PR12, Thm 6] imply [PR12, Thm 4], the description of the chronologically ordered faces of the convex minorant of a Brownian meander. However, as noted in [PR12], a direct proof of [PR12, Thm 6], linking the chronological and Poisson point process descriptions of the convex minorant of a Brownian meander, appears to be out of reach.

3 ε-strong simulation algorithms for convex minorants

As mentioned in the introduction, εSS algorithm is a simulation procedure with random running time, which constructs a random element that is ε-close in the essential supremum norm to the random element of interest, where $\varepsilon > 0$ is an *a priori* specified tolerance level. Moreover, the simulation procedure can be continued retrospectively if, given the value of the simulated random element, the tolerance level ε needs to be reduced. Thus an εSS scheme provides a way to compute the random element of interest to arbitrary precision almost surely, leading to a number of applications (including exact and unbiased algorithms for related random elements) discussed in Subsection 4.1.

We now give a precise definition of an εSS algorithm. Consider a random element Λ taking values in a metric space $(\mathbb{X}, d)$. A simulation algorithm that for any $\varepsilon > 0$ constructs in finitely many steps a random element Λ^ε in $\mathbb{X}$ satisfying (I) and (II) below is termed an εSS algorithm: (I) there exists a coupling $(\Lambda, \Lambda^\varepsilon)$ on a probability space Ω such that the essential supremum $\operatorname{ess\,sup}\{d(\Lambda(\omega), \Lambda^\varepsilon(\omega)) : \omega \in \Omega\}$ is at most ε; (II) for any $m \in \mathbb{N}$, decreasing sequence $\varepsilon_1 > \cdots > \varepsilon_m > 0$, random elements $\Lambda^{\varepsilon_1}, \ldots, \Lambda^{\varepsilon_m}$ (satisfying (I) for the respective $\varepsilon_1, \ldots, \varepsilon_m$) and $\varepsilon' \in (0, \varepsilon_m)$, we can sample $\Lambda^{\varepsilon'}$, given $\Lambda^{\varepsilon_1}, \ldots, \Lambda^{\varepsilon_m}$, which satisfies (I) for ε'. Condition (II), known as the tolerance-enforcement property of εSS, can be seen as a measurement of the realisation of the random element Λ whose error may be reduced in exchange for additional computational effort.

Throughout this paper, the metric d in the definition above is given by the supremum norm on either the space of continuous functions on a compact interval or on a finite dimensional Euclidean space. The remainder of this section is structured as follows. Section 3.1 reduces the problems of constructing εSS algorithms for the finite dimensional distributions of Lévy meanders and the convex minorants of Lévy processes, to constructing an εSS algorithm of the convex minorants of Lévy meanders. In Subsection 3.2 we apply Theorem 2.7 of Section 2 to construct an εSS algorithm for the convex minorant of a Lévy meander under certain technical conditions. In Theorem 3.4 we state a stochastic perpetuity equation (3.2), established in Section 5 using Theorem 2.7, that implies these technical conditions in the case of stable meanders. Subsection 3.2 concludes with the statement of Theorem 3.6 describing the computational complexity of the εSS algorithm constructed in Subsection 3.2.

3.1 εSS of the convex minorants of Lévy processes

In the present subsection we construct εSS algorithms for the convex minorant $C(X)$ and for the finite dimensional distributions of $X^{\mathrm{me},T}$. Both algorithms require the following assumption.

Assumption 3.1 (S)**.** *There is an εSS algorithm for $C((\pm X)^{\mathrm{me},t})$ for any $t>0$.*

In the case of stable processes, an algorithm satisfying Assumption (S) is given in the next subsection. In this subsection we assume that (P) and (S) hold for the process T_cX for some $c\in\mathbb{R}$, where T_c denotes the *linear tilting* functional $T_c: f\mapsto (t\mapsto f(t)+ct)$ for any real function f. We construct an εSS algorithm for the convex minorant $C(X)$, and hence for $(X_T,\underline{X}_T)$, as follows ($\mathcal{L}(\cdot)$ denotes the law of the random element in its argument).

Algorithm 1 εSS of the convex minorant $C(X)$ of a Lévy process X, such that T_cX satisfies Assumptions (P) and (S) for some $c\in\mathbb{R}$.

Require: Time horizon $T>0$, accuracy $\varepsilon>0$ and $c\in\mathbb{R}$.
1: Sample $\beta\sim B(1-\rho,\rho)$ and put $s\leftarrow T\beta$
2: Sample $\varepsilon/2$-strongly $f^{\leftarrow}$ from $\mathcal{L}(C((-T_cX)^{\mathrm{me},s}))$ ▷ Assumption (S)
3: Sample $\varepsilon/2$-strongly $f^{\rightarrow}$ from $\mathcal{L}(C(T_cX^{\mathrm{me},T-s}))$ ▷ Assumption (S)
4: **return** $f_\varepsilon: t\mapsto -ct+f^{\leftarrow}(s-\min\{t,s\})-f^{\leftarrow}(s)+f^{\rightarrow}(\max\{t,s\}-s)$ for $t\in[0,T]$.

Remark 3.2. (i) Note that $(f_\varepsilon(T),\underline{f_\varepsilon}(T))$ is an εSS of $(X_T,\underline{X}_T)$ as $f\mapsto\underline{f}(T)$ is a Lipschitz functional on the space of càdlàg funcitons with respect to the supremum norm. Although $\tau_{[0,T]}(f_\varepsilon)=\inf\{t\in[0,T]: f_\varepsilon(t)=\underline{f_\varepsilon}(T)\}\to\tau_{[0,T]}(X)$ as $\varepsilon\downarrow 0$ by [Kal02, Lem. 14.12], *a priori* control on the error does not follow directly in general. In the case of weakly stable processes, we will construct in steps 2 and 3 of Algorithm 1, piecewise linear convex functions that sandwich $C(X)$, which yield a control on the error of the approximation of τ_T by Proposition 5.4(c).
(ii) The algorithm may be used to obtain an εSS of $-C(-X)$, the concave majorant of X.

Fix $0=t_0<t_1<\ldots<t_m\le t_{m+1}=T$ and recall from Subsection 2.3 above that $X^{\mathrm{me},T}$ follows the law of $(X)_{t\in[0,T]}$ conditional on $\{\underline{X}_T\ge 0\}$. Note that $\mathbb{P}(\underline{X}_T\ge 0)=0$, but $\mathbb{P}(\underline{X}_T\ge 0|\underline{X}_{t_1}\ge 0)>0$ by Assumption (D). Thus, sampling $(X^{\mathrm{me},T}_{t_1},\ldots,X^{\mathrm{me},T}_{t_m})$ is reduced to jointly simulating $(X_{t_1},\ldots,X_{t_m})$ and $\underline{X}_T$ conditional on $\{\underline{X}_{t_1}\ge 0\}$ and rejecting all samples not in the event $\{\underline{X}_T\ge 0\}$. More precisely, we get the following algorithm.

Algorithm 2 ε-strong simulation of the vector $(X^{\mathrm{me},T}_{t_1},\ldots,X^{\mathrm{me},T}_{t_m})$.

Require: Times $0=t_0<t_1<\ldots<t_m\le t_{m+1}=T$ and accuracy $\varepsilon>0$.
repeat
  Put $(\Pi_0,\varepsilon_0,i)\leftarrow(\emptyset,2\varepsilon/(m+1),0)$
  repeat Conditionally on the variables in the set Π_i
    Put $(\varepsilon_{i+1},i)\leftarrow(\varepsilon_i/2,i+1)$
    Sample ε_i-strongly $z_1^{\varepsilon_i}$ from $\mathcal{L}(X^{\mathrm{me},t_1}_{t_1})$ and put $(x_1^{\varepsilon_i},\underline{x}_1^{\varepsilon_i})\leftarrow(z_1^{\varepsilon_i},z_1^{\varepsilon_i})$ ▷ Assumption (S)
    for $k=2,\ldots,m+1$ **do**
      Sample ε_i-strongly $(z_k^{\varepsilon_i},\underline{z}_k^{\varepsilon_i})$ from $(X_{t_k-t_{k-1}},\underline{X}_{t_k-t_{k-1}})$ ▷ Remark 3.2(i)
      Put $(x_k^{\varepsilon_i},\underline{x}_k^{\varepsilon_i})\leftarrow(x_{k-1}^{\varepsilon_i}+z_k^{\varepsilon_i},\min\{\underline{x}_{k-1}^{\varepsilon_i},x_{k-1}^{\varepsilon_i}+\underline{z}_k^{\varepsilon_i}\})$
    end for
    Put $\Pi_i\leftarrow\Pi_{i-1}\cup\{(z_k^{\varepsilon_i},\underline{z}_k^{\varepsilon_i})\}_{k=1}^{m+1}$
  until $\underline{x}_{m+1}^{\varepsilon_i}-(m+1)\varepsilon_i\ge 0$ or $\underline{x}_{m+1}^{\varepsilon_i}+(m+1)\varepsilon_i<0$
until $\underline{x}_{m+1}^{\varepsilon_i}-(m+1)\varepsilon_i\ge 0$
return $(x_1^{\varepsilon_i},\ldots,x_m^{\varepsilon_i})$.

Remark 3.3. (i) All the simulated values are dropped when the condition in line 13 fails. (ii) If the algorithm satisfying Assumption (S) is the result of a sequential procedure, one may remove the explicit reference to ε_i in line 4 and instead run all pertinent algorithms for another step until condition in line 12 holds. This is, for instance, the case for the algorithms we present for stable meanders.

3.2 Simulation of the convex minorant of stable meanders

In the remainder of the paper, we let $Z = (Z_t)_{t\in[0,T]}$ be a stable process with stability parameter $\alpha \in (0, 2]$ and positivity parameter $\mathbb{P}(Z_1 > 0) = \rho \in (0, 1]$, using Zolotarev's (C) form (see e.g. [GCMUB19, App. A]). It follows from [GCMUB19, Eq. (A.1)&(A.2)] that Assumptions (K) and (P) are satisfied by Z. In the present subsection, we will construct an εSS algorithm for the convex minorant of stable meanders, required by Assumption (S) of Subsection 3.1.

The scaling property implies that $(Z^{\mathrm{me},T}_{sT})_{s\in[0,1]} \overset{d}{=} (T^{1/\alpha} Z^{\mathrm{me},1}_s)_{s\in[0,1]}$ and thus

$$(C(Z^{\mathrm{me},T})(sT))_{s\in[0,1]} \overset{d}{=} (C(T^{1/\alpha}Z^{\mathrm{me},1})(s))_{s\in[0,1]} = (T^{1/\alpha}C(Z^{\mathrm{me},1})(s))_{s\in[0,1]}.$$

By the relation in display, it is sufficient to consider the case of the normalised stable meander $Z^{\mathrm{me}} = Z^{\mathrm{me},1}$ in the remainder of the paper.

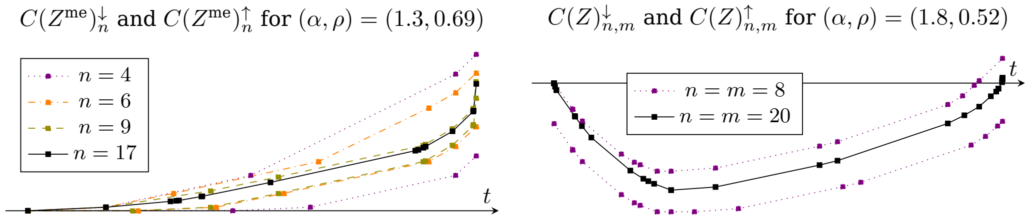
3.2.1 Sandwiching

To obtain an εSS of the convex minorant of a meander, we will construct two convex and piecewise linear functions with finitely many faces that sandwich the convex minorant and whose distance from each other, in the supremum norm, is at most ε. Intuitively, the sandwiching procedure relies on two ingredients: (I) the ability to sample, for each n, the first n faces in the minorant and (II) doing so jointly with a variable $c_n > 0$ that dominates the sum of the heights of all the unsampled faces. Conditions (I) and (II) are, by Proposition 5.2 below, sufficient to sandwich the convex minorant: lower (resp. upper) bound $C(Z^{\mathrm{me}})^{\downarrow}_n$ (resp. $C(Z^{\mathrm{me}})^{\uparrow}_n$) is constructed by adding a final face of height 0 (resp. c_n) and length equal to the sum of the lengths of the remaining faces and sorting all $n+1$ faces in increasing order of slopes. The distance (in the supremum norm) between the convex functions $C(Z^{\mathrm{me}})^{\downarrow}_n$ and $C(Z^{\mathrm{me}})^{\uparrow}_n$ equals c_n (see Proposition 5.2 for details). The εSS algorithm is then obtained by stopping at $-N(\varepsilon)$, the smallest integer n for which the error c_n is smaller than ε (see Algorithm 3 below).

In general, condition (I) is relatively easy to satisfy under the assumptions of Theorem 2.7. Condition (II) however, is more challenging. In the stable case, we first establish a stochastic perpetuity in Theorem 3.4 and use ideas from [GCMUB19] to sample the variables c_n, $n \in \mathbb{N}$, in condition (II) (see Equation (3.3)).

Figure 3(a) above illustrates the output of the εSS Algorithm 3 below for the convex minorant $C(Z^{\mathrm{me}})$. By gluing two such outputs for the (unnormalised) stable meanders $Z^{\leftarrow}$ and $Z^{\rightarrow}$, straddling the minimum of Z over the interval $[0, 1]$ as in (2.1), with n and m faces, respectively, we obtain a convex function $C(Z)^{\downarrow}_{n,m}$ (resp. $C(Z)^{\uparrow}_{n,m}$) that is smaller (resp. larger) than the convex minorant $C(Z)$ of the stable process (see details in Proposition 5.4). Figure 3(b) illustrates how these approximations sandwich the convex minorant $C(Z)$.

A linear tilting can be applied, as in Algorithm 1 above, to obtaina sandwich for the convex minorant of a weakly stable processes for all $\alpha \in (0, 2] \setminus \{1\}$ (see a numerical example in Subsection 4.2.2 below).



(a) Sandwiching $C(Z^{\mathrm{me}})$ (b) Sandwiching $C(Z)$

Figure 3: (A) Sandwiching of the convex minorant $C(Z^{\mathrm{me}})$ using n faces. The lower and upper bounds are numerically indistinguishable for $n=17$. (B) Sandwiching of the convex minorant $C(Z)$ using n and m faces of the convex minorants $C(Z^{\leftarrow})$ and $C(Z^{\rightarrow})$ of the meanders $Z^{\leftarrow}$ and $Z^{\rightarrow}$, respectively. Again the bounds are numerically indistinguishable for $n=m=20$.

3.2.2 The construction of c_n

Since stable processes satisfy Assumptions (P) and (K), we may use Theorem 2.7 and the scaling property of stable laws as stepping stones to obtain a Markovian description of the convex minorants of the corresponding meanders. Let $\mathcal{S}(\alpha,\rho)$, $\mathcal{S}^+(\alpha,\rho)$ and $\mathcal{S}^{\mathrm{me}}(\alpha,\rho)$ be the laws of Z_1, Z_1 conditioned to be positive and Z_1^{me}, respectively, where $(Z_t)_{t\in[0,1]}$ is a stable process with parameters (α,ρ). Recall the definition of the sets $\mathcal{Z}^n=\{k\in\mathbb{Z}:k<n\}$ and $\mathcal{Z}_m^n=\mathcal{Z}^n\setminus\mathcal{Z}^m$ for $n,m\in\mathbb{Z}$.

Theorem 3.4. *Let $((\ell_n^{\mathrm{me}},\xi_n^{\mathrm{me}}))_{n\in\mathbb{N}}$ be the faces of $C(Z^{\mathrm{me}})$ chosen independently at random uniformly on lengths. Define the random variables $L_{n+1}=\sum_{m\in\mathcal{Z}^{n+1}}\ell_{-m}^{\mathrm{me}}$, $U_n=\ell_{-n}^{\mathrm{me}}/L_{n+1}$,*

$$S_n=(\ell_{1-n}^{\mathrm{me}})^{-1/\alpha}\xi_{1-n}^{\mathrm{me}} \qquad \text{and} \qquad M_{n+1}=L_{n+1}^{-1/\alpha}\sum_{m\in\mathcal{Z}^{n+1}}\xi_{-m}^{\mathrm{me}}, \tag{3.1}$$

for all $n\in\mathcal{Z}^0$. Then the following statements hold.
(a) *$((S_n,U_n))_{n\in\mathcal{Z}^0}$ is an iid sequence with common law $\mathcal{S}^+(\alpha,\rho)\times\mathrm{Beta}(1,\rho)$.*
(b) *$(M_n)_{n\in\mathcal{Z}^1}$ is a stationary Markov chain satisfying $M_0=Z_1^{\mathrm{me}}\sim\mathcal{S}^{\mathrm{me}}(\alpha,\rho)$ and*

$$M_{n+1}=(1-U_n)^{1/\alpha}M_n+U_n^{1/\alpha}S_n, \qquad \text{for all } n\in\mathcal{Z}^0. \tag{3.2}$$

(c) *The law of Z_1^{me} is the unique solution to the perpetuity $Z_1^{\mathrm{me}}\overset{d}{=}(1-U)^{1/\alpha}Z_1^{\mathrm{me}}+U^{1/\alpha}S$ for independent $(S,U)\sim\mathcal{S}^+(\alpha,\rho)\times\mathrm{Beta}(1,\rho)$.*

Theorem 3.4, proved in Subsection 5.2 below, enables us to construct a process $(D_n)_{n\in\mathcal{Z}^1}$ that dominates $(M_n)_{n\in\mathcal{Z}^1}$: $D_n\ge M_n$ for $n\in\mathcal{Z}^1$ and can be simulated jointly with the sequence $((S_n,U_n))_{n\in\mathcal{Z}^0}$ (see details in Appendix A). Thus, by (3.1), we may construct the sandwiching convex functions in Algorithm 3 below (see also Subsection 3.2.1 above for an intuitive description) by setting

$$c_{-n}=L_n^{1/\alpha}D_n\ge\sum_{m\in\mathcal{Z}^n}\xi_{-m}^{\mathrm{me}}, \qquad n\in\mathcal{Z}^0. \tag{3.3}$$

3.2.3 The algorithm and its running time

Let $-N(\varepsilon)$ be the smallest $n\in\mathbb{N}$ with $c_{-n}<\varepsilon$ (see (5.8) below for the precise definition).

Remark 3.5. (i) Given the faces $\{(\ell_k^{\mathrm{me}},\xi_k^{\mathrm{me}})\}_{k\in\mathcal{Z}_1^{n+1}}$, the output $(C(Z^{\mathrm{me}})_n^{\downarrow},C(Z^{\mathrm{me}})_n^{\uparrow})$ of Algorithm 3 is defined in Proposition 5.2, see also Lemma 5.1. In particular, Proposition 5.2 requires to sort $(n+1)$ faces, sampled in Algorithm 3. This has a complexity of at most $\mathcal{O}(n\log(n))$ under the *Timsort* algorithm, making the complexity of Algorithm 3

Algorithm 3 ε-strong simulation of the convex minorant $C(Z^{\mathrm{me}})$.

Require: Tolerance $\varepsilon > 0$ and burn-in parameter $m \in \mathbb{N} \cup \{0\}$

1: Sample independently (S_k, U_k) for $k \in \mathcal{Z}_{-m}^0$ from the law $\mathcal{S}^+(\alpha,\rho) \times \mathrm{Beta}(1,\rho)$

2: Sample backwards in time (S_k, U_k, D_k) for $k \in \mathcal{Z}_{-\max\{m+1,|N(\varepsilon)|\}}^{-m}$ ▷ [GCMUB19, Alg. 2]

3: Set $n = \max\{m+1, |N(\varepsilon)|\}$

4: For $k \in \{1,\dots,n\}$, set $\ell_k^{\mathrm{me}} = U_{-k}\prod_{j\in\mathcal{Z}_{1-k}^0}(1-U_j)$ and $\xi_k^{\mathrm{me}} = (\ell_k^{\mathrm{me}})^{1/\alpha} S_{1-k}$

5: Put $c_n = (\prod_{k\in\mathcal{Z}_{-n}^0}(1-U_k))^{1/\alpha} D_{-n}$ and **return** $(C(Z^{\mathrm{me}})_n^{\downarrow}, C(Z^{\mathrm{me}})_n^{\uparrow})$

proportional to $|N(\varepsilon)|\log|N(\varepsilon)|$. Moreover, the burn-in parameter m is conceptually inessential (i.e. we can take it to be equal to zero without affecting the law of the output) but practically very useful. Indeed, since Algorithm 3 terminates as soon as $c_n < \varepsilon$, the inexpensive simulation of the pairs (S_k, U_k) increases the probability of having to sample fewer (computationally expensive) triplets (S_k, U_k, D_k) in line 2 (cf. [GCMUB19, Sec. 5]).
(ii) An alternative to Algorithm 3 is to run forward in time a Markov chain based on the perpetuity in Theorem 3.4(c). This would converge in the L^γ-Wasserstein distance at the rate $\mathcal{O}((1+\gamma/(\alpha\rho))^{-n})$ (see [BDM16, Sec. 2.2.5]), yielding an approximate simulation algorithm for the law $S^{\mathrm{me}}(\alpha,\rho)$.

Note that the running time of Algorithm 3 is completely determined by $\max\{m+1, |N(\varepsilon)|\}$. Applications in Section 4.1 below rely on the exact simulation of $\mathbb{1}_{\{Z_1^{\mathrm{me}}>x\}}$ for arbitrary $x > 0$ via εSS (see Subsection 4.1.1 below), which is based the on sequential refinements of Algorithm 3. Moreover, Algorithms 1 and 2 rely on Algorithm 3. Hence bounding the tails of $N(\varepsilon)$ is key in bounding the running times of all those algorithms. Proposition 5.13 establishes bounds on the tails of $N(\varepsilon)$, which combined with Lemma 5.11 below, implies the following result, proved in Subsection 5.3.2 below.

Theorem 3.6. *The running times of Algorithms 1, 2 and 3 for the εSS of $C(T_cZ)$ (for any $c \in \mathbb{R}$), finite dimensional distributions of Z^{me} and $C(Z^{\mathrm{me}})$, respectively, have exponential moments. The same holds for the exact simulation algorithm of $\mathbb{1}_{\{Z_1^{\mathrm{me}}>x\}}$ for any $x > 0$.*

4 Applications and numerical examples

In Subsection 4.1 we describe applications of εSS paradigm to the exact and unbiased sampling of certain functionals. In Subsection 4.2 we present specific numerical examples. We apply algorithms from Section 3 to estimate expectations via MC methods and construct natural confidence intervals.

4.1 Applications of εSS algorithms

Consider a metric space $(\mathbb{X}, d)$ and a random element Λ taking values in $\mathbb{X}$. For every $\varepsilon > 0$, the random element Λ^ε is assumed to be the output of an εSS algorithm, i.e. it satisfies $d(\Lambda, \Lambda^\varepsilon) < \varepsilon$ a.s. (see Section 3 for definition).

4.1.1 Exact simulation of indicators

One may use εSS algorithms to sample exactly indicators $\mathbb{1}_A(\Lambda)$ for any set $A \subset \mathbb{X}$ with $\mathbb{P}(\Lambda \in \partial A) = 0$, where ∂A denotes the boundary of A in $\mathbb{X}$. Since $d(\Lambda^\varepsilon, \partial A) > \varepsilon$ implies $\mathbb{1}_A(\Lambda) = \mathbb{1}_A(\Lambda^\varepsilon)$, it suffices to sample the sequence $(\Lambda^{2^{-n}})_{n\in\mathbb{N}}$ until $d(\Lambda^{2^{-n}}, \partial A) > 2^{-n}$. Finite termination is ensured because $\{d(\Lambda,\partial A) > 0\} = \bigcup_{n\in\mathbb{N}}\{d(\Lambda,\partial A) > 2^{-n}\} \subset \bigcup_{n\in\mathbb{N}}\{d(\Lambda^{2^{-n-1}},\partial A) > 2^{-n-1}\}$. In particular, line 12 in Algorithm 2 is based on this principle.

4.1.2 Unbiased simulation of finite variation transformations of a continuous functional

Let $f_1 : \mathbb{X} \to \mathbb{R}$ be continuous, $f_2 : \mathbb{R} \to [0,\infty)$ be of finite variation on compact intervals and define $f = f_2 \circ f_1$. The functional $\underline{Z}_T \cdot \mathbb{1}_{\{\underline{Z}_T > b\}}$, for some $b < 0$, is a concrete example defined on the space of continuous functions, since the maps $C(Z) \mapsto \underline{Z}_T$ and $\underline{Z}_T \mapsto \underline{Z}_T \cdot \mathbb{1}_{\{\underline{Z}_T > b\}}$ are continuous and of finite variation, respectively.

By linearity, it suffices to consider a monotone $f_2 : \mathbb{R} \to [0,\infty)$. Let ς be an independent random variable with positive density $g : [0,\infty) \to (0,\infty)$. Then $\Sigma = \mathbb{1}_{(\varsigma,\infty)}(f(\Lambda))/g(\varsigma)$ is simulatable and unbiased for $\mathbb{E}[f(\Lambda)]$. Indeed, it is easily seen that $\mathbb{P}(\Lambda \in \partial f^{-1}(\{\varsigma\})) = 0$. Thus Subsection 4.1.1 shows that the indicator $\mathbb{1}_{(\varsigma,\infty)}(f(\Lambda))$, and hence Σ, may be simulated exactly. Moreover, Σ is unbiased since

$$\mathbb{E}[\Sigma] = \mathbb{E}\left[\mathbb{E}[\Sigma|\Lambda]\right] = \mathbb{E}\left[\int_0^\infty \mathbb{1}_{[0,f(\Lambda))}(s) g(s)^{-1} g(s) ds\right] = \mathbb{E}[f(\Lambda)],$$

and its variance equals $\mathbb{E}[\Sigma^2] - \mathbb{E}[f(\Lambda)]^2 = \mathbb{E}[G(f(\Lambda))] - \mathbb{E}[f(\Lambda)]^2$, where $G(r) = \int_0^t ds/g(s)$. If we use the density $g : s \mapsto \delta(1+s)^{-1-\delta}$, for some $\delta > 0$, then the variance of Σ (resp. $f(\Lambda)$) is $\mathbb{E}\big[\frac{1}{\delta(2+\delta)}((1+f(\Lambda))^{2+\delta} - 1)\big] - \mathbb{E}[f(\Lambda)]^2$ (resp. $\mathbb{E}[f(\Lambda)^2] - \mathbb{E}[f(\Lambda)]^2$). Thus, Σ can have finite variance if $f(\Lambda)$ has a finite $2+\delta$-moment. This application was proposed in [BCD17] for the identity function $f_2(t) = t$ and any Lipschitz functional f_1.

4.1.3 Unbiased simulation of a continuous finite variation function of the first passage time

Let $(\mathcal{X}_t)_{t\in[0,T)}$, $0 < T \le \infty$ be a real-valued càdlàg process such that $\mathcal{X}_0 = 0$ and, for every $t > 0$, there is an εSS algorithm of $\overline{\mathcal{X}}_t = \sup_{s\in[0,t]\cap[0,T)} \mathcal{X}_s$. Fix any $x > 0$ satisfying $\mathbb{P}(\overline{\mathcal{X}}_t = x) = 0$ for almost every $t \in [0,T)$. Then $\sigma_x = \min\{T, \inf\{t \in (0,T) : \mathcal{X}_t > x\}\}$ (using the convention $\inf\emptyset = \infty$) is the first passage time of level x and satisfies the identity $\{t < \sigma_x\} = \{\overline{\mathcal{X}}_{\min\{t,T\}} \le x\}$, for $t \ge 0$. By linearity, it suffices to consider a nondecreasing continuous function $f : [0,T) \to [0,\infty)$ with generalised inverse f^*.

Let ς be as in Subsection 4.1.2 and $f(T) = \lim_{t\uparrow T} f(t)$. By [dLF15, Prop. 4.2], f^* is strictly increasing and $\{\varsigma < f(\sigma_x)\} = \{f^*(\varsigma) < \sigma_x\} = \{f^*(\varsigma) < T, \overline{\mathcal{X}}_{f^*(\varsigma)} \le x\}$, where $\mathbb{P}(\overline{\mathcal{X}}_{f^*(\varsigma)} = x, f^*(\varsigma) < T) = 0$ by assumption. Hence $\Sigma = \mathbb{1}_{(\varsigma,\infty)}(f(\sigma_x))/g(\varsigma)$ is simulated by sampling ς and then, as in Subsection 4.1.1, setting $\Sigma = \mathbb{1}_{[0,x]}(\mathcal{X}_{f^*(\varsigma)})/g(\varsigma)$ if $f^*(\varsigma) < T$ and otherwise putting $\Sigma = 0$. Moreover, by Subsection 4.1.2, Σ is unbiased for $\mathbb{E}[f(\sigma_x)]$. We stress that, unlike the functionals considered in Subsections 4.1.1 and 4.1.2 above, it is not immediately clear how to estimate $\mathbb{E}[f(\sigma_x)]$ using a simulation algorithm for $\overline{\mathcal{X}}_t$, $t \in [0,T)$. In Subsection 4.2.2 we present a concrete example for weakly stable processes.

We end with the following remark. Consider the time the process $\mathcal{X}$ down-crosses (resp. up-crosses) a convex (resp. concave) function mapping $[0,\infty)$ to $\mathbb{R}$ started below (resp. above) $\mathcal{X}_0 = 0$. If one has an εSS algorithm for the convex minorant (resp. concave majorant) of $\mathcal{X}$, then a simple modification of the argument in the previous paragraph yields an unbiased simulation algorithm of any finite variation continuous function of such a first passage time.

4.2 Numerical results

In this subsection we explore three applications of the εSS of stable meanders and their convex minorants. Since Algorithm 3 uses [GCMUB19, Alg. 2] for backward simulation, we specify the values of the parameters $(d,\delta,\gamma,\kappa,m,m^*) = \varpi(\alpha,\rho)$ appearing in [GCMUB19, Sec. 4] and m in Algorithm 3 as follows: $(d^*, r) = (\frac{2}{3\alpha\rho}, \frac{19}{20})$ and

$$\varpi(\alpha,\rho) = \left(d^*, \frac{d^*}{2}, r\alpha, 4+\max\left\{\frac{\log(2)}{3\eta(d^*)}, \frac{1}{\alpha\rho}\right\}, \left\lceil \frac{|\log(\varepsilon/2)|}{\log\left(\frac{\Gamma(1+\rho+1/\alpha)}{\Gamma(1+\rho)\Gamma(1+1/\alpha)}\right)}\right\rceil, 12+\left\lfloor \frac{3\rho}{r}\log \mathbb{E}S^{r\alpha}\right\rfloor^+\right),$$

where $\eta(d) = -\alpha\rho - \mathcal{W}_{-1}(-\alpha\rho d e^{-\alpha\rho d})/d$ is the unique positive root of the equation $dt = \log(1+t/(\alpha\rho))$ (here, $\mathcal{W}_{-1}$ is the secondary branch of the Lambert W function [CGH+96]) and S follows the law $\mathcal{S}^+(\alpha,\rho)$. As usual, $\lfloor x\rfloor = \sup\{n \in \mathbb{Z} : n \le x\}$ and $\lceil x\rceil = \inf\{n \in \mathbb{Z} : n \ge x\}$ denote the floor and ceiling functions and $x^+ = \max\{0, x\}$ for any $x \in \mathbb{R}$. This choice of m satisfies $\mathbb{E}[U^{1/\alpha}]^m \approx \varepsilon/2$ for $\varepsilon < 1$, where $U \sim \mathrm{Beta}(1,\rho)$. We fix $\varepsilon = 2^{-32}$ throughout unless adaptive precision is required (see Subsections 4.1.1–4.1.3).

Figure 4 graphs the empirical distribution function for the running time of Algorithm 3, suggesting the existence of exponential moments of $|N(\varepsilon)|$, cf. Proposition 5.13 below.

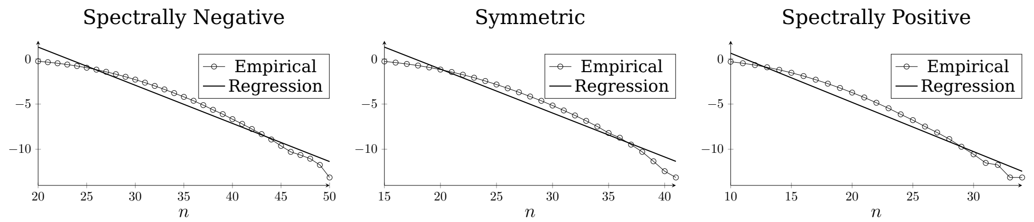


Figure 4: The graphs show the estimated value of $n \mapsto \log \mathbb{P}(|N(\varepsilon)| > n)$ in the spectrally negative, symmetric and positive cases for $\alpha = 1.5$, $\varepsilon = 2^{-32}$ and based on $N = 5 \times 10^5$ samples. The curvature in all three graphs suggests that $|N(\varepsilon)|$ has exponential moments of all orders, a stronger claim than those of Theorem 3.6 (see also Proposition 5.13).

To demonstrate practical feasibility, we first study the running time of Algorithm 3. We implemented Algorithm 3 in the Julia 1.0 programming language (see [GCMUB18b]) and ran it on macOS Mojave 10.14.3 (18D109) with a 4.2 GHz Intel®Core™i7 processor and an 8 GB 2400 MHz DDR4 memory. Under these conditions, generating $N = 10^4$ samples takes approximately 1.30 seconds for any $\alpha > 1$ and all permissible ρ as long as ρ is bounded away from 0. This task much less time for $\alpha < 1$ so long as α and ρ are bounded away from 0. The performance worsens dramatically as either $\alpha \to 0$ or $\rho \to 0$.

This behaviour is as expected since the coefficient in front of M_n in (3.2) of Theorem 3.4 follows the law $\mathrm{Beta}(1, \alpha\rho)$ with mean $\frac{1}{1+\alpha\rho}$, which tends to 1 as $\alpha\rho \to 0$. Hence, the Markov chain decreases very slowly when $\alpha\rho$ is close to 0. From a geometric viewpoint, note that as $\rho \to 0$, the mean length of each sampled face (as a proportion of the lengths of the remaining faces) satisfies $\mathbb{E}\ell_1^{\mathrm{me}} = \frac{\rho}{1+\rho} \to 0$, implying that large faces are increasingly rare. Moreover, as the stability index α decreases, the tails of the density of the Lévy measure become very heavy, making a face of small length and huge height likely. To illustrate this numerically, the approximate time (in seconds) taken by Algorithm 3 to produce $N = 10^4$ samples for certain combinations of parameters is found in the following table:

$\alpha \setminus \rho$	0.95	0.5	0.1	0.05	0.01	0.005
0.5	0.301	0.314	0.690	1.165	4.904	9.724
0.1	0.197	0.242	0.738	1.367	6.257	12.148
0.05	0.229	0.318	1.125	2.137	9.864	20.131

The remainder of the subsection is as follows. In Subsection 4.2.1 we estimate the mean of Z_1^{me} as a function of the stability parameter α in the spectrally negative, symmetric and positive cases. The results are compared with the exact mean, computed in Corollary 5.9 via the perpetuity in Theorem 3.4(c). In Subsection 4.2.2 we numerically analyse the first passage times of weakly stable processes. In Subsection 4.2.3 we estimate the mean of the normalised stable excursion at time $1/2$ and construct confidence intervals.

4.2.1 Marginal of the normalised stable meander Z_1^{me}

Let $\{(\zeta_i^{\varepsilon,\downarrow}, \zeta_i^{\varepsilon,\uparrow})\}_{i\in\mathbb{N}}$ be an iid sequence of ε-strong samples of Z_1^{me}. Put differently, for all $i \in \mathbb{N}$, we have $0 < \zeta_i^{\varepsilon,\uparrow} - \zeta_i^{\varepsilon,\downarrow} < \varepsilon$ and the corresponding sample of Z_1^{me} lies in the interval $(\zeta_i^{\varepsilon,\downarrow}, \zeta_i^{\varepsilon,\uparrow})$. For any continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $\mathbb{E}[|f(Z_1^{\mathrm{me}})|] < \infty$, a Monte Carlo estimate of $\mathbb{E}[f(Z_1^{\mathrm{me}})]$ is given by $\frac{1}{2N}\sum_{i=1}^{N}(f(\zeta_i^{\varepsilon,\downarrow}) + f(\zeta_i^{\varepsilon,\uparrow}))$. If f is nondecreasing we clearly have the inequalities $\mathbb{E}[f(\zeta_1^{\varepsilon,\downarrow})] \le \mathbb{E}[f(Z_1^{\mathrm{me}})] \le \mathbb{E}[f(\zeta_1^{\varepsilon,\uparrow})]$. Thus, a confidence interval (a, b) for $\mathbb{E}[f(Z_1^{\mathrm{me}})]$ may be constructed as follows: a (resp. b) is given by the lower (resp. upper) end of the confidence interval (CI) for $\mathbb{E}[f(\zeta_1^{\varepsilon,\downarrow})]$ (resp. $\mathbb{E}[f(\zeta_1^{\varepsilon,\uparrow})]$). We now use Algorithm 3 to estimate $\mathbb{E}[Z_1^{\mathrm{me}}]$ (for $\alpha > 1$) and $\mathbb{E}[(Z_1^{\mathrm{me}})^{-\alpha\rho}]$ and compare the estimates with the formulae for the expectations from Corollary 5.9. The results are shown in Figure 5 below.

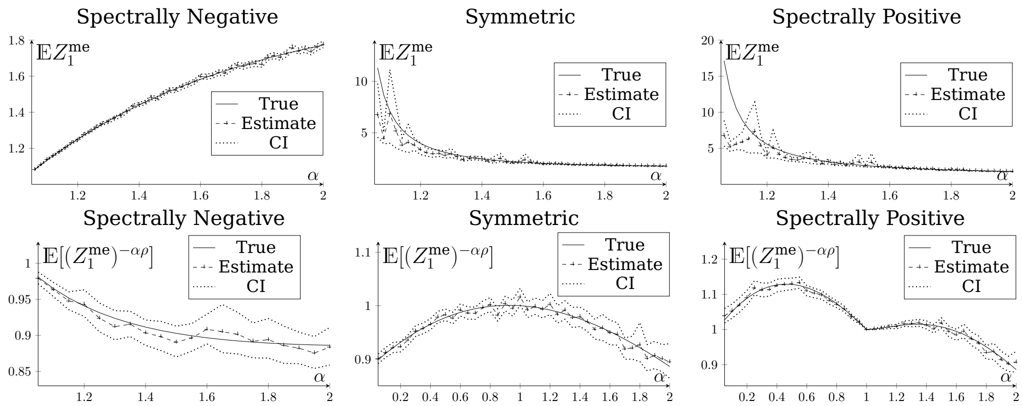


Figure 5: Top (resp. bottom) graphs show the true and estimated means $\mathbb{E}Z_1^{\mathrm{me}}$ (resp. moments $\mathbb{E}[(Z_1^{\mathrm{me}})^{-\alpha\rho}]$) with 95% confidence intervals based on $N = 10^4$ samples as a function of α for the spectrally negative ($\rho = 1/\alpha$ as $\alpha > 1$), symmetric ($\rho = 1/2$) and positive ($\rho = 1 - 1/\alpha$ if $\alpha > 1$ and $\rho = 1$ otherwise) cases. The estimates and confidence intervals of $\mathbb{E}Z_1^{\mathrm{me}}$ are larger and more unstable for values of α close to 1 (except for the spectrally negative case) since the tails of its distribution are at their heaviest.

The CLT is not applicable when the variables have infinite variance and can hence not be used for the CIs of $\mathbb{E}Z_1^{\mathrm{me}}$ (except for the spectrally negative case). Thus, we use bootstrapping CIs throughout, constructed as follows. Given an iid sample $\{x_k\}_{k=1}^n$ and a confidence level $1 - \lambda$, we construct the sequence $\{\mu_i\}_{i=1}^n$, where $\mu_i = \frac{1}{n}\sum_{k=1}^n x_k^{(i)}$ and $\{x_k^{(i)}\}_{k=1}^n$ is obtained by resampling with replacement from the set $\{x_k\}_{k=1}^n$. We then use the quantiles $\lambda/2$ and $1 - \lambda/2$ of the empirical distribution of the sample $\{\mu_i\}_{i=1}^n$ as the CI's endpoints for $\mathbb{E}[x_1]$ with the point estimator $\mu = \frac{1}{n}\sum_{k=1}^n x_k$.

4.2.2 First passage times of weakly stable processes

Define the first passage time $\hat{\sigma}_x = \inf\{t > 0 : \hat{Z}_t > x\}$ of the weakly stable process $\hat{Z} = (\hat{Z}_t)_{t\ge0} = (Z_t + \mu t)_{t\ge0}$ for some $\mu \in \mathbb{R}$ and all $x > 0$. As a concrete example of the unbiased simulation from Subsection 4.1.3 above, we estimate $\mathbb{E}\hat{\sigma}_x$ in the present subsection. To ensure that the previous expectation is finite, it suffices that $\mathbb{E}\hat{Z}_1 = \mu + \mathbb{E}Z_1 > 0$ [Ber96, Ex. VI.6.3], where $\mathbb{E}Z_1 = (\sin(\pi\rho) - \sin(\pi(1-\rho)))\Gamma(1-\frac{1}{\alpha})/\pi$ (see e.g. [GCMUB19, Eq. (A.2)]). Since the time horizon over which the weakly stable process is simulated is random and equal to ς, we chose $g : s \mapsto 2(1+s)^{-3}$ to ensure $\mathbb{E}\varsigma < \infty$. The results presented in Figure 6 used the fixed values $\mu = 1 - \mathbb{E}Z_1$ and $x = 1$.

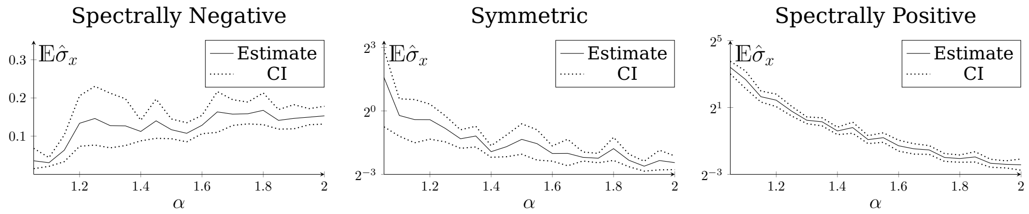


Figure 6: The graphs show estimates of $\mathbb{E}\hat{\sigma}_x$ with 95% CIs based on $N = 4 \times 10^4$ samples as a function of $\alpha \in (1, 2]$ for the spectrally negative, symmetric and positive cases. The estimates are obtained using the procedure in Subsection 4.1.3 with the density function $g : s \mapsto \delta(1+s)^{-1-\delta}$ for $\delta = 2$. The computation of each estimate (employing $N = 4 \times 10^4$ samples) took approximately 290 seconds, with little variation for different values of α.

4.2.3 Marginal of normalised stable excursions

Let $Z^{\text{ex}} = (Z^{\text{ex}}_t)_{t\in[0,1]}$ be a normalised stable excursion associated to the stable process Z with parameters (α, ρ). By [Cha97, Thm 3], if Z has negative jumps (i.e. $\rho \in (0, 1)$ & $\alpha \le 1$ or $\rho \in (1 - \frac{1}{\alpha}, \frac{1}{\alpha}]$ & $\alpha > 1$), the laws of $(Z^{\text{me}}_t)_{t\in[0,1)}$ and $(Z^{\text{ex}}_t)_{t\in[0,1)}$ are *equivalent*: $\mathbb{P}(Z^{\text{ex}} \in A) = \mathbb{E}[(Z^{\text{me}}_1)^{-\alpha}\mathbb{1}_{\{Z^{\text{me}}\in A\}}]/\mathbb{E}[(Z^{\text{me}}_1)^{-\alpha}]$ for any measurable set A in the Skorokhod space $\mathcal{D}[0, 1)$ [Bil99, Ch. 3]. We remark that $Z^{\text{me}}_1 = Z^{\text{me}}_{1-}$ a.s. and $\mathbb{E}[(Z^{\text{me}}_1)^{-\alpha}] < \infty$ since $\alpha < 1 + \alpha\rho$ and $\mathbb{E}[(Z^{\text{me}}_1)^{\gamma}] < \infty$ for all $\gamma \in (-1 - \alpha\rho, \alpha)$ [DS10, Thm 1]. As an illustration of Algorithm 2, we now present a Monte Carlo estimation of $\mathbb{E}Z^{\text{ex}}_{1/2}$ by applying the procedure of Subsection 4.2.1 for the expectations on the right side of $\mathbb{E}Z^{\text{ex}}_{1/2} = \mathbb{E}[Z^{\text{me}}_{1/2}(Z^{\text{me}}_1)^{-\alpha}]/\mathbb{E}[(Z^{\text{me}}_1)^{-\alpha}]$.

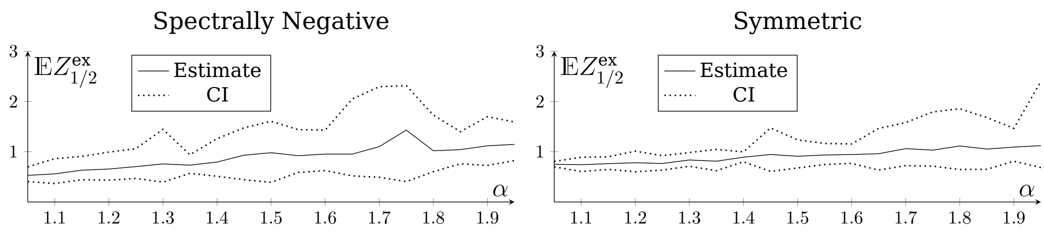


Figure 7: The pictures show the quotient of the Monte Carlo estimates of the expectations on the right side of $\mathbb{E}Z^{\text{ex}}_{1/2} = \mathbb{E}[Z^{\text{me}}_{1/2}(Z^{\text{me}}_1)^{-\alpha}]/\mathbb{E}[(Z^{\text{me}}_1)^{-\alpha}]$ as a function of the stability parameter $\alpha \in (1, 2)$ for $N = 4 \times 10^4$ samples. Computing each estimate (for $N = 4 \times 10^4$) took approximately 160.8 (resp. 123.1) seconds in the spectrally negative (resp. symmetric) case, with little variation in α.

As before, we use the fixed precision of $\varepsilon = 2^{-32}$. The CIs are naturally constructed from the bootstrapping CIs (as in Subsection 4.2.1 above) for each of the expectations $\mathbb{E}[Z^{\text{me}}_{1/2}(Z^{\text{me}}_1)^{-\alpha}]$ and $\mathbb{E}[(Z^{\text{me}}_1)^{-\alpha}]$ and combined to construct a CI for $\mathbb{E}Z^{\text{ex}}_{1/2}$.

5 Proofs and technical results

5.1 Approximation of piecewise linear convex functions

The main aim of the present subsection is to prove Proposition 5.2 and Proposition 5.4, key ingredients of the algorithms in Section 3. Throughout this subsection, we will assume that f is a continuous piecewise linear finite variation function on some compact interval $[a,b]$ with at most countably many faces. More precisely, there exists a set $\{(a_n,b_n) : n \in \mathcal{Z}_1^{N+1}\}$ consisting of $N \in \mathbb{N}\cup\{\infty\}$ pairwise disjoint nondegenerate subintervals of $[a,b]$ such that $\sum_{n=1}^N (b_n-a_n)=b-a$, f is linear on each (a_n,b_n), and $\sum_{n=1}^N |f(b_n)-f(a_n)|<\infty$ (if $N=\infty$ we set $\mathcal{Z}^\infty=\mathbb{Z}$ and thus $\mathcal{Z}_1^\infty=\mathbb{N}$; recall also $\mathcal{Z}^n=\{k\in\mathbb{Z}: k<n\}$ and $\mathcal{Z}_m^n=\mathcal{Z}^n\setminus\mathcal{Z}^m$ for $n,m\in\mathbb{Z}$). A face of f, corresponding to a subinterval (a_n,b_n), is given by the pair (l_n,h_n), where $l_n=b_n-a_n>0$ is its length and $h_n=f(b_n)-f(a_n)\in\mathbb{R}$ its height. Consequently, its slope equals h_n/l_n and the following representation holds (recall $x^+=\max\{0,x\}$ for $x\in\mathbb{R}$):

$$f(t)=f(a)+\sum_{n=1}^N h_n \min\{(t-a_n)^+/l_n,1\}, \qquad t\in[a,b]. \tag{5.1}$$

The number N in representation (5.1) is not unique in general as any face may be subdivided into two faces with the same slope. Moreover, for a fixed f and N, the set of intervals $\{(a_n,b_n): n\in\mathcal{Z}_1^{N+1}\}$ need not be unique. Furthermore we stress that the sequence of faces in (5.1) does not necessarily respect the chronological ordering. Put differently, the sequence $(a_n)_{n\in\mathcal{Z}_1^{N+1}}$ need not be increasing. We start with an elementary but useful result.

Lemma 5.1. *Let $f:[a,b]\to\mathbb{R}$ be a continuous piecewise linear function with $N<\infty$ faces (l_k,h_k), $k\in\mathcal{Z}_1^{N+1}$. Let $\mathcal{K}$ be the set of piecewise linear functions $f_\pi:[a,b]\to\mathbb{R}$ with initial value $f(a)$, obtained from f by sorting its faces according to a bijection $\pi:\mathcal{Z}_1^{N+1}\to\mathcal{Z}_1^{N+1}$. More precisely, defining $a_k^\pi=a+\sum_{j\in\mathcal{Z}_1^k} l_{\pi(j)}$ for any $k\in\mathcal{Z}_1^{N+1}$, f_π in $\mathcal{K}$ is given by*

$$f_\pi(t)=f(a)+\sum_{k=1}^N h_{\pi(k)}\min\left\{\left(t-a_k^\pi\right)^+/l_{\pi(k)},1\right\}, \qquad t\in[a,b].$$

If $\pi^*:\mathcal{Z}_1^{N+1}\to\mathcal{Z}_1^{N+1}$ sorts the faces by increasing slope, $h_{\pi^*(k)}/l_{\pi^*(k)}\le h_{\pi^*(k+1)}/l_{\pi^*(k+1)}$ for $k\in\mathcal{Z}_1^N$, then f_{π^*} is the unique convex function in $\mathcal{K}$ and satisfies $f\ge f_{\pi^*}$ pointwise.

Proof. Relabel the faces (l_k,h_k), $k\in\mathcal{Z}_1^{N+1}$ of f so that they are listed in the chronological order, i.e. as they appear in the function $t\mapsto f(t)$ with increasing t. If every pair of consecutive faces of f is ordered by slope (i.e. $h_i/l_i\le h_{i+1}/l_{i+1}$ for all $i\in\mathcal{Z}_1^N$), then f is convex and $f=f_{\pi^*}$. Otherwise, two consecutive faces of f satisfy $h_i/l_i>h_{i+1}/l_{i+1}$ for some $i\in\mathcal{Z}_1^N$. Swapping the two faces yields a smaller function f_{π_1}, see Figure 8. Indeed, after the swap, the functions f and f_{π_1} coincide on the set $[a,a+\sum_{k\in\mathcal{Z}_1^i} l_k]\cup[a+\sum_{k\in\mathcal{Z}_1^{i+2}} l_k,b]$. In the interval $[a+\sum_{k\in\mathcal{Z}_1^i} l_k, a+\sum_{k\in\mathcal{Z}_1^{i+2}} l_k]$, the segments form a parallelogram whose lower (resp. upper) sides belong to the graph of f_{π_1} (resp. f).

Applying the argument in the preceding paragraph to f_{π_1}, we either have $f_{\pi_1}=f_{\pi^*}$ or we may construct f_{π_2}, which is strictly smaller than f_{π_1} on a non-empty open subinterval of $[a,b]$, satisfying $f_{\pi_1}\ge f_{\pi_2}$. Since the set $\mathcal{K}$ is finite, this procedure necessarily terminates at f_{π^*} after finitely many steps, implying $f\ge f_{\pi^*}$. Since any convex function in $\mathcal{K}$ must have a nondecreasing derivative a.e., it has to be equal to f_{π^*} and the lemma follows. $\square$

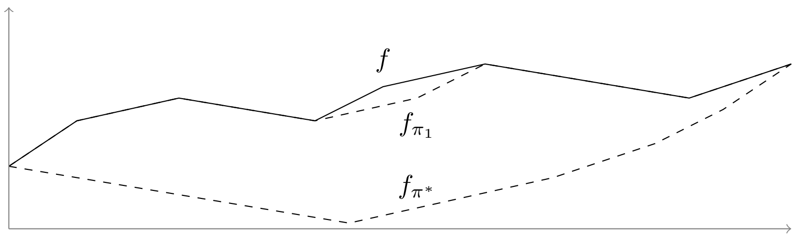


Figure 8: Swapping two consecutive and unsorted faces of f.

A natural approximation of a piecewise linear convex function f can be constructed from the first $n < N$ faces of f by filling in the remainder with a horizontal face. More precisely, for any $n \in \mathcal{Z}_1^N$ let f_n be the piecewise linear convex function with $f_n(a) = f(a)$ and faces $\{(l_k, h_k) : k \in \mathcal{Z}_1^{n+1}\} \cup \{(\tilde{l}_n, 0)\}$, where $\tilde{l}_n = \sum_{k=n+1}^N l_k$. By Corollary 5.6, the inequality $\|f - f_n\|_\infty \le \max\{\sum_{k=n+1}^N h_k^+, \sum_{k=n+1}^N h_k^-\}$ holds, where $\|g\|_\infty = \sup_{t\in[a,b]} |g(t)|$ denotes the supremum norm of a function $g : [a, b] \to \mathbb{R}$ and $x^- = \max\{0, -x\}$ for any $x \in \mathbb{R}$. If $f = C(X)$ is the convex minorant of a Lévy process X, both tail sums decay to zero geometrically fast [GCMUB18a, Thms 1 & 2]. However, it appears to be difficult directly to obtain almost sure bounds on the maximum of the two (dependent!) sums, which would be necessary for an εSS algorithm for $C(X)$. We proceed by "splitting" the problem as follows.

The slopes of the faces of a piecewise linear convex function f may form an unbounded set. In particular, the slopes of the faces accumulating at a could be arbitrarily negative, making it impossible to construct a piecewise linear lower bound with *finitely many faces* starting at $f(a)$. In Proposition 5.2 we focus on functions without faces of negative slope (as is the case with the convex minorants of pre- and post-minimum processes and Lévy meanders in Proposition 2.3 and Theorem 2.7), which makes it easier to isolate the errors. We deal with the general case in Proposition 5.4 below.

Proposition 5.2. *Let $f : [a, b] \to \mathbb{R}$ be a piecewise linear convex function with $N = \infty$ faces (l_n, h_n), $n \in \mathbb{N}$, satisfying $h_n \ge 0$ for all n. Let the constants $(c_n)_{n\in\mathbb{N}}$ satisfy the inequalities $c_n \ge \sum_{k=n+1}^\infty h_k$ and $c_{n+1} \le c_n - h_{n+1}$ for $n \in \mathbb{N}$. There exist unique piecewise linear convex functions $f_n^\downarrow$ and $f_n^\uparrow$ on $[a, b]$, satisfying $f_n^\downarrow(a) = f_n^\uparrow(a) = f(a)$, with faces $\{(l_k, h_k) : k \in \mathcal{Z}_1^{n+1}\} \cup \{(\tilde{l}_n, 0)\}$ and $\{(l_k, h_k) : k \in \mathcal{Z}_1^{n+1}\} \cup \{(\tilde{l}_n, c_n)\}$, respectively, where $\tilde{l}_n = \sum_{k=n+1}^\infty l_k$. Moreover, for all $n \in \mathbb{N}$ the following statements holds:*
(a) $f_{n+1}^\downarrow \ge f_n^\downarrow$, (b) $f_n^\uparrow \ge f_{n+1}^\uparrow$, (c) $f_n^\uparrow \ge f \ge f_n^\downarrow$ *and* (d) $\|f_n^\uparrow - f_n^\downarrow\|_\infty = f_n^\uparrow(b) - f_n^\downarrow(b) = c_n$.

Remark 5.3. (i) Note that if $c_n \to 0$ as $n \to \infty$, Proposition 5.2 implies the sequences $(f_n^\downarrow)_{n\in\mathbb{N}}$ and $(f_n^\uparrow)_{n\in\mathbb{N}}$ converge uniformly and monotonically to f.
(ii) Note that the lower bounds $f_n^\downarrow$ do not depend on c_n and satisfy $\|f - f_n^\downarrow\|_\infty = \sum_{k=n+1}^\infty h_k$. Indeed, set $c_n = \sum_{k=n+1}^\infty h_k$ (for all $n \in \mathbb{N}$) and apply Proposition 5.2(c) & (d) to get

$$\sum_{k=n+1}^\infty h_k = f(b) - f_n^\downarrow(b) \le \|f - f_n^\downarrow\|_\infty \le \|f_n^\uparrow - f_n^\downarrow\|_\infty = \sum_{k=n+1}^\infty h_k.$$

(iii) Given constants $(c'_n)_{n\in\mathbb{N}}$, satisfying $c'_n \ge \sum_{k=n+1}^\infty h_k$, we may construct constants c_n, satisfying $c_n \ge \sum_{k=n+1}^\infty h_k$ and $c_{n+1} \le c_n - h_{n+1}$ for all $n \in \mathbb{N}$ as follows: set $c_1 = c'_1$ and $c_{n+1} = \min\{c'_{n+1}, c_n - h_{n+1}\}$ for $n \in \mathbb{N}$. The condition $c_{n+1} \le c_n - h_{n+1}$ is only necessary for part (b), but is assumed throughout Proposition 5.2 as it simplifies the proof of (c).
(iv) The function f in Proposition 5.2 may have infinitely many faces in a neighbourhood of any point in $[a, b]$. If this occurs at b, the corresponding slopes may be arbitrarily large.
(v) Proposition 5.2 assumes that the slopes of the faces of f are nonnegative. This

condition can be relaxed to all the slopes being bounded from below by some constant $c \le 0$, in which case we use the auxiliary faces $(\tilde{l}_n, c\tilde{l}_n)$ and $(\tilde{l}_n, c\tilde{l}_n + c_n)$ in the construction of $f_n^{\uparrow}$ and $f_n^{\downarrow}$.
(vi) If $n = 0$ and $c_0 \ge c_1 + h_1$, then $\tilde{l}_0 = b - a$ and the functions $f_0^{\downarrow} : t \mapsto f(a)$ and $f_0^{\uparrow} : t \mapsto f(a) + (t-a)c_0$, $t \in [a,b]$, satisfy the conclusion of Proposition 5.2 (with $n = 0$). Moreover, Proposition 5.2 extends easily to the case when f has finitely many faces.

Proof. Note that the set $\mathcal{K}$ in Lemma 5.1 depends only on the value of the function f at a and the set of its faces. Define the set of functions $\mathcal{K}_n^{\uparrow}$ (resp. $\mathcal{K}_n^{\downarrow}$) by the set of faces $\{(l_k, h_k) : k \in \mathcal{Z}_1^{n+1}\} \cup \{(\tilde{l}_n, c_n)\}$ (resp. $\{(l_k, h_k) : k \in \mathcal{Z}_1^{n+1}\} \cup \{(\tilde{l}_n, 0)\}$) and the starting value $f(a)$ as in Lemma 5.1. Let $f_n^{\uparrow}$ (resp. $f_n^{\downarrow}$) be the unique convex function in $\mathcal{K}_n^{\uparrow}$ (resp. $\mathcal{K}_n^{\downarrow}$) constructed in Lemma 5.1.

For each $n \in \mathbb{N}$, $f_{n+1}^{\downarrow}$ and $f_n^{\downarrow}$ share all but a single face, which has nonnegative slope in $f_{n+1}^{\downarrow}$ and a horizontal slope in $f_n^{\downarrow}$. Hence, replacing this face in $f_{n+1}^{\downarrow}$ with a horizontal one yields a smaller (possibly non-convex) continuous piecewise linear function ϕ. Applying Lemma 5.1 to ϕ produces a convex function ϕ_* satisfying $f_{n+1}^{\downarrow} \ge \phi_*$ and $\phi_*(a) = f(a)$ with faces equal to those of $f_n^{\downarrow}$. Since $f_n^{\downarrow}$ is also convex and satisfies $f_n^{\downarrow}(a) = f(a)$, we must have $\phi_* = f_n^{\downarrow}$ implying the inequality in (a).

To establish (b), construct a function ψ by replacing the face $(\tilde{l}_n, c_n)$ in $f_n^{\uparrow}$ with the faces $(\tilde{l}_{n+1}, c_{n+1})$ and (l_{n+1}, h_{n+1}) sorted by increasing slope (note $l_{n+1} + \tilde{l}_{n+1} = \tilde{l}_n$). More precisely, if $(a', a' + \tilde{l}_n) \subset [a,b]$ is the interval corresponding to the face $(\tilde{l}_n, c_n)$ in $f_n^{\uparrow}$, for $t \in [a,b]$ we set

$$\varphi(t) = \begin{cases} h_{n+1} \min\{(t-a')^+/l_{n+1}, 1\} + c_{n+1} \min\{(t-a'-l_{n+1})^+/\tilde{l}_{n+1}, 1\}; & \frac{h_{n+1}}{l_{n+1}} \le \frac{c_{n+1}}{\tilde{l}_{n+1}}, \\ c_{n+1} \min\{(t-a')^+/\tilde{l}_{n+1}, 1\} + h_{n+1} \min\{(t-a'-\tilde{l}_{n+1})^+/l_{n+1}, 1\}; & \frac{h_{n+1}}{l_{n+1}} > \frac{c_{n+1}}{\tilde{l}_{n+1}}. \end{cases}$$

By the inequality $c_n \ge c_{n+1} + h_{n+1}$, the graph of φ on the interval $(a', a' + \tilde{l}_n)$ is below the line segment $t \mapsto c_n(t-a')/\tilde{l}_n$. We then define the continuous piecewise linear function

$$\psi(t) = \begin{cases} f_n^{\uparrow}(t) & t \in [a, a'], \\ f_n^{\uparrow}(a') + \varphi(t) & t \in (a', a' + \tilde{l}_n), \\ f_n^{\uparrow}(t) + h_{n+1} + c_{n+1} - c_n & t \in [a' + \tilde{l}_n, b], \end{cases}$$

which clearly satisfies $f_n^{\uparrow} \ge \psi$. Furthermore, the faces of ψ coincide with those of $f_{n+1}^{\uparrow}$. Thus, applying Lemma 5.1 to ψ yields $\psi \ge f_{n+1}^{\uparrow}$, implying (b).

Recall that a face (l_k, h_k) of f satisfies $l_k = b_k - a_k$ and $h_k = f(b_k) - f(a_k)$ for any $k \in \mathbb{N}$, where $(a_k, b_k) \subset [a,b]$. Let g_n be the piecewise linear function defined by truncating the series in (5.1) at n:

$$g_n(t) = f(a) + \sum_{k=1}^{n} h_k \min\{(t-a_k)^+/l_k, 1\}, \qquad t \in [a,b].$$

By construction, $g_n + \tilde{h}_n \ge f \ge g_n$ and $g_n(b) + \tilde{h}_n = f(b)$, where $\tilde{h}_n = \sum_{k=n+1}^{\infty} h_k$, implying $\|f - g_n\|_\infty = \tilde{h}_n$. Since the set $[a,b] \setminus \bigcup_{k=1}^{n} [a_k, b_k]$ consists of at most $n+1$ disjoint intervals, a representation of g_n exists with at most $2n+1$ faces. Moreover the slopes of g_n over all those intervals are equal to zero. Sorting the faces of g_n by increasing slope yields $f_n^{\downarrow}$. By Lemma 5.1, the second inequality in (c), $f \ge g_n \ge f_n^{\downarrow}$, holds.

We now establish the inequality $f_n^{\uparrow} \ge f$ for all $n \in \mathbb{N}$. First note that $f_n^{\uparrow} \ge f_n^{\downarrow}$ for any $n \in \mathbb{N}$. Indeed, replacing the face $(\tilde{l}_n, c_n)$ in $f_n^{\uparrow}$ with $(0, c_n)$ yields a smaller function with the same faces as $f_n^{\downarrow}$. Hence the convexity of $f_n^{\downarrow}$ and Lemma 5.1 imply the inequality

$f_n^\uparrow \geq f_n^\downarrow$. By (b), $f_n^\uparrow \geq \lim_{k\to\infty} f_k^\downarrow$. Hence, part (c) follows if we show that $\lim_{k\to\infty} f_k^\downarrow = f$ pointwise.

For $k \in \mathbb{N}$ and $n \geq k$, define $a'_{k,n}$ and a'_k by the formulae:

$$a'_{k,n} = a + \sum_{j\in\mathcal{Z}_1^k} l_j \cdot \mathbb{1}_{\{h_j/l_j\}}(h_k/l_k) + \sum_{j\in\mathcal{Z}_1^{n+1}} l_j \cdot \mathbb{1}_{(h_j/l_j,\infty)}(h_k/l_k) + \sum_{j\in\mathcal{Z}_{n+1}^\infty} l_j,$$
$$a'_k = a + \sum_{j\in\mathcal{Z}_1^k} l_j \cdot \mathbb{1}_{\{h_j/l_j\}}(h_k/l_k) + \sum_{j\in\mathcal{Z}_1^\infty} l_j \cdot \mathbb{1}_{(h_j/l_j,\infty)}(h_k/l_k).$$

It is clear that $a'_{k,n} \searrow a'_k$ as $n \to \infty$. Moreover, for $t \in [a,b]$, we have

$$f_n^\downarrow(t) = f(a) + \sum_{k=1}^n h_k \min\{(t-a'_{k,n})^+/l_k, 1\}, \quad f(t) = f(a) + \sum_{k=1}^\infty h_k \min\{(t-a'_k)^+/l_k, 1\}.$$

In other words, for any $k \in \mathbb{N}$ and $k \geq n$, $a'_{k,n}$ (resp. a'_k) is the left endpoint of the interval corresponding to the face (l_k, h_k) in a representation of $f_n^\uparrow$ (resp. f). Thus, for fixed $t \in [a,b]$, the terms $h_k \min\{(t-a'_{k,n})^+/l_k, 1\}$ are a monotonically increasing sequence with limit $h_k \min\{(t-a'_k)^+/l_k, 1\}$ as $n \to \infty$. By the monotone convergence theorem applied to the counting measure we deduce that $f_n^\downarrow \to f$ pointwise, proving (c).

Since $\|f_n^\uparrow - f_n^\downarrow\|_\infty \geq f_n^\uparrow(b) - f_n^\downarrow(b) = c_n$, claim in (d) follows if we prove the reverse inequality. Without loss of generality, the first face of $f_n^\downarrow$ in the chronological order is $(\tilde{l}_n, 0)$. Replace this face with $(\tilde{l}_n, c_n)$ to obtain a piecewise linear function $u_n(t) = f(a) + c_n t/\tilde{l}_n \mathbb{1}_{[0,\tilde{l}_n]}(t) + (f_n^\downarrow(t) + c_n)\mathbb{1}_{(\tilde{l}_n,1]}(t)$. Since u_n has the same faces as $f_n^\uparrow$, Lemma 5.1 implies $u_n \geq f_n^\uparrow$. Hence (d) follows from (c): $\|f_n^\uparrow - f_n^\downarrow\|_\infty \leq \|u_n - f_n^\downarrow\|_\infty = c_n = f_n^\uparrow(b) - f_n^\downarrow(b)$. $\square$

Define $\tau_{[a,b]}(g) = \inf\{t \in [a,b] : \min\{g(t), g(t-)\} = \inf_{r\in[a,b]} g(r)\}$ for any càdlàg function $g : [a,b] \to \mathbb{R}$. Consider now the problem of sandwiching a convex function f with both positive and negative slopes. Splitting f into two convex functions, the pre-minimum $f^\leftarrow$ and post-minimum $f^\rightarrow$ (see Proposition 5.4 for definition), it is natural to apply Proposition 5.2 directly to each of them and attempt to construct the bounds for f by concatenating the two sandwiches. However, this strategy does not yield an upper and lower bounds for f for the following reason: since we may *not* assume to have access to the minimal value $f(s)$ of the function f, the concatenated sandwich cannot be anchored at $f(s)$ (note that we may and do assume that we know the time $s = \tau_{[a,b]}(f)$ of the minimum of f). Proposition 5.4 is the analogue of Proposition 5.2 for general piecewise linear convex functions.

Proposition 5.4. *Let f be a piecewise linear convex function on $[a,b]$ with infinitely many faces of both signs. Set $s = \tau_{[a,b]}(f)$ and let $f^\leftarrow : t \mapsto f(s-t) - f(s)$ and $f^\rightarrow : t \mapsto f(s+t) - f(s)$ be the pre- and post-minimum functions, defined on $[0, s-a]$ and $[0, b-s]$ with sets of faces $\{(l_n^\leftarrow, h_n^\leftarrow) : n \in \mathbb{N}\}$ and $\{(l_n^\rightarrow, h_n^\rightarrow) : n \in \mathbb{N}\}$ of nonnegative slope, respectively. Let the constants $c_n^\leftarrow$ and $c_n^\rightarrow$ be as in Proposition 5.2 for $f^\leftarrow$ and $f^\rightarrow$, respectively. For any $n, m \in \mathbb{N}$, define the functions $f_{n,m}^\uparrow, f_{n,m}^\downarrow : [a,b] \to \mathbb{R}$ by*

$$\begin{aligned} f_{n,m}^\uparrow(t) &= f(a) + [(f^\leftarrow)_n^\downarrow((s-t)^+) - (f^\leftarrow)_n^\downarrow(s-a)] + (f^\rightarrow)_m^\uparrow((t-s)^+), \\ f_{n,m}^\downarrow(t) &= f(a) + [(f^\leftarrow)_n^\downarrow((s-t)^+) - (f^\leftarrow)_n^\downarrow(s-a) - c_n^\leftarrow] + (f^\rightarrow)_m^\downarrow((t-s)^+). \end{aligned} \tag{5.2}$$

For any $c \in \mathbb{R}$, let T_c be the linear tilting defined in Subsection 3.1 above. Set $s_c = \tau_{[a,b]}(T_c f)$ and $s_{n,m} = \tau_{[a,b]}(T_c f_{n,m}^\downarrow)$. Then the following statements hold for any $n, m \in \mathbb{N}$:
(a) *$T_c f_{n,m}^\uparrow \geq T_c f \geq T_c f_{n,m}^\downarrow$;*
(b) *$\|T_c f_{n,m}^\uparrow - T_c f_{n,m}^\downarrow\|_\infty = f_{n,m}^\uparrow(b) - f_{n,m}^\downarrow(b) = c_n^\leftarrow + c_m^\rightarrow$;*
(c) *$s_{n,m} \leq s_c \leq s_{n,m} + \tilde{l}_n^\leftarrow$ (resp. $s_{n,m} - \tilde{l}_m^\rightarrow \leq s_c \leq s_{n,m}$) if $c \geq 0$ (resp. $c < 0$), where we denote $\tilde{l}_n^\leftarrow = \sum_{k=n+1}^\infty l_k^\leftarrow$ (resp. $\tilde{l}_m^\rightarrow = \sum_{k=m+1}^\infty l_k^\rightarrow$);*

(d) $T_c f^{\uparrow}_{n,m} \geq T_c f^{\uparrow}_{n,m+1}$ *and* $T_c f^{\uparrow}_{n,m} \geq T_c f^{\uparrow}_{n+1,m}$*;*
(e) $T_c f^{\downarrow}_{n,m+1} \geq T_c f^{\downarrow}_{n,m}$ *and* $T_c f^{\downarrow}_{n+1,m} \geq T_c f^{\downarrow}_{n,m}$*.*

Remark 5.5. (i) The upper and lower bounds $f^{\uparrow}_{n,m}$ and $f^{\downarrow}_{n,m}$, restricted to $[s,b]$, have the same "derivative" as the corresponding bounds in Proposition 5.2. The behaviour of $f^{\uparrow}_{n,m}$ and $f^{\downarrow}_{n,m}$ on $[a,s]$ differs from that of the bounds in Proposition 5.2. Indeed, the lower bound $f^{\downarrow}_{n,m}$ does not start with value $f(a)$ because the slopes of the faces of f may become arbitrarily negative as t approcheas a. Thus, $f^{\downarrow}_{n,m}$ is defined as a vertical translation of $f^{\uparrow}_{n,m}$ on $[a,s]$.
(ii) Note that all bounds in Proposition 5.4 hold uniformly in $c \in \mathbb{R}$, with the exception of part (c) which depends on the sign of c. Proposition 5.4 extends easily to the case of a function f without infinitely many faces of both signs. Moreover, if either $n = 0$ or $m = 0$, then as in Remark 5.3(vi) above, Proposition 5.4 still holds.

Proof. Since $T_c g_1 - T_c g_2 = g_1 - g_2$ for any functions $g_1, g_2 : [a,b] \to \mathbb{R}$, it suffices to prove the claims (a), (b), (d) and (e) for $c = 0$.
(a) Let $\tilde{h}^{\leftarrow}_n = \sum_{k=n+1}^{\infty} h^{\leftarrow}_k$, then Remark 5.3(ii) gives $\|f^{\leftarrow} - (f^{\leftarrow})^{\downarrow}_n\|_\infty \leq \tilde{h}^{\leftarrow}_n$, so the inequality $c^{\leftarrow}_n \geq \tilde{h}^{\leftarrow}_n$ implies $(f^{\leftarrow})^{\downarrow}_n + \tilde{h}^{\leftarrow}_n \geq f^{\leftarrow} \geq (f^{\leftarrow})^{\downarrow}_n + \tilde{h}^{\leftarrow}_n - c^{\leftarrow}_n$. Note that $f(t) = f(a) + f^{\leftarrow}(s-t) - f^{\leftarrow}(s-a)$ for $t \in [a,s]$. Moreover, since $\tilde{h}^{\leftarrow}_n - f^{\leftarrow}(s-a) = -(f^{\leftarrow})^{\downarrow}_n(s-a)$, (5.2) yields $(f^{\leftarrow})^{\uparrow}_{n,m}(t) \geq f(t) \geq (f^{\leftarrow})^{\downarrow}_{n,m}(t)$ for $t \in [a,s]$. Similarly, Proposition 5.2 and the inequality $(f^{\rightarrow})^{\uparrow}_m \geq f^{\rightarrow} \geq (f^{\rightarrow})^{\downarrow}_m$ show that the inequalities in (a) also hold on $[s,b]$.
(b) The equalities follow from the definition in (5.2) and Proposition 5.2(d).
(c) Note that the minimum of $T_c f$ and $T_c f^{\downarrow}_{n,m}$ is attained after all the faces of negative slope. In terms of the functions f and $f^{\downarrow}_{n,m}$, the minimum takes place after all the faces with slopes less than $-c$. Put differently,

$$s_c = \sum_{k=1}^{\infty} l^{\leftarrow}_k \cdot \mathbb{1}_{(-\infty, h^{\leftarrow}_k / l^{\leftarrow}_k)}(c) + \sum_{k=1}^{\infty} l^{\rightarrow}_k \cdot \mathbb{1}_{(-\infty, -h^{\rightarrow}_k / l^{\rightarrow}_k)}(c)$$

$$s_{n,m} = \sum_{k=1}^{n} l^{\leftarrow}_k \cdot \mathbb{1}_{(-\infty, h^{\leftarrow}_k / l^{\leftarrow}_k)}(c) + \tilde{l}^{\leftarrow}_n \cdot \mathbb{1}_{(-\infty,0)}(c) + \sum_{k=1}^{m} l^{\rightarrow}_k \cdot \mathbb{1}_{(-\infty, -h^{\rightarrow}_k / l^{\rightarrow}_k)}(c) + \tilde{l}^{\rightarrow}_m \cdot \mathbb{1}_{(-\infty,0)}(c).$$

If $c \geq 0$, then all the terms coming from $f^{\rightarrow}$ are 0 and so is $\tilde{l}^{\leftarrow}_n \cdot \mathbb{1}_{(-\infty,0)}(c)$, implying the first claim in (c). A similar analysis for $c < 0$ gives the corresponding claim.
(d) The result follows from the definition in (5.2) and Proposition 5.2(a)&(b).
(e) The result follows from the definition in (5.2) and Proposition 5.2(a)&(b)&(d). □

Corollary 5.6. *Let* $f : [a,b] \to \mathbb{R}$ *be a piecewise linear convex function with faces* $\{(l_k, h_k) : k \in \mathbb{N}\}$*. Pick* $n \in \mathbb{N}$ *and let* $g : [a,b] \to \mathbb{R}$ *be the piecewise linear convex function with faces* $\{(l_k, h_k) : k \in \mathcal{Z}_1^{n+1}\} \cup \{(\tilde{l}_n, 0)\}$ *(recall* $\tilde{l}_n = \sum_{m=n+1}^{\infty} l_m$*), satisfying* $g(a) = f(a)$*. Then the following inequality holds:*

$$\|f - g\|_\infty \leq \max\left\{ \sum_{k=n+1}^{\infty} h^-_k, \sum_{k=n+1}^{\infty} h^+_k \right\}. \tag{5.3}$$

Proof. Let $m_1 = \sum_{k=1}^{n} \mathbb{1}_{(-\infty,-0)}(h_k)$ and $m_2 = n - m_1$. Define $c^{\leftarrow}_{m_1} = \sum_{k=n+1}^{\infty} h^-_k$ and $c^{\rightarrow}_{m_2} = \sum_{k=n+1}^{\infty} h^+_k$. Then, using the notation from Proposition 5.4, the following holds

$$g(t) = f(a) + [(f^{\leftarrow})^{\downarrow}_{m_1}((s-t)^+) - (f^{\leftarrow})^{\downarrow}_{m_1}(s-a)] + (f^{\rightarrow})^{\downarrow}_{m_2}((t-s)^+) \quad \text{for any } t \in [a,b].$$

Moreover, by Propositions 5.2 and 5.4, we have $g + c^{\rightarrow}_{m_2} \geq f^{\uparrow}_{m_1,m_2} \geq f \geq f^{\downarrow}_{m_1,m_2} = g - c^{\leftarrow}_{m_1}$ and (5.3) follows. □

Remark 5.7. The proof of Corollary 5.6 shows that using g to construct lower and upper bounds on f yields poorer estimates than the ones in Proposition 5.4. Indeed, the upper (resp. lower) bound in Proposition 5.4 is smaller than (resp. equal to) $g + c_{m_2}^{\rightarrow}$ (resp. $g - c_{m_1}^{\leftarrow}$).

5.2 Convex minorant of stable meanders

Proof of Theorem 3.4. (a) This is a consequence of Theorem 2.7. Indeed, by the scaling property of the law of ξ_{1-n}^{me}, each S_n has the desired law. Moreover, $(\ell_m^{\mathrm{me}})_{m\in\mathbb{N}}$ is a stick-breaking process based on $\mathrm{Beta}(1,\rho)$. By the definition of the stick-breaking process, the sequence $(U_n)_{n\in\mathcal{Z}^0}$ has the required law. The independence structure is again implied by Theorem 2.7, since the conditional law of $(S_n)_{n\in\mathcal{Z}^0}$, given $(\ell_m^{\mathrm{me}})_{m\in\mathbb{N}}$, no longer depends on the lengths of the sticks.
(b) The recursion (3.2) follows form the definition in (3.1). Since M_n is independent of (U_n, S_n),the Markov property is a direct consequencel of (a) and [Kal02, Prop. 7.6]. The stationarity of $((S_n, U_n))_{n\in\mathcal{Z}^0}$ in (a) and the identity

$$M_n = \sum_{m\in\mathcal{Z}^n}\Bigg(\prod_{k\in\mathcal{Z}_{m+1}^n}(1-U_k)^{1/\alpha}\Bigg)U_m^{1/\alpha}S_m, \quad n\in\mathcal{Z}^1,$$

imply that $(M_n)_{n\in\mathcal{Z}^1}$ is also stationary.
(c) The perpetuity follows from (b). It has a unique solution by [BDM16, Thm 2.1.3]. ☐

Remark 5.8. A result analogous to Theorem 3.4 for stable processes and their convex minorants holds (see [GCMUB19, Prop. 1]). In fact, the proof in [GCMUB19] implies a slightly stronger result, namely, a perpetuity for the triplet $(Z_T, \underline{Z}_T, \tau_{[0,T]}(Z))$.

The following result, which may be of independent interest, is a consequence of Theorem 3.4. Parts of it were used to numerically test our algorithms in Subsection 4.2 above.

Corollary 5.9. *Consider some* $S\sim\mathcal{S}^+(\alpha,\rho)$.
(a) *If* $\alpha>1$, *then* $\mathbb{E}Z_1^{\mathrm{me}} = \frac{\Gamma(1/\alpha)\Gamma(1+\rho)}{\Gamma(\rho+1/\alpha)}\mathbb{E}S = \frac{\Gamma(1/\alpha)\Gamma(1-1/\alpha)}{\Gamma(\rho+1/\alpha)\Gamma(1-\rho)}$.
(b) *For any* (α,ρ) *we have* $\mathbb{E}\big[(Z_1^{\mathrm{me}})^{-\alpha\rho}\big] = \Gamma(1+\rho)/\Gamma(1+\alpha\rho)$.
(c) *For* $\gamma>0$ *let* $k_{\alpha,\gamma} = \big((1+\gamma/\alpha)^{\min\{\gamma^{-1},1\}}-1\big)^{-\max\{\gamma,1\}}$. *Then we have*

$$\rho\mathbb{E}[S^\gamma] \le \mathbb{E}[(Z_1^{\mathrm{me}})^\gamma]\frac{\Gamma(\rho+\gamma/\alpha)}{\Gamma(\rho)\Gamma(1+\gamma/\alpha)} \le \min\{\rho k_{\alpha,\gamma},1\}\mathbb{E}[S^\gamma].$$

(d) *For* $\gamma\in(0,\alpha\rho)$, *we have* $\mathbb{E}\big[(Z_1^{\mathrm{me}})^{-\gamma}\big] \le \Gamma(1+\rho)\Gamma(1-\gamma/\alpha)\mathbb{E}\big[S^{-\gamma}\big]/\Gamma(1+\rho-\gamma/\alpha)$.

Proof. (a) Recall that $\mathbb{E}[V^r] = \frac{\Gamma(\theta_1+r)\Gamma(\theta_1+\theta_2)}{\Gamma(\theta_1+\theta_2+r)\Gamma(\theta_1)}$ for any $V\sim\mathrm{Beta}(\theta_1,\theta_2)$. Taking expectations in (3.2) and solving for $\mathbb{E}Z_1^{\mathrm{me}}$ gives the formula.
(b) Let $V\sim\mathrm{Beta}(\rho,1-\rho)$ be independent of Z^{me} and denote the supremum of Z by $\overline{Z}_1 = \sup_{t\in[0,1]}Z_t$. Then [Ber96, Cor. VIII.4.17] implies that $V^{1/\alpha}Z_1^{\mathrm{me}} \overset{d}{=} \overline{Z}_1$. By Breiman's lemma [BDM16, Lem. B.5.1],

$$1 = \lim_{x\to\infty}\frac{\mathbb{P}\left(V^{-1/\alpha}(Z_1^{\mathrm{me}})^{-1}>x\right)}{\mathbb{E}\big[(Z_1^{\mathrm{me}})^{-\alpha\rho}\big]\mathbb{P}\big(V^{-1/\alpha}>x\big)} = \lim_{x\to\infty}\frac{\mathbb{P}(\overline{Z}_1^{-1}>x)\Gamma(1+\rho)\Gamma(1-\rho)}{\mathbb{E}\big[(Z_1^{\mathrm{me}})^{-\alpha\rho}\big]x^{-\alpha\rho}}.$$

Since [Bin73, Thm 3a] gives $1=\lim_{x\to\infty}\Gamma(1-\rho)\Gamma(1+\alpha\rho)\mathbb{P}\big(\overline{Z}_1^{-1}>x\big)/x^{-\alpha\rho}$, we get (b).
(c) Note $\mathbb{E}\big[\big(Z_1^{\mathrm{me}}\big)^\gamma\big] = \Gamma(\rho)\Gamma(1+\frac{\gamma}{\alpha})\mathbb{E}\big[\overline{Z}_1^\gamma\big]/\Gamma(\rho+\frac{\gamma}{\alpha})$ for $\gamma>-\alpha\rho$ since V and Z_1^{me} are independent and $V^{1/\alpha}Z_1^{\mathrm{me}} \overset{d}{=} \overline{Z}_1$. Hence, we need only prove that

$$\rho\mathbb{E}[S^\gamma] \le \mathbb{E}[\overline{Z}_1^\gamma] \le \min\{\rho k_{\alpha,\rho},1\}\mathbb{E}[S^\gamma]. \tag{5.4}$$

Recall that for a nonnegative random variable ϑ we have $\mathbb{E}[\vartheta^\gamma]=\int_0^\infty \gamma x^{\gamma-1}\mathbb{P}(\vartheta>x)dx$. Since $\mathbb{P}(\overline{Z}_1>x)\ge\mathbb{P}(Z_1^+>x)=\rho\mathbb{P}(S>x)$, we get $\mathbb{E}[\overline{Z}_1^\gamma]\ge\rho\mathbb{E}[S^\gamma]$. Next, fix any $x>0$ and let $\sigma_x=\inf\{t>0:Z_t>x\}$. By the strong Markov property, the process Z' given by $Z'_t=Z_{t+\sigma_x}-Z_{\sigma_x}$, $t>0$, has the same law as Z and is independent of σ_x. Thus, we have

$$\begin{aligned}\mathbb{P}(\overline{Z}_1>x)&=\mathbb{P}(Z_1>x)+\mathbb{P}(\overline{Z}_1>x,Z_1\le x)\le\mathbb{P}(Z_1>x)+\mathbb{P}(\sigma_x<1,Z'_{1-\sigma_x}\le 0)\\&=\mathbb{P}(Z_1>x)+(1-\rho)\mathbb{P}(\sigma_x<1)=\mathbb{P}(Z_1>x)+(1-\rho)\mathbb{P}(\overline{Z}_1>x),\end{aligned}$$

implying $\mathbb{P}(\overline{Z}_1>x)\le\mathbb{P}(S>x)$. Hence the same argument gives $\mathbb{E}[S^\gamma]\ge\mathbb{E}[\overline{Z}_1^\gamma]$. Note

$$k_{\alpha,\gamma}=\begin{cases}\big((1+\gamma/\alpha)^{1/\gamma}-1\big)^{-\gamma} & \text{if } \gamma>1\\ \alpha/\gamma & \text{if } \gamma\le 1.\end{cases}$$

The last inequality $\mathbb{E}[\overline{Z}_1^\gamma]\le\rho k_{\alpha,\gamma}\mathbb{E}[S^\gamma]$ in (5.4) follows from the perpetuity for the law of $\overline{Z}_1$ in [GCMUB19, Eq. (2.1)] and the inequality in the proof of [BDM16, Lem. 2.3.1].
(d) Note that (3.2) and the Mellin transform of S (see [UZ99, Sec. 5.6]) imply

$$\begin{aligned}\mathbb{E}\big[\big(Z_1^{\mathrm{me}}\big)^{-\gamma}\big]&=\mathbb{E}\big[\big(U^{1/\alpha}Z_1^{\mathrm{me}}+(1-U)^{1/\alpha}S\big)^{-\gamma}\big]\le\mathbb{E}\big[\big(1-U\big)^{-\gamma/\alpha}\big]\mathbb{E}\big[S^{-\gamma}\big]\\&=\frac{\Gamma(1-\frac{\gamma}{\alpha})\Gamma(1+\rho)}{\Gamma(1+\rho-\frac{\gamma}{\alpha})}\mathbb{E}\big[S^{-\gamma}\big]=\frac{\Gamma(1+\rho)\Gamma(1-\gamma)}{\Gamma(1+\rho-\frac{\gamma}{\alpha})}\frac{\Gamma(1-\frac{\gamma}{\alpha})\Gamma(1+\frac{\gamma}{\alpha})}{\Gamma(1-\gamma\rho)\Gamma(1+\gamma\rho)}<\infty.\ \square\end{aligned}$$

Remark 5.10. (i) Bernoulli's inequality implies $k_{\alpha,\gamma}\le\alpha^\gamma$ for $\gamma>1$.
(ii) From $V^{1/\alpha}Z_1^{\mathrm{me}}\overset{d}{=}\overline{Z}_1$ we get $\mathbb{E}\overline{Z}_1=\alpha\rho\mathbb{E}S=\frac{\alpha\Gamma(1-1/\alpha)}{\Gamma(\rho)\Gamma(1-\rho)}$ when $\alpha>1$. Similarly, for $\gamma\in(0,\alpha\rho)$, we have $\mathbb{E}\big[\overline{Z}_1^{-\gamma}\big]\le\rho(1+(1-\rho)/(\rho-\gamma/\alpha))\mathbb{E}\big[S^{-\gamma}\big]/(1-\gamma/\alpha)$ by the proof in (d) applied to the perpetuity in [GCMUB19, Thm 1].
(iii) Note that equation (3.2) and the Grincevičius-Grey theorem [BDM16, Thm 2.4.3] give $\lim_{x\to\infty}\frac{1+\rho}{\rho}\mathbb{P}(U^{1/\alpha}S>x)/\mathbb{P}(Z_1^{\mathrm{me}}>x)=1$. Next, Breiman's lemma [BDM16, Lem. B.5.1] gives $\lim_{x\to\infty}(1+\rho)\mathbb{P}(U^{1/\alpha}S>x)/\mathbb{P}(S>x)=1$. Hence, [UZ99, Sec. 4.3] gives the assymptotic tail behaviour $\lim_{x\to\infty}\mathbb{P}(Z_1^{\mathrm{me}}>x)/x^{-\alpha}=\Gamma(\alpha)\sin(\pi\alpha\rho)/(\pi\rho)$ (cf. [DS10]).

5.3 Computational complexity

The aim of the subsection is to analyse the computational complexity of the εSS algorithms from Section 3 and the exact simulation algorithm of the indicator of certain events (see Subsection 4.1.1 above). Each algorithm in Section 3 constructs an approximation of a random element Λ in a metric space $(\mathbb{X},d)$, given by a sequence $(\Lambda_n)_{n\in\mathbb{N}}$ in $(\mathbb{X},d)$ and upper bounds $(\Delta_n)_{n\in\mathbb{N}}$ satisfying $\Delta_n\ge d(\Lambda,\Lambda_n)$ for all $n\in\mathbb{N}$. The εSS algorithm terminates as soon as $\Delta_n<\varepsilon$. Moreover, the computational complexity of constructing the finite sequences $\Lambda_1,\ldots,\Lambda_n$ and $\Delta_1,\ldots,\Delta_n$ is linear in n for the algorithms in Section 3. For $\varepsilon>0$, the runtime of the εSS algorithm is thus proportional to $N^\Lambda(\varepsilon)=\inf\{n\in\mathbb{N}:\Delta_n<\varepsilon\}$ since the element $\Lambda_{N^\Lambda(\varepsilon)}$ is the output of the εSS. Proposition 5.13 below shows that for all the algorithms in Section 3, we have $\Delta_n\to 0$ a.s. as $n\to\infty$, implying $N^\Lambda(\varepsilon)<\infty$ a.s. for $\varepsilon>0$.

The exact simulation algorithm of an indicator $\mathbb{1}_A(\Lambda)$, for some subset $A\subset\mathbb{X}$ satisfying $\mathbb{P}(\Lambda\in\partial A)=0$, has a complexity proportional to

$$B^\Lambda(A)=\inf\{n\in\mathbb{N}:\Delta_n<d(\Lambda_n,\partial A)\},\tag{5.5}$$

since $\mathbb{1}_A(\Lambda)=\mathbb{1}_A(\Lambda_{B^\Lambda(A)})$ a.s. Indeed, if $d(\Lambda,\Lambda_n)\le\Delta_n<d(\Lambda_n,\partial A)$ then $\mathbb{1}_A(\Lambda)=\mathbb{1}_A(\Lambda_n)$. Moreover, $B^\Lambda(A)<\infty$ a.s. since $d(\Lambda_n,\partial A)\to d(\Lambda,\partial A)>0$ and $\Delta_n\to 0$ a.s. The next lemma provides a simple connection between the tail probabilities of the complexities $N^\Lambda(\varepsilon)$ and $B^\Lambda(A)$. It will play a key role in the proof of Theorem 3.6.

Lemma 5.11. *Let $A \subset \mathbb{X}$ satisfy $\mathbb{P}(\Lambda \in \partial A) = 0$. Assume that some positive constants r_1, r_2, K_1 and K_2 and a nonincreasing function $q : \mathbb{N} \to [0,1]$ with $\lim_{n\to\infty} q(n) = 0$ satisfy*

$$\mathbb{P}(d(\Lambda, \partial A) < \varepsilon) \le K_1 \varepsilon^{r_1}, \quad \mathbb{P}(N^\Lambda(\varepsilon) > n) \le K_2 \varepsilon^{-r_2} q(n), \tag{5.6}$$

for all $\varepsilon \in (0,1]$ and $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$, we have

$$\mathbb{P}(B^\Lambda(A) > n) \le (K_1 + 2^{r_2} K_2) q(n)^{r_1/(r_1+r_2)}. \tag{5.7}$$

Proof. Note that $\{d(\Lambda, \partial A) \ge 2\varepsilon\} \subset \{N^\Lambda(\varepsilon) \ge B^\Lambda(A)\}$ for any $\varepsilon > 0$ since $d(\Lambda, \partial A) \ge 2\varepsilon$ and $d(\Lambda, \Lambda_n) \le \Delta_n < \varepsilon$ imply $\Delta_n < \varepsilon < d(\Lambda_n, \partial A)$. Thus, if we define, for each $n \in \mathbb{N}$, $\varepsilon_n = q(n)^{1/(r_1+r_2)}/2 \in (0, 1/2)$, we get

$$\begin{aligned}
\mathbb{P}(B^\Lambda(A) > n) &= \mathbb{P}(B^\Lambda(A) > n, d(\Lambda, \partial A) < 2\varepsilon_n) + \mathbb{P}(B^\Lambda(A) > n, d(\Lambda, \partial A) \ge 2\varepsilon_n) \\
&\le \mathbb{P}(d(\Lambda, \partial A) < 2\varepsilon_n) + \mathbb{P}(N^\Lambda(\varepsilon_n) > n) \\
&\le 2^{r_1} K_1 \varepsilon_n^{r_1} + K_2 \varepsilon_n^{-r_2} q(n) = (K_1 + 2^{r_2} K_2) q(n)^{r_1/(r_1+r_2)}.
\end{aligned}$$

□

5.3.1 Complexities of Algorithm 3 and the exact simulation algorithm of the indicator $\mathbb{1}_{\{Z_1^{\text{me}} > x\}}$

Recall the definition $c_{-n} = L_n^{1/\alpha} D_n$, $n \in \mathcal{Z}^0$, in (3.3). The dominating process $(D_n)_{n\in\mathcal{Z}^1}$, defined in Appendix A (see (A.4) below), is inspired by the one in [GCMUB19]. In fact, the sampling of the process $(D_n)_{n\in\mathcal{Z}^1}$ is achieved by using [GCMUB19, Alg. 2] as explained in the appendix. The computational complexities of Algorithms 2 and 3 are completely determined by

$$N(\varepsilon) = \sup\{n \in \mathcal{Z}^0 : L_n^{1/\alpha} D_n < \varepsilon\}, \quad \varepsilon > 0. \tag{5.8}$$

It is thus our aim to develop bounds on the tail probabilities of $N(\varepsilon)$, which requires the analysis of the sampling algorithm in [GCMUB19, Alg. 2]. We start by proving that the error bounds $(c_m)_{m\in\mathbb{N}}$ are strictly decreasing.

Lemma 5.12. *The sequence $(c_m)_{m\in\mathbb{N}}$, given by $c_m = L_{-m}^{1/\alpha} D_{-m} > 0$, is strictly decreasing: $c_m > c_{m+1}$ a.s. for all $m \in \mathbb{N}$.*

Proof. Fix $n \in \mathcal{Z}^1$ and note that by (A.4)

$$\begin{aligned}
L_n^{1/\alpha} D_n &= e^{\sup_{k\in\mathcal{Z}^{n+1}} W_k} E_n, \quad \text{where} \\
E_n &= \frac{e^{(d-\delta)\chi_n + n\delta}}{1 - e^{\delta - d}} + \sum_{k\in\mathcal{Z}^n_{\chi_n}} e^{(k+1)d} U_k^{1/\alpha} S_k.
\end{aligned}$$

The random walk $(W_k)_{k\in\mathcal{Z}^1}$, the random variables χ_n, $n \in \mathcal{Z}^1$, and the constants d, δ are given in Appendix A below. The pairs (U_k, S_k) are given in Theorem 3.4 (see also Algorithm 3). Since $\sup_{k\in\mathcal{Z}^n} W_k \le \sup_{k\in\mathcal{Z}^{n+1}} W_k$ for all $n \in \mathcal{Z}^1$, it suffices to show that $E_{n-1} < E_n$. From the definition in (A.2) of χ_n it follows that

$$\sum_{k\in\mathcal{Z}^{\chi_n}_{\chi_{n-1}}} e^{(k+1)d} U_k^{1/\alpha} S_k \le \sum_{k\in\mathcal{Z}^{\chi_n}_{\chi_{n-1}}} e^{(k+1)d} e^{\delta(n-k-1)} = \frac{e^{(d-\delta)\chi_n + n\delta}\big(1 - e^{(d-\delta)(\chi_{n-1}-\chi_n)}\big)}{1 - e^{\delta-d}}.$$

The inequality in display then yields

$$\begin{aligned}
E_{n-1} - E_n &= \frac{e^{(d-\delta)\chi_{n-1} + (n-1)\delta} - e^{(d-\delta)\chi_n + n\delta}}{1 - e^{\delta-d}} - e^{nd} U_{n-1}^{1/\alpha} S_{n-1} + \sum_{k\in\mathcal{Z}^{\chi_n}_{\chi_{n-1}}} e^{(k+1)d} U_k^{1/\alpha} S_k \\
&\le e^{(d-\delta)\chi_n + n\delta}\big(e^{(d-\delta)(\chi_{n-1}-\chi_n) - \delta} - 1 + 1 - e^{(d-\delta)(\chi_{n-1}-\chi_n)}\big)/(1 - e^{\delta-d}) \\
&= e^{(d-\delta)\chi_n + n\delta} e^{(d-\delta)(\chi_{n-1}-\chi_n)}\big(e^{-\delta} - 1\big)/(1 - e^{\delta-d}) < 0,
\end{aligned}$$

implying $E_{n-1} < E_n$ and concluding the proof. □

We now analyse the tail of $N(\varepsilon)$ defined in (5.8).

Proposition 5.13. *Pick $\varepsilon \in (0,1)$ and let the constants d, δ, γ and η be as in Appendix A. Define the constants $r = (1-e^{\delta-d})/2 > 0$, $m^* = \lfloor \frac{1}{\delta\gamma} \log \mathbb{E}[S^\gamma] \rfloor + 1$ (here $S \sim \mathcal{S}^+(\alpha,\rho)$) and*

$$K = e^{d\eta} + e^{\delta\gamma}(e^{d\eta}-1) \max\left\{ \frac{\mathbb{E}[S^\gamma]}{(1-e^{-\delta\gamma})(1-e^{-\delta\gamma m^*}\mathbb{E}[S^\gamma])}, e^{\delta\gamma m^*} \right\} > 0. \tag{5.9}$$

Then $|N(\varepsilon)|$ has exponential moments: for all $n \in \mathbb{N}$, we have

$$\mathbb{P}(|N(\varepsilon)| > n) \le (K/r^\eta)\varepsilon^{-\eta} e^{-n\min\{\delta\gamma, d\eta\}} \left(\frac{\mathbb{1}_{\mathbb{R}\setminus\{\delta\gamma\}}(d\eta)}{|e^{\delta\gamma - d\eta} - 1|} + n \cdot \mathbb{1}_{\{\delta\gamma\}}(d\eta) \right) \tag{5.10}$$

Proof of Proposition 5.13. Fix $n \in \mathcal{Z}^0$, put $\varepsilon' = -\log((1-e^{\delta-d})\varepsilon/2) = -\log(r\epsilon) > 0$ and let $R_0 = \sup_{m\in\mathcal{Z}^1} W_m$. Since $L_n^{1/\alpha} = \exp(W_n + nd)$, then by (A.4), we have

$$\begin{aligned} L_n^{1/\alpha} D_n &< e^{nd + \sup_{m\in\mathcal{Z}^{n+1}} W_m} \left(\frac{1}{1-e^{\delta-d}} + \sum_{k\in\mathcal{Z}^n} e^{-(n-k-1)d} S_k \right) \\ &\le e^{R_0} \left(\frac{e^{nd}}{1-e^{\delta-d}} + \sum_{k\in\mathcal{Z}^n} e^{(k+1)d} S_k \right). \end{aligned}$$

Assume that $n \le \chi_m$ for some $m \in \mathcal{Z}^1$, then $n \le \chi_m < m$ and thus

$$\sum_{k\in\mathcal{Z}^n} e^{(k+1)d} S_k \le \sum_{k\in\mathcal{Z}^n} e^{(k+1)d} e^{\delta(m-k-1)} = \frac{e^{\delta m} e^{n(d-\delta)}}{1-e^{\delta-d}} < \frac{e^{md}}{1-e^{\delta-d}}.$$

Hence $L_n^{1/\alpha} D_n < 2\exp(R_0 + md)/(1-e^{\delta-d})$. Thus, the choice $m = \lfloor -(\varepsilon' + R_0)/d \rfloor$ gives $L_n^{1/\alpha} D_n < \varepsilon$ where $\lfloor x \rfloor = \sup\{n \in \mathbb{Z} : n \le x\}$ for $x \in \mathbb{R}$. This yields the bound $|N(\varepsilon)| \le |\chi_{\lfloor -(\varepsilon'+R_0)/d \rfloor}|$. Since $(\chi_n)_{n\in\mathcal{Z}^0}$ is a function of $(S_n)_{n\in\mathcal{Z}^0}$ and R_0 is a function of $(U_n)_{n\in\mathcal{Z}^0}$, the sequence $(\chi_n)_{n\in\mathcal{Z}^0}$ is independent of R_0. By [EG00] (see also [GCMUB19, Rem. 4.3]) there exists an exponential random variable E with mean one, independent of $(\chi_n)_{n\in\mathcal{Z}^0}$, satisfying $R_0 \le \eta^{-1}E$ a.s. Since the sequence $(\chi_n)_{n\in\mathcal{Z}^0}$ is nonincreasing, we have $|\chi_{\lfloor -(\varepsilon'+R_0)/d \rfloor}| \le |\chi_{-J}|$ where $J = \lceil (\varepsilon' + \eta^{-1}E)/d \rceil$. By definition (A.2), for any $n \in \mathbb{N}$ we have

$$\begin{aligned} \mathbb{P}(|N(\varepsilon)| > n) &\le \mathbb{P}(|\chi_{-J}| > n) = \mathbb{P}(J \ge n) + \mathbb{P}(J < n, |\chi_{-J}| > n) \\ &= \mathbb{P}(J \ge n) + \mathbb{1}_{(\lceil \varepsilon'/d \rceil, \infty)}(n) \sum_{k=\lceil \varepsilon'/d \rceil}^{n-1} \mathbb{P}(J = k)\mathbb{P}(|\chi_{-k}| > n) \\ &\le e^{(\varepsilon' - (n-1)d)\eta} + \mathbb{1}_{(\lceil \varepsilon'/d \rceil, \infty)}(n)(e^{d\eta}-1)e^{\varepsilon'\eta} \sum_{k=\lceil \varepsilon'/d \rceil}^{n-1} e^{-kd\eta}\mathbb{P}(|\chi_{-k}| > n). \end{aligned} \tag{5.11}$$

We proceed to bound the tail probabilities of the variables χ_{-k}. For all $n, k \in \mathbb{N}$, by (A.2) and (A.3) below, we obtain

$$\mathbb{P}(|\chi_{-k}| > n + k) = \mathbb{P}(|\chi_0| > n) \le K' e^{-\delta\gamma n}, \quad \text{where} \quad K' = e^{\delta\gamma m^*} \max\{K_0, 1\}$$

and K_0 is defined in (A.3). Thus we find that, for $n > \lceil \varepsilon'/d \rceil$ and $m = n - \lceil \varepsilon'/d \rceil$, we have

$$\begin{aligned}
&\sum_{k=\lceil \varepsilon'/d \rceil}^{n-1} e^{-kd\eta} \mathbb{P}(|\chi_{-k}| > n) \\
&\le \sum_{k=\lceil \varepsilon'/d \rceil}^{n-1} e^{-kd\eta} K' e^{-\delta\gamma(n-k)} = K' e^{-n\delta\gamma} \sum_{k=\lceil \varepsilon'/d \rceil}^{n-1} e^{k(\delta\gamma - d\eta)} \\
&= K' e^{-m\delta\gamma} e^{-\lceil \varepsilon'/d \rceil d\eta} \left(\frac{e^{m(\delta\gamma - d\eta)} - 1}{e^{\delta\gamma - d\eta} - 1} \cdot \mathbb{1}_{\mathbb{R}\setminus\{\delta\gamma\}}(d\eta) + m \cdot \mathbb{1}_{\{\delta\gamma\}}(d\eta) \right) \\
&\le K' e^{-m \min\{\delta\gamma, d\eta\}} e^{-\lceil \varepsilon'/d \rceil d\eta} \left(\frac{\mathbb{1}_{\mathbb{R}\setminus\{\delta\gamma\}}(d\eta)}{|e^{\delta\gamma - d\eta} - 1|} + n \cdot \mathbb{1}_{\{\delta\gamma\}}(d\eta) \right).
\end{aligned}$$

Note that K defined in (5.9) equals $K = e^{d\eta} + (e^{d\eta} - 1)K' e^{\delta\gamma}$. Let $n' = n - \varepsilon'/d$. Using (5.11), $\varepsilon'/d + 1 > \lceil \varepsilon'/d \rceil \ge \varepsilon'/d$ and the inequality in the previous display, we get

$$\begin{aligned}
&\mathbb{P}(|N(\varepsilon)| > n) \\
&\le e^{-n'd\eta} e^{d\eta} + \mathbb{1}_{(\lceil \varepsilon'/d \rceil, \infty)}(n)(K - e^{d\eta}) e^{-n' \min\{\delta\gamma, d\eta\}} \left(\frac{\mathbb{1}_{\mathbb{R}\setminus\{\delta\gamma\}}(d\eta)}{|e^{\delta\gamma - d\eta} - 1|} + n \cdot \mathbb{1}_{\{\delta\gamma\}}(d\eta) \right).
\end{aligned}$$

Since $r\varepsilon < 1$ and $e^{n'd} = r\varepsilon e^{nd}$, the result follows by simplifying the previous display. □

Recall from Theorem 3.4 that $M_0 = Z_1^{\mathrm{me}}$. In applications, we often need to run the chain in Algorithm 3 until, for a given $x > 0$, we can detect which of the events $\{M_0 > x\}$ or $\{M_0 < x\}$ occurred (note that $\mathbb{P}(M_0 = x) = 0$ for all $x > 0$). This task is equivalent to simulating exactly the indicator $\mathbb{1}_{\{M_0 > x\}}$. We now analyse the tail of the running time of such a simulation algorithm.

Proposition 5.14. *For any $x > 0$, let $B(x)$ be the number of steps required to sample $\mathbb{1}_{\{M_0 > x\}}$. Let d, η, δ and γ be as in Proposition 5.13. Then $B(x)$ has exponential moments:*

$$\mathbb{P}(B(x) > n) \le K_0 [e^{-sn}(1 + n \cdot \mathbb{1}_{\{\delta\gamma\}}(d\eta))]^{1/(1+\eta)}, \quad \text{for all } n \in \mathbb{N}, \tag{5.12}$$

where $s = \min\{d\eta, \delta\gamma\}$ and $K_0 > 0$ do not depend on n.

Proof of Proposition 5.14. The inequality in (5.12) will follow from Lemma 5.11 once we identify the constants r_1, K_1, r_2, K_2 and the function $q : \mathbb{N} \to (0, \infty)$ that satisfy the inequalities in (5.6). By [DS10, Lem. 8], the distribution $\mathcal{S}^{\mathrm{me}}(\alpha, \rho)$ has a continuous density f_{me}, implying that the distribution function of $\mathcal{S}^{\mathrm{me}}(\alpha, \rho)$ is Lipschitz at x. Thus we may set $r_1 = 1$ and there exists some $K_1 > f_{\mathrm{me}}(x)$ such that the first inequality in (5.6) holds. Similarly, (5.10) and (5.9) in Proposition 5.13 imply that the second inequality in (5.6) holds if we set $r_2 = \eta$, $K_2 = K/(r^\eta |e^{\delta\gamma - d\eta} - 1|)$ and $q(n) = e^{-sn}(1 + n \cdot \mathbb{1}_{\{\delta\gamma\}}(d\eta))$, where $s = \min\{d\eta, \delta\gamma\}$. Thus, Lemma 5.11 implies (5.12) for $K_0 = K_1 + 2^{r_2} K_2$. □

Remark 5.15. We stress that the constant K_0 is not explicit since the constant K_1 in the proof above depends on the behaviour of the density f_{me} of Z_1^{me} in a neighbourhood of x. To the best of our knowledge even the value $f_{\mathrm{me}}(x)$ is currently not available in the literature.

5.3.2 Proof of Theorem 3.6

The computational complexity of Algorithm 3 is bounded above by a constant multiple of $|N(\varepsilon)| \log |N(\varepsilon)|$, cf. Remark 3.5(i) following Algorithm 3. By Proposition 5.13, its computational complexity has exponential moments. Since Algorithm 1 amounts to running Algorithm 3 twice, its computational complexity also has exponential moments. By Proposition 5.14, the running time of the exact simulation algorithm for the indicator $\mathbb{1}_{\{M_0>x\}}$ has exponential moments. It remains to analyse the runtime of Algorithm 2.

Recall that line 8 in Algorithm 2 requires sampling a beta random variable and two meanders at time 1 with laws Z_1^{me} and $(-Z)_1^{\mathrm{me}}$. The former (resp. latter) meander has the positivity parameter ρ (resp. $1-\rho$). Moreover, we may use Algorithm 3 (see also Remark 3.2) to obtain an εSS of Z_1^{me} and $(-Z)_1^{\mathrm{me}}$ by running backwards the dominating processes (defined in (A.4), see Appendix A) of the Markov chains in Theorem 3.4. Let (d,η,δ,γ) and $(d',\eta',\delta',\gamma')$ be the parameters required for the definition of the respective dominating processes, introduced at the beginning of Appendix A. The εSS algorithms invoked in line 8 of Algorithm 2 require $2m+1$ independent dominating processes to be simulated ($m+1$ of them with parameters (d,η,δ,γ) and m of them with parameters $(d',\eta',\delta',\gamma')$). Denote by $N_k(\varepsilon)$, $k\in\{1,\ldots,2m+1\}$ and $\varepsilon>0$, their respective termination times, defined as in (5.8).

Note that, in the applications of Algorithm 3, the sampled faces need not be sorted (see Remark 3.5), thus eliminating the logarithmic effect described in Remark 3.5(i). The cumulative complexity of executing i times the loop from line 3 to line 12 in Algorithm 2, producing an ε_i-strong sample of $(Z_{t_1},\ldots,Z_{t_m})$ conditioned on $\underline{Z}_{t_1}\geq 0$, only depends on the precision ε_i and not on the index i. Hence, the cumulative complexity is bounded by a constant multiple of $N^\Lambda(\varepsilon)=\sum_{k=1}^{2m+1}|N_k((t_{\lfloor k/2\rfloor+1}-t_{\lfloor k/2\rfloor})^{-1/\alpha}\varepsilon/(2m+1))|$, where we set $\varepsilon=\varepsilon_i$. Let B' denote the sum of the number of steps taken by the dominating processes until the condition in line 12 of Algorithm 2 is satisfied. We now prove that B' has exponential moments.

Note that $N^\Lambda(\varepsilon)\leq(2m+1)\max_{k=\{1,\ldots,2m+1\}}|N_k(T^{-1/\alpha}\varepsilon/(2m+1))|$. Moreover, for any n' independent random variables $\vartheta_1,\ldots,\vartheta_{n'}$, we have

$$\mathbb{P}\bigg(\max_{k\in\{1,\ldots,n'\}}\vartheta_k>x\bigg)=\mathbb{P}\bigg(\bigcup_{k=1}^{n'}\{\vartheta_k>x\}\bigg)\leq\sum_{k=1}^{n'}\mathbb{P}(\vartheta_k>x),\quad x\in\mathbb{R}.$$

Proposition 5.13 implies that the second inequality in (5.6) is satisfied by $N^\Lambda(\varepsilon)$ with $r_2=\max\{\eta,\eta'\}$, $q(n)=e^{-sn}n$ and some $K_2>0$, where $s=\min\{d\eta,\delta\gamma,d'\eta',\delta'\gamma'\}$. Thus, Lemma 5.11 gives $\mathbb{P}(B'>n)\leq K'(e^{-sn}n)^{1/(1+\max\{\eta,\eta'\})}$ for some $K'>0$ and all $n\in\mathbb{N}$.

The loop from line 2 to line 13 of Algorithm 2 executes lines 4 through 12 a geometric number of times R with success probability $p=\mathbb{P}(\underline{Z}_{t_m}\geq 0|\underline{Z}_{t_1}\geq 0)>0$. Hence, the running time B'' of Algorithm 2 can be expressed as $\sum_{i=1}^R B_i'$, where B_i' are iid with the same distribution as B', independent of R. Note that $m:\lambda\mapsto\mathbb{E}[e^{\lambda B'}]$ is finite for any $\lambda<s/(1+\max\{\eta,\eta'\})$. Since m is an analytic function and $m(0)=1$, then there exists some $x^*>0$ such that $m(x)<1/(1-p)$ for all $x\in(0,x^*)$. Hence, the moment generating function of B'' satisfies, $\mathbb{E}e^{\lambda B''}=\mathbb{E}m(\lambda)^R$, which is finite if $\lambda<x^*$, concluding the proof.

A Auxiliary processes and the construction of $\{D_n\}$

Fix constants d and δ satisfying $0<\delta<d<\frac{1}{\alpha\rho}$ and let $\eta=-\alpha\rho-\mathcal{W}_{-1}\big(-\alpha\rho de^{-\alpha\rho d}\big)/d$, where $\mathcal{W}_{-1}$ is the secondary branch of the Lambert W function [CGH+96] (η is only required in [GCMUB19, Alg. 2]). Let $I_k^n=\mathbb{1}_{\{S_k>e^{\delta(n-k-1)}\}}$ for all $n\in\mathcal{Z}^0$ and $k\in\mathcal{Z}^n$. Fix $\gamma>0$ with $\mathbb{E}[S^\gamma]<\infty$ (see [GCMUB19, App. A]), where $S\sim\mathcal{S}^+(\alpha,\rho)$. By Markov's inequality, we have

$$p(n)=\mathbb{P}(S\leq e^{\delta n})\geq 1-e^{-\delta\gamma n}\mathbb{E}[S^\gamma],\quad n\in\mathbb{N}\cup\{0\},\tag{A.1}$$

implying $\sum_{n=0}^\infty(1-p(n))<\infty$. Since the sequence $(S_k)_{k\in\mathcal{Z}^0}$ is iid with distribution $\mathcal{S}^+(\alpha,\rho)$ (as in Theorem 3.4), the Borel-Cantelli lemma ensures that, for a fixed $n\in\mathcal{Z}^0$, the events $\{S_k > e^{\delta(n-k-1)}\} = \{I_k^n = 1\}$ occur for only finitely many $k \in \mathcal{Z}^n$ a.s. For $n \in \mathcal{Z}^1$ let χ_n be the smallest time beyond which the indicators I_k^n are all zero:

$$\chi_n = \min\left\{n-1, \inf\left\{k \in \mathcal{Z}^n : I_k^n = 1\right\}\right\}, \tag{A.2}$$

with the convention $\inf\emptyset = -\infty$. Note that $-\infty < \chi_n < n$ a.s. for all $n \in \mathcal{Z}^0$. Since $\mathcal{Z}^0$ is countable, we have $-\infty < \chi_n < n$ for all $n \in \mathcal{Z}^0$ a.s. Let $m^* = \lfloor \frac{1}{\delta\gamma}\log\mathbb{E}[S^\gamma]\rfloor + 1$ and note that $e^{-\delta\gamma m}\mathbb{E}[S^\gamma] < 1$ for all $m \geq m^*$. Hence, for all $n \in \mathbb{N}\cup\{0\}$, the following inequality holds (cf. [GCMUB19, Sec. 4.1])

$$\mathbb{P}(|\chi_0| > n + m^*) \leq K_0 e^{-\delta\gamma n}, \quad \text{where} \quad K_0 = \frac{e^{-\delta\gamma m^*}\mathbb{E}[S^\gamma]}{(1-e^{-\delta\gamma})(1-e^{-\delta\gamma m^*}\mathbb{E}[S^\gamma])}. \tag{A.3}$$

Indeed, by the inequality (A.1), for every $m \geq m^*$ we have

$$\begin{aligned}
\mathbb{P}(|\chi_0| \leq m) &= \prod_{j=m}^{\infty} p(j) \geq \prod_{j=m}^{\infty}(1-e^{-\delta\gamma j}\mathbb{E}[S^\gamma]) = \exp\left(\sum_{j=m}^{\infty}\log(1-e^{-\delta\gamma j}\mathbb{E}[S^\gamma])\right) \\
&= \exp\left(-\sum_{j=m}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}e^{-\delta\gamma jk}\mathbb{E}[S^\gamma]^k\right) \geq \exp\left(-\sum_{k=1}^{\infty}\frac{e^{-\delta\gamma mk}\mathbb{E}[S^\gamma]^k}{1-e^{-\delta\gamma k}}\right) \\
&\geq \exp\left(-\frac{e^{-\delta\gamma m}\mathbb{E}[S^\gamma]}{(1-e^{-\delta\gamma})(1-e^{-\delta\gamma m}\mathbb{E}[S^\gamma])}\right) \\
&\geq \exp\left(-K_0 e^{-\delta\gamma(m-m^*)}\right) \geq 1 - K_0 e^{-\delta\gamma(m-m^*)}.
\end{aligned}$$

Define the iid sequence $(F_n)_{n\in\mathcal{Z}^0}$ by $F_n = d + \frac{1}{\alpha}\log(1-U_n)$. Note that $d - F_n$ is exponentially distributed with $\mathbb{E}[d-F_n] = \frac{1}{\alpha\rho}$. Let $(W_n)_{n\in\mathcal{Z}^1}$ be a random walk defined by $W_n = \sum_{k\in\mathcal{Z}_n^0} F_k$. Let $(R_n)_{n\in\mathcal{Z}^1}$ be reflected process of $(W_n)_{n\in\mathcal{Z}^1}$ from the infinite past

$$R_n = \max_{k\in\mathcal{Z}^{n+1}} W_k - W_n, \quad n \in \mathcal{Z}^1.$$

For any $n \in \mathcal{Z}^1$ define the following random variables

$$\begin{aligned}
D_n &= \exp(R_n)\left(\frac{e^{(\delta-d)(n-\chi_n)}}{1-e^{\delta-d}} + \sum_{k\in\mathcal{Z}_{\chi_n}^n} e^{-(n-k-1)d}U_k^{1/\alpha}S_k\right), \\
D_n' &= \exp(R_n)\left(\frac{1}{1-e^{\delta-d}} + D_n''\right), \quad \text{where} \quad D_n'' = \sum_{k\in\mathcal{Z}^n} e^{-(n-k-1)d}S_k.
\end{aligned} \tag{A.4}$$

Note that the series in D_n'' is absolutely convergent by the Borel-Cantelli lemma, but D_n' cannot be simulated directly as it depends on an infinite sum. In fact, as was proven in [GCMUB19, Sec. 4], it is possible to simulate $((\Theta_n, D_{n+1}))_{n\in\mathcal{Z}^0}$ backward in time.

Let $\mathcal{A} = (0,\infty)\times(0,1)$ put $\Theta_n = (S_n, U_n)$. Define the update function $\phi : (0,\infty)\times\mathcal{A} \to (0,\infty)$ given by $\phi(x,\theta) = (1-u)^{1/\alpha}x + u^{1/\alpha}s$ where $\theta = (s,u)$. By [GCMUB19, Lem. 2], $M_n \leq D_n \leq D_n'$ for $n \in \mathcal{Z}^1$ and that $((\Theta_n, R_n, D_{n+1}'))_{n\in\mathcal{Z}^0}$ is Markov, stationary, and φ-irreducible (see definition [MT09, p. 82]) with respect to its invariant distribution.

Hence, we may iterate (3.2) to obtain for $m \in \mathcal{Z}^1$ and $n \in \mathcal{Z}^m$,

$$\begin{aligned}
M_m &= \left(\prod_{k\in\mathcal{Z}_n^m}(1-U_k)^{1/\alpha}\right)M_n + \sum_{k\in\mathcal{Z}_n^m}\left(\prod_{j\in\mathcal{Z}_k^m}(1-U_j)^{1/\alpha}\right)U_k^{1/\alpha}S_k \\
&= \underbrace{\phi(\cdots\phi}_{m-n}(M_n,\Theta_n),\ldots,\Theta_{m-1}).
\end{aligned} \tag{A.5}$$

B On regularity

Lemma B.1. *Assume that X is a Lévy process generated by (b, σ^2, ν) and define the function $\overline{\sigma}^2(u) = \sigma^2 + \int_{-u}^{u} x^2 \nu(dx)$. If $\lim_{u \searrow 0} u^{-2} |\log u|^{-1} \overline{\sigma}^2(u) = \infty$, then (K) holds.*

Proof. (the proof is due to Kallenberg [Kal81]) Let $\psi(u) = \log \mathbb{E}[e^{iuX_1}]$ and note that for large enough $|u|$ and fixed $t > 0$ we have

$$-\log\left(\left|e^{t\psi(u)}\right|\right) = \frac{1}{2} t u^2 \sigma^2 + t \int_{-\infty}^{\infty} (1 - \cos(ux)) \nu(dx) \geq \frac{1}{3} t u^2 \overline{\sigma}^2\left(|u|^{-1}\right) \geq 2|\log|u||.$$

Hence, $|e^{t\psi(u)}| = |\mathbb{E}[e^{iuX_t}]| = \mathcal{O}(u^{-2})$ as $|u| \to \infty$, which yields (K). □

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